Canonical Quantization of Lattice Higgs-Maxwell-Chern-Simons Fields: Krein Selfadjointness*

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Abstract

It is shown how techniques from constructive quantum field theory may be applied to indefinite metric gauge theories in Hilbert space for the case of a Higgs-Maxwell-Chern-Simons theory on a lattice. The Hamiltonian operator is shown to be Krein essentially selfadjoint by means of unbounded but Krein unitary transformations relating the Hamiltonian to an essentially maximal accretive operator.

I Introduction

The mathematical study of Higgs gauge theories has been focussed mainly on gauge invariant quantities undertaken using euclidean functional integrals [1, 2, 3]. However in the theoretical physics literature, it is mostly the

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Higgs and gauge fields that are used directly [4, 5, 6, 7]. In this direction we consider a (2+1) dimensional spacetime Maxwell-Chern-Simons field minimally coupled to a charged scalar field in the indefinite metric framework proposed by Wightman and Gårding [8]. By using a representation of the canonically quantized Chern-Simons field in a Hilbert space with a Krein indefinite metric, we show the Chern-Simons field to be a Hilbert normal and Krein selfadjoint operator. Regularizing field operators with a finite periodic lattice allows construction of the Hamiltonian operator. Initially this operator is densely defined and closable but exhibits no immediate spectral properties which would suggest direct implementation of techniques from constructive quantum field theory in obtaining a euclidean path-space representation for a semigroup generated by this Hamiltonian. In this paper, we show the underlying structure of the Hamiltonian to be that of an accretive operator masked by unbounded Krein unitary 'gauge' transformations. This leads to an unbounded semigroup which unlike Fröhlich's theory of unbounded symmetric semigroups is not Hilbert selfadjoint [9]. In a second paper [10], we will obtain the corresponding euclidean theory and examine the relation between the physical states in the Wightman-Gårding framework and Osterwalder-Schrader positivity provided by this unbounded semigroup.

In section II, we describe our representation for the Maxwell-Chern-Simons field which differs from those in [11, 12] and establish regularity of the field that we use. It is convenient to represent these fields in terms of harmonic oscillator variables in section III and the appendix since the Krein 'gauge' transformations then become transparent as well as their properties as unbounded operators. In section IV, it is shown that the real part of the transformed Hamiltonian defines an essentially selfadjoint operator which is bounded below. The 'gauge' transformed Hamiltonian is then shown to be an accretive operator in section V. The bounds from section IV, allow us to prove a quadratic estimate for the transformed Hamiltonian by which it becomes a maximal accretive operator after closure and thus Krein essentially selfadjoint.

II Maxwell-Chern-Simons Fields

We work in $d = s + 1$ dimensional spacetime, where the number of spatial dimensions is $s = 2$. The metric is given by $g_{00} = +1$ and $g_{ij} = -\delta_{ij}$ with the remaining components vanishing. As usual, Latin indices are spatial, ranging from 1 to s, while Greek indices include the time component, 0. We write the inner product of d-vectors k and x alternately as

$$kx = k_\mu x^\mu = k^\mu x_\mu = k_0 x_0 - k \cdot x = k_0 x_0 - k_i x_i$$

where indices are raised or lowered by contraction with g and repeated indices are summed over their range. It should always be clear from context whether k and x refer to d-vectors or s-vectors. We use analogous notation for d-divergences, e.g., $\partial A \equiv \partial_\mu A^\mu$. Our units are such that $\hbar = c = 1$. Finally, we make use of the notation $\omega = \omega(k) = \sqrt{k \cdot k}$ and $\mu = \mu(k) = \sqrt{k \cdot k + m^2} = \sqrt{\omega^2 + m^2}$. Hopefully, our conventions are now clear.

The Maxwell-Chern-Simons (MCS) Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\mu + \frac{\lambda}{2}(\partial A)^2 + \frac{m}{2}\epsilon^{\mu\nu\rho}\partial_\mu A_\nu A_\rho. \tag{1}$$

The term $(\partial A)^2$ fixes the gauge with parameter λ

$$\lambda = \frac{2\xi}{2\xi - 1} = -\frac{\gamma}{1-\gamma} \tag{2}$$

interpolating between the various covariant gauges. Here, $\gamma = \lambda = \xi = 0$ corresponds to the Feynman gauge while $\gamma = 1$, $\lambda \to \infty$, $\xi = 1/2$ all correspond to Landau gauge. The parameters ξ and γ are more convenient than λ for our representation below. The parameter γ is introduced for comparison with other works that start with a different Lagrangian such as [11] while ξ is introduced for comparison with works such as [12, 13].

The Euler-Lagrange equations corresponding to equation (1) are

$$\Box A^\mu - \lambda \partial^\mu(\partial A) + m\epsilon^{\mu\alpha\beta}\partial_\alpha A_\beta = 0. \tag{3}$$

The homogeneous Green's function for (3) can be written

$$\begin{aligned}
&\langle A_\mu(x) A_\nu(y)\rangle = \\
&\quad \tfrac{1}{(2\pi)^2}\int d\Omega_m(k) e^{-ik(x-y)}\left[\tfrac{k_\mu k_\nu}{m^2} - g_{\mu\nu} - \tfrac{i}{m}\epsilon_{\mu\nu\rho}k^\rho\right] \\
&\quad + \tfrac{1}{(2\pi)^2}\int d\Omega_0(k) e^{-ik(x-y)}\left[-\tfrac{k_\mu k_\nu}{m^2} + \tfrac{i}{m}\epsilon_{\mu\nu\rho}k^\rho\right] \\
&\quad - \tfrac{1-2\xi}{(2\pi)^2}\int d\Omega_0(k) e^{-ik(x-y)}\tfrac{1}{2\omega}\left[\delta_{\mu 0}k_\nu + \delta_{\nu 0}k_\mu - i(x_0 - y_0)k_\mu k_\nu - \tfrac{k_\mu k_\nu}{\omega}\right]
\end{aligned} \tag{4}$$

which has support both on the lightcone and the mass m hyperboloid. The canonical momenta associated to (1) are

$$\pi^0 \equiv \frac{\delta\mathcal{L}}{\delta(\partial_0 A_0)} = -\partial_0 A^0 + \lambda\partial A \tag{5}$$

$$\pi^j \equiv \frac{\delta\mathcal{L}}{\delta(\partial_0 A_j)} = -\partial_0 A^j + \frac{m}{2}\epsilon^{0jn}A_n \tag{6}$$

It is convenient to express (3) in terms of differential forms. For this, we define a 1-form $A \equiv A_\mu dx^\mu$. We let $*$ denote the Hodge duality operator associated with our metric g and the orientation given by the volume form $dx^0 \wedge dx^1 \wedge dx^2$, so that $** = +1$. Furthermore, we define the codifferential δ in the usual way on k-forms by

$$\delta = -\det(g)(-1)^{d(k+1)} * d* = -(-1)^{3(k+1)} * d* = (-1)^k * d* \tag{7}$$

so that, for a k-form β, we have

$$\begin{aligned}\delta\beta &= \delta\left(\frac{1}{k!}\beta_{\mu_1\cdots\mu_k}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k}\right)\\ &= -\frac{1}{(k-1)!}\partial^{\mu_1}\beta_{\mu_1\cdots\mu_k}dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_k}\end{aligned}$$

Then, we find that $\Box \equiv \partial^\mu\partial_\mu$ is given by

$$\Box = -(\delta d + d\delta) \tag{8}$$

in analogy to the Laplace-Beltrami operator in the Riemannian case. With this notation, we may write (3) as

$$\delta dA - m * dA = (\lambda - 1)d\delta A \tag{9}$$

From this standpoint, it becomes easy to prove the following decomposition theorems which we shall utilize in the process of canonical quantization.

Theorem II.1 (MCS Decomposition) *Let A be a 1-form satisfying (9). Then, there exist 1-forms V and S satisfying the equations*

$$dV = m * V \qquad *dS = -\frac{(\lambda-1)}{m}d\delta S \tag{10}$$

such that $A = V + S$.

Conversely, given 1-forms V and S satisfying (10), the 1-form $A = V + S$ satisfies (9).

Proof:
For the first statement, define

$$V \equiv \frac{1}{m} * dA + \frac{1}{m^2}(\lambda - 1) d\delta A$$

so that

$$\begin{aligned} *dV &= \frac{1}{m} * d * dA = \frac{1}{m} \delta dA \\ &= \frac{1}{m} \left(m * dA + (\lambda - 1) d\delta A \right) \\ &= mV \end{aligned}$$

where we have used (9). Now, define the 1-form $S \equiv A - V$. This satisfies

$$\begin{aligned} *dS &= *dA - *dV = *dA - mV \\ &= *dA - \left(*dA + \frac{\lambda - 1}{m} d\delta A \right) \\ &= -\frac{\lambda - 1}{m} d\delta A \end{aligned}$$

Then, the result (10) follows by noting that

$$\delta A = \delta S + \delta V = \delta S + \delta \left(\frac{*dV}{m} \right) = \delta S$$

For the converse, we note that equations (10) imply

$$\begin{aligned} \delta dV = (-1)^2 * d * dV = + * d(mV) = m^2 V \\ \delta V = (-1)^1 * d * V = - * d * \left(\frac{*dV}{m} \right) = 0 \\ \delta dS = (-1)^2 * d * dS = + * d \left(-\frac{\lambda - 1}{m} d\delta S \right) = 0 \end{aligned}$$

Then, we simply substitute $A = V + S$ into equation (9) to find

$$\begin{aligned} \text{LHS} &= \delta d(V + S) - m * d(V + S) \\ &= m^2 V - m * dV - m * dS \\ &= -m \left(-\frac{\lambda - 1}{m} d\delta S \right) \\ &= (\lambda - 1) d\delta S \\ \text{RHS} &= (\lambda - 1) d\delta (V + S) = (\lambda - 1) d\delta S \end{aligned}$$

and note that RHS = LHS. Q.E.D.

Notice that the gauge dependence, as parameterized by λ, has been separated into the field S in equations (10). For this reason, we often refer to V as the physical MCS field. We can say more about V and S. It has long been known that there are certain necessary conditions that a valid decomposition of a solution of (3) into the sum of a massive and a massless vector field must satisfy[14, 15]. These conditions follow easily from our theorem, so we state them as a corollary.

Corollary II.2 *The vector field V_μ corresponding to the 1-form V satisfies*

$$\begin{aligned} (\Box + m^2)V_\mu &= 0 \\ \partial V &= 0 \end{aligned}$$

which are the Proca field equations. The vector field S_μ corresponding to the 1-form S satisfies

$$\begin{aligned} \Box S_\mu &= \partial_\mu(\partial S) \\ \Box(\partial S) &= 0, \end{aligned}$$

provided $\lambda \neq 1$.

One should note that, although V_μ satisfies the Proca field equations, it is not a true Proca field since it also satisfies the additional constraint (10). Thus, one expects the quantized version of V_μ to propagate one less physical mode than a $2+1$- dimensional Proca field. We shall see that this is indeed the case. In fact, even at the classical level, we have the following result.

Proposition II.3 (Proca Decomposition) *Let U be a 1-form satisfying the Klein-Gordon equation*

$$(\Box + m^2)U = 0$$

for $m > 0$. Then, it is both necessary and sufficient that U satisfy the decomposition

$$U = V^+ + V^- + d\phi \tag{11}$$

where $V^\pm$ satisfy $*dV^\pm = \pm mV^\pm$ and ϕ is a scalar solution to the Klein-Gordon equation. If U is additionally a Proca field (i.e., $\delta U = 0$), then we may choose $\phi \equiv 0$.

Proof:
For necessity, define

$$V^{\pm} \equiv -\frac{1}{2}\left(\frac{1}{m^2}d\delta U \mp \frac{1}{m} * dU - U\right) \quad \text{and} \quad \phi \equiv \frac{1}{m^2}\delta U \tag{12}$$

The remainder then follows by straightforward calculations. Q.E.D.

The well-known bosonization of the MCS field exhibiting its single, massive, physical degree of freedom (see, e.g., [4]) also follows easily from theorem II.1. Thus, we state it as a corollary as well.

Corollary II.4 *A vector field satisfying*

$$\epsilon_{\mu\nu\rho}\partial^{\nu}V^{\rho} = mV_{\mu} \tag{13}$$

(i.e., $*dV = mV$ as per (10)) must necessarily be of the form

$$\begin{aligned} V_0 &= \frac{\sqrt{-\triangle}}{m}\phi \\ V_j &= -\left[\frac{\partial_0\partial_j}{m\sqrt{-\triangle}} - \frac{\epsilon_{jn}\partial_n}{\sqrt{-\triangle}}\right]\phi \end{aligned}$$

for some scalar field ϕ

Proof:
Taking the time derivative of (13) for $\mu = j$, we find

$$\begin{aligned} m\partial_0 V_j &= -\epsilon_{jn}\partial_n\partial_0 V_0 + \epsilon_{jn}\partial_0\partial_0 V_n \\ &= -\epsilon_{jn}\partial_n\partial_0 V_0 + \epsilon_{jn}(\triangle - m^2)V_n \end{aligned}$$

and noting that

$$m^2 V_n = m\epsilon_{nk}(\partial_0 V_k - \partial_k V_0)$$

we obtain the necessary condition

$$\triangle V_j = m\epsilon_{jn}\partial_n V_0 - \partial_0\partial_j V_0$$

from which the result follows by defining

$$\phi \equiv \frac{m}{\sqrt{-\triangle}}V_0$$

Q.E.D.

We proceed now to the quantization. Using the intuition gained from theorem II.1, we seek a massive vector field $V_\mu(x)$ satisfying

$$(\Omega_R, V_\mu(x)V_\nu(y)\Omega_R)_R = \frac{1}{(2\pi)^2}\int d\Omega_m(k)\left[\frac{k_\mu k_\nu}{m^2} - g_{\mu\nu} - i\frac{\epsilon_{\mu\nu\lambda}k^\lambda}{m}\right] e^{-ik(x-y)} \tag{14}$$

and a commuting massless vector field $S_\mu(x)$ satisfying

$$\begin{aligned}(\Omega_R, S_\mu(x)S_\nu(y)\Omega_R)_R = {}& \\ &\frac{1}{(2\pi)^2}\int d\Omega_0(k)e^{-ik(x-y)}\left[-\frac{k_\mu k_\nu}{m^2} + \frac{i}{m}\epsilon_{\mu\nu\rho}k^\rho\right] \\ -\frac{1-2\xi}{(2\pi)^2}\int d\Omega_0(k)e^{-ik(x-y)}\frac{1}{2\omega}&\left[\delta_{\mu 0}k_\nu + \delta_{\nu 0}k_\mu - i(x_0-y_0)k_\mu k_\nu - \frac{k_\mu k_\nu}{\omega}\right]\end{aligned} \tag{15}$$

so that

$$(\Omega_R, V_\mu(x)V_\nu(y)\Omega_R)_R + (\Omega_R, S_\mu(x)S_\nu(y)\Omega_R)_R = \langle A_\mu(x)A_\nu(y)\rangle \tag{16}$$

given in equation (4). First, we consider the massive field $V_\mu(x)$.

Guided by proposition II.3, we define

$$V_\mu(x) = \frac{1}{\sqrt{2}}\left(-\frac{\partial_\mu\partial_\alpha}{m^2} - g_{\mu\alpha} + \frac{1}{m}\epsilon_{\mu\alpha\beta}\partial^\beta\right)U^\alpha(x) \tag{17}$$

$$\begin{aligned} &= \frac{1}{2\pi\sqrt{2}}\int d\Omega_m(k)\left[b^\alpha(k)\left(\frac{k_\alpha k_\mu}{m^2} - g_{\mu\alpha} - \frac{i}{m}\epsilon_{\alpha\mu\beta}k^\beta\right)e^{-ikx}\right. \\ &\left. + b^{\dagger\,\alpha}(k)\left(\frac{k_\alpha k_\mu}{m^2} - g_{\mu\alpha} + \frac{i}{m}\epsilon_{\alpha\mu\beta}k^\beta\right)e^{ikx}\right]\end{aligned} \tag{18}$$

where U is a massive vector field given by

$$U_\mu(x) = \frac{1}{2\pi}\int d\Omega_m(k)[e^{-ikx}b_\mu(k) + e^{ikx}b^\dagger_\mu(k)] \tag{19}$$

with $d\Omega_m(k)$ denoting the invariant measure on the positive mass m hyperboloid and b, $b^\dagger$ satisfying the commutation relations

$$\left[b_\mu(k), b^\dagger_\nu(k')\right] = -g_{\mu\nu}\, 2\mu(k)\delta(k-k') \tag{20}$$

This is clearly a covariant field that, by construction, satisfies the field equations $\epsilon_{\mu\nu\rho}\partial^\nu V^\rho(x) = mV_\mu(x)$. Furthermore, one checks that (14) is satisfied under the usual assumption $b_\mu(k)\Omega_R = 0$.

In fact, all is made rigorous by realizing the expressions as bilinear forms on the Proca Fock space $\mathcal{H}_R^P$ given by

$$\mathcal{H}_R^P \equiv \Gamma_{\text{sym}}(L^2(R^d, d\Omega_m(k)) \otimes C^d),$$

i.e., the symmetric Fock space over $\mathcal{H}_R^{P(1)} \equiv L^2(R^d, d\Omega_m(k)) \otimes C^d$. Given $\Phi \in \mathcal{H}_R^P$, we denote its projection onto the n-particle subspace $\mathcal{H}_R^{P(n)}$ by $\Phi^{(n)}$. With this notation, the inner product on $\mathcal{H}_R^P$ is given by

$$(\Phi, \Psi)_R = \sum_{n=0}^{\infty} (\Phi^{(n)}, \Psi^{(n)})_R^{(n)}$$

with

$$\begin{aligned}(f, g)_R^{(n)} &= \\ &\prod_{j=1}^{n} \left[\int d\Omega_m(k_j)\right] \overline{f_{\mu_1\cdots\mu_n}(k_1, \cdots, k_n)} g_{\mu_1\cdots\mu_n}(k_1, \cdots, k_n)\end{aligned} \tag{21}$$

for $f, g \in \mathcal{H}_R^{P(n)}$ with $n > 0$ and $(f, g)_R^{(0)} = \overline{f}g$ for $f, g \in \mathcal{H}_R^{P(0)}$. We denote adjoints with respect to $(\cdot, \cdot)_R$ by a superscript $*$ and refer to them as Hilbert adjoints.

We introduce the usual Fock creation and annihilation bilinear forms on $\mathcal{H}_R^P$ by

$$\begin{aligned}[c_\mu^*(k)\Phi]_{\mu_1\cdots\mu_n}^{(n)}(k_1, \cdots, k_n) &= \\ &\tfrac{1}{\sqrt{n}} \textstyle\sum_{j=1}^{n} \delta_{\mu\mu_j} \delta(k - k_j) 2\mu(k_j) \Phi_{\mu_1\cdots\widehat{\mu_j}\cdots\mu_n}^{(n-1)}(k_1, \cdots, \widehat{k_j}, \cdots, k_n)\end{aligned} \tag{22}$$

$$\begin{aligned}[c_\mu(k)\Phi]_{\mu_1\cdots\mu_n}^{(n)}(k_1, \cdots, k_n) &= \\ &\sqrt{n+1}\Phi_{\mu\mu_1\cdots\mu_n}^{(n+1)}(k, k_1, \cdots, k_n)\end{aligned} \tag{23}$$

where a hat over a variable or an index means to omit it. These are well-defined (as bilinear forms) on vectors $\Phi \in D(N^{1/2}) \subset \mathcal{H}_R$ where N is the selfadjoint number operator. In fact, $c_\mu(k)$ defines an operator on this domain[16]. These formally satisfy the canonical commutation relations

$$[c_\mu(k), c_\nu^*(k')] = \delta_{\mu\nu} \; 2\mu(k) \; \delta(k - k'). \tag{24}$$

We also need to introduce the indefinite metric on $\mathcal{H}_R^P$ which is given by $\{\Phi,\Psi\}_R = (\Phi,\eta\Psi)_R$. The action of η on the n-particle subspace is given by

$$(f,\eta g)_R^{(n)} = \prod_{j=1}^{n}\left[\int d\Omega_m(k_j)\ (-g)_{\mu_j\nu_j}\right]\overline{f_{\mu_1\cdots\mu_n}(k_1,\cdots,k_n)}g_{\nu_1\cdots\nu_n}(k_1,\cdots,k_n) \tag{25}$$

which in turn determines its action on $\mathcal{H}_R^P$. In short, η is the second quantization of the operator

$$f_\mu(k)\mapsto(-g_{\mu\nu}f_\nu)(k)$$

on $\mathcal{H}_R^{P(1)}$ [16, 17]. With respect to the indefinite inner product $\{\cdot,\cdot\}_R$, $\mathcal{H}_R^P$ is a Krein space [18]. This follows from the observations that $\eta=\eta^*$ and $\eta^2=1$. We denote adjoints with respect to the Krein metric by a superscript $\dagger$. Clearly, it is also true that $\eta=\eta^\dagger$.

Finally, we can define b, $b^\dagger$ as Krein creation and annihilation forms:

$$b^\dagger_\mu(k) = g_{\mu\nu}c^*_\nu(k) \qquad b_\mu(k) = -c_\mu(k) \tag{26}$$

These, then, formally satisfy equation (20). Furthermore, as bilinear forms, we have the relations

$$\begin{aligned} N &= \int d\Omega_m(k)c^*_\mu(k)c_\mu(k) \\ &= \int d\Omega_m(k)b^\dagger_\mu(k)(-g_{\mu\nu})b_\nu(k) \end{aligned}$$

Calculating the commutator, we obtain

$$[V_\mu(x),V_\nu(y)] = -i\left(\partial_\mu\partial_\nu + g_{\mu\nu} - \frac{\epsilon_{\mu\nu\lambda}\partial^\lambda}{m}\right)\Delta_m(x-y) \tag{27}$$

Note that this is not a local expression, despite the facts that the Pauli-Jordan function $\Delta_m(x)$ given by

$$\Delta_m(x) = \frac{-i}{(2\pi)^s}\int d^dk\,\mathrm{sgn}\,(k_0)\,\delta(k^2-m^2)e^{-ikx}. \tag{28}$$

vanishes for spacelike separations and differential operators are, in general, local. This subtlety arises from the fact that a time derivative of $\Delta_m(x)$

evaluated at time zero yields a delta function (see, e.g., appendix I of [19]). Thus, we have the following non-vanishing equal-time commutators:

$$[V_0(t,x), V_n(t,y)] = \frac{i\partial_n}{m^2}\delta(x-y) \tag{29}$$

$$[V_j(t,x), V_n(t,y)] = -i\epsilon_{jn}\delta(x-y) \tag{30}$$

This is analogous to the situation that arises in canonical quantization of the Proca field without use of an indefinite metric[19].

In order to exhibit the bosonization (and, hence, the single physical mode of [4, 12]) within this framework, we note that (18) can be written

$$V_\mu(x) = \frac{1}{2\pi\sqrt{2}}\int d\Omega_m(k)\left[(Lgb)_\mu(k)e^{-ikx} + (\bar{L}gb^\dagger)_\mu(k)e^{ikx}\right] \tag{31}$$

where

$$L_{\mu\nu} \equiv \frac{k_\mu k_\nu}{m^2} - g_{\mu\nu} - \frac{i}{m}\epsilon_{\mu\nu\lambda}k^\lambda \tag{32}$$

$$= \left(\frac{\sqrt{\mu^2+\omega^2}}{m}\hat{w}_3 - \hat{w}_2\right)\otimes\left(\frac{\sqrt{\mu^2+\omega^2}}{m}\hat{w}_3 - \hat{w}_2\right)^* \tag{33}$$

using the basis

$$\begin{aligned}
\hat{w}_1 &= \frac{1}{\sqrt{\mu^2+\omega^2}}\begin{pmatrix}\mu\\ -k_1\\ -k_2\end{pmatrix}\\
\hat{w}_2 &= \frac{i}{\omega}\begin{pmatrix}0\\ k_2\\ -k_1\end{pmatrix}\\
\hat{w}_3 &= \frac{1}{\omega\sqrt{\mu^2+\omega^2}}\begin{pmatrix}\omega^2\\ \mu k_1\\ \mu k_2\end{pmatrix}
\end{aligned} \tag{34}$$

which is orthonormal in the C^3 Euclidean inner product.

Then, we can define

$$a(k) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\mu^2+\omega^2}}{m}\hat{w}_3 - \hat{w}_2\right)^*_\alpha g_{\alpha\beta}b_\beta(k) \tag{35}$$

$$a^\dagger(k) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\mu^2+\omega^2}}{m}\hat{w}_3 + \hat{w}_2\right)^*_\alpha g_{\alpha\beta}b^\dagger_\beta(k) \tag{36}$$

where we've used $\bar{\hat{w}}_2 = -\hat{w}_2$. Hence,

$$\left[a(k), a^\dagger(k')\right] = 2\mu\delta(k-k') \tag{37}$$

and suppressing the vector index we have the representation

$$\begin{aligned} V(x) &= \frac{1}{2\pi}\int d\Omega_m(k)\left[\left(\frac{\sqrt{\omega^2+\mu^2}}{m}\hat{w}_3 - \hat{w}_2\right)a(k)e^{-ikx}\right. \\ &\quad \left. + \left(\frac{\sqrt{\omega^2+\mu^2}}{m}\hat{w}_3 + \hat{w}_2\right)a^\dagger(k)e^{ikx}\right] \end{aligned} \tag{38}$$

for which the time evolution is implemented by

$$H_0^V = \int d\Omega_m(k)\mu(k)a^\dagger(k)a(k) \tag{39}$$

From this expression for V_μ, it is clear that the physical field is both Krein and Hilbert symmetric. It also has a dense set of analytic vectors, thereby making its closure selfadjoint.

Next, we turn to the ghost field S_μ. We wish to realize S_μ on the Maxwell Fock space $\mathcal{H}_R^M$ given as the symmetric Fock space over the single particle space

$$\mathcal{H}_R^{M(1)} \equiv L^2(R^d, d\Omega_0)\otimes C^d \tag{40}$$

where $d\Omega_0$ is the invariant measure on the forward light cone. As before, we equip $\mathcal{H}_R^M$ with an indefinite metric $\{\cdot,\cdot\}_R \equiv (\cdot, \eta\cdot)_R$ where $\eta = -g$ on $\mathcal{H}_R^{M(1)}$. Then, we will have a realization of the Maxwell-Chern-Simons field $A_\mu \equiv V_\mu\otimes 1 + 1\otimes S_\mu \equiv V_\mu + S_\mu$ on $\mathcal{H}_R = \mathcal{H}_R^P\otimes\mathcal{H}_R^M$. For this, we first choose the following C^3-orthonormal vectors

$$\hat{v}_1 = \frac{1}{\omega\sqrt{2}}\begin{pmatrix}\omega \\ k_1 \\ k_2\end{pmatrix} \tag{41}$$

$$\hat{v}_2 = \frac{i}{\omega}\begin{pmatrix}0 \\ k_2 \\ -k_1\end{pmatrix} \tag{42}$$

$$\hat{v}_3 = \frac{1}{\omega\sqrt{2}}\begin{pmatrix}\omega \\ -k_1 \\ -k_2\end{pmatrix} \tag{43}$$

Next, we define bilinear forms

$$a_R(k) = (\hat{v}_3)_\alpha g^{\alpha\beta} a_\beta(k) \tag{44}$$

$$a_Q(k) = (\hat{v}_1)_\alpha g^{\alpha\beta} a_\beta(k) \tag{45}$$

$$a_Q^\dagger(k) = (\hat{v}_1)_\alpha g^{\alpha\beta} a_\beta^\dagger(k) \tag{46}$$

$$a_R^\dagger(k) = (\hat{v}_3)_\alpha g^{\alpha\beta} a_\beta^\dagger(k) \tag{47}$$

in terms of the Maxwell Krein forms a, $a^\dagger$ satisfying

$$\left[a_\mu(k), a_\nu^\dagger(k')\right] = -g_{\mu\nu} 2\omega(k)\delta(k-k'). \tag{48}$$

These differ slightly from analogous forms given in [11, 12], but are more convenient for our purposes. They satisfy the algebra

$$\left[a_Q(k), a_R^\dagger(k')\right] = -2\omega\delta(k-k') = \left[a_R(k), a_Q^\dagger(k')\right] \tag{49}$$

with the remaining commutators vanishing.

Then, we can define the field

$$\begin{aligned} S(x;\gamma) &= \frac{1}{2\pi}\int d\Omega_0(k)\left\{e^{-ikx}\left[\frac{\omega}{m}\hat{v}_1 a_R(k)\right.\right. \\ &+ \left.\left(\frac{\omega}{m}\hat{v}_1 - \sqrt{2}\hat{v}_2 + (1-\gamma)\left(\frac{m}{2\omega}\hat{v}_3 - ix_0 m\hat{v}_1\right)\right) a_Q(k)\right] \\ &+ e^{ikx}\left[\frac{\omega}{m}\hat{v}_1^* a_R^\dagger(k) + \left(\frac{\omega}{m}\hat{v}_1^* - \sqrt{2}\hat{v}_2^*\right.\right. \\ &+ \left.\left.\left.(1-\gamma)\left(\frac{m}{2\omega}\hat{v}_3^* + ix_0 m\hat{v}_1^*\right)\right) a_Q^\dagger(k)\right]\right\} \end{aligned} \tag{50}$$

where we have again suppressed the vector index. Clearly, this expression simplifies considerably in Landau gauge $\gamma = 1$. One can check that (50) yields the correct two-point expression (15) on $\mathcal{H}_R^M$.

The time evolution of S_μ is implemented by the Hamiltonian

$$\begin{aligned} &H_0^S(\gamma) = \\ &\int d\Omega_0(k)\omega\left[-a_Q^\dagger(k)a_R(k) - a_Q^\dagger(k)a_R(k) - (1-\gamma)\tfrac{m^2}{\omega^2}a_Q^\dagger(k)a_Q(k)\right] \\ &\qquad = \int d\Omega_0(k)\omega a_\mu^\dagger(k)\left(-g^{\mu\nu} - (1-\gamma)\tfrac{m^2 k^\mu k^\nu}{2\omega^4}\right)a_\nu(k) \\ &\qquad = \int d\Omega_0(k)\omega c_\mu^*(k)\left(\delta_\mu^\nu + (1-\gamma)\tfrac{m^2 k_\mu k^\nu}{2\omega^4}\right)c_\nu(k) \end{aligned} \tag{51}$$

It is Krein symmetric, but not Hilbert symmetric unless one chooses Landau gauge $\gamma = 1$. However, standard arguments show its closure to be Hilbert normal and Krein selfadjoint. By techniques similar to the Maxwell case [13], we find

Theorem II.5

1. *$[0,\infty) \subset \sigma(H_0^S)$*

2. *H_0^S is accretive*

We note that the field S_μ is also non-local. We have, in fact, the non-vanishing equal-time commutation relations

$$[S_0(t,x), S_n(t,y)] = \frac{-i}{m^2}\partial_n\delta(x-y) \tag{52}$$

$$[S_j(t,x), S_n(t,y)] = \frac{i}{m}\epsilon_{jn}\delta(x-y) \tag{53}$$

However, the full MCS field $A_\mu = V_\mu + S_\mu$ is local since

$$[A_0(t,x), A_n(t,y)] = [V_0(t,x), V_n(t,y)] + [S_0(t,x), S_n(t,y)] = 0 \tag{54}$$

$$[A_j(t,x), A_n(t,y)] = [V_j(t,x), V_n(t,y)] + [S_j(t,x), S_n(t,y)] = 0 \tag{55}$$

where we have used equations (29) and (30).

We regularize the above fields by restricting them to a finite spatial volume $V \subset (\delta Z)^s$ with periodic boundary conditions (thus, V is a product of finite cyclic groups). We denote the volume of V by $|V|$. We denote the dual group of V by Γ. The measure on V is taken to be

$$\frac{1}{\sqrt{|V|}}\sum_{x\in V}\delta^s \tag{56}$$

while that on Γ is

$$\frac{1}{\sqrt{|V|}}\sum_{k\in\Gamma} \tag{57}$$

with Fourier transform

$$\tilde{f}(k) = \frac{1}{\sqrt{|V|}}\sum_{x\in V}\delta^s e^{-ik\cdot x}f(x) \tag{58}$$

We choose to use midpoint derivatives on the lattice so that

$$k_n = \frac{2}{\delta}\sin\left(\frac{k_n\delta}{2}\right) \tag{59}$$

and

$$\omega^2 = k_n k_n = \sum_{n=1}^{s} \frac{4}{\delta^2} \sin^2\left(\frac{k_n \delta}{2}\right) \tag{60}$$

With these conventions, the regularized MCS field becomes

$$\begin{aligned}
A(x) &= \frac{1}{\sqrt{|V|}} \sum_{k\in\Gamma_0} \frac{1}{2\mu} \left[\left(\frac{\sqrt{\omega^2+\mu^2}}{m} \hat{w}_3 - \hat{w}_2 \right) a(k) e^{-ikx} \right. \\
&+ \left. \left(\frac{\sqrt{\omega^2+\mu^2}}{m} \hat{w}_3 + \hat{w}_2 \right) a^\dagger(k) e^{ikx} \right]_{k_0=\mu} \\
&+ \frac{1}{\sqrt{|V|}} \sum_{k\in\Gamma_0} \frac{1}{2\omega} \left\{ e^{-ikx} \left[\frac{\omega}{m} \hat{v}_1 a_R(k) \right. \right. \\
&+ \left. \left(\frac{\omega}{m} \hat{v}_1 - \sqrt{2}\hat{v}_2 + (1-\gamma) \left(\frac{m}{2\omega} \hat{v}_3 - i x_0 m \hat{v}_1 \right) \right) a_Q(k) \right] \\
&+ e^{ikx} \left[\frac{\omega}{m} \hat{v}_1^* a_R^\dagger(k) + \left(\frac{\omega}{m} \hat{v}_1^* - \sqrt{2}\hat{v}_2^* \right. \right. \\
&+ \left. \left. \left. (1-\gamma) \left(\frac{m}{2\omega} \hat{v}_3^* + i x_0 m \hat{v}_1^* \right) \right) a_Q^\dagger(k) \right]_{k_0=\omega} \right\}
\end{aligned} \tag{61}$$

in which our infrared regularization is to sum over $\Gamma_0 \equiv \Gamma \backslash \{0\}$. This field is defined as a distribution-valued unbounded operator on the lattice Fock space $\mathcal{H}_R = \mathcal{H}_R^P \otimes \mathcal{H}_R^M$. Here, $\mathcal{H}_R^P$ is the symmetric Fock space over $L^2(\Gamma_0) \otimes C^d$ with measure

$$\frac{1}{\sqrt{|V|}} \sum_{k\in\Gamma_0} \frac{1}{2\mu} \tag{62}$$

while $\mathcal{H}_R^M$ is the symmetric Fock space over $L^2(\Gamma_0) \otimes C^d$ with measure

$$\frac{1}{\sqrt{|V|}} \sum_{k\in\Gamma_0} \frac{1}{2\omega} \tag{63}$$

In the following sections, we shall loosely use Γ to indicate whichever momentum sums are appropriate.

III Krein Gauge Transformations

The formal Lagrangian density for the MCS-Higgs model will be

$$\mathcal{L} = \mathcal{L}^{\,\mathrm{cs}} + \mathcal{L}^{\,\mathrm{boson}} \tag{64}$$

where $\mathcal{L}^{\text{cs}}$ is given by equation (1) and the boson term is

$$\mathcal{L}^{\text{boson}} = \sum_{j=1}^{2} \left[D_\mu(\phi_j) D^\mu(\phi_j) - m_0^2 (\phi_j)^2 \right] - V(\phi_1(x), \phi_2(x)) \tag{65}$$

for two neutral scalar fields $\phi_j, j = 1, 2$ with a potential $V(\phi_1(x), \phi_2(x)) = \lambda_4(\phi_1^2(x) + \phi_2^2(x))^2 + \lambda_2(\phi_1^2(x) + \phi_2^2(x))$ with $m_0 \geq 0, \lambda_4 > 0$ and covariant derivative $D_\mu(\phi) = (\partial_\mu + ieA_\mu)(\phi)$. The usual canonical construction produces a Hamiltonian operator on the lattice of the form

$$H(V, \delta) = H_0{}^{\text{cs}} + H_0{}^{\text{boson}} + H_{\,int} + V(\phi_1, \phi_2) \tag{66}$$

with the free MCS-Hamiltonian in equations (39) and (51) and a free boson-Hamiltonian with mass m_0. The interaction terms contain the MCS field as

$$\begin{aligned}
H_{\text{int}} &= H_{\text{el}} + H_{\text{mag}} \\
H_{\text{el}} &= \sum_{x \in V} \delta^s \left[e \left\{ \pi_1(x)\phi_2(x) - \pi_2(x)\phi_1(x) \right\} A_0(x) \right] \\
H_{\text{mag}} &= \sum_{x \in V} \delta^s \left[e \left\{ \partial_\ell \phi_2(x)\phi_1(x) - \partial_\ell \phi_1(x)\phi_2(x) \right\} A_\ell(x) \right. \\
&+ \left. \frac{e^2}{2} \left\{ (\phi_1^2(x) + \phi_2^2(x) \right\} A_\ell^2(x) \right] \\
V(\phi_1, \phi_2) &= \sum_{x \in V} \delta^s V(\phi_1(x), \phi_2(x))
\end{aligned} \tag{67}$$

Since in our representation the MCS-field is a normal operator, the 'magnetic' part of the Hamiltonian has lost the positivity which would normally arise from the spatial covariant derivatives while the 'electric' part has both real and imaginary terms in its numerical range. Usually, to maintain positivity of the Hamiltonian operator for a gauge theory of this type, quantization would be carried out in either the Coulomb or axial gauges. However, such gauges are not particularly amenable for renormalization issues arising when taking the continuum limit of the lattice theory. In order to understand the structure of $H(V, \delta)$ above and make transparent the Krein transformations that we use below, we resort to transforming the annihilation and creation operators into harmonic oscillator coordinates and momenta. Our particular conventions are standard for the boson terms and about which we shall say little while those for the MCS-field are given in more detail. Both are described in the appendix.

From equation (A13) in appendix A, the Fourier components of $A_0(x)$ contain both skew-symmetric and symmetric terms. The first Krein transformation removes the symmetric terms in $A_0(x)$ by means of

$$T_1 = \sum_{p\in\Gamma'} (-i)\sqrt{\frac{\omega}{\mu}} q^{\rm cs}_{\ell,2}(p) p^{\rm cs}_{\ell,0}(p). \tag{68}$$

while the second Krein transformation

$$T_2 = \sum_{p\in\Gamma'} (-i) q^{\rm cs}_{\ell,0}(p) p^{\rm cs}_{\ell,1}(p). \tag{69}$$

removes the skew-symmetric terms from $A'_\ell(x)$ after a Krein unitary transformation using T_1. Each of the expressions above is realized as a sum of commuting essentially selfadjoint operators for which the finite particle vectors D_F are analytic vectors. Each is also skew-symmetric with respect to the Krein metric so will be the infinitesimal generator of a Krein unitary but unbounded operator. The number of particle estimate for each T_j on a k-particle vector increases as $O((n+k+1)^n)$ in the analytic vector calculation which leads to geometric convergence in t as below.

Lemma III.1 *The operators $\widetilde{T_j}, j = 1, 2$ are self-adjoint and $\dagger$-skew-symmetric. Further each $exp[t\widetilde{T_j}]$ is selfadjoint and $\dagger$-unitary for real t and satisfy*

$$e^{t\widetilde{T_2}} e^{t\widetilde{T_1}}\, \widetilde{A_0(x)}\, e^{-t\widetilde{T_1}} e^{-t\widetilde{T_2}}\Phi = (A_0(x) + t[T_1, A_0(x)]) = A'_0(x;t)\Phi \tag{70}$$

$$e^{t\widetilde{T_2}} e^{t\widetilde{T_1}}\, \widetilde{A_\ell(x)}\, e^{-t\widetilde{T_1}} e^{-t\widetilde{T_2}}\Phi = (A_\ell(x) + t[T_1, A_\ell(x)] + t[T_2, A_\ell(x)])\,\Phi = A'_\ell(x;t)\Phi \tag{71}$$

for all $\Phi \in D_F$ and complex t with $|t| < t_0$.

To begin the proof notice that $\eta\, T_j\, \eta = T_j$ on vectors in D_F so by closure this relation is valid for $\widetilde{T_j}$. Nelson's theorem on analytic vectors shows $\eta\, e^{tT_j}\, \eta = e^{tT_j}$ proving the first statement.

For their Krein unitary property, consider $\Phi, \Psi \in D(e^{z\widetilde{T_j}})$ for suitable complex z and sequences of vectors Φ_n, Ψ_n which have compact support with respect to the spectrum of $\widetilde{T_j}$ and converge strongly to Φ, Ψ. Then notice $T_j^\dagger = -T_j = \eta\, T_j^*\, \eta = \eta\, T_j\, \eta$ as bounded operators on Φ_n, Ψ_n so consequently

$$\{e^{z\widetilde{T_j}}\Phi, e^{z'\widetilde{T_j}}\Psi\} \;=\; \lim_{n\to\infty}\{e^{z\widetilde{T_j}}\Phi_n, e^{z'\widetilde{T_j}}\Psi_n\}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} (\Phi_n, e^{\overline{z}\widetilde{T_j}} \eta e^{z'\widetilde{T_j}} \Psi_n) \\
&= \lim_{n\to\infty} (\Phi_n, \eta e^{(z'-\overline{z})\widetilde{T_j}} \Psi_n) \\
&= \lim_{n\to\infty} \{\Phi_n, e^{(z'-\overline{z})\widetilde{T_j}} \Psi_n\} \\
&= \{\Phi, e^{(z'-\overline{z})\widetilde{T_j}} \Psi\}
\end{aligned}
$$

For the commutator identities, set $\Phi_N = \sum_{n=0}^{N} \frac{z^n T_1^n}{n!} \Phi$ which converges strongly to $e^{z\widetilde{T_1}}\Phi$ for $\Phi \in D_F$ and $|z| < z_0$. Each Φ_N is a finite particle vector belonging to the domain of $A_\mu(x)^k$ for any positive integer k and $\mu = 0, 1, 2$ so we may calculate the expression

$$
\begin{aligned}
A_0(x)\Phi_N &= \sum_{n=0}^{N} \frac{z^n A_0(x) T_1^n}{n!} \Phi \\
&= \sum_{n=0}^{N} \frac{z^n T_1^n}{n!} A_0(x)\Phi + z \sum_{n=0}^{N-1} \frac{z^n T_1^n}{n!} [T_1, A_0(x)]\Phi
\end{aligned}
$$

Taking the limit in N shows the right-hand side converges for $|z| < z_0$ with $e^{z\widetilde{T_1}}\Phi$ in the domain of the closure of $A_0(x)$ and $\widetilde{A_0(x)} e^{z\widetilde{T_1}}\Phi$ in the range of $e^{z\widetilde{T_1}}$ leading to equation (70) since T_2 commutes with $A_0'(x)$. A similar calculation results in equation (71). A short calculation produces expressions for the gauge transformed fields which for $t = 1$ are

$$
A_0'(x) = \frac{1}{\sqrt{|V|}} \sum_{k\in\Gamma} \frac{\sqrt{2\omega}}{m} \left\{ q^{\,\mathrm{cs}}_{1,0}(k)\cos(k\cdot x) + q^{\,\mathrm{cs}}_{2,0}(k)\sin(k\cdot x) \right\} \tag{72}
$$

$$
\begin{aligned}
A_\ell'(x) &= \frac{1}{\sqrt{|V|}} \sum_{k\in\Gamma} \left\{ \left(\frac{2\epsilon_{\ell n} k_n}{\omega\sqrt{2\omega}} q^{\,\mathrm{cs}}_{2,1}(k) - \frac{2\mu\, k_\ell}{m\omega\sqrt{2\mu}} p^{\,\mathrm{cs}}_{2,2}(k) \right) \cos(k\cdot x) \right. \\
&+ \left. \left(-\frac{2\epsilon_{\ell n} k_n}{\omega\sqrt{2\omega}} q^{\,\mathrm{cs}}_{1,1}(k) + \frac{2\mu\, k_\ell}{m\omega\sqrt{2\mu}} p^{\,\mathrm{cs}}_{1,2}(k) \right) \sin(k\cdot x) \right\}
\end{aligned} \tag{73}
$$

which are respectively skew-symmetric and symmetric.

Turning our attention to the Hamiltonian we find a completely analogous result in

Lemma III.2 *For all $\Phi \in D_F$ and complex t with $|t| < t_0$*

$$\begin{aligned} e^{t\widetilde{T}_2} e^{t\widetilde{T}_1}\, \widetilde{H}\, e^{-t\widetilde{T}_1} e^{-t\widetilde{T}_2}\Phi &= H'(V,\delta;t)\Phi \\ H'(V,\delta;t) &= H_0^{\mathrm{boson}} + H_0^{\mathrm{cs}}{}'(t) + H'_{\mathrm{int}}(t) \end{aligned} \tag{74}$$

where $H'_{\mathrm{int}}(t)$ is $H_{\mathrm{int}}(t)$ with $A_\mu(x)$ replaced by $A'_\mu(x;t)$ and the gauge transformed Chern-Simons part of the free Hamiltonian given by

$$\begin{aligned} H_0^{\mathrm{cs}}{}'(t) &= \sum_{k\in\Gamma} \left\{ \frac{(\mu^2 - t^2\,\omega^2)}{2} q^{\,\mathrm{cs}}_{\ell,2}(k)^2 - \mu \right. \\ &+ \frac{\omega}{2}\left(q^{\,\mathrm{cs}}_{\ell,1}(k)^2 - 4\right) + \frac{\left(\sqrt{\mu}\, p^{\,\mathrm{cs}}_{\ell,2}(k) - t\sqrt{\omega}\, p^{\,\mathrm{cs}}_{\ell,1}(k)\right)^2}{2} \\ &+ t\,\omega\, q^{\,\mathrm{cs}}_{\ell,0}(k)\left(q^{\,\mathrm{cs}}_{\ell,1}(k) + \sqrt{\omega/\mu}\, q^{\,\mathrm{cs}}_{\ell,2}(k)\right) \\ &- t\sqrt{\omega\,\mu}\, p^{\,\mathrm{cs}}_{\ell,2}(k) p^{\,\mathrm{cs}}_{\ell,0}(k) + \omega\,(1-t)\, p^{\,\mathrm{cs}}_{\ell,0}(k) p^{\,\mathrm{cs}}_{\ell,1}(k) \\ &+ (1-t^2)\frac{\omega}{2}\left(p^{\,\mathrm{cs}}_{\ell,0}(k)^2 - q^{\mathrm{cs}}_{\ell,0}(k)^2\right) \\ &+ \left. \frac{\omega}{2}(1-t^2) p^{\,\mathrm{cs}}_{\ell,1}(k)^2 \right\} \end{aligned} \tag{75}$$

The statements for the domains of the unbounded operators are the same as for Lemma III.1 while calculations use the relation $e^{t\widetilde{T}_2} e^{t\widetilde{T}_1}\, S^2\, e^{-t\widetilde{T}_1} e^{-t\widetilde{T}_2}\Phi = \left\{e^{t\widetilde{T}_2} e^{t\widetilde{T}_1}\, S\, e^{-t\widetilde{T}_1} e^{-t\widetilde{T}_2}\right\}^2 \Phi$ for $|t| < t_0$ which is valid on D_F after taking closures. In each of Lemmas III.1 and III.2 the left-hand side is analytic in a disc $|t| < t_0$ while the right-hand side is an entire function of t and provides a unique analytic continuation of the operator on the left-hand side to the entire complex plane. However it is straight forward to find a dense domain D_0 which is a common core for all these operators and their powers on which both sides of equations (70), (71) and (74) are valid as entire functions in t. Consider a k-finite particle vector Φ_k realized as a normalized Hermite function $\Phi_k(\{q^{\,\mathrm{cs}}_{\ell,2}\}, \{q^{\,\mathrm{cs}}_{\ell,1}\}, q')$ in the q-harmonic oscillator variables with the coordinates appearing in $\{T_j | j = 1,2\}$ denoted explicitly. Let $\chi \in C_0^\infty(R_+)$ be positive, one for $0 \le t \le 1/2$ and zero for $t \ge 1$ say. Form the cutoff functions $\chi_{j\,,N}(q) = \chi(\sum_{k\in\Gamma,\,j} q^{\,cs}_{\ell,j}(k)^2/N^2)$ for N=1,2,... which have support inside a ball of radius N. Then by taking products and convolutions in different variables, we find

Lemma III.3 *Let D_0 denote the linear span of vectors formed by*

$$\Phi_{k,N}(q) = \chi_{2,\,N}(q)\,(\chi_{1,\,N} * \Phi_k)(q) \tag{76}$$

as k and N vary over the positive integers. For each of the operators and their powers appearing in Lemmas III.1 and III.2, D_0 forms a common dense core. Moreover each of the relations in equations (70), (71) and (74) are valid as entire analytic functions of t. In particular the corresponding operator relations are densely defined on D_0 for t=1.

That D_0 is dense and a common core is clear as also the analyticity. Upon re-examining convergence of the exponential series in equations (70), (71) and (74) the $q^{\,\mathrm{cs}}_{\ell,2}(k)$ and $p^{\,\mathrm{cs}}_{\ell,1}(k)$ operators are bounded by N when acting on vectors in D_0 while the N-particle estimates applied to the operators $q^{\,\mathrm{cs}}_{\ell,0}(k)$ and $p^{\,\mathrm{cs}}_{\ell,0}(k)$ acting on Φ_k provide a bound $O((n+k+1)^{n/2})$. Vectors in D_0 are now entire vectors for each of the Krein gauge transformations.

The gauge transformations in Lemma III.2 are 'natural' for the Hamiltonian $H(V,\delta)$, as may be seen from

$$\widetilde{H}\, e^{itT_j}\Phi_{k,\,N} = e^{itT_j}\left\{H + it[H,T_j] + (it)^2/2[[H,T_j],T_j]\right\}\Phi_{k,\,N} \tag{77}$$

with $\Phi_{k,\,N} \in D_0$ and $-\infty < t < \infty$. By using a Riemann approximation and then taking strong limits, the right-hand side converges and defines convergence of the left-hand side to the relation

$$\begin{aligned}\widetilde{H}\int_{-\infty}^{\infty} F(\lambda)dE_j(\lambda)\Phi_{k,\,N} &= \int_{-\infty}^{\infty}\widetilde{F(t)}\widetilde{H}e^{itT_j}\Phi_{k,N}dt \\ &= \int_{-\infty}^{\infty} F(\lambda)dE_j(\lambda)H\Phi_{k,N} \\ &+ \int_{-\infty}^{\infty} F'(\lambda)dE_j(\lambda)[H,T_j]\Phi_{k,N} \\ &+ \int_{-\infty}^{\infty} F''(\lambda)/2\, dE_j(\lambda)[[H,T_j],T_j]\Phi_{k,N}\end{aligned} \tag{78}$$

where $\{E_j(\lambda)\}$ is the spectral measure associated with the essentially self-adjoint operator T_j. Choosing $F \in C_0^2(R)$ shows that a large class of the bounded functions of $\widetilde{T_j}$ belong to the domain of $\widetilde{H}$. As we shall show in section V, the Hamiltonians $H(V,\delta)$ and $H'(V,\delta;1)$ give Krein equivalent dynamics.

IV Real Part of the Hamiltonian

Following the Krein gauge transformations in the previous section, all operators in the remainder of this article will take their physical values at $t = 1$.

The transformed Hamiltonian is thereby split into two parts. One part defines a symmetric operator while the other part defines a skew-symmetric operator. We now show that the symmetric part is bounded below and in fact is essentially selfadjoint, hence the transformed Hamiltonian is accretive. The transformed lattice Hamiltonian operator may be recast as

$$H'(V,\delta) = H_0^{\mathrm{cs}\,\prime} + H_0^{\mathrm{boson}} + V(\phi_1,\phi_2) + H'_{\mathrm{int}}. \tag{79}$$

To exploit positivity in the real part of the Hamiltonian, it is convenient to remove the gradient terms from H_0^{boson} and rewrite this as

$$H_0^{\mathrm{boson}} = H_0^{\mathrm{boson}\,\prime} + \sum_{x\in V}\frac{\delta^s}{2}[\nabla\phi_j(x)]^2 \tag{80}$$

in which

$$H_0^{\mathrm{boson}\,\prime} = \sum_{k\in\Gamma}\frac{1}{2}[p_{j,\ell}^2(k) + m_0^2 q_{j,\ell}^2(k) - 4\mu(k)] \tag{81}$$

is again a free massive harmonic oscillator which on the lattice acts in the same way as a number operator for determining domains. By reincorporating the gradient terms into the covariant derivative, the real part of $H'(V,\delta)$ takes the form

$$\mathrm{Re}H'(V,\delta) + c_1 I = X + Y \tag{82}$$

where c_1 is a positive constant to be chosen later and

$$X = H_0^{\mathrm{cs}\,\prime\prime} + H_0^{\mathrm{boson}\,\prime} + c_1 I, \quad Y = Y_{\mathrm{deg}} + Y_+ \tag{83}$$

with

$$Y_{\mathrm{deg}} = \sum_{k\in\Gamma}[-\sqrt{\mu\omega}\, p_{\ell,1}^{\mathrm{cs}}(k) p_{\ell,2}^{\mathrm{cs}}(k)] \tag{84}$$

and

$$Y_+ = \sum_{x\in V}\frac{\delta^s}{2}[\{\partial_\ell\phi_1(x) - e\phi_2(x)A_\ell(x)\}^2 + \{\partial_\ell\phi_2(x) + e\phi_1(x)A_\ell(x)\}^2] + V(\phi_1,\phi_2). \tag{85}$$

The operator $H_0^{\mathrm{cs}\,\prime\prime}$ denotes the real part of $H_0^{\mathrm{cs}\,\prime}$ with the degeneracy producing term Y_{deg} removed. As a sum of squares of symmetric operators which have dense analytic vectors, Y_+ clearly defines a symmetric operator which is bounded below since the Higgs potential is taken to be bounded below. The degenerate quadratic term Y_{deg} will be treated as a perturbation which is of the same size as the Chern-Simons number operator. Following the notation used in Reed and Simon [16, pages 174 – 176 and Theorem X.14 on page 164], it is a straight forward matter to use the Konrady trick for Y_+ with $Z = c_2 X^2$ to obtain the quadratic estimate

$$Z^2 \leq (X + Y + Z)^2 + c_3 I \tag{86}$$

for some positive constant c_3. The degenerate quadratic piece changes none of the estimates for the commutator $[X, [X, Y]]$ and is easily bounded in the term $XY_{\text{deg}}X$ by completing the square as

$$\begin{aligned} Y_{\text{deg}} &= \sum_{k\in\Gamma} \frac{1}{2}[-\mu\, p^{\text{cs}}_{\ell,2}{}^2(k) - \omega\, p^{\text{cs}}_{\ell,1}{}^2(k) + (\sqrt{\mu} p^{\text{cs}}_{\ell,2}(k) - \sqrt{\omega} p^{\text{cs}}_{\ell,1}(k))^2] \\ &= -Y_{\text{deg},1} + Y_{\text{deg},+} \end{aligned} \tag{87}$$

whereupon $XY_{\text{deg},+}X \geq 0$ and may be discarded with the other positive terms in obtaining the quadratic estimate above. The terms in $Y_{\text{deg},1}$ are cancelled by terms that appear in $H_0^{\text{cs}\,\prime\prime}$ with the result $H_0^{\text{cs}\,\prime\prime} - Y_{\text{deg},1} \geq 0$. Now we find

$$2c_2X^3 + 2c_2XYX \geq 2c_2X[X + Y_+ + Y_{\text{deg},+} - Y_{\text{deg},1}]X \geq 2c_2X[H_0^{\text{boson},\prime} + c_1 I]X$$

so

$$2c_2X^3 + 2c_2XYX + c_2[X, [X, Y]] \geq c_2(2c_1 - c_3)X^2 \geq 0 \tag{88}$$

provided we choose $c_1 \geq c_3/2$. The quadratic estimate and Wüst's Theorem allow us to conclude that $\text{Re}H'(V, \delta) + c_1 I = X + Y$ is essentially selfadjoint and bounded below. It is important to notice that the term $2c_2XH_0^{\text{boson},\prime}X$ is a large positive operator and only it's positivity is needed for the estimate (86). In fact we have the sharper estimate

$$(X + Y + Z)^2 \geq Z^2 + c_2(2c_1 - c_3)X^2 + 2c_2XH_0^{\text{boson},\prime}X \geq Z^2 \tag{89}$$

on $D_F \times D_F$ which is used in the next section.

V Skew-Symmetric Part of the Hamiltonian

The Krein transformations in section III were chosen to obtain a Hamiltonian of the form

$$H'(V, \delta) = \text{Re}H'(V, \delta) + S \tag{90}$$

in which S is a densely defined skew-symmetric operator given by

$$S = S_1 + S_2$$

where

$$\begin{aligned} S_1 &= \sum_{x\in V} \delta^s e\,[\pi_1\phi_2 - \pi_2\phi_1](x)A_0'(x) && (91)\\ S_2 &= \sum_{k\in\Gamma} \Bigg[\omega(k)q^{\rm cs}_{\ell,0}(k)\left\{q^{\rm cs}_{\ell,1}(k) + \sqrt{\frac{\omega}{\mu}}\,q^{\rm cs}_{\ell,2}(k)\right\}\\ &- \sqrt{\omega\,\mu}\,p^{\rm cs}_{\ell,0}(k)p^{\rm cs}_{\ell,2}(k)\Bigg] && (92)\end{aligned}$$

As $\mathrm{Re}H'(V,\delta)$ is essentially selfadjoint and bounded below, Chernoff's extension of Wüst's theorem [20] will show that $H'(V,\delta)$ is essentially maximal accretive given the estimate

$$\| Zu \| \leq \| (X+Y+Z+S)u \| + c_5 I \| u \|$$

for vectors u in the core domain D_F or equivalently

$$(Z)^2 \leq (X+Y+Z)^2 + [X+Y+Z,S] + S^*S + c_5 I \qquad (93)$$

as a quadratic form on $D_F \times D_F$. Using the estimate of the last section, it will then be sufficient to verify

$$c_2(2c_1-c_3)X^2 + 2c_2 X H_0^{\rm boson\,\prime} X + [X+Y+Z,S] + S^*S + c_5 I \geq 0 \qquad (94)$$

which follows from:

Lemma V.1 *On the domain D_F, the commutator $[X+Y+Z,S]$ is Kato-tiny relative to $XH_0^{\rm boson\,\prime}X$.*

To prove the Lemma we calculate individual terms in the commutator. From $[Z,S] = c_2[X^2,S] = c_2\,(X[X,S]+[X,S]X)$ since X acts as a free Hamiltonian, $[X,S]$ is bounded by $H_0^{\rm boson\,\prime}X^{1/2}$ while [Z,S] is bounded by $H_0^{\rm boson\,\prime}X^{3/2}$. Both of these are Kato-tiny relative to $2c_2XH_0^{\rm boson\,\prime}X$. In X+Y let us recombine the gradient boson terms in Y with the boson terms in X to replace $H_0^{\rm boson\,\prime}$ by $H_0^{\rm boson}$ leading to $X+Y = H_0^{\rm cs\,\prime} + H_0^{\rm boson} + c_1 I + Y_{\rm deg} + Y_0$

where the 'magnetic part' of the Chern-Simons interaction now appears with the boson potential in

$$
\begin{aligned}
Y_0 &= \sum_{x\in V} \frac{\delta^s}{2} \left[-e\left\{\partial_\ell\phi_1(x)\phi_2(x) - \partial_\ell\phi_2(x)\phi_1(x)\right\} A'_\ell(x) \right. \\
&+ \left. \frac{e^2}{2}\left\{\phi_1^2(x) + \phi_2^2(x)\right\} A'_\ell(x)^{\,2} \right] \\
&+ V(\phi_1,\phi_2) \\
&= Y_{\mathrm{mag}} + V(\phi_1,\phi_2). \qquad (95)
\end{aligned}
$$

Clearly $[Y_{\mathrm{deg}}, S_1] = 0$ with the remaining terms for this part of the commutator given by

$$
\begin{aligned}
[Y_{\mathrm{deg}}, S_2] &= \sum_{k\in\Gamma} \left[-i\sqrt{\mu}\omega^{3/2} q^{\mathrm{cs}}_{\ell,0}(k) p^{\mathrm{cs}}_{\ell,2}(k) \right. \\
&- \left. i\omega^2 q^{\mathrm{cs}}_{\ell,0}(k) p^{\mathrm{cs}}_{\ell,1}(k) \right] \qquad (96)
\end{aligned}
$$

Consequently on the core domain, $-[Y_{\mathrm{deg}}, S] \le c_6 X$ for c_1 large enough as the commutator is quadratic in Chern-Simons harmonic oscillators. It is useful to look at the commutators in $[Y_{\mathrm{mag}}, S]$ separately. For $[Y_{\mathrm{mag}}, S_1]$ the fields $A_0{}'$ and $A_\ell{}'$ commute so it is only necessary to calculate the boson field commutators. Lattice quantization preserves enough locality for the boson fields that for terms quadratic in the individual boson fields, the relation $[\phi_j^2(x), \pi_j(y)] = 2i\delta_{P,V}(x-y)\phi_j(x)$ with the periodic lattice $\delta_{P,V}$ shows their contribution to $[Y_{\mathrm{mag}}, S_1]$ vanishes. As we use a symmetric $k \leftrightarrow -k$ summation range, it is possible to effect an integration by parts of sorts, nevertheless the remaining terms in this commutator are not zero and we arrive at

$$
\begin{aligned}
[Y_{\mathrm{mag}}, S_1] &= \sum_{x\in V} \delta^s ie^2 [\partial_\ell\phi_1(x)\phi_1(x) + \partial_\ell\phi_2(x)\phi_2(x)] A'_\ell(x) A'_0(x) \\
&+ \sum_{x\in V} \delta^{2s}(-ie^2)[\partial_{x,\ell}\delta_{P,V}(x-y)\phi_1(x)\phi_1(y)] A'_\ell(x) A'_0(x) \\
&+ \sum_{x\in V} \delta^{2s}(-ie^2)[\partial_{x,\ell}\delta_{P,V}(x-y)\phi_2(x)\phi_2(y)] A'_\ell(x) A'_0(x).
\end{aligned}
\qquad (97)
$$

Using $H_0^{\mathrm{boson}\,\prime}$ to bound $\phi_j(x)\phi_j(y)$ and X to bound $A'_\ell(x)A'_0(x)$ the commutator $-[Y_{\mathrm{mag}}, S_1] \le c_7 H_0^{\mathrm{boson}\,\prime} X$, which is also Kato-tiny compared to $H_0^{\mathrm{boson}\,\prime} X^2$. Up to multiplicative terms quadratic in the boson fields, the size of the commutator $[Y_{\rm mag}, S_2]$ is determined by $[A'_\ell, S_2]$. From the the Landau gauge expression for $A'_\ell(x)$ in section III, it is straight forward to arrive at the expression

$$\begin{aligned}[A'_\ell(x), S_2] &= \frac{1}{\sqrt{|V|}} \sum_{k\in\Gamma} \left[\left\{ -i\frac{\sqrt{2\omega}}{m} k_\ell \, q^{\rm cs}_{2,0}(k) \right\} \cos(k\cdot x) \right. \\ &+ \left. \left\{ i\frac{\sqrt{2\omega}}{m} k_\ell \, q^{\rm cs}_{1,0}(k) \right\} \sin(k\cdot x) \right] \end{aligned} \tag{98}$$

which is dominated by $H_0^{\rm boson\,\prime} X^{1/2}$. Hence the quadratic term $[A'_\ell{}^2(x), S_2]$ requires an additional factor $X^{1/2}$ to produce the estimate $-[Y_{\rm mag}, S_2] \leq c_8 H_0^{\rm boson\,\prime} X$ which is also Kato-tiny relative to $H_0^{\rm boson\,\prime} X^2$.

Finally $[V(\phi_1,\phi_2), S] = 0$. This is immediate for $[V(\phi_1,\phi_2), S_2] = 0$ while the commutator identities

$$\begin{aligned}[\phi_j(x)^4, \pi_j(y)] &= 4i\delta_{P,V}(x-y)\phi_j(x)^3 \\ [\phi_1^2(x)\phi_2^2(x), \pi_1(y)] &= 2i\delta_{P,V}(x-y)\phi_1(x)\phi_2^2(x)\end{aligned}$$

readily show that $[V(\phi_1,\phi_2), S_1] = 0$ as well. Collecting together these estimates for the various commutators leads to the inequality

$$(X+Y+Z)^2 + [X+Y+Z, S] \geq Z^2 - c_5 I \tag{99}$$

for some constant $c_5 > 0$ provided c_2 is large enough and $c_1 > c_3/2$. As $(X+Y+Z)^*$ and Z^* are densely defined on the finite particle vectors, Chernoff's theorem applied to $X+Y+S = X+Y+Z+S-Z$ produces:

Theorem V.2 *After Krein gauge transformations, the transformed Hamiltonian $H'(V,\delta) = {\rm Re}H'(V,\delta) + S$ is maximal accretive after closure on the finite particle domain D_F and hence Krein essentially selfadjoint.*

Krein selfadjointness follows by a simple argument which can be found in [21, Lemma 2.2, page 5].

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A Harmonic Oscillator Coordinates

We choose standard harmonic oscillator coordinates for the time zero boson field and its conjugate momentum, yielding (see, e.g., [22, chapter II, section B]):

$$\begin{aligned}
\phi_j(x) &= \frac{1}{\sqrt{|V|}} \sum_{k\in\Gamma} \frac{1}{2\mu} \left[e^{ik\cdot x} a_j(k) + e^{-ik\cdot x} a_j^*(k)\right] \\
&= \frac{1}{\sqrt{2|V|}} \sum_{k\in\Gamma} \frac{1}{2\mu} \left[q_{1,j}(k)\cos(k\cdot x) + q_{2,j}(k)\sin(k\cdot x)\right] \qquad \text{(A1)} \\
\pi_j(x) &= \frac{1}{\sqrt{|V|}} \sum_{k\in\Gamma} \frac{-i}{2} \left[e^{ik\cdot x} a_j(k) - e^{-ik\cdot x} a_j^*(k)\right] \\
&= \frac{1}{\sqrt{2|V|}} \sum_{k\in\Gamma} \frac{1}{2\mu} \left[p_{1,j}(k)\cos(k\cdot x) + p_{2,j}(k)\sin(k\cdot x)\right] \qquad \text{(A2)}
\end{aligned}$$

for $j = 1,2$. The free boson Hamiltonian with Wick ordering then becomes

$$H_0^{\text{boson}} = \sum_{k\in\Gamma} \frac{1}{2} a_j^*(k) a_j(k) = \sum_{k\in\Gamma} \frac{1}{2}\left[p_{\ell,j}^2 + \mu^2 q_{\ell,j}^2\right] \qquad \text{(A3)}$$

In order to define harmonic oscillator coordinates for the MCS field $A_\mu = V_\mu + S_\mu$ in Landau gauge ($\gamma = 1$), we first express the MCS field in terms of Fock operators b_μ defined by

$$b_0(k) = \frac{c_0(k)}{\sqrt{2\omega}}, \quad b_1(k) = \frac{k_n c_n(k)}{\omega\sqrt{2\omega}}, \quad b_2(k) = \frac{a(k)}{\sqrt{2\mu}} \qquad \text{(A4)}$$

so that b_0, b_1 represent the ghost degrees of freedom while b_2 corresponds to the single physical mode. In terms of these, we have for the time zero MCS field

$$A_\mu = \frac{1}{\sqrt{|V|}} \sum_{k\in\Gamma} \left[M_{\mu\nu} b_\nu(k) e^{ik\cdot x} + \overline{M}_{\mu\alpha}(-g)_{\alpha\beta} b_\beta^*(k) e^{-ik\cdot x}\right] \qquad \text{(A5)}$$

with

$$M = \begin{pmatrix} -\frac{\omega}{m\sqrt{2\omega}} & 0 & -\frac{\omega}{m\sqrt{2\mu}} \\ -\frac{1}{\sqrt{2\omega}}\left(\frac{k_1}{m} - \frac{ik_2}{\omega}\right) & \frac{-ik_2}{\omega\sqrt{2\omega}} & -\frac{1}{\sqrt{2\mu}}\left(\frac{\mu k_1}{m\omega} - \frac{ik_2}{\omega}\right) \\ -\frac{1}{\sqrt{2\omega}}\left(\frac{k_2}{m} + \frac{ik_1}{\omega}\right) & \frac{ik_1}{\omega\sqrt{2\omega}} & -\frac{1}{\sqrt{2\mu}}\left(\frac{\mu k_2}{m\omega} + \frac{ik_1}{\omega}\right) \end{pmatrix} \qquad \text{(A6)}$$

Note the appearance of the indefinite metric in the second term of (A5). We define Krein operators $a_\mu = -b_\mu$ and $a_\mu^\dagger = g_{\mu\beta} b_\beta^*$. It is in terms of these that we define our harmonic oscillator coordinates for the MCS field:

$$q_{1,\mu}^{\mathrm{cs}}(k) = \frac{1}{2}\left[a_\mu(k) + a_\mu^\dagger(k) + a_\mu(-k) + a_\mu^\dagger(-k)\right] \quad (A7)$$

$$q_{2,\mu}^{\mathrm{cs}}(k) = \frac{i}{2}\left[a_\mu(k) - a_\mu^\dagger(k) - a_\mu(-k) + a_\mu^\dagger(-k)\right] \quad (A8)$$

$$p_{1,\mu}^{\mathrm{cs}}(k) = \frac{i}{2}\left[a_\mu(k) - a_\mu^\dagger(k) + a_\mu(-k) - a_\mu^\dagger(-k)\right] \quad (A9)$$

$$p_{2,\mu}^{\mathrm{cs}}(k) = \frac{1}{2}\left[-a_\mu(k) - a_\mu^\dagger(k) + a_\mu(-k) + a_\mu^\dagger(-k)\right] \quad (A10)$$

These satisfy the relations

$$q_{\ell,\mu}^{\mathrm{cs}}(-k) = (-1)^{\ell+1} q_{\ell,\mu}^{\mathrm{cs}}(k) \quad \text{and} \quad p_{\ell,\mu}^{\mathrm{cs}}(-k) = (-1)^{\ell+1} p_{\ell,\mu}^{\mathrm{cs}}(k) \quad (A11)$$

Thus, upon restriction to "allowed" momenta $k, k' \in \Gamma'$, we have the commutation relations

$$[q_{\ell,\mu}^{\mathrm{cs}}(k), p_{n,\nu}^{\mathrm{cs}}(k')] = i g_{\mu\nu} \delta_{\ell n} \delta_{k,k'} \quad (A12)$$

This leads to the following expressions for the time zero MCS field in Landau gauge

$$\begin{aligned}
A_0(x) &= \frac{1}{\sqrt{|V|}} \sum_{k\in\Gamma'} \left\{ \cos(k\cdot x) \left[\frac{2\omega}{m\sqrt{2\omega}} q_{1,0}^{\mathrm{cs}}(k) + \frac{2\omega}{m\sqrt{2\mu}} q_{1,2}^{\mathrm{cs}}(k) \right] \right. \quad (A13) \\
&+ \left. \sin(k\cdot x) \left[\frac{2\omega}{m\sqrt{2\omega}} q_{2,0}^{\mathrm{cs}}(k) + \frac{2\omega}{m\sqrt{2\mu}} q_{2,2}^{\mathrm{cs}}(k) \right] \right\} \\
A_\ell(x) &= \frac{1}{\sqrt{|V|}} \sum_{k\in\Gamma'} \left\{ \cos(k\cdot x) \left[-\frac{2k_\ell}{m\sqrt{2\omega}} p_{2,0}^{\mathrm{cs}}(k) - \frac{2\mu k_\ell}{m\omega\sqrt{2\mu}} p_{2,2}^{\mathrm{cs}}(k) \right. \right. \\
&- \left. \frac{2\epsilon_{\ell n} k_n}{\omega} \left(\frac{q_{2,0}^{\mathrm{cs}}(k)}{\sqrt{2\omega}} - \frac{q_{2,1}^{\mathrm{cs}}(k)}{\sqrt{2\omega}} + \frac{q_{2,2}^{\mathrm{cs}}(k)}{\sqrt{2\mu}} \right) \right] \\
&+ \sin(k\cdot x) \left[\frac{2k_\ell}{m\sqrt{2\omega}} p_{1,0}^{\mathrm{cs}}(k) + \frac{2\mu k_\ell}{m\omega\sqrt{2\mu}} p_{1,2}^{\mathrm{cs}}(k) \right. \\
&+ \left. \left. \frac{2\epsilon_{\ell n} k_n}{\omega} \left(\frac{q_{1,0}^{\mathrm{cs}}(k)}{\sqrt{2\omega}} - \frac{q_{1,1}^{\mathrm{cs}}(k)}{\sqrt{2\omega}} + \frac{q_{1,2}^{\mathrm{cs}}(k)}{\sqrt{2\mu}} \right) \right] \right\}
\end{aligned}$$

For the free MCS Hamiltonian in Landau gauge with Wick ordering, this leads to

$$H_0^{\mathrm{cs}} = \sum_{k\in\Gamma'} \left\{ -\frac{\omega}{2} \left(p_{\ell,0}^{\mathrm{cs}}(k) p_{\ell,0}^{\mathrm{cs}}(k) + q_{\ell,0}^{\mathrm{cs}}(k) q_{\ell,0}^{\mathrm{cs}}(k) \right) \right.$$

$$
\begin{aligned}
&+ \quad \frac{\omega}{2}\Big(p^{\mathrm{cs}}_{\ell,1}(k)p^{\mathrm{cs}}_{\ell,1}(k)+q^{\mathrm{cs}}_{\ell,1}(k)q^{\mathrm{cs}}_{\ell,1}(k)\Big) \qquad \text{(A14)}\\
&+ \quad \frac{\mu}{2}\Big(p^{\mathrm{cs}}_{\ell,2}(k)p^{\mathrm{cs}}_{\ell,2}(k)+q^{\mathrm{cs}}_{\ell,2}(k)q^{\mathrm{cs}}_{\ell,2}(k)\Big)\Big\}
\end{aligned}
$$

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