On a new fixed point of the renormalization group operator for area-preserving maps

K. Fuchss,[1] A. Wurm,[2] and P.J. Morrison[1]

[1] *Department of Physics and Institute for Fusion Studies, The University of Texas at Austin, Austin, TX 78712*

[2] *Department of Physical & Biological Sciences, Western New England College, Springfield, MA 01119*

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The breakup of the shearless invariant torus with winding number $\omega = \sqrt{2} - 1$ is studied numerically using Greene's residue criterion in the standard nontwist map. The residue behavior and parameter scaling at the breakup suggests the existence of a new fixed point of the renormalization group operator (RGO) for area-preserving maps. The unstable eigenvalues of the RGO at this fixed point and the critical scaling exponents of the torus at breakup are computed.

Area-preserving nontwist maps are low-dimensional models of physical systems whose Hamiltonians locally violate a nondegeneracy condition (see below) as described, e.g., in Refs. 1–3. Some applications are the study of magnetic field lines in toroidal plasma devices[4–10] and stellarators[11, 12] (plasma physics), and traveling waves,[13] coherent structures, self-consistent transport[14] (fluid dynamics), and particle accelerators.[15] Nontwist regions have also been shown to appear generically in the phase space of area-preserving maps that have a tripling bifurcation of an elliptic fixed point.[16, 17] Additional references can be found in Refs. 1, 3.

Of particular interest from a physics perspective is the breakup of invariant tori, consisting of quasiperiodic orbits with irrational winding number,[**?**] that often correspond to transport barriers in the physical system, i.e., their existence determines the long-time stability of the system. In nontwist maps, the invariant tori that appear to be the most resilient to perturbations are the so-called *shearless* tori, which correspond to local extrema in the winding number profile of the map.

Invariant tori at breakup exhibit scale invariance under specific phase space re-scalings, which are observed to be universal for certain classes of area-preserving maps. To interpret these results, a renormalization group framework has been developed (see, e.g., Refs. 2, 18–20). For twist maps, it is well understood which fixed point, cycle, or strange attractor of the renormalization group operator (RGO) is encountered within a given class of maps, depending on properties of the winding number of the critical torus (see Ref. 21 for a recent review). For nontwist maps, however, only results for the single class of shearless critical noble tori, i.e., shearless critical tori with winding numbers that have a continued fraction expansion tail of 1's, are known. The result reported in this letter represents the first new fixed point for nontwist maps.

A tool for studying the breakup of a torus with given winding number is Greene's residue criterion, originally introduced in the context of twist maps.[22] This method is based on the numerical observation that the breakup of an invariant torus with irrational winding number ω is determined by the stability of nearby periodic orbits. Some aspects of this criterion have been proved for nontwist maps.[23]

To study the breakup, one considers a sequence of periodic orbits with winding numbers q_n/p_n converging to ω, $\lim_{n\to\infty} q_n/p_n = \omega$. The elements of the sequence converging the fastest are the convergents of the continued fraction expansion of ω, i.e., $[n] := q_n/p_n = [a_0, a_1, \ldots, a_n]$, where

$$\omega = [a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}. \qquad (1)$$

The stability of the corresponding orbits is determined by their residues, $R_n = [2 - \mathrm{Tr}(DM^{p_n})]/4$, where Tr is the trace and DM^{p_n} is the linearization of the p_n times iterated map M about the periodic orbit: An orbit is elliptic for $0 < R_n < 1$, parabolic for $R_n = 0$ and $R_n = 1$, and hyperbolic otherwise. The convergence or divergence of the residue sequence associated with the chosen periodic orbit sequence then determines whether the torus exists or not, respectively: If the ω-torus exists, $\lim_{n\to\infty} |R_n| = 0$; if the ω-torus is destroyed, $\lim_{n\to\infty} |R_n| = \infty$. At the breakup, different scenarios can be encountered, depending on the class of maps and winding number of the invariant torus considered.

In nontwist maps, the residue criterion was first used in Ref. 1 to study the breakup of the shearless torus of inverse golden mean $1/\gamma = (\sqrt{5} - 1)/2 = [0, 1, 1, 1, \ldots]$ winding number in the standard nontwist map. The residue sequence was discovered to converge to a six-cycle. Similar studies were conducted for other noble shearless tori of winding numbers $\omega = 1/\gamma^2$ (Refs. 2, 24), $\omega = [0, 2, 2, 1, 1, 1, \ldots]$ (Ref. 25), and $\omega = [0, 1, 11, 1, 1, 1, \ldots]$ (Ref. 26), and the same six-cycle was found for all tori in the same symmetry class.

In this letter we study the breakup of the $\omega = \sqrt{2} - 1 = [0, 2, 2, 2, 2, \ldots]$ shearless torus, which is an example of a non-noble winding number and leads to the discovery of a new fixed point of the renormalization group operator of area-preserving maps. In contrast to twist maps,[27] the periodicity of the elements of the continued fraction expansion has not been linked to the periodicity of the critical residue sequence and therefore our new result would not have been predicted. The numerical methods we use and their accuracy are discussed in Refs. 2, 3, 26, 28 and we refer the reader to these publications for details.

As our specific model we use the *standard nontwist map* (SNM) M as introduced in Ref. 13,

$$\begin{aligned} x_{i+1} &= x_i + a\left(1 - y_{i+1}^2\right) \\ y_{i+1} &= y_i - b \sin\left(2\pi x_i\right) , \end{aligned} \qquad (2)$$

where $(x, y) \in \mathbb{T} \times \mathbb{R}$ are phase space coordinates and $a, b \in \mathbb{R}$ are parameters. This map is area-preserving and violates the *twist condition*, $\partial x_{i+1}(x_i, y_i)/\partial y_i \neq 0$, along a curve in phase space. Although the SNM is not generic due to its symmetries, it models the essential features of nontwist systems with a local, approximately quadratic extremum of the winding number profile.

One important characteristic of nontwist maps is the existence of multiple orbit chains (*up* and *down* orbits) of the same winding number, which can undergo bifurcations when the map parameters a and b are changed. When two invariant tori collide, the winding number profile shows a local extremum and the orbit at collision is referred to as the *shearless* torus. For a given winding number these collisions occur along bifurcation curves $b_\omega(a)$ in parameter space.[1]

In order to study a shearless invariant torus, its bifurcation curve is found numerically by approximating it by the bifurcation curves, $b_{[n]}(a)$, of nearby periodic orbits with winding numbers that are the continued fraction convergents of ω. Greene's residue criterion can then be used to determine where on $b_\omega(a)$ the shearless torus still exists: At parameter values a and the best known approximation to $b_\omega(a)$, the residues of all periodic orbits of convergents that have not collided, here the orbits $[n]$ with even n, are computed. Their limiting behavior for $n \to \infty$ reveals the status of the torus. By repeating the procedure for various values of a, with alternating residue convergence to 0 and ∞, the parameter values of the shearless torus breakup, $(a_c, b_\omega(a_c))$, can be determined to high precision.

The study of the breakup of tori with non-noble winding numbers is difficult because, due to numerical limitations, only periodic orbits for a small number of elements of the continued fraction expansion can be found. Therefore one cannot numerically distinguish between a torus that has 2's as all entries in the continued fraction expansion and one that has 2's until one reaches the numerical limit, and then 1's for the tail (i.e., a noble number).

To make a definite prediction for the $\omega = \sqrt{2} - 1$ torus we study the breakup of a series of sixteen invariant tori T_i, starting with $\omega = [0, 1, 1, 1, \ldots]$ up to

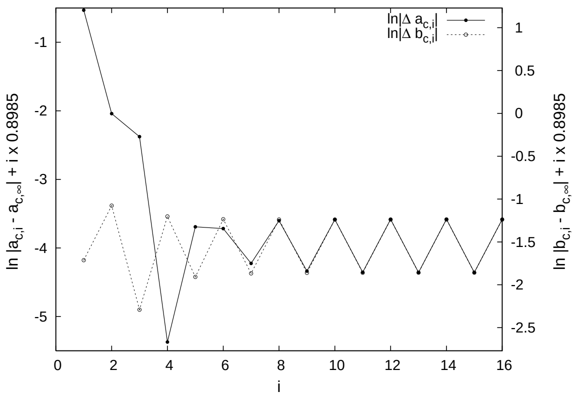

FIG. 1: 2-cycle of $a_{c,i}$ and $b_{c,i}$ differences in approximating the critical shearless torus $\omega = \sqrt{2} - 1$ by noble tori T_i. The average (negative) slope has been added to the data.

$\omega = [0, 2, \ldots, 2, 1, \ldots]$ with fifteen 2's in the continued fraction expansion.

The breakup parameters $(a_{c,i}, b_{c,i})$ for the tori T_i are found to converge exponentially to the critical parameters $(a_{c,\infty}, b_{c,\infty})$ of the $\omega = \sqrt{2} - 1$ torus. Plotting the logarithms of the differences $|a_{c,i} - a_{c,\infty}|$ and $|b_{c,i} - b_{c,\infty}|$, corrected for their average slopes of $c_a = -0.8986 \pm 0.0031$ and $c_b = -0.8984 \pm 0.0002$, respectively (see Fig. 1), one observes a period-two oscillation. This result should be compared to Fig. 5 in Ref. 29, where a similar study was conducted for the (one-parameter) standard twist map. The plot of parameter vs. i showed a straight line with negative slope.

Figure 2 depicts the behavior of a few of the critical residues (residues at breakup) on the s_1 symmetry line[**?**] as a function of $[n]$. The values for different tori are shifted by 4 along the y-axis to avoid overlap. The results show that the more 2's are included, the further the familiar six-cycle for noble winding numbers gets pushed towards higher n values. We conjecture that the emerging pattern for small n values represents the critical residue pattern of the $\omega = [0, 2, 2, \ldots]$ torus. The critical residues at

$$(a_{c,\infty}, b_{c,\infty}) = (0.446710414656, 0.838135537624831489)$$

along s_1 converge to the single value $R_c = 0.621723$ for the down orbits, and the single value $R_c = -0.909118$ for the up orbits. The same result is found along the other symmetry lines.

As discussed in Refs. 1, 26, close to the critical breakup value, the $b_{[n]}(a)$ obey a scaling law

$$b_{[n]} = b_{c,\infty} + B(n)\ \delta_1^{-n} , \qquad (3)$$

where $B(n)$ is numerically found to be periodic in n as $n \to \infty$ with period 4. This period is the same as the

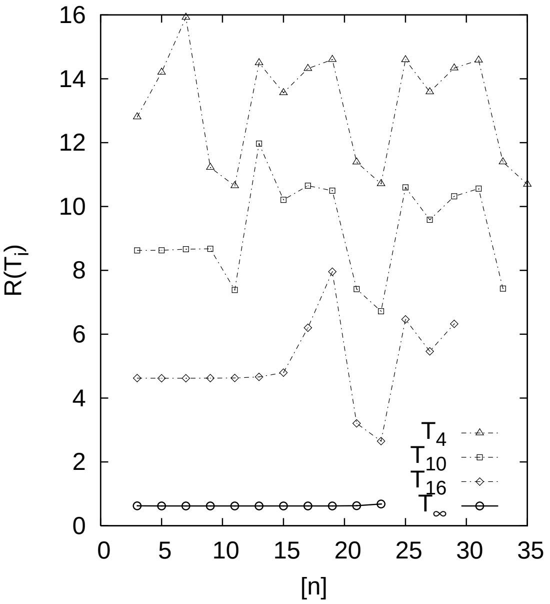

FIG. 2: Residue behavior at the shearless T_i torus breakup for down orbits on s_1. For clarity the residues of the tori have been shifted upward by 4. (See text for definition of T_∞.)

period of the critical fixed cycle of the renormalization group operator $\mathcal{R}$ (RGO), i.e., a critical fixed point of $\mathcal{R}^4$.[19]

The scaling can be observed by plotting

$$\ln\left(b_{[n+1]} - b_{[n]}\right) = \tilde{B}(n) - n \ln \delta_1 \ ,$$

where $\tilde{B}(n) = \ln(B(n+1)/\delta_1 - B(n))$ is also periodic in n. This is shown (on s_1) in Fig. 3, where for clarity only the offsets of $\ln\left(b_{[n+1]} - b_{[n]}\right)$ about the average slope are shown. The slope was calculated from the last 16 difference values by averaging the last 12 slopes $\left[\ln(b_{[n+5]} - b_{[n+4]}) - \ln(b_{[n+1]} - b_{[n]})\right]/12$, with $n = 6, \ldots, 17$. The periodicity of $\tilde{B}(n)$ makes it possible to obtain a better approximation for $b_{c,\infty}$ (see Ref. 1)

$$b_{c,\infty} \approx b_{[22]} + \frac{\left(b_{[22]} - b_{[18]}\right)\ \left(b_{[22]} - b_{[21]}\right)}{\left(b_{[19]} - b_{[18]}\right) - \left(b_{[22]} - b_{[21]}\right)} \ .$$

This value is used to find the critical residue pattern labeled T_∞ in Fig. 2. Compared to the critical residue six-cycle for noble winding numbers,[1] the four-cycle of $b_{[n]}$ differences indicates that the $\omega = \sqrt{2} - 1$ breakup exhibits a residue two-cycle along each symmetry line (only half of the periodic orbits exist at breakup). As discussed above, in each two cycle, the numerical values of the residues are found to be the same.

As in previous studies, the critical torus exhibits invariance under local re-scaling of the phase space in the neighborhood of the symmetry lines. Following

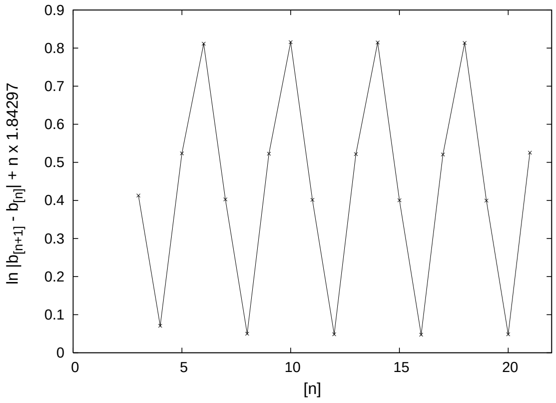

FIG. 3: 4-cycle of $b_{[n]}$ differences in approximating the critical shearless $\omega = \sqrt{2} - 1$ torus at $a_{c,\infty} = 0.446710414656$ on s_1.

Refs. 2, 19, we compute the scaling factors α and β such that the torus in the vicinity of its intersection with s_3 is invariant under $(x', y') \to (\alpha^N x', \beta^N y')$. (The exponent N is the length of the critical cycle, $N = 4$ for $\omega = \sqrt{2} - 1$ and $N = 12$ for noble winding numbers.) These factors are found from the limiting behavior of convergent periodic orbits: Denoting by $(\hat{x}'_{n,\pm}, \hat{y}'_{n,\pm})$ the symmetry line coordinates[19] of the point on the up (+) or down (−) orbit of the $[n]$th convergent that is located closest to $(0, 0)$, we compute (see Table I)

$$\alpha_\pm^N = \lim_{n\to\infty} \left| \frac{\hat{x}'_{n,\pm}}{\hat{x}'_{n+N,\pm}} \right| \ , \quad \beta_\pm^N = \lim_{n\to\infty} \left| \frac{\hat{y}'_{n,\pm}}{\hat{y}'_{n+N,\pm}} \right| \ . \tag{4}$$

The scaling invariance of the torus at breakup and of the nearby periodic orbits is illustrated in Fig. 4.

As shown in Ref. 19, the numerical data obtained can be used to compute the unstable eigenvalues, δ_1 and δ_2, of the RGO $\mathcal{R}$ by

$$\frac{1}{\delta_1^N} = \lim_{n\to\infty} \left(\frac{b_{[n+N]}\,(a_c) - b_c}{b_{[n]}\,(a_c) - b_c} \right) \tag{5}$$

and

$$\frac{1}{\delta_2^N} = \lim_{n\to\infty} \left(\frac{a_{c\,[n+N]} - a_c}{a_{c\,[n]} - a_c} \right) \ , \tag{6}$$

where $a_{c[n]}$ is the a value at which the ω-torus breaks up along the $b_{[n](a)}$ bifurcation curve.

Our results are displayed in Table I. For comparison we also show the corresponding values for the breakup of tori with noble numbers.

In summary, we found a new critical cycle of the renormalization group operator $\mathcal{R}$, a fixed point of $\mathcal{R}^4$, which governs the breakup of the $\omega = \sqrt{2} - 1$ shearless invariant torus. In analogy with the breakup of tori with noble winding numbers, we expect this result to be the same for all shearless tori (in the same universality class) with continued fraction expansion tails consisting of 2's.

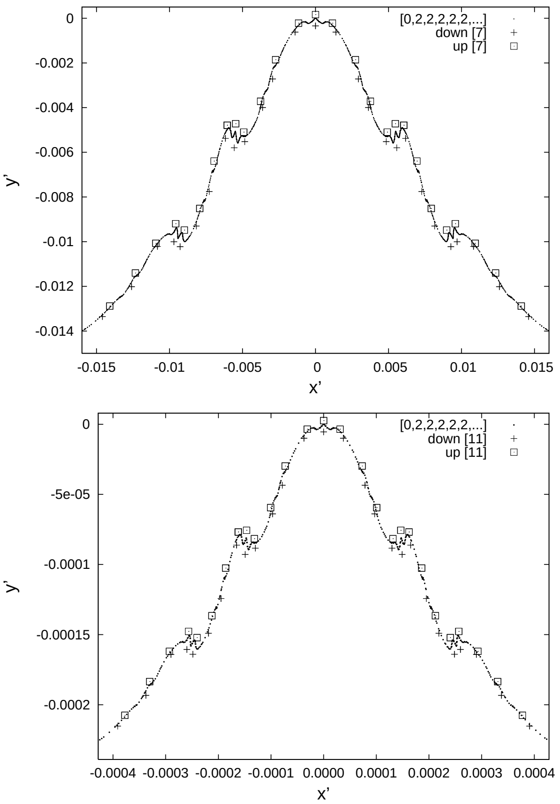

FIG. 4: Two levels of magnification in symmetry line coordinates (x', y') of the $\omega = \sqrt{2} - 1$ torus at breakup. Also shown are the nearby up and down orbits of the [7]th (top) and [11]th (bottom) continued fraction convergents.

tail	$[\ldots, 1, 1, 1, \ldots]$	$[\ldots, 2, 2, 2, \ldots]$
cycle	12	4
α	1.6179	2.4725
β	1.6579	2.8146
δ_1	2.680	6.311
δ_2	1.584	2.455

TABLE I: Universal breakup values for noble tori ($N = 12$) and $\omega = \sqrt{2} - 1$ ($N = 4$). Values for noble tori are from Ref. 26.

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[] An *orbit* of an area-preserving map M is a sequence of points $\{(x_i, y_i)\}_{i=-\infty}^{\infty}$ such that $M(x_i, y_i) = (x_{i+1}, y_{i+1})$. The *winding number* ω of an orbit is defined as the limit $\omega = \lim_{i\to\infty}(x_i/i)$, when it exists. Here the x-coordinate is "lifted" from $\mathbb{T}$ to $\mathbb{R}$. A *periodic orbit* of period n is an orbit $M^n(x_i, y_i) = (x_i + m, y_i)$, $\forall\, i$, where m is an integer. Periodic orbits have rational winding numbers $\omega = m/n$. An *invariant torus* is a one-dimensional set C, a curve, that is invariant under the map, $C = M(C)$. Orbits belonging to such a torus generically have irrational winding number.

[] A map M is called *reversible* if it can be decomposed as $M = I_1 \circ I_2$ with $I_i^2 = 0$. The fixed point sets of I_i are one-dimensional sets, called the *symmetry lines* of the map. For the SNM the symmetry lines are $s_1 = \{(x, y)|x = 0\}$, $s_2 = \{(x, y)|x = 1/2\}$, $s_3 = \{(x, y)|x = a(1 - y^2)/2\}$, and $s_4 = \{(x, y)|x = a(1 - y^2)/2 + 1/2\}$.