ALGEBRO-GEOMETRIC CONSTRAINTS ON SOLITONS WITH RESPECT TO QUASI-PERIODIC BACKGROUNDS

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Abstract. We investigate the algebraic conditions the scattering data of short-range perturbations of quasi-periodic finite-gap Jacobi operators have to satisfy. As our main result we provide the Poisson-Jensen-type formula for the transmission coefficient in terms of Abelian integrals on the underlying hyperelliptic Riemann surface and give an explicit condition for its single-valuedness. In addition, we establish trace formulas which relate the scattering data to the conserved quantities in this case.

1. Introduction

Solitons are a key feature of completely integrable wave equations and there are usually two ways of constructing them: One is via the inverse scattering transform by choosing an arbitrary number of eigenvalues and setting the reflection coefficient equal to zero. The other is by inserting the eigenvalues using commutation methods. This works fine in case of a constant background and the eigenvalues can be chosen arbitrarily. However, in case of a nontrivial background, e.g., a periodic solution, it turns out that the eigenvalues need to satisfy certain restrictions. This was probably first observed in [14], where they showed that adding one eigenvalue to the two-gap Weierstrass solution of the Korteweg-de Vries (KdV) equation preserves the asymptotics on one side, but gives a phase shift on the other side. The general case was solved in [8]. In particular, this shows that the eigenvalues and reflection coefficients can no longer be prescribed independently if one wants to stay in the class of short-range perturbations of a given quasi-periodic background. It turns out that these constraints are related to the fact that the resolvent set of the background operator is not simply connected in the (quasi-)periodic case. In this case we have to reconstruct the transmission coefficient from its boundary values on this non simply connected domain which is only possible in terms of multi-valued functions in general, see [25]. Hence one needs to impose algebraic constraints on the scattering data to obtain a single-valued transmission coefficient. It seems that this was first emphasized in [4].

The aim of the present paper is to make this reconstruction explicit in terms of Abelian integrals on the underlying hyperelliptic Riemann surface for the case of Jacobi operators (respectively the Toda equation). However, similar results apply to one dimensional Schrödinger operators (respectively the KdV equation). This will then allow us to derive an explicit condition for single-valuedness and to establish trace formulas which relate the scattering data to conserved quantities for the Toda

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hierarchy. In particular, these trace formulas are extensions of well-known sum rules (see e.g. [3], [11], [15], [17], [18], [26]) which have attracted an enormous amount of interest recently.

To archive this aim we will first compute the Green function, harmonic measure, and Blaschke factors for our domain. This case seems to be hard to find in the literature; the only example we could find is the elliptic case in the book by Akhiezer [1]. See however also [21], [22], where similar questions are investigated. An appendix computes the phase shift caused by the insertion of solitons via commutation methods.

2. Notation

To set the stage, let $\mathbb{M}$ be the Riemann surface associated with the following function

$$R_{2g+2}^{1/2}(z), \qquad R_{2g+2}(z) = \prod_{j=0}^{2g+1} (z - E_j), \qquad E_0 < E_1 < \cdots < E_{2g+1}, \tag{2.1}$$

$g \in \mathbb{N}$. $\mathbb{M}$ is a compact, hyperelliptic Riemann surface of genus g. We will choose $R_{2g+2}^{1/2}(z)$ as the fixed branch

$$R_{2g+2}^{1/2}(z) = -\prod_{j=0}^{2g+1} \sqrt{z - E_j}, \tag{2.2}$$

where $\sqrt{.}$ is the standard root with branch cut along $(-\infty, 0)$.

A point on $\mathbb{M}$ is denoted by $p = (z, \pm R_{2g+2}^{1/2}(z)) = (z, \pm)$, $z \in \mathbb{C}$, or $p = \infty_\pm$, and the projection onto $\mathbb{C} \cup \{\infty\}$ by $\pi(p) = z$. The points $\{(E_j, 0), 0 \le j \le 2g+1\} \subseteq \mathbb{M}$ are called branch points and the sets

$$\Pi_\pm = \{(z, \pm R_{2g+2}^{1/2}(z)) \mid z \in \mathbb{C}\backslash\Sigma\} \subset \mathbb{M}, \qquad \Sigma = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}], \tag{2.3}$$

are called upper and lower sheet, respectively. Note that the boundary of $\Pi_\pm$ consists of two copies of Σ corresponding to the two limits from the upper and lower half plane.

Let $\{a_j, b_j\}_{j=1}^g$ be loops on the Riemann surface $\mathbb{M}$ representing the canonical generators of the fundamental group $\pi_1(\mathbb{M})$. We require a_j to surround the points E_{2j-1}, E_{2j} (thereby changing sheets twice) and b_j to surround E_0, E_{2j-1} counterclockwise on the upper sheet, with pairwise intersection indices given by

$$a_j \circ a_k = b_j \circ b_k = 0, \qquad a_j \circ b_k = \delta_{jk}, \qquad 1 \le j, k \le g. \tag{2.4}$$

The corresponding canonical basis $\{\zeta_j\}_{j=1}^g$ for the space of holomorphic differentials can be constructed by

$$\underline{\zeta} = \sum_{j=1}^{g} \underline{c}(j) \frac{\pi^{j-1} d\pi}{R_{2g+2}^{1/2}}, \tag{2.5}$$

where the constants $\underline{c}(.)$ are given by

$$c_j(k) = C_{jk}^{-1}, \qquad C_{jk} = \int_{a_k} \frac{\pi^{j-1} d\pi}{R_{2g+2}^{1/2}} = 2 \int_{E_{2k-1}}^{E_{2k}} \frac{z^{j-1} dz}{R_{2g+2}^{1/2}(z)} \in \mathbb{R}.$$

The differentials fulfill

$$\int_{a_j} \zeta_k = \delta_{j,k}, \qquad \int_{b_j} \zeta_k = \tau_{j,k}, \qquad \tau_{j,k} = \tau_{k,j}, \qquad 1 \le j,k \le g. \tag{2.6}$$

For further information we refer to [6], [20, App. A].

3. Algebro-geometric constraints

We are motivated by scattering theory for the pair (H, H_q) of two Jacobi operators, where H is a short-range perturbation of a quasi-periodic finite-gap operator H_q associated with the Riemann surface introduced in the previous section (see [20, Ch. 9]). One key quantity is the transmission coefficient $T(z)$. It is meromorphic in Π_+ with finitely many simple poles in $\Pi_+ \cap \mathbb{R}$ precisely at the eigenvalues of the perturbed operator H. Since

$$|T(\lambda)|^2 + |R_\pm(\lambda)|^2 = 1, \qquad \lambda \in \Sigma, \tag{3.1}$$

it can be reconstructed from the reflection coefficients $R_\pm(\lambda)$ once we show how to reconstruct $T(z)$ from its boundary values $|T(\lambda)|^2 = 1 - |R_\pm(\lambda)|^2$, $\lambda \in \partial\Pi_+$. Rather than enter into more details here, see [4] (respectively [24]), we will focus on the reconstruction procedure only.

We begin by deriving a formula for the Green function of Π_+:

Lemma 3.1. *The Green function of Π_+ with pole at p_0 is given by*

$$g(z, z_0) = -\mathrm{Re} \int_{E_0}^{p} \omega_{p_0 \tilde{p}_0}, \quad p = (z,+),\ p_0 = (z_0,+), \tag{3.2}$$

where $\tilde{p}_0 = \overline{p_0}^*$ (i.e., the complex conjugate on the other sheet) and ω_{pq} is the normalized Abelian differential of the third kind with poles at p and q.

Proof. First of all observe $\omega_{p_0\tilde{p}_0} = \omega_{p_0 E_0} - \omega_{\tilde{p}_0 E_0}$ and set

$$\omega_{p_0 E_0} = r_\pm(z, z_0) dz \tag{3.3}$$

on $\Pi_\pm$. Since $\omega_{p_0 E_0}$ is continuous on the branch cuts, the corresponding values of $r_\pm$ must match up, that is,

$$\lim_{\varepsilon \downarrow 0} r_+(\lambda + \mathrm{i}\varepsilon, z_0) = \lim_{\varepsilon \downarrow 0} r_-(\lambda - \mathrm{i}\varepsilon, z_0), \quad \lambda \in \Sigma. \tag{3.4}$$

Moreover,

$$\omega_{\tilde{p}_0 E_0} = \overline{r_\mp(\overline{z}, z_0)} dz \tag{3.5}$$

on $\Pi_\pm$. Hence,

$$\omega_{p_0\tilde{p}_0} = \lim_{\varepsilon \downarrow 0} (r_+(\lambda + \mathrm{i}\varepsilon, z_0) - \overline{r_-(\lambda - \mathrm{i}\varepsilon, z_0)} d\lambda = 2\mathrm{i}\,\mathrm{Im}(r(\lambda, z_0)) d\lambda, \quad \lambda \in \Sigma, \tag{3.6}$$

where $r(\lambda, z_0) = \lim_{\varepsilon\downarrow 0} r_+(\lambda + \mathrm{i}\varepsilon, z_0)$, shows that $\omega_{p_0,\tilde{p}_0}$ is purely imaginary on the boundary of Π_+. Together with the fact that the a-periods of $\omega_{p_0\tilde{p}_0}$ vanish this shows $\int_{E_0}^{p} \omega_{p_0\tilde{p}_0}$ is purely imaginary on $\partial\Pi_+$. Hence $g(z, z_0)$ vanishes on $\partial\Pi_+$ and since it has the proper singularity at z_0 by construction, we are done. □

Clearly, we can extend $g(z, z_0)$ to a holomorphic function on $\mathbb{M}\backslash\{z_0\}$ by dropping the real part. By abuse of notation we will denote this function by $g(p, p_0)$ as well. However, note that $g(p, p_0)$ will be multi-valued with jumps in the imaginary part across b-cycles. We will choose the path of integration in $\mathbb{C}\backslash[E_0, E_{2g+1}]$ to guarantee a single-valued function.

From the Green's function we obtain the Blaschke factor and the harmonic measure (see e.g., [23]). Since we are mainly interested in the case where the poles are on the real line (since $T(z)$ has all poles on the real line), we note the following relation which will be needed later on:

Lemma 3.2. *For ρ with $\pi(\rho) \in \mathbb{R}\backslash\Sigma$ we have*

$$g(p,\rho) = \int_{E_0}^{p} \omega_{\rho\rho^*} = \int_{E(\rho)}^{\rho} \omega_{pp^*}, \tag{3.7}$$

where $E(\rho)$ is E_0 if $\rho < E_0$, either E_{2j-1} or E_{2j} if $\rho \in (E_{2j-1}, E_{2j})$, $1 \le j \le g$, and E_{2g+1} if $\rho > E_{2g+1}$.

Proof. By symmetry of the Green's function this holds at least when taking real parts. Since both quantities are real for $\pi(p) < E_0$ it holds everywhere. □

Now we come to the Blaschke factor

$$B(p,\rho) = \exp\Big(g(p,\rho)\Big) = \exp\Big(-\int_{E_0}^{z} \omega_{\rho\rho^*}\Big), \qquad \pi(\rho) \in \mathbb{R}, \tag{3.8}$$

and first show that it can be written in terms of theta functions.

Lemma 3.3. *The Blaschke factor is given by*

$$B(p,\rho) = \frac{\theta(\underline{A}_{p_0}(\rho^*) + \underline{\hat{\alpha}}_{p_0}(\mathcal{D}) - \underline{\Xi}_{p_0})}{\theta(\underline{A}_{p_0}(\rho) + \underline{\alpha}_{p_0}(\mathcal{D}) - \underline{\Xi}_{p_0})} \frac{\theta(\underline{A}_{p_0}(p) - \underline{A}_{p_0}(\rho) - \underline{\alpha}_{p_0}(\mathcal{D}) + \underline{\Xi}_{p_0})}{\theta(\underline{A}_{p_0}(p) - \underline{A}_{p_0}(\rho^*) - \underline{\hat{\alpha}}_{p_0}(\mathcal{D}) + \underline{\Xi}_{p_0})}, \tag{3.9}$$

where $\mathcal{D}$ is any divisor of degree $g-1$ such that $\mathcal{D}_\rho + \mathcal{D}$ and $\mathcal{D}_{\rho^*} + \mathcal{D}$ are nonspecial.

In addition, it satisfies

$$B(E_0,\rho) = 1, \quad \textit{and} \quad B(p^*,\rho) = B(p,\rho^*) = B(p,\rho)^{-1}; \tag{3.10}$$

it is real-valued for $\pi(p) \in (-\infty, E_0)$.

Proof. Both the Blaschke factor and the quotient of theta functions are multi-valued meromorphic functions. Invoking the bilinear relations shows that we have the same jumps $\int_{\tilde{\rho}}^{\rho} \zeta_\ell$ around b-cycles. Hence their quotient is single-valued. Moreover both have the same divisor $\mathcal{D}_\rho - \mathcal{D}_{\tilde{\rho}}$ and hence the quotient is holomorphic and thus constant.

To see that $B(p^*,\rho)B(p,\rho) = 1$, note that this function has no jumps and no poles and hence is constant. Since it is one at E_0 it is one everywhere. □

Next we compute the harmonic measure of $\partial\Pi_+$.

Lemma 3.4. *The harmonic measure of $\partial\Pi_+$ with pole at λ is given by*

$$\mu(p,\lambda)d\lambda = \frac{1}{\pi}\mathrm{Im}\,\omega_{pE_0}(\lambda) = -\frac{1}{\pi}\mathrm{Im}\left(\int_{E_0}^{p} \omega_{\lambda,0}\right) d\lambda, \tag{3.11}$$

where $\omega_{\lambda,0}$ is the normalized Abelian differential of the second kind with a second-order pole at λ.

Proof. All we have to do is to compute $(2\pi)^{-1} \lim_{\varepsilon\downarrow 0} \frac{\partial}{\partial\varepsilon} g(z, \lambda \pm \mathrm{i}\varepsilon)$ (where the sign is chosen according to which side of Σ one is interested):

$$-\frac{\partial}{\partial\varepsilon}\mathrm{Re}\int_{E_0}^{\lambda\pm\mathrm{i}\varepsilon} \omega_{p\tilde{p}}\Big|_{\varepsilon=0} = \mathrm{Im}\,\omega_{p\tilde{p}} = 2\mathrm{Im}\,\omega_{pE_0} \tag{3.12}$$

since $\omega_{p\tilde{p}} = \omega_{pE_0} - \omega_{\tilde{p}E_0}$.

The other formula follows similarly:

$$-\frac{\partial}{\partial\varepsilon}\mathrm{Re}\int_{E_0}^{p}\omega_{p_0\tilde{p}_0}\Big|_{\varepsilon=0}=-2\mathrm{Im}\int_{E_0}^{p}\omega_{\lambda,0} \tag{3.13}$$

since $\frac{\partial}{\partial z_0}\omega_{p_0 E_0}=-\omega_{z_0,0}$. □

Note that

$$\omega_{pE_0}=\left(\frac{-1}{2(\lambda-E_0)}+o(1)\right)d\lambda$$

for λ near E_0 and that the imaginary part has no singularity for $\lambda\in\partial\Pi_+$.

Now we can characterize the scattering data ([4], [5]):

Theorem 3.5. *Let $T(z)$ be meromorphic in Π_+ with simple poles at $\{\rho_j\}_{j=1}^q\subseteq\mathbb{R}\backslash\Sigma$ such that T is continuous up to the boundary with the only possible simple zeros at the branch points.*

Then $T(z)$ can be recovered from the boundary values $\ln|T(\lambda)|$, $\lambda\in\Sigma$, via the Poisson-Jensen-type formula

$$T(z)=\left(\prod_{j=1}^{q}B(p,\rho_j)^{-1}\right)\exp\left(\frac{1}{2\pi\mathrm{i}}\int_{\partial\Pi_+}\ln|T|^2\omega_{pE_0}\right),\quad p=(z,+), \tag{3.14}$$

where we have identified ρ_j with $(\rho_j,+)$ and defined $T(p)=\lim_{\varepsilon\downarrow 0}T(\lambda\pm\varepsilon)$, $p=(\lambda,\pm)\in\partial\Pi_+$.

Proof. The formula for $T(z)$ holds by [25, Thm. 1], when taking absolute values. Since both sides are analytic and have equal absolute values, they can only differ by a constant of absolute value one. But both sides are positive at $z=\infty$ and hence this constant is one. □

Remark 3.6. *A few remarks are in order:*

(i) *The integrand in (3.14) is not integrable at E_0 and the integral has to be understood as a principal value. Otherwise, one can move the singularity away from $\partial\Pi_+$ which just alters the value by a constant.*

(ii) *In scattering theory one has $|T(p^*)|=|T(p)|$, $p\in\partial\Pi_+$, and under this assumption we have*

$$T(z)=\left(\prod_{j=1}^{q}\exp\left(-\int_{E(\rho_j)}^{\rho_j}\omega_{pp^*}\right)\right)\exp\left(\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\ln(1-|R_\pm|^2)\omega_{pp^*}\right),$$

where $E(\rho)$ is defined in Lemma 3.2 and the integral over Σ is taken on the upper sheet.

(iii) *The Abelian differential is explicitly given by*

$$\omega_{pq}=\left(\frac{R_{2g+2}^{1/2}+R_{2g+2}^{1/2}(p)}{2(\pi-\pi(p))}-\frac{R_{2g+2}^{1/2}+R_{2g+2}^{1/2}(q)}{2(\pi-\pi(q))}+P_{pq}(\pi)\right)\frac{d\pi}{R_{2g+2}^{1/2}},$$

where $P_{pq}(z)$ is a polynomial of degree $g-1$ which has to be determined from the normalization $\int_{a_\ell}\omega_{pp^*}=0$. In particular,

$$\omega_{pp^*}=\left(\frac{R_{2g+2}^{1/2}(p)}{\pi-\pi(p)}+P_{pp^*}(\pi)\right)\frac{d\pi}{R_{2g+2}^{1/2}}.$$

In inverse scattering theory one uses (3.1) to reconstruct T from the reflection coefficient R_+ (or R_-). Since neither the Blaschke factors nor the outer function in (3.14) are in general single-valued on Π_+, we are naturally interested in when T is single-valued for given $R_\pm$.

Theorem 3.7. *The transmission coefficient T defined via (3.14) is single-valued if and only if the eigenvalues ρ_j and the reflection coefficient $R_\pm$ satisfy*

$$\sum_j \int_{\rho_j^*}^{\rho_j} \zeta_\ell - \frac{1}{2\pi \mathrm{i}} \int_{\partial\Pi_+} \ln(1-|R_\pm|^2)\zeta_\ell \in \mathbb{Z}. \tag{3.15}$$

Proof. For $T(z)$ to be single-valued we need $\lim_{\varepsilon\downarrow 0} T(x-\mathrm{i}\varepsilon) = \lim_{\varepsilon\downarrow 0} T(x+\mathrm{i}\varepsilon)$ for every x in a spectral gap. If $x \in (E_{2\ell-1}, E_{2\ell})$ is in the ℓ'th gap, the path of integration from E_0 to λ from above and back from λ to E_0 from below just gives the b-cycle b_ℓ. Hence,

$$\lim_{\varepsilon\downarrow 0} \frac{T(x+\mathrm{i}\varepsilon)}{T(x-\mathrm{i}\varepsilon)} = \exp\left(\sum_{j=1}^q \int_{b_\ell} \omega_{\rho_j,\tilde\rho_j} - \frac{1}{2\pi\mathrm{i}} \int_{\partial\Pi_+} \ln(1-|R_\pm(\lambda)|^2) \int_{b_\ell} \omega_{\lambda,0} d\lambda \right). \tag{3.16}$$

Evaluating the b_ℓ-cycle using the usual bilinear relations finally yields

$$\lim_{\varepsilon\downarrow 0} \frac{T(x+\mathrm{i}\varepsilon)}{T(x-\mathrm{i}\varepsilon)} = \exp\left(2\pi\mathrm{i} \sum_{j=1}^q \int_{\tilde\rho_j}^{\rho_j} \zeta_\ell - \int_{\partial\Pi_+} \ln(1-|R_\pm|^2)\zeta_\ell \right) \tag{3.17}$$

and if the limit is supposed to be one, we are lead to (3.15). □

4. Trace formulas

The transmission coefficient also plays a central role in the inverse scattering transform. Since it turns out to be the perturbation determinant of the pair (H, H_q), in the sense of Krein ([12], [13]), its asymptotic expansion provides the conserved quantities of the Toda hierarchy ([5], [19], [16]),

$$\frac{d}{dz} \ln T(z) = -\sum_{k=1}^\infty \frac{\tau_k}{z^{k+1}}, \qquad \tau_k = \mathrm{tr}(H^k - (H_q)^k). \tag{4.1}$$

Relating this expansion with the one obtained by expanding (3.14) near $z = \infty$, one obtains the usual trace formulas (also known as Case-type sum rules, [3]).

Next, let ω_j be the meromorphic differential

$$\omega_0 = \omega_{\infty_+\infty_-}, \qquad \omega_k = \omega_{\infty_+;k-1} - \omega_{\infty_-;k-1}, \tag{4.2}$$

where $\omega_{p,k}$ is the Abelian differential of the second kind with a pole of order $k+2$ at p. Note that ω_k is of the form

$$\omega_k = \frac{P_k(\pi)}{R_{2g+2}^{1/2}} d\pi, \tag{4.3}$$

where $P_k(z)$ is a monic polynomial of degree $g+k$ whose coefficients have to be determined from the fact that the a-cycles vanish and from the behavior at $\infty_\pm$ (see [20, Eq. (13.30)]).

Theorem 4.1. *The following trace formulas are valid:*

$$\begin{aligned} \ln(T(\infty)) &= -\sum_{j=1}^{q} \int_{E(\rho_j)}^{\rho_j} \omega_{\infty_+ \infty_-} + \frac{1}{\pi \mathrm{i}} \int_{\Sigma} \ln |T|\, \omega_{\infty_+ \infty_-}, \\ \frac{1}{k+1} \tau_{k+1} &= -\sum_{j=1}^{q} \int_{E(\rho_j)}^{\rho_j} \omega_k + \frac{1}{\pi \mathrm{i}} \int_{\Sigma} \ln |T|\, \omega_k, \end{aligned} \tag{4.4}$$

where $E(\rho)$ is defined in Lemma 3.2 and the integral over Σ is taken on the upper sheet.

Proof. By $\frac{d^k}{dz^k} \omega_{p(z) E_0}|_{z=0} = k!\, \omega_{p_0, k-1}$, where z is a coordinate centered at p_0, we have

$$\omega_{p E_0} = \omega_{\infty_+ E_0} + \sum_{k=1}^{\infty} z^k \omega_{p, k-1}, \qquad p = (\frac{1}{z}, +), \tag{4.5}$$

and the claim follows. □

5. Appendix: N-soliton solutions

In this section we investigate commutation methods when applied to a quasi-periodic finite-gap background solution. In particular, we compute the phase shift (in the Jacobian variety) introduced by the solitons. As pointed out in the introduction, this can be found for the case of one-dimensional Schrödinger operators in [8] (see also [10] for the elliptic case). The case of Jacobi operators seems to be missing and hence we provide the corresponding results to fill this gap. We want to be rather brief and refer to [9] or [20, Ch. 11] for further details in this connection.

We start by inserting an eigenvalue using the single commutation method.

Lemma 5.1. *Let H be a quasi-periodic finite-gap operator associated with the Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ and let H_σ, $-1 < \sigma < 1$, be the commuted operator as in* [20, Sect. 11.2]. *Then we have*

$$H_\sigma \sim \left\{ \begin{array}{ll} H_{+1} & \text{as } n \to -\infty \\ H_{-1} & \text{as } n \to +\infty \end{array} \right. , \tag{5.1}$$

where $H_{\pm 1}$ are the quasi-periodic finite-gap operators associated with the Dirichlet divisors $\mathcal{D}_{\underline{\hat{\mu}}_{\pm 1}}$ defined via

$$\underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}_{\pm 1}}) = \underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}}) - \underline{A}_{p_0}(p_1) - \underline{A}_{p_0}(\infty_+), \qquad p_1 = (\lambda_1, \pm). \tag{5.2}$$

Proof. That $H_{\pm 1}$ is associated with the divisor $\mathcal{D}_{\underline{\hat{\mu}}_{\pm 1}}$ is shown in [20, Ch. 11.5] and the asymptotics follow from the corresponding asymptotics for $u_{\sigma_1}(\lambda_1, n)$. □

Similarly, one obtains the following result for the double commutation method.

Lemma 5.2. *Let H be a quasi-periodic finite-gap operator associated with the Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ and let H_γ, $0 < \gamma < \infty$ be the doubly commuted operator as in* [20, Sect. 11.6]. *Then we have*

$$H_\gamma \sim \left\{ \begin{array}{ll} H_0 & \text{as } n \to -\infty \\ H_\infty & \text{as } n \to +\infty \end{array} \right. , \tag{5.3}$$

where $H_0 = H$ and H_∞ is the quasi-periodic finite-gap operator associated with the Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}_\infty}$ defined via

$$\underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}_{\lambda_1}}) = \underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + 2\underline{A}_{p_0}(\hat{\lambda}_1), \qquad \hat{\lambda}_1 = (\lambda_1, +). \tag{5.4}$$

Proof. Since the double commutation method can be obtained via two single commutation steps (see [20, Sect. 11.5]), the result is a consequence of our previous lemma. □

Clearly, if we add k eigenvalues $\lambda_1, \dots, \lambda_k$, then the asymptotics at $+\infty$ are given by

$$\underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}_{\lambda_1,\dots,\lambda_k}}) = \underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + 2\sum_{j=1}^{k} \underline{A}_{p_0}(\hat{\lambda}_j), \qquad \hat{\lambda}_j = (\lambda_j, +). \tag{5.5}$$

In particular, by choosing at least one eigenvalue in each gap, we can attain any prescribed asymptotics in the given isospectral class by Lemma 9.1 in [20].

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