Randomly forced nonlinear PDEs and statistical hydrodynamics in 2 space dimensions

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0 Introduction

This short book is based on a lecture–course on the randomly forced two–dimensional Navier–Stokes equation (2D NSE) and two–dimensional statistical hydrodynamics which the author taught at ETH-Zürich during the winter term of the year 2004/2005. The goal of the course was to review the recent progress in the qualitative theory of randomly forced nonlinear PDE (especially, the 2D NSE), and discuss applications of the corresponding results to the 2D statistical hydrodynamics, including the 2D turbulence. The book, as well as the lecture-course, is aimed on people with some background in PDE, or in probability, or in physics. For the benefit of the last two groups of readers we included in the book a section on the deterministic 2D NSE.

Due to the luck of time, the lectures and this book do not include all relevant material. The author restricted himself to the results, related to his current scientific interests — the statistical hydrodynamics of randomly forced two–dimensional fluids. So some important relevant topics are not represented in the book. Probably, the most serious omissions are the results on the free NSE with random initial data. Concerning them we refer the reader to the books [VF88] and [FMRT01]. Some important results on randomly forced 2D fluids also are not covered by the book. With the exception of very short Section 6.5 we avoid the randomly forced 3D NSE since not much is known there, and what is known differs in spirit from the 2D results we are interested in. See [Fla05].

The book contains only rigorously proven theorems. Connections with the (heuristic) theory of turbulence are reduced to short discussions on relevance of the obtained results to the theory of turbulence, made at the ends of the main sections. There we show that the theorems form a rigorous mathematical foundation for the theory of 2D space-periodic turbulence. In particular, the results obtained imply that

i) when time grows, statistical characteristics of a turbulent flow stabilise to characteristics, independent from the initial velocity field (Sections 6, 7);

ii) for any characteristic of a turbulent flow, its time-average equals the ensemble-average (Section 8);

iii) in large time-scale the turbulent flow is a Gaussian process (Section 9).

In the last two sections of the book we prove and discuss some recent results, which seem to be unknown to experts in turbulence. Namely, we show that

iv) when the coefficient of kinematic viscosity decays to zero and the random force, applied to the fluid, is scaled to keep the energy of the fluid of order one, the solution of the 2D NSE converges in distribution to a random field such that each its realisation satisfies the free 2D Euler equation (Section 10),

v) stationary in space and time solutions of randomly forced 2D NSE satisfy infinitely many explicit algebraic relations (i.e. "space-periodic 2D turbulence is integrable"; Section 11).

The results i)–v) follow from rigorous analysis of the randomly forced 2D NSE

$$\dot u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \eta(t,x)\,, \quad \operatorname{div} u = 0\,, \tag{0.1}$$

where η is a random field. Usually the equation is supplemented by the periodic boundary conditions

$$x \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2\,.$$

Eq. (0.1) is the main object, studied in this book. Sections 1–5 contain preliminaries, and the rest of the book – the new results on the equation (which imply the assertions i)–v) above).

Most of the results in Sections 6–9 hold true for eq. (0.1) in a bounded domain with suitable boundary conditions (say, Dirichlet), or in a two-dimensional compact Riemann surface, e.g. in a sphere (if the action of the Laplacian on vector-fields $u(x)$ is defined accordingly). More generally, the results hold for solutions of many nonlinear dissipative equations in bounded domains (or in a torus), perturbed by a random force. In particular, for the reaction-diffusion equation

$$\dot u - \nu\Delta u + u^3 = \eta(t,x)\,; \tag{0.2}$$

or for the Ginsburg–Landau equation

$$\dot u - \nu\Delta u + i|u|^2 u = \eta(t,x), \tag{0.3}$$

where $u(t,\,x)\in\mathbb{C}$ and $\dim x \le 3$; or for the equation

$$\dot u - (\nu + i)\Delta u + i|u|^2 u = \eta(t,x), \tag{0.4}$$

where $u(t,\,x)\in\mathbb{C}$, $\dim x \le 4$. From time to time we briefly discuss these equations and properties of their solutions, similar to those of 2D NSE.

In contrast, the results of Section 10 only hold for eq. (0.1) with *some* boundary conditions. For example, they do not apply to (0.1) with the Dirichlet boundary conditions, but they hold for the equation on a Riemann surface. Moreover, the results are valid for some other equations. In particular — for eq. (0.4).

The results of Section 11 are the most rigid: they only hold for the 2D NSE (0.1) under the periodic boundary conditions (so only the periodic 2D turbulence is integrable, cf. v) above).

In this book we do not discuss properties of equations (0.2)–(0.4) which have no proven analogies for the 2D NSE (e.g., see [Kuk97, Kuk99] for a study of (0.3) when $\nu\to 0$). Similar, we do not touch the problem of Burgers turbulence, described by the randomly forced Burgers equation (see [EKMS00]).

We consider two classes of ransom forces η: they are either Gaussian random fields, smooth in x and white as functions of t, or they are kick–processes as functions of t, smooth in x. In the former case the equations define stochastic (in Ito's sense) differential equations in function spaces, while in the latter case they define Markov chains in function spaces. All our results, apart from those in Section 11, hold for both classes of the forces. We think that this is important since it indicates that the results obtained for the 2D NSE (0.1) are not properties of a specific model, but of the 2D statistical hydrodynamics.

Notations. We define $\mathbb{Z}^2_0 = \mathbb{Z}^2\backslash\{0\}$. For a Banach space X we set

$$\mathfrak{B}_r(X) = \{x\in X \mid \|x\|_X \le r\}\,.$$

By $\mathcal{D}(\xi)$ we denote the distribution of a random variable ξ. Each metric space M is provided with the σ-algebra of its Borel sets $\mathcal{B}(\mathcal{M})$ (so 'measurable' means 'Borel-measurable'). We denote by $C_b(M)$ the space of bounded continuous function on M, by $\mathcal{M}(M)$ — the set of finite signed Borel measures, and by $\mathcal{P}(M)$ — the probability Borel measures on M. For $f \in C_b(M)$ and $\mu \in \mathcal{M}(M)$ we define $(f,\mu) = (\mu, f) = \int_M f(u)\,\mu(du)$. By I_Q we denote the characteristic function of a set Q.

We adopt the Einstein rule of summation over repeated indexes.

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1 Function spaces

1.1 Function spaces for functions of x

Let Q be an open domain of $\mathbb{R}^d$ or the torus $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$.

Lebesgue spaces.

We denote with $L_p(Q;\mathbb{R}^n), 1 \le p \le \infty$, the usual Lebesgue spaces of vector-valued functions, abbreviate $L_p(Q;\mathbb{R}) = L_p(Q)$, and denote the L_2 scalar-product by $\langle\cdot,\cdot\rangle$.

Sobolev spaces $W^{m,p}(Q;\mathbb{R}^n)$.

Let $C_c^\infty(Q;\mathbb{R}^n)$ be the space of infinitely differentiable maps $\phi : Q \to \mathbb{R}^n$ with compact support in Q. Suppose u, v are locally integrable functions on Q and $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multiindex. We say that v is the α^{th}-weak partial derivative of u, written $D^\alpha u = v$, provided $\int_Q u D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_Q v\phi\, dx$, for all test functions $\phi \in C_c^\infty(Q)$. Here $|\alpha| \overset{\text{def}}{=} \alpha_1 + \ldots + \alpha_d$.

Let $m \in \mathbb{N}$ and $1 \le p \le \infty$. The space $W^{m,p}(Q,\mathbb{R}^n)$ consists of all locally integrable functions $u : Q \to \mathbb{R}^n$ such that for each multiindex α with $|\alpha| \le m$, $D^\alpha u$ exists in the weak sense and belongs to $L_p(Q;\mathbb{R}^n)$. We shall only use these spaces with $p = 2$ and adopt the notations:

$$W^{m,2}(Q;\mathbb{R}^n) = H^m(Q;\mathbb{R}^n)\,, \quad H^m(Q;\mathbb{R}^1) = H^m(Q)\,.$$

If $u \in H^m(Q;\mathbb{R}^n)$, we define its norm to be:

$$\|u\|_m = \Big(\sum_{|\alpha|\le m} |D^\alpha u|_2^2\Big)^{1/2}$$

When analysing eq. (0.1), we will mostly restrict ourselves to the case where the domain Q is the torus $\mathbb{T}^2 \overset{\text{def}}{=} \mathbb{R}^2/\mathbb{Z}^2$.

(S1) If $Q = \mathbb{T}^d$, then $u \in L_2$ can be written as $u(x) = \sum_{s\in\mathbb{Z}^d} u_s e^{is\cdot x}$. It is then possible to define $H^m(\mathbb{T}^d;\mathbb{R}^n)$ even for $m \in \mathbb{R}$. For this purpose, we define a norm which is equivalent to $\|\cdot\|_m$ in (1.1) if $m \in \mathbb{N}$:

$$\|u\|_m^2 = \sum_{s\in\mathbb{Z}^d} \big(1 + |s|^2\big)^m |u_s|^2, \quad m \in \mathbb{R}$$

Now for $m \ge 0$ we set $H^m(\mathbb{T}^d;\mathbb{R}^n) = \{u \in L^2(\mathbb{T}^d;\mathbb{R}^n) : \|u\|_m^2 < \infty\}$, and for $m < 0$ define the space $H^m(\mathbb{T}^d;\mathbb{R}^n)$ as the closure of $L_2(\mathbb{T}^d;\mathbb{R}^n)$ in the $\|\cdot\|_m$-norm.

Lemma 1.1. *For any $r \in \mathbb{R}$ and any multiindex α the linear map D^α is continuous from $H^r(\mathbb{T}^d;\mathbb{R}^n)$ to $H^{r-|\alpha|}(\mathbb{T}^d;\mathbb{R}^n)$. Accordingly, the map $\Delta : H^r(\mathbb{T}^d;\mathbb{R}^n) \to H^{r-2}(\mathbb{T}^d;\mathbb{R}^n)$ is continuous.*

(S2) If $\langle u\rangle \overset{\text{def}}{=} \int_{\mathbb{T}^d} u(x)\,dx = 0$, then $u_0 = 0$. Therefore in the space

$$H_0^m(\mathbb{T}^d;\mathbb{R}^n) = \{u \in H^m(\mathbb{T}^d;\mathbb{R}^n) \mid \langle u\rangle = 0\}$$

the norm can be equivalently defined by the relation

$$\|u\|_m^2 = \sum_{s\neq 0} |s|^{2m}|u_s|^2 \,.$$

In particular, $\|u\|_1^2 = |\nabla u|_2$.

(S3) *Sobolev Embeddings*

Let Q be an open subset of $\mathbb{R}^d$ with a Lipschitz boundary, or the torus $\mathbb{T}^d$.

1. If $m \le \frac{d}{2}$ and $2 \le q \le \frac{2d}{d-2m}$, $q<\infty$, then
$$H^m(Q;\mathbb{R}^n) \subset L_q(Q;\mathbb{R}^n) \tag{1.1}$$
2. If $m > \frac{d}{2}+\alpha$, $0\le\alpha<1$, then
$$H^m(Q;\mathbb{R}^n) \subset C^\alpha(Q;\mathbb{R}^n) \tag{1.2}$$
$C^\alpha(Q)$, $\alpha>0$, denotes the space of α-Hölder continuous functions and C^0 is the space of continuous functions.
3. If Q is an open bounded subset of $\mathbb{R}^d$ with Lipschitz boundary, or if $Q=\mathbb{T}^d$, then the embedding (1.2) is compact, and the embedding (1.1) is compact as far as $q<\frac{2d}{d-2m}$. Besides, in this case
$$H^{m_1}(Q;\mathbb{R}^n) \Subset H^{m_2}(Q;\mathbb{R}^n) \quad \text{if } \; m_1 > m_2. \tag{1.3}$$

Exercise 1.2. *Prove* (1.2) *for* $\alpha=0$ *and* $Q=\mathbb{T}^d$.

Solution: We have to show that $H^m(Q;\mathbb{R}^n)\subset C^0(Q;\mathbb{R}^n)$ if $m>\frac{d}{2}$. It is clear that $u=\sum_{s\in\mathbb{Z}^d} u_s e^{is\cdot x}$ with $\sum|u_s|<\infty$ is continuous. So it suffice to check that $\sum|u_s|<\infty$ for $u\in H^m(Q;\mathbb{R}^n)$ with $m>\frac{d}{2}$. We have:

$$\sum_{s\in\mathbb{Z}^d} |u_s|(1+|s|^2)^{-m/2}(1+|s|^2)^{m/2}$$
$$\le \left(\sum_{s\in\mathbb{Z}^d} |u|^2(1+|s|^2)^m\right)^{1/2} \left(\sum_{s\in\mathbb{Z}^d}(1+|s|^2)^{-m}\right)^{1/2} .$$

The first factor on the r.h.s. is finite since $u\in H^m(Q;\mathbb{R}^n)$. The second factor is finite because $2m>d$ and

$$\sum_{s\in\mathbb{Z}^d}(1+|s|^2)^{-m} \le 1 + \sum_{s\in\mathbb{Z}^d\setminus\{0\}} |s|^{-2m} \le c + c'\int_{|x|>1} |x|^{-2m}dx < \infty. \quad \square$$

Example: If $Q=\mathbb{T}^2$, then

$$H^1(Q)\Subset L_q(Q) \quad \forall q<\infty, \tag{1.4}$$

and

$$H^{\frac12}(Q)\subset L_4(Q). \tag{1.5}$$

(S4) The spaces $H^m(\mathbb{T}^d;\mathbb{R}^n)$ and $H^{-m}(\mathbb{T}^d;\mathbb{R}^n)$ are dual:

$$\forall u\in C^\infty(\mathbb{T}^d;\mathbb{R}^n), \quad \|u\|_m = \sup_{v\in C^\infty, \|v\|_{-m}\le 1} \langle u,v\rangle \,.$$

Exercise 1.3. *Prove this relation.*

In particular, the scalar product in L^2 extends to a continuous bilinear map $H^m(\mathbb{T}^d;\mathbb{R}^n)\times H^{-m}(\mathbb{T}^d;\mathbb{R}^n)\to\mathbb{R}$.

(S5) *Interpolation inequality.* Let $a<b$ and $0\le\theta\le1$. Then

$$\|u\|_{\theta a+(1-\theta)b} \le \|u\|_a^\theta \cdot \|u\|_b^{1-\theta}$$

Proof (for the case $Q=\mathbb{T}^d$ and $\langle u\rangle=0$). We have

$$\begin{aligned}
\|u\|^2_{\theta a+(1-\theta)b} &= \sum_{s\neq0}|u_s|^2|s|^{2(\theta a+(1-\theta)b)}\\
&= \sum_{s\neq0}|u_s|^{2\theta}|s|^{2\theta a}|u_s|^{2(1-\theta)}|s|^{2(1-\theta)b}\\
&\le \left(\sum_{s\neq0}|u_s|^2|s|^{2a}\right)^\theta \left(\sum_{s\neq0}|u_s||s|^{2b}\right)^{1-\theta} ,
\end{aligned}$$

where in the last step we use the Hölder inequality.

Example: In the Example in (S3), using the interpolation with $a=0, b=1, \theta=\frac12$, we get that $|u|_4\le c\,\|u\|_{\frac12}\le c\sqrt{|u|_2\,\|u\|_1}$. This is the *Ladyzhenskaya inequality.*

1.2 Functions of t and x

The solutions of the equations, mentioned in the introduction, are functions depending on time t and space x. We fix $T>0$ and view $u(t,x)$ with $0\le t\le T$ as a map

$$[0,T]\longrightarrow \text{“space of functions of } x\text{”}, \qquad t\mapsto u(t,\cdot)\,.$$

Accordingly, we can define spaces $L_p\big(0,T;L_q(Q)\big) \overset{\text{def}}{=} L_p\big([0,T],L_q(Q)\big)$, $L_p\big(0,T;H^k(Q)\big) \overset{\text{def}}{=} L_p\big([0,T],H^k(Q)\big)$, $C\big(0,T;L_q(Q)\big) \overset{\text{def}}{=} C\big([0,T],L_q(Q)\big)$, and so on. Fubini's theorem implies that

$$L_p\big(0,T;L_p(Q)\big) = L_p\big([0,T]\times Q\big)\,,$$

if $p<\infty$. Discussion of these spaces see in [Lio69].

We shall denote

$$C^\infty = \{u(t,x) \in C^\infty\} \text{ or } C^\infty = \{u(x) \in C^\infty\},$$

depending on the context.

Exercise 1.4. *Let $Q = \mathbb{R}^d$ and $T = 1$. Consider the heat kernel:*

$$u(t,x) = (2\sqrt{\pi t})^{-d} \exp\left(-\frac{|x|^2}{4t}\right).$$

Prove that

$$|u|_{L_p(0,1;L_2)} = \begin{cases} \infty, & pd \geq 4 \\ < \infty, & \textit{otherwise} \end{cases}$$

2 The deterministic 2D Navier-Stokes Equation

In the forthcoming, we write "2D NSE" for the "two-dimensional Navier–Stokes equation", and often abbreviate 2D NSE to NSE. We will consider 2D NSE with periodic boundary conditions. That is, we assume that the space-variable x is a point in the torus $\mathbb{T}^2 \stackrel{\text{def}}{=} \mathbb{R}^2/2\pi\mathbb{Z}^2$. The Navier–Stokes equation in a bounded two-dimensional domain under the Dirichlet boundary conditions can be studied very similar. In an unbounded domain, e.g. in the whole plane, the equation becomes a bit more difficult since in this case the Sobolev embeddings in (S3) are not compact.

The 2D NSE on the torus is the following system of three equations for three unknown functions: two components of the vector-function $u(t,x) = \left(u^1(t,x), u^2(t,x)\right)^t$ (the velocity) and the scalar function $p(t,x)$ (the pressure), where $x \in \mathbb{T}^2$ and $t \in \mathbb{R}$:

$$\begin{cases} \dot{u}(t,x) - \nu\Delta u(t,x) + (u(t,x)\cdot\nabla)u(t,x) + \nabla p(t,x) = \tilde{f}(t,x) \\ \operatorname{div} u(t,x) = 0\,. \end{cases} \tag{2.1}$$

Usually we study the equation for $t \geq 0$ and supplement it with the initial condition at $t = 0$:

$$u(\cdot,0) = u_0(\cdot)\,.$$

Standard references are, e.g., [Lio69, CF88] and [BV92].

2.1 Leray decomposition

Let $u \in L_2(\mathbb{T}^2;\mathbb{R}^2)$, then u can be written as a Fourier series: $u(x) = \sum u_s e^{is\cdot x}$, with $u_s \in \mathbb{C}^2$ and $u_{-s} = \bar{u}_s$. If $u(x) \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$, then $\operatorname{div} u(x) = \sum_{s\in\mathbb{Z}^2} is\cdot u_s e^{is\cdot x}$. Denote with H the space

$$H = \text{the closure in } L_2(\mathbb{T}^2;\mathbb{R}^2) \text{ of } \{u(x) \in C^\infty(\mathbb{T}^2;\mathbb{R}^2) |\ \operatorname{div} u = 0, \langle u\rangle = 0\}.$$

Then it holds that

$$H = \{u(x) = \sum_{s\in\mathbb{Z}_0^2} u_s e^{is\cdot x} \in L_2(\mathbb{T}^2;\mathbb{R}^2) \mid u_{-s} = \bar{u}_s,\ s\cdot u_s = 0\}\,, \tag{2.2}$$

where $\mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus \{0\}$. The norm in H will be denoted with $|\cdot|$, and the inner product with $\langle\cdot,\cdot\rangle$.

Exercise 2.1. *H can be defined as $H = \{u \in L_2 | \langle u\rangle = 0, div\ u = 0\}$, where the derivatives are viewed in the sense of generalised functions.*

We next introduce a basis of H. Let us define $\mathbb{Z}_+^2 = \{(s_1,s_2) \mid (s_1 > 0) \text{ or } (s_1 = 0, s_2 > 0)\}$. Then

$$\mathbb{Z}_0^2 = \mathbb{Z}_+^2 \cup (-\mathbb{Z}_+^2)\,, \quad \mathbb{Z}_+^2 \cap -\mathbb{Z}_+^2 = \emptyset\,,$$

and we define the following set of vectors $\{e_s \mid s\in\mathbb{Z}_0^2\}$:

$$e_s=\begin{cases} c_s s^\perp \sin(s\cdot x), & s\in\mathbb{Z}_+^2\,,\\ c_s s^\perp \cos(s\cdot x), & s\in-\mathbb{Z}_+^2\,,\end{cases}$$

where $c_s=\frac{1}{\sqrt{2}\pi|s|}$, and if $s=(s_1,s_2)^t$, then $s^\perp=(-s_2,s_1)^t$. The set $\{e_s\}$ is a Hilbert basis of H, formed by eigen–vectors of $-\Delta$:

$$-\Delta e_s=|s|^2 e_s \quad \forall\, s\,.$$

We further introduce the space

$$\nabla H^1 \stackrel{\text{def}}{=} \{\nabla f(x) | f\in H^1(\mathbb{T}^2)\}\,.$$

Equivalently,

$$\nabla H^1=\{\sum_{s\in\mathbb{Z}_0^2} s a_s e^{is\cdot x}\in L_2(\mathbb{T}^2;\mathbb{R}^2)\}\,, \tag{2.3}$$

so ∇H^1 is a closed subspace of H.

The relations (2.2) and (2.3) immediately imply the following classical result due to Helmholtz, which is a common tool to study the Navier–Stokes equation since the works of Leray:

Lemma 2.2. *The space $L_2(\mathbb{T}^2;\mathbb{R}^2)$ admits the following decomposition in a direct sum of three closed orthogonal subspaces*

$$L_2(\mathbb{T}^2;\mathbb{R}^2)=H\oplus\nabla H^1\oplus\mathbb{R}^2\,,$$

where $\mathbb{R}^2$ stands for the space of constant vector fields.

The orthogonal projection $\Pi: L_2(\mathbb{T}^2;\mathbb{R}^2)\to H$ is called the *Leray projection.* Note that

$$\Pi(\nabla p)=0, \quad \Pi(constant)=0\,.$$

Let $(u(t,x),p(t,x))$ be a smooth solution of (2.1). Let us denote $u(0,x)=u_0(x)$ and assume that

$$\langle u_0(x)\rangle=0,\ \langle\tilde f(t,x)\rangle=0 \text{ for all } t\ge 0\,.$$

Then, integrating the first equation of (2.1) over space, we obtain

$$\frac{d}{dt}\langle u\rangle-\langle\Delta u\rangle+\langle(u\cdot\nabla)u\rangle+\langle\nabla p\rangle=\langle\tilde f\rangle\,.$$

Since $\langle\Delta u\rangle=\langle\nabla p\rangle=\langle\tilde f\rangle=0$ and for $l=1,2$ we have

$$\langle((u\cdot\nabla)u)^l\rangle=\int u^j(\frac{\partial}{\partial x^j}u^l)dx=-\int \text{div}\ u\cdot u^l dx=0\,,$$

then $\frac{d}{dt}\langle u\rangle=0$. Hence, if $u_0\in H$, then $u(t)\in H$ for all $t\ge 0$.

Now, apply the projection Π to (2.1). Since $\Pi u=u$, we find that

$$\dot u-\nu\Pi\Delta u+\Pi(u\cdot\nabla)u=\Pi\tilde f\,.$$

With the notation

$$Lu\stackrel{\text{def}}{=}-\Pi\Delta u=-\Delta u \text{ and } B(u)\stackrel{\text{def}}{=}\Pi(u\cdot\nabla)u, \text{ for } u\in H; \qquad f\stackrel{\text{def}}{=}\Pi\tilde f\,,$$

we are lead to the equation

$$\begin{cases}\dot u+\nu Lu+B(u)=f(t)\,,\\ u(t)\in H\,.\end{cases} \tag{2.4}$$

Lemma 2.3. *If $u(t,x)\in C^\infty(\mathbb{T}^2)$ satisfies (2.4), then there exists $p(t,x)\in C^\infty(\mathbb{T}^2)$ such that (2.1) holds.*

Proof. Denote $\dot u(t,x)-\nu\Delta u(t,x)+B(u(t,x))-\tilde f(t,x)=-\xi(t,x)$. Then $\Pi\xi=0$. So, by Lemma 2.2, $\xi(t,x)=\nabla p(t,x)+C(t)$. For the same reasons as above, $C(t)\equiv 0$. □

Below we study eq. (2.4) instead of (2.1).

Notations: We denote $B(u,v)=\Pi(u\cdot\nabla)v$ (so $B(u)=B(u,u)$). For $r\ge 0$ we define the space H^r as $H^r:=H\cap H^r(\mathbb{T}^2,\mathbb{R}^2)$, and for $r<0$ – as $H^r:=$ closure of H in $H^r(\mathbb{T}^2,\mathbb{R}^2)$. Then

$$H^r=\Big\{u=\sum_{s\in\mathbb{Z}_0^2} u_s\, e_s(x)\,|\,\|u\|_r^2=\sum|s|^{2r}|u_s|^2<\infty\Big\}\,.$$

For $u\in H^r$ we have

$$\|u\|_r^2=\langle L^r u,u\rangle\,,$$

since $u\in H^r(\mathbb{T}^2,R^2)$ with $\langle u\rangle=0$ satisfies $\|u\|_r^2=\langle(-\Delta)^r u,u\rangle$ (see (S2)), and $Le_s=|s|^2 e_s$. In particular,

$$\|u\|_1^2=\langle Lu,u\rangle=|\nabla u|_{L_2}\,,\quad \langle Lu,v\rangle=\langle\nabla u,\nabla v\rangle \tag{2.5}$$

for $u,v\in H^1$.

We note that $H^{r_1}\subset H^{r_2}$ if $r_1\ge r_2$, that

$$\bigcap_r H^r=H\cap C^\infty$$

(this follows from (1.2)), and that the linear space $\cap_r H^r$ is dense in each space H^s.

Lemma 2.4. *("Bounded poly-linear map is continuous")*
If X_1, X_2 are Banach spaces and $F : X_1 \times \cdots \times X_1 \to X_2$ is a poly-linear map such that

$$\|F(u_1, \ldots, u_r)\|_{X_2} \leq C\|u_1\|_{X_1} \ldots \|u_r\|_{X_1},$$

then F is continuous. Moreover, if $V_1 \subset X_1$ is a dense linear subspace and the inequality above holds for $u_j \in V_1$, then F extends to a poly-linear continuous map $X_1 \times \cdots \times X_1 \to X_2$.

The proof of this result is straight-forward.

2.2 Properties of the nonlinearity B

(B1) If $u,\ v,\ w \in C^\infty \cap H$, then

i) $\langle B(u,v), v\rangle = 0$,

ii) $\langle B(u,v), w\rangle = -\langle B(u,w), v\rangle$.

Proof. i) Integrating by parts we have:

$$\begin{aligned}\langle B(u,v), v\rangle = \int_{\mathbb{T}^2} u^j \Big(\frac{\partial}{\partial x^j}\, v^l\Big)\, v^l\, dx = & \frac{1}{2}\int_{\mathbb{T}^2} u^j\, \frac{\partial}{\partial x^j}\, |v|^2\, dx \\ = & -\frac{1}{2}\int_{\mathbb{T}^2} (\operatorname{div} u)|u|^2 dx = 0\,.\end{aligned}$$

ii) Apply i) with $v := v + w$. □

(B2) If $u,\ v,\ w \in C^\infty \cap H$, then

i) $|\langle B(u,w), v\rangle| = |\langle B(u,v), w\rangle| \leq C\|u\|_{1/2}\|v\|_{1/2}\|w\|_1$,

ii) $\|B(u,v)\|_{-1} \leq C\|u\|_{1/2}\, \|v\|_{1/2}\,.$

Proof. i) implies ii) by the duality, see (S4). Now we prove i):

$$\begin{aligned}|\langle B(u,v), w\rangle| \overset{(B1)}{=} & \ |-\langle B(u,w), v\rangle| \leq C\int |u|\, |\nabla w|\, |v| dx \\ \overset{\text{Hölder}}{\leq} & \ C_1\, |\nabla w|_{L_2}\, |u|_{L_4}\, |v|_{L_4} \overset{(1.5)}{\leq} C_2\|w\|_1\, \|u\|_{1/2}\, \|v\|_{1/2}\,.\end{aligned}$$

□

So by Lemma 2.4 and (B2 ii), B extends to a bilinear continuous map,

$$B : H^{1/2} \times H^{1/2} \to H^{-1}\,.$$

(B3) If $u, v \in H \cap C^\infty$, then $\|B(u,v)\|_{-3} \leq C|u|\,|v|$. So by Lemma 2.4, B extends to a continuous map $B : H \times H \to H^{-3}$.

The proof is left as an exercise.

Let us define the space

$$\mathcal{H} = \{u \in L_2(0,T;H^1) \,|\, \dot u \in L_2(0,T;H^{-1})\},$$

$$\|u\|^2_{\mathcal{H}} = \int_0^T \big(\|u(t)\|_1^2 + \|\dot u(t)\|_{-1}^2\big)\, dt\,.$$

Here for $u \in L_2(0,T;H^1)$ and $\xi \in L_2(0,T;H^{-1})$ we say that $\dot u = \xi$, if there exists $\eta \in H^{-1}$ such that $u(t) = \eta + \int_0^t \xi(s)\, ds$, for a.a. $t \in [0,T]$. Note that $\mathcal{H}$ is a Hilbert space.

Lemma 2.5. *$\mathcal{H}$ is continuously and compactly embedded in $C(0,T;H)$. In particular, for any $u \in \mathcal{H}$ and each $0 \leq t \leq T$, $u(t)$ is a well defined vector in H.*

Proof. The embedding $\mathcal{H} \subset C(0,T;H)$ is an exercise. For its compactness see [Lio69]. □

(B4) For any $u, v \in \mathcal{H}$ and any $0 \leq t \leq T$ we have

$$\begin{aligned}&\int_0^T \langle Lu(t),\, v(t)\rangle\, dt = \int_0^t \langle \nabla u(t),\, \nabla v(t)\rangle\, dt\,,\\ &\int_0^t \langle \dot u, u\rangle\, ds = \frac{1}{2}(|u(t)|^2 - |u(0)|^2)\,.\end{aligned}$$

Proof. The first relation follows from (2.5). To establish the second we first prove it for smooth u and v, and then argue by continuity. □

Noting that $t \mapsto \langle \dot u, u\rangle \in L_1(0,T)$, we see that the second relation in (B4) means that $|u(t)|^2$ is an absolutely continuous function and

$$\frac{d}{dt}\, |u(t)|^2 = 2\langle \dot u, u\rangle, \quad \forall u \in \mathcal{H}\,.$$

(B5) i) B defines a three-linear continuous map

$$\mathcal{H} \times \mathcal{H} \times \mathcal{H} \to L_1[0,T]\,, \qquad (u_1, u_2, u_3) \longmapsto \langle B\big(u_1(t),\, u_2(t)\big), u_3(t)\rangle.$$

ii) If $u_1, u_2, u_3 \in \mathcal{H}$ and $u_2 = u_3$, then $\langle B(u_1, u_2), u_2\rangle = 0$ in $L_1[0,T]$.

Proof. i) Note first that $\mathcal{H} \subset L_4(0,T;H^{1/2})$. Indeed, since $\|u\|^4_{1/2} \leq |u|^2\, \|u\|_1^2$, then

$$\|u\|^4_{L_4(0,T;H^{1/2})} = \int_0^T \|u(s)\|^4_{1/2}\, ds \leq \sup_t\, |u(t)|^2 \int_0^T \|u(t)\|_1^2\, dt \leq \|u\|^4_{\mathcal{H}}\,.$$

Hence, for $u_1, u_2, u_3 \in \mathcal{H} \cap C^\infty$ we have

$$\int_0^T |\langle B(u_1,u_2),u_3\rangle|\,ds \overset{(B2)}{\le} C\int_0^T \|u_3\|_1\,\|u_1\|_{1/2}\,\|u_2\|_{1/2}\,ds \overset{\text{Hölder}}{\le}$$
$$C\Big(\int_0^T \|u_3\|_1^2\,ds\Big)^{1/2}\Big(\int_0^T \|u_1\|_{1/2}^4\,ds\Big)^{1/4}\Big(\int_0^T \|u_2\|_{1/2}^4\,ds\Big)^{1/4}$$
$$\le C\,\|u_3\|_{\mathcal{H}}\|u_1\|_{\mathcal{H}}\|u_2\|_{\mathcal{H}}\,.$$

Now i) follows from Lemma 2.4.

If $u_1(t), u_2(t), u_3(t) \in C^\infty \cap H$ for all t, then ii) holds by (B1). Now ii) follows by continuity. □

2.3 The existence and uniqueness theorem

To simplify notations, we assume that $\nu = 1$.

Theorem 2.6. *Let $f \in L_2(0,T;H^{-1})$ and $u_0 \in H$. Then*

a) *(2.4) has a unique solution $u \in \mathcal{H}$ such that $u(0) = u_0$.*

b) *This solution satisfies*

$$\sup_{0\le t\le T}\Big(|u(t)|^2 + \int_0^t \|u(s)\|_1^2\,ds\Big) \le |u_0|^2 + \int_0^T \|f(t)\|_{-1}^2\,dt. \tag{2.6}$$

Proof.
Uniqueness. Let $u_1, u_2 \in \mathcal{H}$ be two solutions, equal to u_0 at $t=0$. Denote $u = u_2 - u_1$. Then

$$\dot u + Lu + B(u_2,u_2) - B(u_2,u_1) + B(u_2,u_1) - B(u_1,u_1) = 0\,,$$

or

$$\dot u + Lu + B(u_2,u) + B(u,u_1) = 0\,.$$

By (B5 i) we can multiply this relation in H by $u(s)$ and integrate over ds from 0 to t. Using (B5 ii) we have for all $t \ge 0$:

$$\int_0^t \big(\langle \dot u, u\rangle + \langle Lu,u\rangle + \langle B(u,u_1),u\rangle\big)\,ds = 0.$$

Since $\langle Lu,u\rangle = \|u\|_1^2$, then using (B4) we obtain the following equality in $L_1(0,T)$:

$$\frac{d}{dt}\,|u(t)|^2 + 2\|u\|_1^2 = -2\langle B(u,u_1),u\rangle\,.$$

By (B2 i) and the interpolation inequality,

$$|\langle B(u,u_1),u\rangle| \le C\,\|u_1\|_1\;\|u\|_{\frac12}^2 \;\le C\,\|u_1\|_1\;|u|\;\|u\|_1 \le \frac12\,\|u\|_1^2 + C_1\,\|u_1\|_1^2\,|u|^2\,.$$

Therefore,

$$\frac{d}{dt}\,|u|^2 \le C\,|u|^2\,\|u_1\|_1^2\,.$$

By Gronwall's lemma,

$$|u(t)|^2 \le e^{\int_0^t C\|u_1(s)\|_1^2\,ds}|u(0)|^2\,. \tag{2.7}$$

So $u(t) \equiv 0$ since $u(0) = 0$.

Existence: Step 1 (a-priori estimate).

Let $u(t,x)$ be a *smooth* solution of (2.4). Let us consider the function $u \longmapsto \frac12\,|u|^2$ and calculate its derivative along trajectories of (2.4):

$$\frac12\,\frac{d}{dt}\,|u(t)|^2 = \langle u(t),\,\dot u(t)\rangle = \langle u, -Lu - B(u) + f(t)\rangle \overset{(B1)}{=}$$
$$= -\|u\|_1^2 + \langle u,f\rangle \le -\|u\|_1^2 + \|u\|_1\,\|f\|_{-1} \le -\frac12\,\|u\|_1^2 + \frac12\,\|f\|_{-1}^2\,.$$

So,

$$\frac{d}{dt}\,|u(t)|^2 + \|u\|_1^2 \le \|f\|_{-1}^2\,,$$

and for $0 \le t \le T$ we have

$$|u(t)|^2 + \int_0^t \|u(s)\|_1^2\,ds \le |u_0|^2 + \int_0^t \|f(s)\|_{-1}^2\,ds\,. \tag{2.8}$$

Hence,

$$|u|_{L_\infty(0,T;H)}^2,\;|u|_{L^2(0,T;H^1)}^2 \le |u_0|^2 + |f|_{L_2(0,T;H^{-1})}^2 =: C(u_0,f)\,. \tag{2.9}$$

Due to (B2) and the interpolation inequality,

$$\|B(u)\|_{-1} \le C\,\|u\|_{1/2}^2 \le C\,|u|\;\|u\|_1\,.$$

Expressing $\dot u$ in terms of u from (2.4) and using (2.9), we get that

$$|\dot u|_{L^2(0,T:H^{-1})} \le C'\big(C(u_0,f) + C^2(u_0,f)\big)\,.$$

So,

$$|u|_{\mathcal{H}} \le C_1(u_0,f)\,. \tag{2.10}$$

Step 2: Galerkin approximations

For the basis $\{e_s, s \in \mathbb{Z}_0^2\}$, we set

$$H_{(N)} = \mathrm{span}\{e_s,\;|s| \le N\}\,.$$

Let $P_N : H \to H_{(N)}$ be the orthogonal projection. Clearly,

$$H_{(N)} \subset C^\infty \cap H,\;\dim H_{(N)} < \infty,\;\text{ and }\;L : H_{(N)} \to H_{(N)}\,.$$

Let us apply P_N to (2.4):

$$P_N \dot{u} + P_N\, Lu + P_N\, B(u) = P_N\, f\,.$$

A curve $u(t) \in H_{(N)}$ satisfies this relation iff

$$\dot{u} + Lu + P_N\, B(u) = P_N\, f\,. \tag{2.11}$$

This is an ODE in $H_{(N)}$, defined by a smooth vector field, with an L_2–r.h.s. Let us supplement it with the initial condition:

$$u(0) = P_N\, u_0\,.$$

Then the equation has a unique solution $u = u_N$, which exists till $t = T$, or till it blows up at time $T_N \le T$:

$$u_N \in C([0, T_N], H_{(N)}),\ \dot{u}_N \in L_2([0, T_N], H_{(N)})\,,$$

$$0 < T_N \le T, \quad T_N < T \implies |u_N(t)| \underset{t \to T_N}{\longrightarrow} \infty \tag{2.12}$$

A-priori estimate for $\boldsymbol{u_N}$.
Consider the derivative of $\frac{1}{2}\,|u(t)|^2$ along trajectories of (2.11):

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\,|u(t)|^2 = \langle u, \dot{u}\rangle = \langle u, -Lu - P_N\, B(u) + P_N\, f\rangle \\ = \langle u, -Lu - B(u) + f(t)\rangle\,.\end{aligned}$$

Note that

$$\boxed{\text{This is the same relation as for solutions of (2.4) obtained in Step 1.}}$$

Hence,

$$u_N(t) \;\; \text{satisfies (2.8), (2.10) uniformly in } N \text{ (with } T := T_N).$$

Due to (2.12), it means that $T_N = T$ for all N. Therefore the Galerkin approximations satisfy

$$\begin{cases} \{u_N(t),\ 0 \le t \le T\} \subset \mathcal{H}\,, \\ \|u_N\|_{\mathcal{H}} \le C_1\,, \\ u_N(0) = P_N\, u_0\,. \end{cases}$$

Step 3: Transition to the limit
By the Banach-Alaoglu Theorem a closed ball in $\mathcal{H}$ is compact in the weak topology. Hence, there is a subsequence $\{N_j\}$ such that $u_{N_j} \rightharpoonup u \in \mathcal{H}$ as $j \to \infty$. Since the linear maps

$$\mathcal{H} \to L_2(0, T; H^{-1}),\ u \longmapsto \dot{u}$$
$$\mathcal{H} \to L_2(0, T; H^{-1}),\ u \longmapsto Lu$$

are continuous, then

$$\dot{u}_{N_j} \rightharpoonup \dot{u} \quad \text{in } \; L_2(0, T; H^{-1}) \tag{2.13}$$

$$L\, u_{N_j} \rightharpoonup Lu \;\text{ in }\; L_2(0, T; H^{-1})\,. \tag{2.14}$$

Now consider the term $\{B(u_{N_j})\}$. By Lemma 2.5,

$$u_{N_j} \to u \;\text{ in }\; C(0, T; H)\,. \tag{2.15}$$

Noting that by (B3) the map

$$C(0, T; H) \longrightarrow C(0, T; H^{-3}),\ u(t) \longmapsto B(u(t))$$

is continuous, we have

$$B(u_{N_j}) \longrightarrow B(u) \quad \text{in } C(0, T; H^{-3})\,. \tag{2.16}$$

Finally, due to (2.15),

$$u_{N_j}(0) \longrightarrow u(0) \;\text{ in }\; H\,. \tag{2.17}$$

Let us take any m and apply P_m to (2.11) with $N_j \ge m$ in place of N. Since $P_m \circ P_{N_j} = P_m$, we have

$$P_m\, \dot{u}_{N_j} + P_m\, Lu_{N_j} + P_m\, B(u_{N_j}) = P_m f\,.$$

Send N_j to ∞. Due to (2.13)–(2.16),

$$P_m\, \dot{u} + P_m\, Lu + P_m\, B(u) = P_m f\,.$$

(This equality holds in the space $L_2(0, T; H^{-3})$). Hence, since m is arbitrary, we have

$$\dot{u} + Lu + B(u) = f.$$

Due to (2.17),

$$u(0) = \lim u_{N_j}(0) = \lim P_{N_j} u_0 = u_0\,.$$

Now the assertion a) is proved.

To prove b) we note that (2.9) and (2.15) imply the estimate

$$|u|^2_{L^\infty(0,T;H)} \le C(u_0, f)\,.$$

Since $u_{N_j} \rightharpoonup u$ in $\mathcal{H}$, then $u_{N_j} \rightharpoonup u$ in $L_2(0, T; H^1)$. As in a Hilbert space a norm of the weak limit is no bigger than the liminf of the norms, then

$$|u|^2_{L^2(0,T;H^1)} \le C(u_0, f)\,.$$

This proves (2.6) with the extra factor 2 in the r.h.s. To get rid of this factor, one has to repeat the arguments, used in the proof of the Proposition 2.11 below. $\square$

Remark 2.7. *Our proof shows that any sequence of Galerkin solutions u_{N_j} contains a subsequence that converges to a solution. By the uniqueness, this solution is $u(t,x)$. Hence, the whole sequence converges to u:*

$$u_N \underset{N\to\infty}{\rightharpoonup} u \text{ in } \mathcal{H} \tag{2.18}$$

(u_N is a solution of (2.11)). So, $u_N \to u$ in $C(0,T;H)$ by Lemma 2.5.

Proposition 2.8. *Let $u(t,x)$ be a solution as in Theorem 2.6. Then one can find $p(t,x) \in L_2(0,T;L_2(\mathbb{T}^2)) = L_2((0,T)\times\mathbb{T}^2)$ such that (u,p) satisfies (2.1).*

Exercise 2.9. *Prove the Proposition.*

The presented proof of the existence and uniqueness of a solution applies to the 2D NSE in a bounded domain with the Dirichlet boundary conditions. Similar arguments apply to (0.4), but not to (0.3). To handle the latter equation (with $d \ge 2$) one has to use the Maximum Principle, which holds for solutions of the equation, but not for solutions of its Galerkin approximations. So in this case one should argue differently.

2.4 Improving the smoothness of solutions

Lemma 2.10. *If $u \in H \cap C^\infty$, then $\langle B(u), \Delta u\rangle = 0$.*

Proof. We have

$$\begin{aligned}\langle B(u),\Delta u\rangle &= \int u^i \frac{\partial u^j}{\partial x^i}\frac{\partial^2 u^j}{\partial x^k \partial x^k}\,dx\\ &= \int u^i \frac{\partial^2 u^j}{\partial x^i\partial x^k}\frac{\partial u^j}{\partial x^k}\,dx + \int \frac{\partial u^i}{\partial x^k}\frac{\partial u^j}{\partial x^i}\frac{\partial u^j}{\partial x^k}\,dx.\end{aligned}$$

The first integral in the r.h.s. vanishes since its integrand equals $u(x)\cdot\nabla\Big(\frac{\partial u^j}{\partial x^k}\frac{\partial u^j}{\partial x^k}\Big)$ and $\operatorname{div} u = 0$. The integrand in the second integral equals $\operatorname{tr}\big(U^2U^t\big)$, where U is the matrix of du (i.e., of the linearisation of the mapping $x \mapsto u(x)$). As $\operatorname{tr} U = \operatorname{div} u = 0$, then the matrix U^2 is proportional to the identity. Therefore the integrand vanishes identically, and the lemma is proved. ☐

We note that the lemma's assertion does not hold for smooth divergence–free functions in a domain, vanishing on the domain's boundary. So Lemma 2.10 does not apply to study the 2D NSE under the Dirichlet boundary conditions.

Let $u_0 \in H^1$ and $f \in L_2(0,T;H)$. Consider the functional: $u \mapsto \frac{1}{2}\|u\|_1^2 = \frac{1}{2}\langle u, Lu\rangle$. If $u(t,x)$ is a smooth solution, then by the lemma

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_1^2 &= \langle Lu,\dot u\rangle = -\|u\|_2^2 + \langle Lu, f\rangle\\ &\le -\|u\|_2^2 + \|u\|_2\,|f| \le -\frac{1}{2}\|u\|_2^2 + \frac{1}{2}|f|^2.\end{aligned}$$

Integrating over time we get that for any $t \le T$,

$$\varphi(u) := \|u(t)\|_1^2 + \int_0^t \|u(s)\|_2^2\,ds \le \|u_0\|_1^2 + \int_0^t |f(s)|^2 ds =: \widetilde{C}(u_0,f) \le \infty\,. \tag{2.19}$$

As in Step 2, we see that

$$\text{this estimate also holds for Galerkin approximations } u_N(t). \tag{2.20}$$

Proposition 2.11. *The estimate (2.19) holds for the solution u as in Theorem 2.6.*

Proof. By (2.18), $u_N \underset{N\to\infty}{\longrightarrow} u$ in $C(0,T;H)$. So $\varphi(P_m u_N) \underset{N\to\infty}{\longrightarrow} \varphi(P_m u)$ since $\varphi\circ P_m$ is a continuous functional on $C(0,T;H)$. Since $\varphi(P_m u_N) \le \widetilde{C}$ for all N due to (2.20), then for each m we have $\varphi(P_m u) \le \widetilde{C}$. As $\varphi(P_m u) \nearrow \varphi(u)$, then the assertion follows. ☐

The arguments, used to prove the Proposition above, are general. They may be stated as a **Principle:**

If an estimate holds for all Galerkin approximations, then it also holds for the solutions as in Theorem 2.6. (2.21)

Often it means that:

If an estimate holds for smooth solutions, then it also holds for solutions as in Theorem 2.6. (2.22)

In particular, the a-priori estimate (2.8) implies (2.6).

Exercise 2.12. Let φ be a smooth functional, defined on some Sobolev space H^r, $r \ge 0$. Find under what assumptions on φ the equation obtained by differentiating φ along (2.11) is the same as for differentiating φ along (2.4). [For such functionals the a-priori estimate which φ implies for (2.11) is the same as the one which it implies for (2.4); so (2.22) follows from (2.21)].

The smoothing property.
The *smoothing property* states that if in (2.4) $u_0 \in H$, but $f(t,x) \in C^\infty$, then $u(t,x) \in C^\infty$ for $t > 0$. We start with

Theorem 2.13. *If $u_0 \in H$ and $f \in L_2(0,T;H)$, then $u(s) \in H^2$ for a.a. $s > 0$, and for each $0 < t \le T$ we have*

$$t\,\|u(t)\|_1^2 + \int_0^t s\|u(s)\|_2^2\,ds \le |u_0|^2 + \int_0^t s\,|f(s)|^2\,ds + \int_0^t \|f(s)\|_{-1}^2\,ds\,. \tag{2.23}$$

Sketch of the proof. Take the functional $\varphi(t,u) = t\,\|u(t)\|_1^2$. Then for a smooth solution $u(t,x)$, we have

$$\frac{d}{dt}\,(t\|u\|_1^2) = \|u\|_1^2 + 2t\langle Lu, \dot u\rangle = \|u\|_1^2 + 2t\langle Lu, -Lu - B(u) + f\rangle$$
$$\overset{\text{Lemma 2.10}}{=} \|u\|_1^2 - 2t\,\|u\|_2^2 + 2t\langle Lu, f(t)\rangle$$

With

$$2t\langle Lu, f\rangle \le t\,\|u\|_2^2 + t\,|f|^2\,,$$

we obtain that

$$\frac{d}{dt}\,(t\|u\|_1^2) \le \|u\|_1^2 - t\,\|u\|_2^2 + t\,|f|^2\,.$$

Integrating from 0 to t we get

$$t\,\|u(t)\|_1^2 + \int_0^t s\|u(s)\|_2^2\,ds \le \int_0^t s\,|f(s)|^2\,ds + \int_0^t \|u(s)\|_1^2\,ds\,.$$

Using (2.6) we see that u satisfies (2.23). This estimate also holds for Galerkin approximations $u_N(t,x)$. As in the proof of the proposition it implies that the estimate holds for solutions of (2.4) (cf. (2.22)). □

Remark 2.14. We used that $\langle B(u), Lu\rangle = 0$, but this relation **is not** really needed for the proof (instead we may estimate this term, using Lemma 2.16) below.

Now assume (only for simplicity!) that $f = 0$. Then

$$\dot u + Lu + B(u) = 0, \quad u(0) = u_0\,. \tag{2.24}$$

Theorem 2.15. *If $u_0 \in H$ and u is a solution of (2.24), then*

$$t^m\|u(t)\|_m^2 + \int_0^t s^m\|u\|_{m+1}^2\,ds \le C_m(|u_0|^2 + |u_0|^{4m+2})\,. \tag{2.25}$$

Proof. Take $\varphi_m(t,u) = t^m\|u\|_m^2 = t^m\langle L^m u, u\rangle$. Then for a smooth solution u we have

$$\begin{aligned}\frac{d}{dt}\,\varphi_m &= m\,t^{m-1}\|u\|_m^2 + 2t^m\langle L^m u, \dot u\rangle \\ &= m\,t^{m-1}\|u\|_m^2 - 2\,t^m\|u\|_{m+1}^2 - 2t^m\langle L^m u, Bu\rangle\,.\end{aligned} \tag{2.26}$$

Lemma 2.16. *For $u \in H \cap C^\infty$ the following inequality holds:*

$$|\langle L^m u, Bu\rangle| \le \frac12\,\|u\|_{m+1}^2 + C_m\,\|u\|_1^{2(m+1)}|u|^{2m}.$$

Postponing the lemma's proof, we continue to prove the theorem. Due to (2.26) and the lemma, we have:

$$\frac{d}{dt}\,t^m\|u\|_m^2 + t^m\|u\|_{m+1}^2 \le m\,t^{m-1}\|u\|_m^2 + 2t^m\,C_m\|u\|_1^{2(m+1)}|u|^{2m}\,.$$

By Theorems 2.6 and 2.13 with $f = 0$, $t^m\|u\|_1^{2(m+1)}|u|^{2m} \le \|u\|_1^2\,|u_0|^{2m}|u_0|^{2m}$.

So, integrating (2.26) we get that

$$\begin{aligned} t^m\,\|u(t)\|_m^2 &+ \int_0^t s^m\|u(s)\|_{m+1}^2 \\ &\le \int_0^t s^{m-1}m\|u(s)\|_m^2 ds + C'_m|u_0|^{4m}\int_0^t \|u(s)\|_1^2 ds \\ &\le \int_0^t s^{m-1}m\|u(s)\|_m^2 ds + C'_m|u_0|^{4m+2}\,.\end{aligned}$$

Now the estimate (2.25) for smooth solutions follows by induction, where the base of induction with $m = 0$ is provided by Theorem 2.6. To complete the proof we again use Principle (2.22) (justifying this as before).

It remains to prove the lemma.

Proof of Lemma 2.16. We first show that

$$|\langle L^m u, B(u)\rangle| \le C\,\|u\|_{m+1}^{\frac{4m-1}{2m}}\,\|u\|_1^{\frac{m+1}{2m}}\,|u|^{1/2}\,. \tag{2.27}$$

To prove the inequality we note that

$$\langle B(u), L^m u\rangle = \sum_{|\alpha|=m} C_\alpha\,\langle D^\alpha B(u), D^\alpha u\rangle,$$

and

$$\langle D^\alpha B(u), D^\alpha u\rangle = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\langle B(D^{\alpha-\beta}u, D^\beta u), D^\alpha u\rangle\,.$$

Since $\langle B(u, D^\alpha u), D^\alpha u\rangle = 0$, then using (B2) we get that

$$\begin{aligned}\langle D^\alpha\,B(u), D^\alpha u\rangle &\le C\sum_{\beta<\alpha}\|D^{\alpha-\beta}u\|_{1/2}\|D^\beta u\|_1\|D^\alpha u\|_{1/2} \\ &\le C\sum_{\beta<\alpha}\|u\|_{1/2+m-|\beta|}\,\|u\|_{1+|\beta|}\|u\|_{m+\frac12}\end{aligned} \tag{2.28}$$

where $0 \le |\beta| < m$. Note that all the numbers $\frac12 + m - |\beta|$, $1 + |\beta|$ and $m + \frac12$ lie between 1 and $m + \frac12$. For any $a \in [1, m + \frac12]$ we have

$$\|u\|_a \le |u|^{1-\frac{a}{m+1}}\,\|u\|_{m+1}^{\frac{a}{m+1}} := X \quad\text{and}\quad \|u\|_a \le \|u\|_1^{1-\frac{a-1}{m}}\,\|u\|_{m+1}^{\frac{a-1}{m}} := Y\,.$$

Take any term in the sum in the r.h.s. of (2.28). Estimating each its factor by $X^\theta Y^{1-\theta}$ with a suitable θ, depending on the factor (and using that $\|u\|_1 \le \|u\|_{m+1}$), we get that

$$|\langle D^\alpha B(u), D^\alpha u\rangle| \le C\,\|u\|_{m+1}^{\frac{4m-1}{2m}}\,\|u\|_1^{\frac{m+1}{2m}}|u|^{1/2}.$$

Therefore, (2.27) follows.

Now, applying the Young inequality

$$AB \le p^{-1}A^p + q^{-1}B^q \quad \text{if } 1<p<\infty \text{ and } \frac{1}{p}+\frac{1}{q}=1\,, \tag{2.29}$$

where $p=\frac{4m}{4m-1}$, $A=\delta\|u\|_{m+1}^{\frac{4m-1}{2m}}$ and $B=\delta^{-1}C\,\|u\|_1^{\frac{m+1}{2m}}|u|^{1/2}$ with $\delta \ll 1$, we get the lemma's assertion. □

A version of the estimate (2.25) (with another r.h.s. and with $0\le t\le T$) holds for a solution of (2.4), where $f\in L_2(0,T;H^s)$ for all s.

Due to Theorem 2.15, solutions of (2.24) with $u_0\in H$ are smooth in t and x for $t>0$. In fact, they are analytic. See [DG95].

2.5 The NS semigroup

For $\tau>0$, we set

$$S_\tau : H \longrightarrow H, \quad u_0 \longmapsto u(\tau)\,, \tag{2.30}$$

where $u(t)$ is a solution of (2.24), equal to u_0 at $t=0$. Since $u(t)\in H$ is continuous in t (see Lemma 2.5), then these are well defined maps. Obviously,

$$S_0=\mathrm{id}\,,\quad S_{t_1}\circ S_{t_2}=S_{t_1+t_2}\,.$$

So $\{S_t\}$ is a semi-group of non-linear transformations of H, called the *Navier–Stokes semigroup* or, for short, the *NS semigroup.*

Properties of the NS semigroup:

(NS1) $$|S_t u_0| \le e^{-t}|u_0|\,.$$

Proof. Multiplying (2.24) by u in H, we get:

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 = -\langle Lu,u\rangle = -\|u\|_1^2 \le -|u|^2\,.$$

That is, $\frac{d}{dt}|u|^2 \le -2|u|^2$. By Gronwall's lemma, $|u(t)|^2 \le |u_0|^2 e^{-2t}$. So (NS1) follows. □

Similar, using Lemma 2.10 we get that

(NS1′) $$\|S_t u_0\|_1 \le e^{-t}\|u_0\|_1\,.$$

The maps S_t are uniformly Lipschitz on bounded sets:

(NS2) $$|S_t(u_{10})-S_t(u_{20})| \le e^{C|u_{10}|^2}|u_{10}-u_{20}|.$$

Proof. Let $u_1(t)$ and $u_2(t)$ be solutions such that $u_1(0)=u_{10}$, $u_2(0)=u_{20}$. We denote $w_0=u_{20}-u_{10}$, $w(t)=u_2(t)-u_1(t)$. Subtracting from the equation for u_2 the equation for u_1, we get:

$$\dot w + Lw + B(u_2,w)+B(w,u_1)=0\,.$$

Calculating $(d/dt)|w(t)|^2$ and transforming the result as in the proof of uniqueness in Theorem 2.6, we get that $w(t)$ satisfies the estimate (2.7). So

$$|w(t)|^2 \le e^{C\int_0^t \|u_1(s)\|_1^2\,ds}|w_0|^2 \le e^{C|u_{10}|^2}|w_0|^2$$

(we use (2.6) with $f=0$), and we have got (NS2). □

(NS3) The maps $S_t : H\to H^1$ are Lipschitz uniformly on bounded sets, and

$$\|S_t(u_{10})-S_t(u_{20})\|_1 \le C_1(t^{-a}\vee 1)\,e^{C_2(|u_{10}|^2+|u_{20}|^2)}|u_{10}-u_{20}|\,,$$

for suitable constants a, C_1, C_2. See [KS01b], Lemma 2.4.

(NS4) a) $S_t : H\to H\cap C^\infty \quad \forall t>0$.

b) The maps S_t are embeddings.

Proof. a) Take any $u\in H$. By Theorem 2.15, $S_t(u)\in H^m$ for any m. But $\cap H^m = H\cap C^\infty$.

b) See [CF88]. □

2.6 Singular forces

Consider the equation (2.4), where $f=\dot\zeta(t,x)$, $\zeta\in L_\infty(0,T;H^1)$ and ζ is continuous at 0. By definition, $u(t,x)$ solves the initial value problem

$$\dot u + Lu + B(u) = \dot\zeta(t,x),\ u(0)=u_0,$$

if $u\in L_\infty(0,T;H)$, $u(t)$ is continuous at 0 as a curve in H, $u(0)=u_0$ and

$$u(t)-u_0+\int_0^t \big(Lu(s)+B(u(s))\big)ds = \zeta(t)-\zeta(0) \tag{2.31}$$

for a.a. $0\le t\le T$. [1] (Note that due to (B3) the r.h.s. and the l.h.s. belong to $L_\infty(0,T;H^{-3})$, so we can compare them). Making the substitution

$$u(t)=w(t)+\zeta(t)\,,$$

we obtain for $w(t)$ an equation which we write as

$$\dot w + Lw + B(\zeta,w)+B(w,\zeta)+B(w,w) = -B(\zeta,\zeta)-L(\zeta) =: g(t)\,, \tag{2.32}$$

and

$$w(0)=w_0-\zeta_0\,.$$

Due to (B2) we have $g\in L_\infty(0,T;H^{-1})$.

[1] So defined solutions of the initial value problem usually are called *mild solutions.*

Theorem 2.17. *Equation (2.31) has a unique solution $u(t,x)$ such that $u-\zeta\in\mathcal{H}$. Besides, $u_N\to u$ in $L_\infty(0,T;\mathcal{H})$, where u_N is a solution of the N-th Galerkin approximation to (2.31).*

Proof. Write $u=w+\zeta$, where w satisfies (2.32), and repeat for w the proof of Theorem 2.6. $\square$

Define

$$L^0_\infty(0,T;H^1)=\{\zeta\in L_\infty(0,T;H^1)\mid \zeta \text{ is continuous at } t=0\}$$

Proposition 2.18. *The map*

$$H\times L^0_\infty(0,T;H^1)\to C(0,T;H)\,,\quad (u_0,\zeta)\mapsto u-\zeta\,,$$

where u is a solution of (2.31) with initial data u_0, is locally bounded and locally Lipschitz-continuous.

Proof. We only show the local boundedness of the map since the proof of the local Lipschitz property is similar. Let

$$|u_0|+|\zeta|_{L_\infty(0,T;H^1)}\le R\,. \tag{2.33}$$

We substitute $u=\zeta+w$ and obtain for w equation (2.32). Multiplying the equation with w yields:

$$\frac{1}{2}\frac{d}{dt}|w|^2+\|w\|_1^2+\underbrace{(B(w,\zeta),w)}_{I_1}=-\underbrace{(L\zeta,w)}_{I_2}-\underbrace{(B(\zeta),w)}_{I_3}\,.$$

We have from (B2) that

$$|I_1|\le C\,\|\zeta\|_1\,|w|\,\|w\|_1\le \tfrac{1}{4}\,\|w\|_1^2+c\,\|\zeta\|_1^2\,|w|^2$$

and

$$|I_3|\le c\,\|w\|_1\,|\zeta|\,\|\zeta\|_1\le \tfrac{1}{4}\,\|w\|_1^2+c\,\|\zeta\|_1^2\,|\zeta|^2.$$

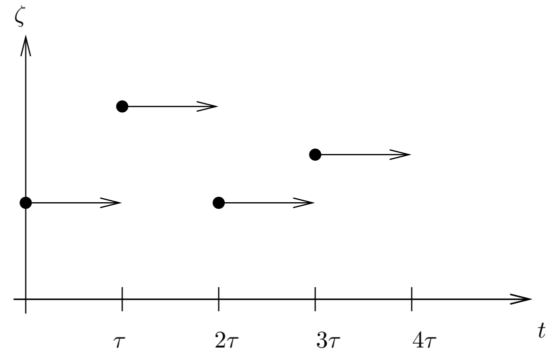
Besides,

$$|I_2|\le \|w\|_1\,\|\zeta\|_1\le \tfrac{1}{4}\,\|w\|_1^2+c\,\|\zeta\|_1^2\,.$$

Using (2.33) we obtain:

$$\frac{d}{dt}|w|^2+\tfrac{1}{2}\,\|w\|_1^2-c\,\|\zeta\|_1^2\,|w|^2\le C\big[\,\|\zeta\|_1^2+|\zeta|^2\,\|\zeta\|_1^2\,\big]\le C(R^2+R^4)$$

for almost all $t\ge 0$. Now Gronwall's lemma implies that $|w(t)|^2\le C(R)$ for all $0\le t\le T$. $\square$



Example 2.19. (Kick-force). Let $\tau>0$ and

$$\mathcal{L}=(\tau\mathbb{Z}\cap(0,T)).$$

Let $\zeta:[0,T]\to H^1$ be a piecewise constant curve, continuous from the right, and discontinuous only at the points of $\mathcal{L}$: Denote by $\eta_j\in H^1, j\ge 0$ the jump of $\zeta(t)$ at τj. Then

$$f=\dot\zeta=\sum_{\tau j\in\mathcal{L}}\eta_j\,\delta_{\tau j}\,.$$

This is a *kick force*. Now we describe the solution u as in Theorem 2.17.

– For $t\in(j\tau,(j+1)\tau), j\ge 0$, $u(t)$ satisfies the free Navier–Stokes equation, i.e. (2.4) with $f=0$, so

$$u(\tau_j+t)=S_t(u(\tau_j)),\quad t\in(j\tau,(j+1)\tau), j\ge 0.$$

– At $t=\tau_j, j\ge 1$, $u(t)$ has the same jump as ζ. So

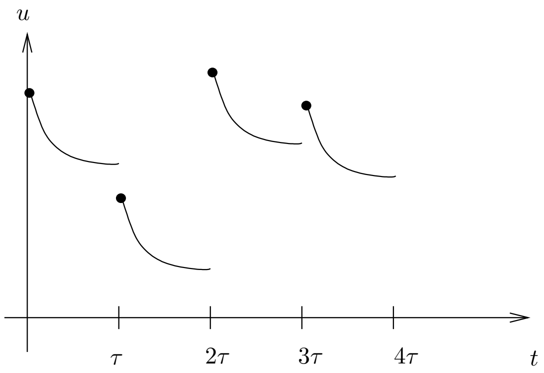
$$u((j+1)\tau)=S_\tau(u(j\tau))+\eta_{j+1},\ j\ge 0. \tag{2.34}$$

In particular, $u(t)$ is continuous at $[0,T]\backslash\mathcal{L}$ and is continuous from the right everywhere.

Example 2.20. Let $\zeta\in C(0,T;H^1)$. Since $u-\zeta\in\mathcal{H}$, then

$$u\in C(0,T;H)\cap L_2(0,T;H^1)\,.$$

Simple analysis of the proof of uniqueness in Theorem 2.6 shows that if $\zeta\in C(0,T;H^1)$, then (2.31) has a unique solution $u(t)$, as smooth as above.



2.7 Some hydrodynamical terminology

Behaviour of incompressible fluid, occupying a domain $O \subset \mathbb{R}^d$ or the d-torus $\mathbb{T}^d$, $d = 2$ or 3, is described by the d+1 equations (2.1), supplemented by suitable boundary conditions, if necessary. There $u(t,x)$ denotes the velocity of the fluid, $p(t,x)$ the pressure and ν the kinematic viscosity. The quantity $\frac{1}{2}\,|u(t)|^2 = \frac{1}{2}\int |u(t,x)|^2 dx$ is the *energy of the fluid* (at time t).

Assume $f = 0$ and formally multiply (2.1) with $u(t)$ in L_2:

$$\frac{1}{2}\,\frac{d}{dt}\,|u(t)|^2 + \nu \int |\nabla u|^2 dx = 0.$$

Accordingly, the quantity $\varepsilon := \nu \int |\nabla u|^2 dx$ is called the *rate of dissipation of energy.*

The *Reynolds number* of the flow is defined as

$$R = \frac{\langle \text{characteristic scale for } x\rangle \cdot \langle \text{characteristic scale for } u\rangle}{\nu}\,.$$

The terms in the numerator are ambiguous. For the purposes of these Lectures we understand them as follows:

$\langle$characteristic scale for $x\rangle$ = diameter of domain O (or the period of torus),
$\langle$characteristic scale for $u\rangle = \frac{1}{\sqrt{2}}\,|u|_H = \sqrt{E}\,.$
If u depends on a random parameter, then we modify this definition as
$\langle$characteristic scale for $u\rangle = \big(\mathbf{E}\,E\big)^{1/2}$,
where $\mathbf{E}$ stands for the mathematical expectation.

In the **2D case** we denote with $\xi = \text{curl }\ u = \dfrac{\partial u^2}{\partial x^1} - \dfrac{\partial u^1}{\partial x^2}$ the *vorticity* of the flow, and call $\Omega = \frac{1}{2}\int \xi^2 dx$ the *enstrophy.*

Exercise 2.21. Show that $\Omega = \frac{1}{2}\int |\nabla u|^2 dx$

Applying the operator *curl* to (2.1), we obtain

$$\dot{\xi} - \nu\Delta\xi + \text{curl}\,\big((u\cdot\nabla)u\big) = \text{curl }\ f.$$

It is easy to see that curl $\big((u\cdot\nabla)u\big) = (u\cdot\nabla)\xi$.

If we assume that $f = 0$ and multiply the previous equation by ξ in L_2, we get:

$$\frac{1}{2}\,\frac{d}{dt}\,|\xi|^2_{L_2} + \nu\int |\nabla\xi|^2\,dx = 0\,.$$

We deduce that $\nu\int |\nabla\xi|^2\,dx$ is the *rate of dissipation of enstrophy.*

Exercise 2.22. Show that $\int |\nabla\xi|^2 dx = |\Delta u|^2_{L_2}$.

3 Random kick-forces

3.1 Ingredients for the constructions

We consider a sequence $b_s \ge 0$, $s \in \mathbb{Z}_0^2$, of real numbers with

$$B_0 = \sum b_s^2 < \infty \, .$$

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and $\{\xi_{sk} \mid s \in \mathbb{Z}_0^2,\ k \in \mathbb{Z}\}$ be a family of independent random variables, such that $|\xi_{sk}(\omega)| \le 1$ for all $s \in \mathbb{Z}_0^2, k \in \mathbb{Z}, \omega \in \Omega$, and

$$\mathcal{D}(\xi_{sk}) = p_s(r)\, dr, \quad s \in \mathbb{Z}_0^2,\ k \in \mathbb{Z}.$$

Here p_s, $s \in \mathbb{Z}_0^2$, are Lipschitz functions with support in $[-1,1]$, and $p_s(0) \neq 0$.

Remark 3.1. In fact, it suffices that each density $p_s(r)$ has finite total variation and $\int_{-\varepsilon}^{\varepsilon} p_s\, dr > 0$, for each $\varepsilon > 0$.

The *kick number* k, $k \in \mathbb{Z}$, is the random variable $\eta_k : \Omega \to H$, defined by

$$\eta_k = \sum_{s \in \mathbb{Z}_0^2} b_s\, \xi_{sk}\, e_s \, ,$$

where $\{e_s\}$ is the orthonormal basis of H, defined in Section 2.1. Note that $\{\eta_k, k \in \mathbb{Z}\}$, are i.i.d. random variables in H. The kicks η_k possess the following obvious properties:

$$\begin{aligned} \mathbf{E}|\eta_k|^2 &= \sum_s b_s^2 d_s =: D_0 < B_0 \, , \text{where } d_s = \mathbf{E}\,\xi_{sk}^2 \le 1; \\ \mathbf{E}\,\|\eta_k\|_1^2 &= \sum_s b_s^2\, |s|^2\, d_s =: D_1 \le \infty\, ; \\ |\eta_k(\omega)|^2 &= \sum_s b_s^2\, \xi_{sk}^2(\omega) \le B_0, \quad \text{for all } k \in \mathbb{Z},\ \omega \in \Omega. \end{aligned} \tag{3.1}$$

Since the density p_s of ξ_{sk} is continuous and does not vanish around 0, we have

$$\mathbf{P}(\,|\eta_k| \le \varepsilon) > 0, \quad \text{for all } \varepsilon > 0\, .$$

Consider any $\tau > 0$. The *(random) kick-force* $\eta(t) \in H$, $t \in \mathbb{R}$, is then defined by

$$\eta(t) = \eta^\omega(t) = \sum_{k \in \mathbb{Z}} \eta_k^\omega\, \delta(t - \tau k)\, . \tag{3.2}$$

As a function of t, this is a random vector–valued measure that takes values in H. Then (cf. Example 2.19) $\eta(t) = \frac{\partial}{\partial t}\zeta(t)$, where the function $\zeta(t) = \zeta^\omega(t)$ is continuous from the right, constant in $\mathbb{R}\backslash\tau\mathbb{Z}$, and its jump at $t = j\tau$ is η_j.

3.2 The kicked NSE

We consider the equation

$$\dot u + Lu + B(u) = \eta^w(t)\, , \tag{3.3}$$

with $\eta(t)$ as in (3.2). A random process $u^\omega(t)$ is a solution if it satisfies (3.3) for a.e. $\omega \in \Omega$ in the sense of Section 2.6. We choose $\tau = 1$ to simplify notations. Then (see Example 2.19),

$$u(k) = S\big(u(k-1)\big) + \eta_k\, , \quad k \in \mathbb{Z}, \tag{3.4}$$

where $S = S_1$ is the time-one shift along trajectories of the free equation. For any $0 \le t < 1$,

$$u(k+t) = S_t\big(u(k)\big)\, . \tag{3.5}$$

We will be mainly interested in $u(t)$, $t \in \mathbb{Z}$. By

$$u(k; u_0)\, ,\ k \ge 0,\ \ u_0 \in H,$$

we denote the solution of (3.4), equal u_0 at $t = 0$.

Basic estimate. We have for any $k \in \mathbb{Z}$ and $m \ge 0$

$$\begin{aligned} |u(k)| = \big|S\big(u(k-1)\big) + \eta_k\big| &\overset{(NS1)}{\le} \sqrt{B_0} + e^{-1}|u(k-1)| \\ &\le \sqrt{B_0} + e^{-1}\big(\sqrt{B_0} + e^{-1}\,|u(k-2)|\big) \\ &< \cdots < \sqrt{B_0}\,(1 + e^{-1} + \dots) + e^{-m}\,|u(k-m)|\, . \end{aligned}$$

So,

$$|u(k)| \le \sqrt{B_0}\, \frac{e}{e-1} + e^{-m}\,|u(k-m)|, \quad \text{for all } m \ge 0, \omega \in \Omega. \tag{3.6}$$

In particular,

$$|u(k;0)| \le \frac{e}{e-1}\, \sqrt{B_0}\, , \quad \text{for all } k \ge 0, \omega \in \Omega. \tag{3.7}$$

For $r \in \mathbb{R}$ we denote

$$B_r = \sum |s|^{2r}\, b_s^2 \le \infty\, .$$

Repeating the arguments above and using the property (NS1′) instead of (NS1) we get that

$$\|u(k)\|_1 \le \sqrt{B_1}\, \frac{e}{e-1} + e^{-m}\|u(k-m)\|_1, \quad \text{for all } m \ge 0,\ \omega \in \Omega\, , \tag{3.8}$$

if $B_1 < \infty$.

We may write (3.4) as

$$u(k) = F_k(u(k-1), w)\, ,$$

where the map $F_k : H \times \Omega \to H$ is measurable and locally Lipschitz in $u \in H$. So (3.4) is a Random Dynamical System (RDS). See the books [Kif86, Arn98]. Every RDS defines a Markov chain:

The Markov chain. For $u_0 \in H$, $0 \le k \in \mathbb{Z}$ and $\Gamma \in \mathcal{B}(H)$ we define

$$P_k(u_0, \Gamma) = \mathbf{P}(u(k; u_0) \in \Gamma).$$

P is called the *transition function* for (3.4). It satisfies the *Chapman-Kolmogorov relation*: for $k, n \ge 0$,

$$P_{k+n}(u_0, \Gamma) = \int_H P_k(u_0, du)\, P_n(u, \Gamma).$$

So (3.4) defines a Markov chain $\{u(k)\}_{k \ge 0}$ in H. (This is almost obvious, or see in [Kif86, Arn98]). Due to (3.6) with $k = m$,

$$\int |u|\, P_k(u_0, du) = \mathbf{E}\, |u(k)| \le \sqrt{B_0}\, \frac{e}{e-1} + e^{-k} |u_0| \,, \tag{3.9}$$

where $u(t)$ is a solution such that $u(0) = u_0$.

The Markov semigroups. We next introduce the two Markov semigroups, one on $C_b(H)$, another on $\mathcal{P}(H)$:

i) $\mathcal{B}_k : C_b(H) \to C_b(H)$, $\quad (\mathcal{B}_k f)(v) = \int_H P_k(v, dz)\, f(z)$,

for $0 \le k \in \mathbb{Z}$, $v \in H$ (see Exercise 3.3 below).

ii) $\mathcal{B}_k^* : \mathcal{P}(H) \to \mathcal{P}(H)$, $\quad (\mathcal{B}_k^* \mu)(\Gamma) = \int_H P_k(v, \Gamma)\, \mu(dv)$,

for $0 \le k \in \mathbb{Z}$, $\Gamma \in \mathcal{B}(H)$.

Let $u_0 \in H$ be a random variable, such that $\mathcal{D}(u_0) = \mu$. Then

$$\mathcal{D}\, u(k; u_0) = \mathcal{B}_k^* \mu, \quad \forall k \ge 0\,.$$

In particular, taking $k = 1$ and $\mu = \delta_v$, $v \in H$, we have

$$\mathcal{B}_1^* \delta_v = P_1(v, \cdot) = \mathcal{D}(S(v) + \eta_1). \tag{3.10}$$

It is immediate that each operator $\mathcal{B}_k^*$ extends to a **linear** map $\mathcal{M}(H) \to \mathcal{M}(H)$, defined by the same relation ii).

The duality. For the two semigroups the duality relations hold:

$$(\mathcal{B}_k f, \mu) = (f, \mathcal{B}_k^* \mu), \quad \text{for all } f \in C_b,\ \mu \in \mathcal{M} \text{ and } k \ge 0$$

(the relations do not mean that these are dual semigroups since the spaces $C_b(H)$ and $\mathcal{M}(H)$ are not dual.)

Example 3.2. For $u \in H$, let δ_u be the delta-measure at u. Then $\mathcal{B}_k^*(\delta_u) = P_k(u, \cdot)\,$.

Exercise 3.3. a) Prove that for any $k \ge 1$ the map $H \to \mathcal{P}(H)$, $u \longmapsto P_k(u, \cdot)$, is continuous, where $\mathcal{P}(H)$ is endowed with the weak topology. Show that this is equivalent to the fact that $(\mathcal{B}_k f) \in C_b(H)$ if $f \in C_b(H)$. This property of a Markov chain is called the *Feller property*.

b) Prove that the maps $\mathcal{B}_k^* : \mathcal{P}(H) \to \mathcal{P}(H)$ are continuous (in the weak topology).

3.3 Stationary measures

Definition 3.4. *A measure $\mu \in \mathcal{P}(H)$ is a stationary measure for (3.4) if $\mathcal{B}_1^* \mu = \mu$. (Then $\mathcal{B}_k^* \mu = \mu$ for each $k \ge 0$).*

Theorem 3.5. *A stationary measure μ exists.*

Proof. For simplicity, assume that $B_1 < \infty$. (For $B_1 = \infty$, the proof is slightly longer, see [KS00]).

Let us take in (3.4) $u(0) = 0$, denote $\mu_k = \mathcal{D}(u(k))$ and set

$$\overline{\mu}_k = \frac{1}{k} \sum_{j=0}^{k-1} \mu_j\,.$$

Note that $\mathcal{B}_1^* \mu_j = \mu_{j+1}$. We define $\mathfrak{B}_r(H^s) = \{u \,|\, \|u\|_s \le r\}$. Let $r_1 = \frac{e}{e-1} \sqrt{B_1}$. By (3.8), we see that

$$\mu_j\big(\mathfrak{B}_{r_1}(H^1)\big) = 1, \quad \text{for all } j \ge 0.$$

Hence $\overline{\mu}_k\big(\mathfrak{B}_{r_1}(H^1)\big) = 1$ for all $k \ge 0$. But from (1.3) we know that $H^1 \Subset H$, and thus the set $\mathfrak{B}_{r_1}(H^1)$ is compact in H. It follows from the Prokhorov Theorem (see [Bil99, Shi96, Dud89]) that the set $\{\overline{\mu}_k, k \ge 1\}$ is compact in $\mathcal{P}(H)$, where the latter is provided with the weak topology. So there exists a subsequence $\{k_m\}$ such that

$$\overline{\mu}_{k_m} \rightharpoonup \mu \in \mathcal{P}(H)\,.$$

We claim that μ is a stationary measure. Indeed, for any $f \in C_b(H)$, we have

$$\begin{aligned}
(f, \mathcal{B}_1^* \mu) &\overset{\text{Exerc. 3.3}}{=} \lim_{m\to\infty} (f, \mathcal{B}_1^* \overline{\mu}_{k_m}) = \lim_{m\to\infty} \frac{1}{k_m} \sum_{j=0}^{k_m-1} (f, \mathcal{B}_1^* \mu_j) \\
&= \lim_{m\to\infty} \frac{1}{k_m} \sum_{j=1}^{k_m} (f, \mu_j) = \lim_{m\to\infty} \Big((f, \overline{\mu}_{k_m}) + \frac{1}{k_m} \big((f, \mu_{k_m}) - (f, \mu_0) \big) \Big) \\
&= (f, \mu)\,.
\end{aligned}$$

As a consequence, $\mathcal{B}_1^* \mu = \mu$. $\square$

Remark 3.6. *The weighted sum, defining the measure $\overline{\mu}_k$, is called the Bogolyubov–Krylov ansatz. If $B_1 = \infty$, but $B_0 < \infty$, then still the set of measures $\{\overline{\mu}_k\}$ is weakly precompact in $\mathcal{P}(H)$ and every limiting measure is stationary, see [KS00].*

Exercise 3.7. Use (3.7) to prove that $\mu\big(\mathfrak{B}_{r_0}(H)\big) = 1$, with $r_0 = \frac{e}{e-1} \sqrt{B_0}$. If $B_1 < \infty$, then also $\mu\big(\mathfrak{B}_{r_1}(H^1)\big) = 1$.

Let $u(t)$, $t \ge 0$, be a solution of (3.3), such that $\mathcal{D}\, u(0) = \mu$, where μ is a stationary measure. Then $\mathcal{D}\, u(k) = \mu$ for all integer $k \ge 0$, i.e., $\{u(k)\}$ is a stationary process. Due to (3.5), the process $\{u(t), t \ge 0\}$ is 1–periodic in distribution:

$$\mathcal{D}\big(u(t+1)\big) = \mathcal{D}\,\big(u(t)\big) \quad \forall t \ge 0.$$

3.4 More estimates

Let us assume that

$$\mathbf{E}\,\xi_{sk} = 0 \quad \text{for all } s \in \mathbb{Z}_0, k \in \mathbb{Z}.$$

Then $\mathbf{E}\,\eta_k = 0$. Consider the following scaled NSE:

$$\dot{u} + \nu L u + B(u) = \sqrt{\nu}\,\eta(t), \quad 0 < \nu \le 1\,. \tag{3.11}$$

(This equation coincides with (3.3) if $\nu = 1$). As before, $\tau = 1$. The numbers D_0 and D_1 are defined as in (3.1).

Exercise 3.8. Let $u(t)$, $t \ge 0$, be a solution of (3.11). Then for all $k \ge 0$,

$$\mathbf{E}\,|u(k+1)|^2 + 2\nu \int_k^{k+1} \mathbf{E}\,\|u(s)\|_1^2\,ds = \mathbf{E}\,|u(k)|^2 + \nu D_0\,.$$

If $D_1 < \infty$, then also

$$\mathbf{E}\,\|u(k+1)\|_1^2 + 2\nu \int_k^{k+1} \mathbf{E}\,\|u(s)\|_2^2\,ds = \mathbf{E}\,\|u(k)\|_1^2 + \nu D_1\,.$$

Note that if u is a stationary solution, then these relations imply that

$$2\int_k^{k+1} \mathbf{E}\,\|u(s)\|_1^2\,ds = D_0, \quad 2\int_k^{k+1} \mathbf{E}\,\|u(s)\|_2^2\,ds = D_1\,. \tag{3.12}$$

If $u(t,x)$ is a solution of (3.11), then

$$u(k+1) = S_1^{(\nu)}\,u(k) + \sqrt{\nu}\,\eta_{k+1}\,, \tag{3.13}$$

where $S_1^{(\nu)}$ is the time-one shift along trajectories of the free equation (3.11). Now the property (NS1) takes the form

$$|S_1^{(\nu)} u| \le e^{-\nu}\,|u|\,. \tag{3.14}$$

Lemma 3.9. *There exist ν-independent constants C_m, $m \in \mathbb{N}$, such that for any m solutions of (3.13) satisfy*

$$\mathbf{E}\,|u(k+1)|^{2m} \le (1-\nu)\,\mathbf{E}\,|u(k)|^{2m} + C_m\nu\,. \tag{3.15}$$

If $D_1 < \infty$, then also

$$\mathbf{E}\,\|u(k+1)\|_1^{2m} \le (1-\nu)\,\mathbf{E}\,\|u(k)\|_1^{2m} + C'_m\nu$$

for some constants C'_m. In particular, if $u(0) = 0$, then

$$\mathbf{E}\,|u(k)|^{2m} \le C_m \ \text{ for all } k \ge 0, \quad \text{and} \tag{3.16}$$

$$\mathbf{E}\,\|u(k)\|_1^{2m} \le C'_m \ \text{ for all } k \ge 0, \text{ if } D_1 < \infty. \tag{3.17}$$

If $u(t)$, $t \ge 0$ is a solution of (3.11), equal zero at $t = 0$, then (3.16) and (3.17) hold for any real $k \ge 0$.

Proof. Let $u(k)$ be a solution of (3.13). Then

$$\mathbf{E}|u(k+1)|^{2m} = \mathbf{E}\,|S_1^{(\nu)} u(k) + \sqrt{\nu}\,\eta_{k+1}|^{2m}\,. \tag{3.18}$$

We denote $u = S_1^{(\nu)} u(k)$ and $\eta = \eta_{k+1}$. We claim that for any $c > 0$ we can find $C_m > 0$ such that

$$I_\nu := \mathbf{E}\,|u + \sqrt{\nu}\,\eta|^{2m} \le (1 + c\nu)\,\mathbf{E}\,|u|^{2m} + C_m\nu\,. \tag{3.19}$$

Indeed, we have

$$\begin{aligned} I_\nu &= \mathbf{E}(|u|^2 + \nu\,|\eta|^2 + 2\sqrt{\nu}\,\langle u, \eta\rangle)^m \\ &= \mathbf{E}\,|u|^{2m} + 2m\sqrt{\nu}\ \mathbf{E}(|u|^{2m-2}\,\langle u, \eta\rangle) + \nu J_\nu, \end{aligned}$$

where $J_\nu = O(1)$ as $\nu \to 0$. The second term in the r.h.s. vanishes since u is independent of η and $\mathbf{E}\,\eta = 0$. Let us estimate the third term. We have

$$|J_\nu| \le \nu^{m-1}\,\mathbf{E}\,|\eta|^{2m} + C'_m \sum_{2m-2 \ge n \ge 2} |u|^n\,|\eta|^{2m-n}\,. \tag{3.20}$$

By the Young inequality (2.29), for any $\kappa > 0$ and any $n \in [2, 2m-2]$ we have

$$|u|^n\,|\eta|^{2m-n} \le \frac{n}{2m}\,\kappa^{\frac{2m}{n}}\,|u|^{2m} + \frac{2m-n}{2m}\,\kappa^{-\frac{2m}{2m-n}}\,|\eta|^{2m}\,.$$

Using in (3.20) this inequality with various $\kappa = \kappa_n$, we get that for any $c > 0$ there is a C_m such that

$$|J_\nu| \le c\,\mathbf{E}\,|u|^{2m} + C_m\,\mathbf{E}\,|\eta|^{2m}\,.$$

Now (3.19) follows since $\mathbf{E}\,|\eta|^{2m} \le B_0^m$.

Due to (3.19) and (3.14), we obtain from (3.18)

$$\begin{aligned} \mathbf{E}\,|u(k+1)|^{2m} &\le (1 + c\nu)\,\mathbf{E}\,|S_1^{(\nu)}\,u(k)|^{2m} + C_m\nu \\ &\le (1 + c\nu)\,e^{-2m\nu}\,\mathbf{E}\,|u(k)|^{2m} + C_m\nu \le (1-\nu)\,\mathbf{E}\,|u(k)|^{2m} + C_m\nu \end{aligned}$$

(since for c small enough we have $(1 + c\nu)\,e^{-2\nu} \le 1 - \nu$). Now (3.15) is proven. Since the estimate (3.14) also holds for the norm $\|\cdot\|_1$, then repeating the same arguments we get the required bound for $\mathbf{E}\,\|u(k+1)\|_1^{2m}$.

The last two estimates of the lemma are straightforward consequences of the first two. These estimates and (3.5) imply the last assertion since the maps $S_t^{(\nu)}$ decrease the norms in H and in H^1. □

4 White-forced equations

4.1 White in time forces

For $x \in \mathbb{T}^2$ and $t \geq 0$, we consider $\eta(t,x) = \dfrac{d}{dt}\zeta(t,x)$, where

$$\zeta(t,x) = \sum_{s\in\mathbb{Z}_0^2} b_s\,\beta_s(t)\,e_s(x), \quad \zeta(0) = 0\,.$$

Here $\{\beta_s, s \in \mathbb{Z}_0^2\}$ are standard independent Wiener processes, and

$$B_1 := \sum |s|^2\, b_s^2 < \infty\,.$$

Basic inequalities for the random field $\boldsymbol{\zeta}$. The following relations are obvious:

$$\mathbf{E}\,|\zeta(t)|^2 = \sum_s b_s^2\,\mathbf{E}\,\beta_s^2(t) = t\sum b_s^2 = tB_0\,,$$

$$\mathbf{E}\,\|\zeta(t)\|_k^2 = \sum_s b_s^2\,\mathbf{E}\,\beta_s^2(t)\,\|e_s\|_k^2 = t\sum_s b_s^2\,|s|^{2k} = t\,B_k\,,$$

$$\mathbf{E}\,|\zeta|^2_{L_2(0,T;H)} = \mathbf{E}\int_0^t |\zeta(s)|^2 ds = \frac{1}{2}\,t^2 B_0\,,$$

where we denote

$$B_k = \sum_s |s|^{2k}\, b_s^2 \leq \infty.$$

The Wiener processes $\beta_s(t)$ are a.s. continuous. The process $\zeta(t)$ also is a.s. continuous in the corresponding spaces:

Lemma 4.1. *Assume that $B_k < \infty$. Then the process $\zeta(t)$, $0 \leq t < \infty$, is continuous in H^k a.s., and*

$$\mathbf{E}\max_{0\leq s\leq t}\|\zeta(s)\|_k^p \leq t^p\,C(p,B_k)\,, \tag{4.1}$$

for each $p > 1$.

Proof. Set $\zeta_N(t) = \sum_{|s|\leq N} b_s\,\beta_s(t)\,e_s$. Clearly, $\zeta_N(\cdot) \in C([0,T];H^k)$ a.s., for any fixed constant T. By the Doob inequality [IW89, Kry03, Øk03], for $M \geq N$ we have

$$\mathbf{E}\max_{0\leq s\leq T}\|(\zeta_M - \zeta_N)(s)\|_k^2 \leq 4\,\mathbf{E}\,\|(\zeta_M-\zeta_N)(T)\|_k^2 =$$

$$4T\sum_{M\geq|s|>N} b_s^2\,|s|^{2k} \leq 4T\sum_{|s|>N} b_s^2\,|s|^{2k}\,.$$

Since $B_k < \infty$, then a.s. $\{\zeta_N(\cdot)\}$ is a Cauchy-sequence in $C([0,T];H^k)$. Hence, the limit $\zeta(t) = \sum b_s\,\beta_s(t)\,e_s$ exists and belongs to $C([0,T];H^k)$.

The estimate (4.1) follows from the Doob inequality

$$\mathbf{E}\max_{0\leq s\leq t}\|\zeta(s)\|_k^p \leq \left(\frac{p}{p-1}\right)^p\mathbf{E}\,\|\zeta(t)\|_k^P\,,$$

since $\mathbf{E}\|\zeta(t)\|_k^p \leq t^p C(p,B_k)$ (say, by the Fernique theorem, see [Bog98]). □

It is convenient to redefine ζ on a null-set to achieve

$$\zeta^\omega(\cdot) \in C([0,\infty),H^k) \ \text{ for all } \omega, \text{ if } B_k < \infty\ . \tag{4.2}$$

Homogeneity. The random field $\zeta(x,t)$ is called *homogeneous* (in the space-variable x), if for each $\xi \in \mathbb{T}^2$ the field $T_\xi\zeta(t,x) = \zeta(t,x+\xi)$ has the same distribution as $\zeta(t,x)$.

Exercise 4.2. Show that $\zeta(t,s)$ is homogeneous in x if and only if

$$b_s = b_{-s} \ \text{ for all } s \in \mathbb{Z}_0^2.$$

4.2 The white-forced 2D NSE

The *white-forced NSE* is

$$\dot u + Lu + B(u) = \frac{d}{dt}\zeta(t,x)\,, \tag{4.3}$$

where ζ is the process as in Section 4.1. We supplement (4.3) with an initial condition:

$$u(0) = u_0 \in H\,. \tag{4.4}$$

Due to (4.2), Theorem 2.17 and Example 2.20 applied with any $T < \infty$, and we obtain:

Theorem 4.3. *The problem has a unique solution such that*

$$u \in C([0,\infty),H)\cap L_{2,\,\mathrm{loc}}([0,\infty),H^1) \ \textit{ for all } \omega\,. \tag{4.5}$$

Moreover for any $T > 0$ and any ω,

$$u_N \underset{N\to\infty}{\longrightarrow} u \text{ in } C([0,T],H)\,, \tag{4.6}$$

where $u_N(t)$ is a Galerkin approximation. Besides

$$u(t) = F(t,u_0,\{\zeta(s), 0\leq s\leq t\})\,. \tag{4.7}$$

Amplification. Same is true if $u_0 \in H$ is a random variable.

Below either u^0 is non-random, or $u^0 \in H$ is a random variable, independent of $\zeta(\cdot)$.

Measurability, measures and weak solutions. In $(\Omega,\mathcal{F},\mathbf{P})$, we consider the filtration $\mathcal{F}_t \subset \mathcal{F}, t \geq 0$, generated by all zero-sets and by the random variables

$$\zeta(s),\ 0 \leq s \leq t\,.$$

A random process $y(t) = y^\omega(t)$, $t \geq 0$, is called *$\mathcal{F}_t$-adapted* if $y(t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.

Due to (4.7), the random process $u(t)$, interpreted as a random variable

$$\Omega \ni \omega \longmapsto u^\omega \in C([0,\infty), H)$$

has the form

$$\omega \longmapsto \zeta^\omega \longmapsto \Phi(\zeta^\omega)\,, \tag{4.8}$$

where

$$\Phi : C([0,\infty), H^1) \to C([0,\infty), H)\,,$$
$$\Phi\big(y(\cdot)\big) = Y(\cdot), \quad Y(t) = F(t, u_0, \{y(s), 0 \leq s \leq t\})\,.$$

If we consider the process u^ω for $t \in [0,T]$, then the corresponding random variable $\omega \longmapsto u^\omega|_{[0,T]}$ has the form

$$\omega \longmapsto \zeta^\omega\,|_{[0,T]} \longmapsto \Phi_T(\zeta^\omega\,|_{[0,T]})\,, \tag{4.9}$$

where

$$\Phi_T : C(0,T; H^1) \to C(0,T; H)\,.$$

The map Φ and the maps Φ_T agree:

$$\Phi_T(\zeta\,|_{[0,T]}) = \Phi(\zeta)\,|_{[0,T]}\,.$$

By Proposition 2.18, the maps Φ_T are locally Lipschitz. In particular, they are continuous. Accordingly, the map Φ is also continuous.

By Theorem 4.3, $u^\omega(t)$ is a random process with continuous trajectories. Due to (4.9)

$$u(t) \text{ is } \mathcal{F}_t\text{-adapted}\,. \tag{4.10}$$

For any $t_0 \geq 0$, let us denote $u^{t_0}(t) = u(t_0 + t)$. Then u^{t_0} satisfies (4.3), (4.4) with $u_0 = u(t_0)$ and $\zeta(t)$ replaced by $\zeta(t+t_0) - \zeta(t_0)$. Applying (4.9) to u^{t_0} we see that

$$u(t)\,,\ t \geq t_0\,, \text{ is a function of } u_{t_0} \text{ and } \zeta(s) - \zeta(t_0),\ t_0 \leq s \leq t\,. \tag{4.11}$$

The properties (4.10), (4.11) imply

Proposition 4.4. *The solution $u(t)$ is an $\mathcal{F}_t$-adapted Markov process with continuous trajectories.*

We omit details of the proof.

Let $dW = \mathcal{D}(\zeta)$ be the distribution of ζ in $C([0,\infty), H^1)$ (i.e., the Wiener measure, corresponding to the process ζ). Then, by (4.8),

$$\mathcal{D}\big(u(\cdot)\big) = \Phi \circ (dW)\,. \tag{4.12}$$

Here $\Phi \circ (dW)$ denotes the push-forward of dW by the map Φ.

Definition 4.5. *A process $\widetilde{u}(t)$ as in* (4.5) *is a weak solution of (4.3), (4.4) if it satisfies (4.3), (4.4) with the process $\zeta(t)$ replaced by some process $\zeta'(t)$ such that $\mathcal{D}(\zeta') = \mathcal{D}(\zeta)$.*

Proposition 4.6. *The distribution of a weak solution $\mathcal{D}(\widetilde{u})$ is uniquely defined.*

Proof. The measure $\mathcal{D}(\widetilde{u})$ satisfies (4.12). So it is independent of the choice of $\widetilde{u}$. $\square$

4.3 Estimates for solutions

The N-th Galerkin approximation to equation (4.3) is the equation

$$\dot{u} = \overbrace{-Lu - \Pi_N\, B(u)}^{V_N(u)} + \frac{d}{dt} \sum_{|s| \leq N} b_s\, \beta_s(t)\, e_s\,, \tag{4.13}$$
$$u \in H_N = \operatorname{span}\{e_s \,\big|\, |s| \leq N\};\ u(0) = u_{0,N} = \Pi_N\, u_0 \in H_N\,.$$

We assume that $u_0 = u_0^\omega$ is independent of $\zeta(t)$, $t \geq 0$, and $\mathbf{E}\,|u_0|^2 < \infty$.

For the same reasons as in Section 4.2, this equation has a solution $u = u_N(t)$. This random process also is a solution of (4.13), understood as a stochastic differential equation. That is,

$$du = V_N(u)\,dt + \sum_{|s| \leq N} b_s\, e_s\, d\beta_s\,.$$

The solutions $u_N(t)$ exists till a blow–up time $T^\omega \leq \infty$. It is a classical result that due to a-priori estimate (4.16) below $T^\omega = \infty$ a.e. (this happens by a reason, similar to that in Section 2.3), see [Kha80, IW89, Kry03, Øk03].

Let $f(t,u)$ be a C^2-smooth function on $\mathbb{R} \times H_N$. Then the *Ito formula* applies to the process $f = f(t, u(t))$ (see [Kha80, IW89, Kry03, Øk03]), and we have

$$\begin{aligned} \frac{d}{dt}\, f(t,u(t)) &= (\partial_t f)(t, u(t)) + \langle \nabla_u f,\, V_N(u) \rangle \\ &\quad + \frac{1}{2} \sum_{|s| \leq N} \frac{\partial^2 f}{\partial u_s^2}\, b_s^2 + \Big\langle \nabla_u f,\, \sum_{|s| \leq N} b_s \dot{\beta}_s\, e_s \Big\rangle\,. \end{aligned} \tag{4.14}$$

Choose $f(t,u) = |u|^2$. Then

$$\frac{d}{dt}\,|u|^2 = 2\,\underbrace{\langle u, -Lu - B(u) \rangle}_{\overset{B1\ i)}{=} -2\|u\|_1^2} + \underbrace{\Big(\sum_{|s| \leq N} b_s^2 \Big)}_{B_{0N}} + 2\Big\langle u,\, \sum_{|s| \leq N} b_s\, \dot{\beta}_s \Big\rangle\,. \tag{4.15}$$

By taking expectations, we find:

$$\frac{d}{dt}\,\mathbf{E}\,|u|^2 = -2\,\mathbf{E}\,\|u\|_1^2 + B_{0N} \leq -2\,\mathbf{E}\,|u|^2 + B_0\,.$$

Therefore, by Gronwall's lemma,

$$\mathbf{E}\,|u(t)|^2 \leq -\frac{1}{2} e^{-2t}\, B_0 + \frac{1}{2} B_0 + e^{-2t}\,\mathbf{E}\,|u_0|^2 \leq \frac{1}{2} B_0 + e^{-2t}\,\mathbf{E}\,|u_0|^2\,. \tag{4.16}$$

Integrating (4.15) we see that

$$|u(T)|^2 + 2 \int_0^T \|u\|_1^2\, ds = |u_0|^2 + B_{0N}\, T + M(T), \tag{4.17}$$

where $M(t) = 2\int_0^t \left\langle u(\tau), \sum_s b_s\, d\beta_s(\tau)\right\rangle$ is a martingale. Taking the expectations, we obtain

$$\mathbf{E}\int_0^T \|u(s)\|_1^2 ds \le \frac{1}{2}(T\,B_0 + \mathbf{E}\,|u_0|^2)\,. \tag{4.18}$$

Due to (4.17) with T replaced by $t \le T$,

$$\sup_{0\le t\le T} |u(t)|^2 \le |u_0|^2 + B_0 T + \sup_{0\le t\le T} M(t)\,. \tag{4.19}$$

The Burkholder-Davis-Gundy inequality (with $p=1$) states that:

$$\mathbf{E}\sup_{0\le t\le T} |M(t)| \le N_1\,\mathbf{E}\langle M\rangle_T^{1/2} \le N_1\sqrt{\mathbf{E}\langle M\rangle_T}, \tag{4.20}$$

where $\langle M\rangle_T = \int_0^T \left(\sum u_s^2(\tau) b_s^2\right) d\tau$, see [Kry03]. We have

$$\begin{aligned}\mathbf{E}\langle M_T\rangle = \int_0^T \mathbf{E}\Big(\sum u_s^2(\tau)\, b_s^2\Big)\, d\tau &\le b_{\max}^2 \int_0^T \mathbf{E}\,|u(\tau)|^2\, d\tau \\ &\overset{(4.18)}{\le} \frac{1}{2}\, b_{\max}^2\Big(B_0 T + \mathbf{E}\,|u_0|^2\Big)\,.\end{aligned}$$

So,

$$\sqrt{\mathbf{E}\langle M_T\rangle} \le \frac{b_{\max}}{\sqrt{2}}\Big(\sqrt{B_0 T} + \sqrt{\mathbf{E}\,|u_0|^2}\Big)\,.$$

Assume that $T\ge 1$. Then (4.19) and (4.20) imply:

$$\mathbf{E}\sup_{0\le t\le T} |u(t)|^2 \le C_1\big(T + \mathbf{E}\,|u_0|^2 + \sqrt{\mathbf{E}\,|u_0|^2}\big)\,. \tag{4.21}$$

We remind that the estimates (4.16)–(4.21) are obtained for $u = u_N$, a solution of (4.13).

Due to (4.6), for any M and every ω, we have:

$$\Pi_M\, u_N \underset{N\to\infty}{\longrightarrow} \Pi_M\, u \text{ in } L_2(0,T;H^1)\,.$$

So, by (4.18) and the Fatou Lemma,

$$\mathbf{E}\int_0^T \|\Pi_M\, u(s)\|_1^2 ds \le \varliminf_{N\to\infty} \mathbf{E}\int_0^T \|\Pi_M\, u_N\|_1^2\, ds \le \frac{1}{2}(TB_0 + \mathbf{E}\,|u_0|^2)\,.$$

Now, applying Fatou again, we obtain that

$$\mathbf{E}\int_0^T \|u\|_1^2\, ds \le \varliminf_{M\to\infty} \mathbf{E}\int_0^T \|\Pi_M u\|^2\, ds \le \frac{1}{2}(T\,B_0 + \mathbf{E}\,|u_0|^2)\,.$$

In a similar way, we can pass to the limits in the estimates (4.16) and (4.21) and obtain:

Theorem 4.7. *Let $u_0 \in H$ be a random variable, independent of $\zeta(\cdot)$, and such that $\mathbf{E}\,|u_0|^2 < \infty$. Then the solution u, constructed in Theorem 4.3, satisfies (4.16), (4.18) and (4.21).*

Exercise 4.8. Apply Ito's formula to $\varphi(u) = \|u\|_1^2$ to get:

$$\mathbf{E}\,\|u(t)\|_1^2 \le \tfrac{1}{2}B_1 + e^{-2t}\,\mathbf{E}\,\|u_0\|_1^2,$$

$$\mathbf{E}\int_0^T \|u(s)\|_2^2\, ds \le \frac{1}{2}(T\,B_1 + \mathbf{E}\,\|u_0\|_1^2)\,,$$

$$\mathbf{E}\sup_{0\le t\le T} \|u(t)\|_1^2 \le C\Big(T + \mathbf{E}\,\|u_0\|_1^2 + \sqrt{\mathbf{E}\,\|u_0\|_1^2}\Big)\,,\ C = C(b_{\max}, B_1)\,.$$

Exercise 4.9. Apply Ito's formula to $\exp\big(\sigma|u|^2\big)$, $\sigma < (2b_{\max}^2)^{-1}$, to get

$$\mathbf{E}\,e^{\sigma|u(t)|^2} \le e^{-\sigma t}\,\mathbf{E}\,e^{\sigma|u_0|^2} + C\,.$$

Similar, apply Ito's formula to $\exp(\sigma\|u\|_1^2)$ to get that

$$\mathbf{E}\,e^{\sigma\|u(t)\|_1^2} \le e^{-\sigma t}\,\mathbf{E}\,e^{\sigma\|u_0\|_1^2} + C\,.$$

Exercise 4.10. * [2] If $B_k = \sum |s|^{2k} b_s^2 < \infty$, apply Ito's formula to functionals $t^m\|u(t)\|_m^2$ and argue by induction to see that

$$\mathbf{E}\big(\sup_{0\le t\le T} t^k\,\|u(t)\|_k^2\big)^p \le C\,\mathbf{E}\,e^{\sigma|u_0|^2},\ \text{ with } C = C(k,p,\sigma,T)\,,$$

for any $k\,,p\in\mathbb{N}$ (see [KS03]).

4.4 Stationary measures

The semigroups

$$\begin{aligned}\mathcal{B}_t &: C_b(H) \to C_b(H),\ \ t\ge 0\,,\\ \mathcal{B}_t^* &: \mathcal{P}(H) \to \mathcal{P}(H),\ \ \ \ t\ge 0\,,\end{aligned}$$

are defined as in Section 3.2. That is, for $t\ge 0, v\in H$ and Γ a measurable subset of H:

$$(\mathcal{B}_t f)(v) = \int_H P_t(v,du)\, f(u) = \mathbf{E}\, f\big(u(t;v)\big)\,, \tag{4.22}$$

$$(\mathcal{B}_t^*\mu)(\Gamma) = \int_H P_t(v,\Gamma)\,\mu(dv)\,,$$

$$(\mathcal{B}_t^*\mu) = \mathcal{D}\big(u(t;u_0)\big),\ \text{ if } \mathcal{D}(u_0) = \mu\,. \tag{4.23}$$

[2] As usual, the asterisk indicates a more difficult exercise (in the D. Knuth's book "The Art of Computer Programming" this notion is illustrated by the following example: *Prove that for an integer $n\ge 3$ the equation $x^n + y^n = z^n$ has no integer solutions.*)

Here $P_t(v,du)$ is the Markov transition function, and $u(t;u_0)$ is a solution of (4.3), equal u_0 at $t=0$ (in (4.23) u_0 is a random variable).

We note that by Proposition 4.6 relations (4.22) and (4.23) remain valid if $u(t;u_0)$ is a weak solution.

A measure $\mu \in \mathcal{P}(H)$ is called a *stationary measure* if $\mathcal{B}_t^* \mu = \mu$ for all $t \ge 0$.

Theorem 4.11. *A stationary measure exists.*

Proof. Let $u(t)$ be a solution such that $u(0)=0$. We set $\mu_t = \mathcal{D}(u(t))$. Due to the first estimate in Exercise 4.8 and the Chebychev inequality we have

$$\mu_t\big(H\backslash \mathfrak{B}_R(H^1)\big) = \mathbf{P}(\|u(t)\|_1 > R) \le B_1\, R^{-2}\,.$$

For $\varepsilon > 0$, choosing $R_\varepsilon = (B_1/\varepsilon)^{1/2}$, we see that

$$\mu_t\big(H\backslash \mathfrak{B}_{R\varepsilon}(H^1)\big) < \varepsilon, \text{ for all } t \ge 0.$$

Setting $\overline{\mu}_t = \frac{1}{t}\int_0^t \mu_s\, ds$, $t \ge 1$, we see that $\overline{\mu}_t$ also satisfies this estimate. Since by (1.3), $\mathfrak{B}_{R_1}(H^1) \Subset H$, then the Prokhorov theorem applies, and the set $\{\overline{\mu}_t,\ t\ge 1\}$ is precompact in $\mathcal{P}(H)$. Choosing a converging sequence $\overline{\mu}_{t_j} \rightharpoonup \mu$, we see as in the proof of Theorem 3.5 that μ is a stationary measure. □

The constructed measure μ has finite second moment:

$$\int |u|^2\, \mu(du) \le B_0\,.$$

Indeed, by Theorem 4.7, $\int |u|^2 \mu_t(du) \le B_0$ for each t. Hence, the measure $\overline{\mu}_t$ satisfies the same inequality. For every $M < \infty$ we have

$$\int (|u|^2 \wedge M)\, \mu(du) = \lim \int (|u|^2 \wedge M)\, \overline{\mu}_t(du) \le B_0\,.$$

Now the Fatou lemma implies the desired estimate.

Exercise 4.12. Use Exercise 4.10 to prove that if $B_k < \infty$, then the constructed stationary measure μ satisfies

$$\int \|u\|_k^p\, \mu(du) < \infty \qquad \forall\, p \ge 0.$$

4.5 High-frequency random kicks

Consider the kick-force

$$\eta_\varepsilon(t) = \sqrt{\varepsilon}\sum_k \eta_k^\omega\, \delta(t-\varepsilon k), \quad \eta_k = \sum_{s\in\mathbb{Z}_0^2} b_s\, \xi_{sk}\, e_s\,.$$

The random variables are as in Section 3, and in addition

$$\mathbf{E}\,\xi_{sk} = 0, \quad B_1 = \sum_{s\in\mathbb{Z}_0^2} |s|^2\, b_s^2 < \infty\,.$$

We set

$$\zeta_\varepsilon(t) = \int_{0+}^t \eta_\varepsilon(\tau)\, d\tau\,,$$

and normalise ζ_ε to be continuous from the right.

We want to compare the force η_ε with the white-force η,

$$\eta = \frac{d}{dt}\,\zeta, \quad \zeta = \sum_{s\in\mathbb{Z}_0^2} b_s\, \beta_s(t)\, e_s(x)\,.$$

Fixing $u_0 \in H$, we denote by $u_\varepsilon(t;u_0)$ the solution of (3.3) with $\zeta = \zeta_\varepsilon, u_\varepsilon(0,u_0) = u_0$, and by $u(t;u_0)$ – the solution of (4.3) with $u(0;u_0)=u_0$.

Theorem 4.13. *For any finite time-interval* $[0,T]$*, we have*

$$\mathcal{D}\big(u_\varepsilon(\cdot\,;u_0)\big) \rightharpoonup \mathcal{D}\big(u(\cdot\,;u_0)\big) \ \text{ in } \mathcal{P}\big(L_\infty(0,T;H)\big)$$

and

$$\mathcal{D}(u_\varepsilon(T;u_0)) \rightharpoonup \mathcal{D}(u(T;u_0)) \ \text{ in } \mathcal{P}(H)$$

as $\varepsilon \to 0$.

Sketch of the proof. Let $\zeta_\varepsilon'(t)$ be a piecewise-linear continuous process, equal $\zeta_\varepsilon(t)$ for $0 \le t \in \varepsilon\,\mathbb{Z}$. Then

$$\mathcal{D}(\zeta_\varepsilon') \underset{\varepsilon\to 0}{\rightharpoonup} \mathcal{D}(\zeta) \text{ in } \mathcal{P}\big(C(0,T;H^1)\big)\,,$$

by the Donsker principle, see [Bil99, Dur91]. Let us fix any sequence $\varepsilon_j \to 0$. Due to the Skorokhod embedding theorem (see [Shi96], Section III.8 or [IW89], Section 1, Theorem 2.7), there exist processes $\widetilde{\zeta}'_{\varepsilon_j}$ and $\widetilde{\zeta}$, defined on the same probability space, such that

$$\mathcal{D}(\widetilde{\zeta}'_{\varepsilon_j}) = \mathcal{D}(\zeta'_{\varepsilon_j}),\ \ \mathcal{D}(\widetilde{\zeta}) = \mathcal{D}(\zeta) \quad \text{and} \quad \widetilde{\zeta}'_{\varepsilon_j} \underset{j\to\infty}{\longrightarrow} \widetilde{\zeta} \text{ a.s. in } C(0,T;H^1)\,. \tag{4.24}$$

Let $\widetilde{\zeta}_{\varepsilon_j}$ be a process which is continuous from the right, constant on the intervals $(k\varepsilon,(k+1)\varepsilon), k \ge 0$ and equal to $\widetilde{\zeta}'_{\varepsilon_j}$ for $t \in \varepsilon\,\mathbb{Z}$. Then

$$\mathcal{D}(\widetilde{\zeta}_{\varepsilon_j}) = \mathcal{D}(\zeta_{\varepsilon_j})\,.$$

Define the map Φ_T as in (4.9). Then

$$\begin{aligned} \mathcal{D}\big(u_{\varepsilon_j}(\cdot\,;u_0)\big) &= \Phi_T \circ \big(\mathcal{D}(\widetilde{\zeta}_{\varepsilon_j})\big)\,, \\ \mathcal{D}\big(u(\cdot\,;u_0)\big) &= \Phi_T \circ \big(\mathcal{D}(\widetilde{\zeta})\big)\,, \end{aligned} \tag{4.25}$$

in $\mathcal{P}(L_\infty(0,T;H))$.

Since the Brownian motion is Hölder-continuous, we have from (4.24) that

$$|\widetilde{\zeta}_{\varepsilon_j}(\cdot) - \widetilde{\zeta}(\cdot)|_{L_\infty(0,T;H^1)} \xrightarrow[\varepsilon_j\to 0]{\text{in Prob.}} 0\,. \tag{4.26}$$

By Proposition 2.18 the map

$$\Phi_T : L^0_\infty(0,T;H^1) \to L_\infty(0,T;H) \tag{4.27}$$

is locally Lipschitz. (We recall that L^0_∞ stands for the linear subspace of L_∞, formed by functions, continuous at $t=0$). Now (4.25)–(4.27) imply the first theorem's assertion.

Proof of the second is left as an exercise. □

Both the theorem's assertion and its proof become more natural by evoking the Skorokhod spaces (formed by discontinuous functions with jumps, see [Bil99, LS89]). Guided by *Occam's Razor*,[3] we decided not to introduce this notion.

[3] "Entities are not to be multiplied beyond necessity."

5 Preliminaries from the measure theory

Everywhere below M is a complete separable metric space (e.g. $M=H$, or M is a closed ball in $\mathbb{R}^n$).

5.1 Weak convergence of measures and Lipschitz-dual distance

A sequence of measures $\mu_n \in \mathcal{P}$ *weakly converges* to $\mu\in\mathcal{P}$, denotes $\mu_n \rightharpoonup \mu$, if $(\mu_n, f) \to (\mu, f)$ for each $f \in C^b(M)$.

For any $f\in C(M)$, $\mathrm{Lip}(f) \le \infty$ is the Lipschitz constant of f. We define the *Lipschitz-norm* of f as

$$|f|_L = \sup_M |f(m)| + \mathrm{Lip}(f)\,,$$

and set

$$O = \{f \in C^b(M) |\; |f|_L \le 1\}\,. \tag{5.1}$$

Proposition 5.1. *A sequence of measures μ_n weakly converges to μ if and only if $(\mu_n,f)\to(\mu,f)$ for each $f\in O$.*

Proof. The 'only if' part of the statement is obvious. If M is compact, then the 'if' part follows since any continuous function on M may be approximated in the sup–norm by bounded Lipschitz functions. For the non-compact case see [Dud89, Shi96]. □

For $\mu,\nu\in\mathcal{P}(M)$, define

$$\|\mu-\nu\|_L^* = \sup_O |(f,\mu)-(f,\nu)|\,. \tag{5.2}$$

This is the *Lipschitz-dual distance* on $\mathcal{P}(M)$.

Exercise 5.2. Let $M=H$ (with the usual distance) and $\|\cdot\|_L^*$ be the corresponding Lipschitz-dual distance. Now, for $0<d\le 1$ we set

$$\mathrm{dist}(u_1,u_2) = |u_1-u_2| \wedge d\,.$$

It is immediate that this is a distance which defines on H the same topology. Let $\|\cdot\|_{L,d}^*$ be the corresponding Lipschitz-dual distance. Prove that

$$\|\mu-\nu\|_{L,d}^* \le \|\mu-\nu\|_L^* \le \frac{2}{d}\,\|\mu-\nu\|_{L,d}^*\,. \tag{5.3}$$

Theorem 5.3. a) *The space $\mathcal{P}(M)$, given the distance dist$(\mu,\nu) = \|\mu-\nu\|_L^*$, is complete.*

b) *The weak convergence in $\mathcal{P}(M)$ is equivalent to the convergence in the Lipschitz-dual distance.*

Proof. (when M is compact). a) Let L be the space of Lipschitz functions on M with the norm $|\cdot|_L$. Let L^* be the dual space. The norm in L^* is given by (5.2). Clearly $\mathcal{P}(M) \subset L^*$. Let $\{\mu_n\}$ be a sequence in $\mathcal{P}(M)$ which is a Cauchy sequence in the norm $\|\cdot\|_L^*$. Then there is $\mu \in L^*$ such that

$$\mu_n \to \mu \in L^* .$$

We have to show that $\mu \in \mathcal{P}(M)$. Since $C(M)^* = \mathcal{M}(M)$ (the space of signed measures on M), we have to check that

$$\mu \geq 0 \ \text{ and } \ \mu \in C(M)^* . \tag{5.4}$$

Exercise 5.4. *Prove that (5.4) follows from the following condition:*

$$0 \leq (\mu, f) \leq 1 \quad \forall f \in L,\ 0 \leq f \leq 1 . \tag{5.5}$$

□

Since (5.5) holds for μ replaced by any μ_n, then it also holds for μ.
b) The implication '$\Longleftarrow$' follows from Proposition 5.1, and '$\Longrightarrow$' holds since for all $\varepsilon > 0$ the ball $\{f \mid |f|_L \leq 1\}$ has a finite ε-net in $C_b(M)$.

For the case when M is a separable metric space, see [Dud89, Shi96]. □

Note that the relation (5.2) also defines a distance on the space $\mathcal{M}(M)$. But this distance **is not complete**.

5.2 Variational distance

For $\mu, \nu \in \mathcal{P}(M)$ we define

$$\|\mu - \nu\|_{\text{var}} = \sup_{\Gamma \subset B(M)} |\mu(\Gamma) - \nu(\Gamma)| = \sup \big(\mu(\Gamma) - \nu(\Gamma)\big) \leq 1 . \tag{5.6}$$

This is the *variational distance* between the two measures.

Example 5.5. Let $M = \mathbb{R}$, $\mu = \delta_0$ and $\nu_n = \delta_{1/n}$, $n \geq 1$. Then $\|\mu - \nu_n\| = 1$, but $\|\mu - \nu_n\|_L^* = \frac{1}{n}$.

Exercise 5.6. *Prove that if μ and ν are absolutely continuous with respect to some measure $\rho \in \mathcal{P}(M)$, and $\mu = p_\mu(m)\,\rho(dm)$, $\nu = p_\nu(m)\,\rho(dm)$, then*

$$\|\mu - \nu\|_{\text{var}} = \frac{1}{2} \int_M |p_\mu(m) - p_\nu(m)|\,\rho(dm) . \tag{5.7}$$

Note that μ and ν are absolutely continuous with respect to $\rho = \frac{1}{2}(\mu + \nu)$. So '(5.7) always applies'.

Due to (5.7) for any continuous bounded function f we have

$$\begin{aligned} |(\mu, f) - (\nu, f)| &= \Big| \int_M \big(p_\mu(m) - p_\nu(m)\big)\, f(m)\,\rho(dm) \Big| \\ &\leq |f|_{C_b(M)} \int |p_\mu - p_\nu|\,\rho(dm)[1ex] \leq 2|f|_{C_b(M)}\, \|\mu - \nu\|_{\text{var}} . \end{aligned} \tag{5.8}$$

Hence, the space $\mathcal{P}(M)$, provided with the variational distance, is continuously embedded in $C_b(M)^*$. Besides, using (5.8) in (5.2) we see that

$$\|\mu - \nu\|_L^* \leq 2\,\|\mu - \nu\|_{\text{var}} .$$

5.3 Coupling

Definition 5.7. *A pair of random variables ξ_1, ξ_2, defined on the same probability space, is called a coupling for given measures $\mu_1, \mu_2 \in \mathcal{P}(M)$ if $\mathcal{D}(\xi_j) = \mu_j$, $j = 1, 2$.*

Given a coupling (ξ_1, ξ_2), let us consider the random variable $\xi = \xi_1 \times \xi_2 \in M \times M$. Denoting $m = \mathcal{D}(\xi)$, we have

$$\mu_1 = \pi_1 \circ m, \quad \mu_2 = \pi_2 \circ m,$$

where $\pi_1(m_1, m_2) = m_1$, $\pi_2(m_1, m_2) = m_2$. An equivalent definition of the coupling is that this is a measure m on $\big(M \times M,\, \mathcal{B}(M \times M)\big)$ which satisfies the two relations above. Indeed, if such a measure exists, we can take $(M \times M, \mathcal{B}, m)$ for the probability space, and choose $\xi_1 = \pi_1$, $\xi_2 = \pi_2$.

If (ξ_1, ξ_2) is a coupling for (μ_1, μ_2), then for any $\Gamma \in \mathcal{B}(M)$ we have

$$\begin{aligned} \mu_1(\Gamma) - \mu_2(\Gamma) &= \mathbf{E}\big(I_\Gamma(\xi_1) - I_\Gamma(\xi_2)\big) \\ &= \mathbf{E}\Big(I_{\{\xi_1 \neq \xi_2\}}\big(I_\Gamma(\xi_1) - I_\Gamma(\xi_2)\big)\Big) \leq \mathbf{P}(\xi_1 \neq \xi_2) . \end{aligned}$$

Therefore,

$$\mathbf{P}(\xi_1 \neq \xi_2) \geq \|\mu_1 - \mu_2\|_{\text{var}} .$$

A coupling (ξ_1, ξ_2) is called *maximal* if

$$\mathbf{P}\{\xi_1 \neq \xi_2\} = \|\mu_1 - \mu_2\|_{\text{var}} .$$

The following result is often known as the *Dobrushin lemma*:

Lemma 5.8. *For any two measures $\mu_1, \mu_2 \in \mathcal{P}(M)$ a maximal coupling (ξ_1, ξ_2) exists.*

For a proof see, say, [Lin92] and the Appendix in [KS01a].

This lemma shows that a coupling can be used as a tool to study the variational distance between measures. A coupling is also suited to study the Lipschitz-dual distance:

Exercise 5.9. If the measures $\mu_1, \mu_2 \in \mathcal{P}(M)$ admit a coupling (ξ_1, ξ_2) such that $\mathbf{P}\{\|\xi_1 - \xi_2\| > \varepsilon\} \le \theta$, for some $\varepsilon, \theta > 0$, then $\|\mu_1 - \mu_2\|_L^* \le 2\theta + \varepsilon$.

Proof. For any $f \in O$ (see (5.1)) we have

$$|(\mu_1 - \mu_2, f)| \le \big|\mathbf{E}\, I_Q\big(f(\xi_1) - f(\xi_2)\big)\big| + \big|\mathbf{E}\, I_{Q^c}\big(f(\xi_1) - f(\xi_2)\big)\big|,$$

where $Q = \{\mathrm{dist}(\xi_1, \xi_2) > \varepsilon\}$. This estimate implies that $|(\mu_1 - \mu_2, f)| \le 2\theta + \varepsilon$, so the assertion follows. □

5.4 Kantorovich functionals

Let f_K be any measurable function on $M \times M$ such that

$$f_K(m_1, m_2) = f_K(m_2, m_1) \ge \mathrm{dist}(m_1, m_2) \quad \forall\, m_1, m_2 .$$

We define the *Kantorovich functional*, corresponding to f_K, as the following function $\mathcal{K}$ on $\mathcal{P}(M) \times \mathcal{P}(M)$:

$$\mathcal{K}(\mu_1, \mu_2) = \inf\{\mathbf{E}\, f_K(\xi_1, \xi_2)\},$$

where the infimum is taken over all couplings (ξ_1, ξ_2) for (μ_1, μ_2).

Lemma 5.10. *For any $\mu_1, \mu_2 \in \mathcal{P}(M)$,*

$$\|\mu_1 - \mu_2\|_L^* \le \mathcal{K}(\mu_1, \mu_2). \tag{5.9}$$

Proof. Let ξ_1, ξ_2 be any coupling for μ_1, μ_2, and $g \in O$. Then

$$(\mu_1 - \mu_2, g) = \mathbf{E}\big(g(\xi_1) - g(\xi_2)\big) \le \mathbf{E}\,\mathrm{dist}(\xi_1, \xi_2) \le \mathbf{E}\, f_K(\xi_1, \xi_2).$$

Taking first the supremum in $g \in O$ and next the infimum in (ξ_1, ξ_2) we recover (5.9). □

Let us modify (5.2) and introduce the new distance in $\mathcal{P}(M)$, called the *Kantorovich distance*:

$$\|\mu - \nu\|_K = \sup_{\mathrm{Lip}(f) \le 1} |(f, \mu) - (f, \nu)|.$$

It also possesses the properties, specified in Theorem 5.3. In addition, now (5.9) with $f_K(m_1, m_2) = \mathrm{dist}\,(m_1, m_2)$ strengthens as follows:

$$\|\mu - \nu\|_K = \inf\{\mathbf{E}\,\mathrm{dist}(\xi_1, \xi_2)\},$$

where the infimum is taken over all couplings (ξ_1, ξ_2) for (μ, ν). Moreover, the infimum is attained at some coupling (ξ_1, ξ_2). This is the assertion of the celebrated *Kantorovich Theorem*, see [KA77] and [Dud89]. [4]

[4] In his research on the mass-transfer problem L. Kantorovich interpreted $\mathbf{E}\,\mathrm{dist}(\xi_1, \xi_2)$ as the work, needed to transport mass points $\xi_1(\omega)$ to the points $\xi_2(\omega)$, and used the equality above to estimate the work via the distance between the measures μ and ν (we use the equality other way round, i.e. as a tool to estimate this distance). This research is an important part of the study of the optimal production for which in 1975 L. Kantorovich was given the Nobel prise in economics.

6 Uniqueness of a stationary measure: kick-forces

6.1 The main lemma

Consider the randomly kicked NSE (3.3) and the corresponding discrete time RDS:

$$u(k) = S\big(u(k-1)\big) + \eta_k, \qquad \eta_k = \sum_s b_s\, \xi_{sk}\, e_s . \tag{6.1}$$

In this section we prove that the system (6.1) is *exponentially mixing* in the sense that distributions of all its solutions converge exponentially fast to a stationary measure. Next we derive from this result some immediate consequences. The proof of the exponential mixing is the most technical part of the book. To simplify its reading we conclude this section with an appendix, containing a summary of the proof.

Let us denote the underlying probability space $(\Omega_1, \mathcal{F}_1, \mathbf{P}_1)$.

Recall that the time-one transition function for the corresponding Markov chain is $P_1(u, \cdot) = \mathcal{D}(S(u) + \eta_1)$ (see (3.10)).

Lemma 6.1. *There exists a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\forall R \ge 1$, there exists $N = N(R) \ge 1$ with the following property: If*

$$b_s \ne 0 \quad \forall |s| \le N(R), \tag{6.2}$$

then for any $u_1, u_2 \in \mathfrak{B}_R(H) =: \mathfrak{B}_R$ the measures $\mu_1 = P_1(u_1, \cdot)$, $\mu_2 = P_1(u_2, \cdot)$ admit a coupling (V_1, V_2), where $V_j = V_j(u_1, u_2; \omega)$ $(j = 1, 2)$, and
(a) *the maps $V_1, V_2 : \mathfrak{B}_R \times \mathfrak{B}_R \times \Omega \to H$ are measurable,*
(b) *denoting $d = |u_1 - u_2|$ we have*

$$\mathbf{P}\Big\{|V_1 - V_2| \ge \frac{1}{2}\, d\Big\} \le C_*\, d,$$

where C_* depends on R, B_0 and $b_1, \dots, b_N$.

Proof. Let us denote by P_N and Q_N the orthogonal projectors

$$P_N : H \longrightarrow \mathrm{span}\{e_s,\ |s| \le N\} = H_{(N)},$$

$$Q_N : H \longrightarrow \overline{\mathrm{span}\{e_s,\ |s| > N\}}.$$

We are looking for random variables V_1, V_2 in the following form

$$V_1 = S(u_1) + \xi_1, \quad V_2 = S(u_2) + \xi_2, \tag{6.3}$$

where $\xi_1, \xi_2 \in H$ are random variables on Ω, distributed as η_1. Clearly (V_1, V_2) is a coupling for (μ_1, μ_2). To define the random variables ξ_1, ξ_2 we specify their projections $P_N\, \xi_j$ and $Q_N\, \xi_j$ $(j = 1, 2)$, where N is to be chosen below.

We construct the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ for ξ_1, ξ_2 as $\Omega = \Omega_1 \times \Omega_2$, where the space $(\Omega_2, \mathcal{F}_2, \mathbf{P}_2)$ is defined below, and extend η_1 in the natural way to a random variable $\overline{\eta}_1$ on $(\Omega, \mathcal{F}, \mathbf{P})$. We set

$$Q_N\, \xi_1 = Q_N\, \xi_2 = Q_N\, \overline{\eta}_1 .$$

To define $P_N\,\xi_1$, $P_N\,\xi_2$ let us denote $\nu_j = P_N \circ \mu_j$, $j = 1, 2$. We claim that

$$\|\nu_1 - \nu_2\|_{\mathrm{var}} \le C_*\, d\,. \tag{6.4}$$

Indeed, let us denote $v_j = P_N\, S(u_j)$, $j = 1, 2$. Then, due to (S2),

$$|v_1 - v_2| \le C(R)\, d\,. \tag{6.5}$$

Since $b_s \ne 0$ for $|s| \le N$ and $\mathcal{D}(\xi_{sk}) = p_s(r)dr$, where the functions p_s are Lipschitz, then

$$\mathcal{D}(P_N\,\eta_1) = q(x)dx, \quad x \in \mathbb{R}^{N_*}\,.$$

Here $N_* = \dim H_{(N)}$ and q is a Lipschitz function. Hence,

$$\mathcal{D}\,\nu_j = \mathcal{D}(v_j + P_N\,\eta_1) = q(x - v_j)dx\,,$$

and (6.4) follows from (6.5) and (5.3).

Due to (6.4) and the Dobrushin lemma, the measures ν_1, ν_2 admit a maximal coupling (Ξ_1, Ξ_2), defined on a probability space $(\Omega_2, \mathcal{F}_2, \mathbf{P}_2)$, and depending on the parameter (u_1, u_2). I.e., $\Xi_j = \Xi_j(\omega_2; u_1, u_2)$ and

$$\mathbf{P}_2\big(\Xi_1 \ne \Xi_2\big) = \|\nu_1 - \nu_2\|_{\mathrm{var}} \le C_*\, d\,. \tag{6.6}$$

Moreover, the maps $\Xi_j : \Omega_2 \times \mathfrak{B}_R \times \mathfrak{B}_R \to H_{(N)}$ are measurable (see [KS01a]).

Denoting by $\overline{\Xi}_j$, $j = 1, 2$, the natural extension of Ξ_j to Ω, we set

$$P_N\,\xi_j = \overline{\Xi}_j - P_N\, S(u_j)\,.$$

Since $P_N\,\xi_j$ and $Q_N\,\xi_j$ are specified, we now can construct the random variable V_1, V_2 (see (6.3)). Clearly, $\overline{\Xi}_1 = \overline{\Xi}_2$ implies that $P_N\, V_1 = P_N\, V_2$, so that in this case

$$V_1 - V_2 = Q_N\, V_1 - Q_N\, V_2 = Q_N\, S(u_1) - Q_N\, S(u_2)\,.$$

Using (S3), we have for $\overline{\Xi}_1 = \overline{\Xi}_2$,

$$|Q_N\, S(u_1) - Q_N\, S(u_2)| \le N^{-1}\, \|Q_N\, S(u_1) - Q_N\, S(u_2)\|_1 \le N^{-1}\, C_1(R)\, d\,.$$

Choosing $N \ge 2\, C_1(R)$ we see that $|V_1 - V_2| \le \frac{1}{2}\, d$, if $\overline{\Xi}_1 = \overline{\Xi}_2$. Now the assertion (b) follows from (6.6). Since the maps $\overline{\Xi}_1, \overline{\Xi}_2$ are measurable, then V_1, V_2 are measurable as well, and the lemma is proved. □

We call the measurable maps V_1, V_2 the *coupling maps.*

Exercise 6.2. *Prove that (6.4) still holds if $p_s(r)$ are functions of bounded total variation.*

Proof. Assume first that q is smooth. Then

$$\begin{aligned}
&\int_{H_N} |q(x - v_1) - q(x - v_2)|\, dx \\
&\le |v_1 - v_2| \int_{H_N} \int_0^1 |\nabla q(x - \theta v_1 + (1-\theta) v_2)\,|\, d\theta\, dx \\
&= |v_1 - v_2| \int_{H_N} |\nabla q(x)|\, dx \le |v_1 - v_2| \sum_{j=1}^{N_*} \int_R |\partial_{x_j} p_j(x_j)\, dx_j| \\
&= |v_1 - v_2| \sum \operatorname{var} q_j = |v_1 - v_2| \sum b_j^{-1} \operatorname{var} p_j\,,
\end{aligned}$$

where $q(x) = \prod_{j=1}^{N_*} q_j(x_j)$ with $q_j(x_j) = b_j^{-1}\, p_j(x_j/b_j)$. If the functions p_j are not smooth, we approximate them by smooth functions and go to the limit in the estimate above. □

6.2 Weak solution of (6.1)

Definition 6.3. *A process $\{u(k) \in H,\ k \ge k_0\}$, defined on some probability space, is called a weak solution of (6.1), if it satisfies (6.1) with the process of kicks $\{\eta_k\}$ replaced by some other process $\{\eta'_k\}$, distributed as $\{\eta_k\}$.*

Let $\{u_1(k)\}$ and $\{u_2(k)\}$ be two weak solutions defined on the same probability space. Possibly, the corresponding two processes $\{\eta'_k\}$ are different (but they may be correlated). We set

$$d(k) = |u_1(k) - u_2(k)|, \quad R(k) = |u_1(k)| + |u_2(k)|\,.$$

Let $u(k)$ be $u_1(k)$ or $u_2(k)$. Then a.s. we have

$$|u(k+1)| \le e^{-1}\,|u(k)| + |\eta_{k+1}| \le e^{-1}\,|u(k)| + \frac{1}{2}\,\widetilde{C} \tag{6.7}$$

where $\widetilde{C} = 2\sqrt{B_0}$, see Section 3.2 (this constant also depends on the viscosity ν, which now is assumed to be one). Accordingly, $\mathbf{E}\, R(k+1) \le e^{-1}\,\mathbf{E}\, R(k) + \widetilde{C}$. Let us choose any number

$$R_0 \ge 1 \vee \frac{2e}{e-1}\,\widetilde{C}.$$

Then

$$\mathbf{E}\, R(k) \ge R_0 \Longrightarrow \mathbf{E}\, R(\ell) \le \frac{1}{2}\,\mathbf{E}\, R(k), \quad \forall \ell \ge k+1\,. \tag{6.8}$$

Similarly, (6.7) implies that

$$R(k) \le R_0 \Longrightarrow R(\ell) \le R_0, \ \text{a.s.}\ \forall \ell \ge k\,. \tag{6.9}$$

Let us fix some $0 < d_0 \le 1$. Let $\{u_1(k)\}$ and $\{u_2(k)\}$ be two weak solutions of (6.1), corresponding to the same kicks $\{\eta'{}_k\}$. Assume that $R(0) \le R_0$. Then, $|u_1(0)|$,

$|u_2(0)| \le R_0$. So, if $0 = \eta'_1 = \cdots = \eta'_T$, then $|u_j(T)| \le \exp(-T) R_0$, and $|u_j(T)| \le \frac{1}{4} d_0$ for $j = 1, 2$, if

$$T = [\ln(4 R_0 d_0^{-1})] + 1 .$$

Since $\mathbf{P}(|\eta'_k| \le \varepsilon) > 0$ for each $\varepsilon > 0$ (see Section 3.1), we have that

$$\mathbf{P}(d(T) \le d_0) \ge \theta > 0 , \tag{6.10}$$

where θ depends on R_0 and d_0.

6.3 The theorem

Let us denote by $\mathcal{P}_1(H)$ the class of measures μ with the finite first moment

$$M_1(\mu) = \int |u| \, \mu(du) < \infty.$$

Theorem 6.4. *Let the kicks $\{\eta_k\}$ satisfy the assumptions of Section 3.1. Then there exists $N \ge 1$, depending on ν and B_0, such that if*

$$b_s \ne 0 \quad \forall |s| \le N ,$$

then for any $\mu_1, \mu_2 \in \mathcal{P}_1(H)$ and each $k \in \mathbb{N}$ we have

$$\|\mathcal{B}_k^* \mu_1 - \mathcal{B}_k^* \mu_2\|_L^* \le C\big(1 + M_1(\mu_1) + M_1(\mu_2)\big)\, \kappa^k , \tag{6.11}$$

where $\kappa < 1$ and $C \ge 1$ depend on ν, B_0 and $\{b_s, |s| \le N\}$.

Proof. **Step 1** (coupling). Let us denote

$$\mu_j(n) = \mathcal{B}_n^*(\mu_j), \quad j = 1, 2, \ n \ge 0 .$$

To estimate the distances $\|\mu_1(n) - \mu_2(n)\|_L^*$ we shall construct special couplings $\big(U_1(1), U_2(1)\big)$, $\big(U_1(2), U_2(2)\big), \ldots$ for the measures $\big(\mu_1(1), \mu_2(1)\big), \ldots$. The construction starts with any coupling (U_1, U_2) for (μ_1, μ_2), defined on a space $(\Omega^0, \mathcal{F}^0, \mathbf{P}^0)$ which is a copy of the space $(\Omega, \mathcal{F}, \mathbf{P})$ from Lemma 6.1. Below we use other copies $(\Omega^j, \mathcal{F}^j, \mathbf{P}^j)$ of this space as well.

Let us apply Lemma 6.1 with $R = R_0$ and denote

$$d_0 = 1 \wedge \frac{1}{16\, C_*} .$$

Let $V_1(u_1, u_2; \omega^1)$, $V_2(u_1, u_2; \omega^1)$ be the coupling-maps as in Lemma 6.1. They are defined for $|u_1|, |u_2| \le R_0$. We define maps $\widehat{V}_1, \widehat{V}_2$ as follows:

$$\widehat{V}_j(u_1, u_2; \omega^1) = \begin{cases} V_j, \text{ if } |u_1 - u_2| \le d_0, \ |u_1| + |u_2| \le R_0 \\ S(u_j) + \eta(\omega^1) \ \text{otherwise} . \end{cases}$$

Here $\eta(\omega^1)$ is a random variable, distributed as the kick η_1. Now we define the coupling $\big(U_1(1), U_2(1)\big)$ on the probability space $\Omega^0 \times \Omega^1$ (equipped with the product σ-algebra and the product measure), by the following relation:

$$U_j^{\omega^0, \omega^1}(1) = \widehat{V}_j(U_1(\omega^0), \ U_2(\omega^0); \omega^1) .$$

Then, by the definition of the coupling maps (V_1, V_2),

$$U_j^{\omega^0, \omega^1}(1) = S(U_j^{\omega^0}) + \eta_{1,j}^{\omega^1} , \quad j = 1, 2 ,$$

where $\mathcal{D}(\eta_{1,1}) = \mathcal{D}(\eta_{1,2}) = \mathcal{D}(\eta_1)$. In particular,

$$\mathcal{D}\big(U_j^{\omega^0, \omega^1}(1)\big) = \mu_j(1) .$$

We iterate this procedure T times, using successively coupling maps $\widehat{V}_j(u_1, u_2; \omega^2), \ldots$, $\widehat{V}_j(u_1, u_2; \omega^T)$ $(j = 1, 2)$, where $\omega^\ell \in \Omega^\ell$. In this way for $\ell = 1, 2, \ldots, T$ we get couplings $\big(U_1(\ell), U_2(\ell)\big)\big(u_1, u_2; (\omega^0, \omega^1, \ldots, \omega^\ell)\big)$ such that

$$U_j(\ell) = S\big(U_j(\ell - 1)\big) + \eta_{\ell, j} , \quad \mathcal{D}(\eta_{\ell, j}) = \eta_1 .$$

We view the couplings as random variables, defined on the same probability space

$$\Omega_T = \Omega^0 \times \cdots \times \Omega^T .$$

Then the kicks $\{\eta_{\ell, j}\}$ also are defined on Ω_T. The processes $\{U_1(\ell), \ell = 0, \ldots, T\}$ and $\{U_2(\ell), \ell = 0, \ldots, T\}$ are weak solutions of (6.1). As before, we denote $d(k) = |U_1(k) - U_2(k)|$ and $R(k) = |U_1(k)| + |U_2(k)|$, $0 \le k \le T$.

Below we also write the probability space Ω_T as

$$\Omega_T = \Omega^0 \times \Omega', \ \ \Omega' = \Omega^1 \times \cdots \times \Omega^T .$$

Let us make some observations about the weak solutions U_1 and U_2:

i) If $d(0) = |U_1 - U_2| \le d_0$ and $R(0) \le R_0$, then by Lemma 6.1 the probability (in Ω') that $d(T) \le 2^{-T} d_0$ is

$$\ge 1 - C_*\, d(1 + 2^{-1} + \cdots + 2^{-T+1}) \ge 1 - 2\, C_*\, d .$$

ii) If $d(0) > d_0$ and $R(0) \le R_0$, then by (6.10) $\mathbf{P}^{\Omega'}(d(T) \le d_0) \ge \theta$.

Step 2 (Kantorovich functional). Let us introduce in H the distance

$$\mathrm{dist}(u, v) = |u - v| \wedge d_0 ,$$

and denote by $\|\cdot\|_{L, d_0}^*$ the corresponding Lipschitz-dual distance (see Exercise 5.2 in Section 5.1). Consider the following function $f_K(u_1, u_2)$:

$$f_K(u_1, u_2) = \begin{cases} d, & \text{if } d \le d_0, \ R \le R_0, \\ 2d_0, & \text{if } d > d_0, \ R \le R_0, \\ R, & \text{if } R > R_0, \end{cases}$$

where $d=|u_1-u_2|$, $R=|u_1|+|u_2|$. Since $R_0>2d_0$, then $f_K(u_1,u_2)\ge \mathrm{dist}(u_1,u_2)$. Let $\mathcal{K}$ be the corresponding Kantorovich functional.

Let us take any coupling (U_1,U_2) for (μ_1,μ_2) and apply the construction from Step 1 to get the couplings $\big(U_1(\ell),U_2(\ell)\big)$, $1\le\ell\le T$. We wish to estimate $\mathbf{E}\, f_K\big(U_1(T),U_2(T)\big)$ in terms of $A=\mathbf{E}\, f_K(U_1,U_2)$. We abbreviate

$$f_K(0)=f_K(U_1,U_2),\ \ f_K(\ell)=f_K\big(U_1(\ell),U_2(\ell)\big)\,,$$

and consider the three cases, accordingly to the partition of Ω^0 to three events:

a) $\omega^0\in Q_1=\{R(0)>R_0\}$. Then $f_K(0)=R(0)$. Applying (6.8) and using that $2d_0\le\frac12 R_0<\frac12 R(0)$ we see that in this case

$$\mathbf{E}^{\Omega'} f_K(T)\le\frac12\, f_K(0)\,.$$

b) $\omega^0\in Q_2=\{d(0)>d_0,\ R(0)\le R_0\}$.

Now $f_K(0)=2d_0$. By (6.9), $R(T)\le R_0$ a.s., and by ii) with probability $\ge\theta$, we have $f_K(T)\le d_0$. Hence, now

$$\mathbf{E}^{\Omega'} f_K(T)\le(1-\theta)\,2d_0+\theta d_0=2d_0\Big(1-\frac12\,\theta\Big)=\Big(1-\frac12\,\theta\Big)\,f_K(0)\,.$$

c) $\omega^0\in Q_3=\{d(0)\le d_0,\ R\le R_0\}$.

Then $f_K(0)=d(0)$. By (6.9), $R(T)\le R_0$ a.s., and by i) with probability greater than $(1-2\,C_*\,d(0))$, we have $f_K(T)\le 2^{-T}\,d_0$. So,

$$\begin{aligned}\mathbf{E}^{\Omega'}\, f_K(T)&\le 2^{-T}\,d(0)\big(1-2\,C_*\,d(0)\big)+2C_*\,d(0)2d_0\\ &\le d(0)(2^{-T}+4C_*\,d_0)\le d(0)\Big(\frac12+\frac14\Big)=\frac34\,f_K(0)\,.\end{aligned}$$

Using the estimates in a)–c) we get

$$\begin{aligned}\mathbf{E}\, f_K(T)&=\mathbf{E}^{\Omega^0}\Big(I_{Q_1}\mathbf{E}^{\Omega'}\, f_K(T)+I_{Q_2}\,\mathbf{E}^{\Omega'}\, f_K(T)+I_{Q_3}\,\mathbf{E}^{\Omega'}\, f_K(T)\Big)\\ &\le\mathbf{E}^{\Omega^0}\Big(I_{Q_1}\frac12\, f_K(0)+I_{Q_2}\Big(1-\frac12\,\theta\Big)\,f_K(0)+I_{Q_3}\,\frac34\,f_K(0)\Big)\\ &\le\widetilde\kappa\,\mathbf{E}\, f_K(0)\,,\end{aligned}$$

where $\widetilde\kappa=\big(1-\frac12\,\theta\big)\vee\frac34<1$. If $j=kT$, then iterating these arguments we have

$$\mathbf{E}\, f_K(j)\le\widetilde\kappa^k\,\mathbf{E}\, f_K(0)\,.$$

If $j\in[1,T-1]$, then the arguments in a) and c) with T replaced by j remain unchanged, while in b) we cannot anymore claim that $\mathbf{P}\{d(j)\le d_0\}>0$. So (6.10) does not hold, and in this case $\mathbf{E}\, f_K(j)\le\mathbf{E}\, f_K(0)$.

Finally, for any integer $t=kT+j$, $0\le j<T$, we have

$$\mathbf{E}\, f_K(t)\le\widetilde\kappa^k\mathbf{E}\, f_K(0)\le C\,\kappa^t\,\mathbf{E}\, f_K(0)\,,$$

with $\kappa=\widetilde\kappa^{1/T}<1$ and $C=\widetilde\kappa^{-j/T}>1$.

Step 3 (end of the proof). Let us choose for the original coupling (U_1,U_2) the coupling, formed by independent random variables. Then

$$\mathbf{E}\, f_K(U_1,U_2)\le\mathbf{E}(2d_0+|U_1|+|U_2|)\le 1+M_1(\mu_1)+M_1(\mu_2)$$

since $d_0<\frac12$. Therefore,

$$\mathcal{K}\big(\mu_1(t),\mu_2(t)\big)\le\mathbf{E}\, f_K(t)\le C\big(1+M_1(\mu_1)+M_1(\mu_2)\big)\,\kappa^t\,. \tag{6.12}$$

Combining this estimate with (5.3) and Lemma 5.10 we get that

$$\|\mu_1(t)-\mu_2(t)\|_L^*\le\frac{2C}{d_0}\,\big(1+M_1(\mu_1)+M_1(\mu_2)\big)\kappa^t\,.$$

Re-denoting $2C/d_0$ by C we get the theorem's assertion. □

The theorem which we have proved has many important corollaries, discussed in the next section. Before that we derive some consequences which follow not from the theorem, but from its proof.

Due to (6.12) we can construct a coupling $\big(U_1(t),\,U_2(t)\big)$ for $\big(\mu_1(t),\mu_2(t)\big)$ such that

$$\mathbf{E}\, f_K\big(U_1(t),U_2(t)\big)\le 2C\,\kappa^t\,K_0,\quad K_0=1+M_1(\mu_1)+M_1(\mu_2)\,.$$

Since for $|U_1-U_2|>d_0$ we have $f_K\ge 2d_0$ and for $|U_1|+|U_2|\ge R_0$ we have $f_K\ge R_0>d_0$, then

$$P_d:=\mathbf{P}\big\{|U_1(t)-U_2(t)|>d\ \text{ or }\ |U_1(t)|+|U_2(t)|\ge R_0\big\}\le d^{-1}\,2C\,\kappa^t\,K_0\,,$$

for any $d\le d_0$. Since $P_d\le P_{d_0}$ for $d>d_0$, then

$$P_d\le d^{-1}(2C\,d_0^{-1})\,\kappa^t\,K_0=d^{-1}\,C_1\,\kappa^t\,K_0$$

for $d\le 1$. Choosing

$$d=\delta(t)=(C_1\,\kappa^t\,K_0)^{1/2},$$

we get the following

Corollary 6.5. *The measures $\mu_1(t),\mu_2(t)$ admit a coupling $\big(U_1(t),U_2(t)\big)$ such that*

$$\mathbf{P}\big\{|U_1(t)-U_2(t)|\ge\delta(t)\ \text{ or }\ |U_1(t)|+|U_2(t)|\ge R_0\big\}\le\delta(t) \tag{6.13}$$

for all integer $t\ge 0$.

Denoting the event in (6.13) by Ω_t, it follows from Borel-Cantelli's Lemma that

$$\mathbf{P}\Big(\bigcap_{k\ge0}\ \bigcup_{l\ge k}\Omega_l\Big)=0.$$

Hence, there exists a random variable $T'\ge 0$, a.s. finite, and such that

$$|U_1(\ell)-U_2(\ell)|<\delta(\ell)\ \text{ and }\ |U_1(\ell)|+|U_2(\ell)|<R_0,\quad\forall\ell\ge T'\,. \tag{6.14}$$

We shall need this relation later.

Definition 6.6. *The Prokhorov distance between $\mu_1, \mu_2 \in \mathcal{P}(H)$ is*

$$\mathrm{dist}_{\mathrm{Prok}}(\mu_1, \mu_2) = \inf\{\varepsilon > 0 \,|\, \mu_1(A) \le \mu_2(A+\varepsilon) + \varepsilon, \quad \forall A \in \mathcal{B}(H)\},$$

where $A+\varepsilon$ stands for the ε–neighbourhood of A in H.

This is a distance on $\mathcal{P}(H)$ which satisfies all assertions of Theorem 5.3, see [Dud89, Shi96]. Moreover, $\|\mu_1 - \mu_2\|_L^* \le 2\,\mathrm{dist}_{\mathrm{Prok}}(\mu_1, \mu_2)$.

Exercise 6.7. Show that (6.13) implies that

$$\mathrm{dist}_{\mathrm{Prok}}\big(\mu_1(t), \mu_2(t)\big) \le \delta(t)\,.$$

We note that the last estimate can be obtained as a corollary of Theorem 6.4 since

$$\mathrm{dist}_{\mathrm{Prok}}(\mu_1, \mu_2) \le \|\mu_1 - \mu_2\|_L^* + \sqrt{\|\mu_1 - \mu_2\|_L^*}\,,$$

see [Shi96], Section III.7.

6.4 Corollaries from the theorem

By Theorem 3.5, equation (6.1) has a stationary measure $\mu \in \mathcal{P}_1(H)$. Applying Theorem 6.4 we obtain

Corollary 6.8. *There exists a stationary measure $\mu \in \mathcal{P}_1(H)$ such that for each $\nu \in \mathcal{P}_1(H)$ we have*

$$\|\mathcal{B}_k^*\nu - \mu\|_L^* \le C\big(1 + M_1(\nu)\big)\,\kappa^k\,.$$

In particular,

$$P_k(u,\cdot) \underset{k\to\infty}{\rightharpoonup} \mu \quad \forall u \in H\,.$$

Therefore,

$$\mathcal{B}_k f(u) = \big(f, P_k(u,\cdot)\big) \to (f,\mu) \quad \forall\, u \in H\,, \tag{6.15}$$

for each $f \in C_b(H)$.

Corollary 6.9. *The system (6.1) has a unique stationary measure $\mu \in \mathcal{P}(H)$.*

Proof. If $\nu \in \mathcal{P}(H)$ is a stationary measure, then due to (6.15) for any $f \in C_b(H)$ we have

$$(\nu, f) = (\mathcal{B}_k^*\nu, f) = \big(\nu, \mathcal{B}_k f\big) \underset{k\to\infty}{\longrightarrow} \big(\nu, (\mu, f)\mathbf{1}\big) = (\mu, f)\,.$$

Hence, $\nu = \mu$. □

Let $\nu \in \mathcal{P}_1(H)$ and $u(k;\nu)$ be a solution such that $\mathcal{D}\,u(0) = \nu$. Then $\mathcal{D}\,u(k;\nu) = \mathcal{B}_k^*\,\nu \underset{k\to\infty}{\rightharpoonup} \mu$. Now we shall get a stronger version of this result.

Denote $\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$, $\mathcal{A}_+ = H^{\mathbb{Z}_+} = \{u_j \in H, j \ge 0\}$. Provide $\mathcal{A}_+$ with the Tikhonov topology and a corresponding norm. For $r \ge 0$ consider the shift-map $T_r : \mathcal{A}_+ \longmapsto \mathcal{A}_+$, $(u_0, u_1, \dots) \longmapsto (u_r, u_{r+1}, \dots)$. Finally, let us denote $u(k;\mu) = U(k)$, where μ is the stationary measure.

Exercise 6.10. Prove that for any $\nu \in \mathcal{P}(H)$, $\mathcal{D}(T_r\,u(\cdot,\nu)) \underset{r\to\infty}{\rightharpoonup} \mathcal{D}\big(U(\cdot)\big)$ weakly in $\mathcal{P}(\mathcal{A}_+)$.
Hint: 1) By Proposition 5.1 it is sufficient to check that $\mathbf{E}\,f(T_r\big(u(\cdot,\nu)\big) \rightharpoonup \mathbf{E}\,f\big(U(\cdot)\big)$, where f is a bounded Lipschitz functional on $\mathcal{A}_+$.
2) It suffices to check this for f which depend only on finitely many coordinates u_j.
3) To prove this use Exercise 3.3. □

Exercise 6.11. *By Exercise 3.7,* $\mathrm{supp}\,\mu \subset \mathfrak{B}_{r_0}(H)$, $r_0 = \frac{e}{e-1}\sqrt{B_0}$. *Use Theorem 2.15 to prove that if $B_k = \sum |s|^{2k}\, b_s^2 < \infty$, $k \ge 1$, then* $\mathrm{supp}\,\mu \subset \mathfrak{B}_{r_k}(H^k)$, *where* $r_k = C_k(r_0 + r_0^{2(k+1)} + \sqrt{B_k})\,.$
Hint: *Use that for each j, $\|\eta_j\|_k^2 \le B_k$, a.s.*

Corollary 6.12. *If $B_k < \infty$ for all $k \ge 0$, then* $\mu(H \cap C^\infty) = 1\,.$

Let $B_k < \infty$ and $f \in C(H^k)$ be a function, which is Lipschitz on bounded subsets of H^k. Then, due to Theorem 2.15, $\mathcal{B}_r f(u) = (P_r(u,\cdot), f)$ is well defined.

Proposition 6.13. *If $B_{k+1} < \infty$, then*

$$|\mathcal{B}_r f(u) - \big(f,\mu\big)| \le C\,C_f\,\kappa^{\frac{r}{2(k+1)}}, \quad r \ge 1\,.$$

Here $C = C(k, |u|)$ and C_f is the Lipschitz-norm of f restricted to a ball $\mathfrak{B}_R(H^{k+1})$, where $R = R(k, |u|)$.

For a proof see Section 3 in [KS03].

Example 6.14. For any two points $x_1, x_2 \in \mathbb{T}^2$, we set $f_{ij}(u) = u_i(x_1)u_j(x_2)$, $i,j = 1,2$. Since $H^2 \subset C(\mathbb{T}^2)$, then f is a locally Lipschitz function on H^2. If $B_3 < \infty$, then for any solution of (6.1) with (say) a deterministic initial data u_0 we have

$$\mathbf{E}\,u_i(k,x_1)\,u_j(k,x_2) \to \int_H u_i(x_1)\,u_j(x_2)\,\mu(du)$$

exponentially fast. The l.h.s is called the *correlation tensor of the solution*, evaluated at time k, and the r.h.s. is called the *correlation tensor of the stationary measure.* The convergence above is one of postulates of statistical hydrodynamic, taken there for granted. See [Bat82].

6.5 3D NSE with small random kicks

Consider the 3D NSE, written as

$$\dot u + Lu + B(u) = \eta(t,x).$$

All objects in this equation are natural 3D analogues of those in Section 3. As before,

$$u(k+1) = S(u(k)) + \eta_{k+1}. \tag{6.16}$$

It is known that $\|S(u_1)\|_2 \le C\|u_1\|_2$ and $\|S(u_1)-S(u_2)\|_2 \le C\|u_1-u_2\|_2$ with some $C<1$, if $\|u_1\|_2, \|u_2\|_2$ are small enough. Therefore if B_2 (defined as in Section 3.2) is sufficiently small, then (6.16) defines a random dynamical system (and a Markov chain) in a small ball $\mathfrak{B}_\epsilon(H^2)$. A natural analogy of Theorem 6.4 holds true for (6.16) with $u \in \mathfrak{B}_\epsilon(H^2)$. In fact, the proof much simplifies since now the map S is a contraction.

6.6 Stationary measures and random attractors

In this section we discuss random attractors for the randomly kicked NSE, interpreted as the system (6.1).

First we put the system (6.1) in the classical framework of RDS [Arn98]. To do this we take infinitely many copies $(\Omega_j, F_j, \mathbf{P}_j)$, $j\in\mathbb{Z}$, of the probability space, on which are defined the random variables η_k, and define $(\Omega, \mathcal{F}, \mathbf{P})$ as the product of these spaces (so $\Omega = \Pi_{j\in\mathbb{Z}}\Omega_j$, etc). Introducing the shift operators $\theta_k:\Omega\to\Omega$, $k\in\mathbb{Z}$, which send $\varpi=(\omega_l,\, l\in\mathbb{Z})$ to $\varpi'=(\varpi'_l,\, l\in\mathbb{Z})$, $\omega'_l=\omega_{l+k}$, we write the system (6.1) as

$$u(k+1)=\phi\big(u(k),\,\theta_k\varpi\big), \tag{6.17}$$

where $\phi(u,\varpi)=S(u)+\eta_0(\omega_0)$. Clearly, (6.17) defines in H the same Markov chain as (6.1). For $k\ge 0$ we denote $\phi_k(\varpi)u=u(k)$, where $(u(l),\, l\ge 0)$ is a solution of (6.17), such that $u(0)=u$.

A closed (compact) set $A_\varpi\subset H$, depending on a random parameter $\varpi\in\Omega$, is called a *closed (compact) random set* if for any open set $U\subset H$ we have

$$\{\varpi \mid A_\varpi\cap U\ne\emptyset\}\in\mathcal{F}.$$

A random compact set A_ϖ is called a *random point attractor (in the sense of convergence in probability)* if for each $u\in H$ the sequence of random variables $d(\phi_k(\varpi)\,u,\, A_{\theta_k}\varpi)$[5] converges to zero in probability. I.e.,

$$\lim_{k\to\infty}\mathbf{P}\{d(\phi_k(\varpi)\,u,\,A_{\theta_k\varpi})>\delta\}=0$$

for each $\delta>0$. A random point attractors A_ϖ is said to be *minimal* if for any random point attractor A'_ϖ we have $A_\varpi\subset A'_\varpi$ for a.a. ϖ. Clearly, if a minimal random attractor exists, then it is unique.

There are other types of random attractors, considered in modern mathematical literature, see [Arn98, CDF97, Cra01]. Among them the random attractors, defined above, are the smallest.

Let μ be a stationary measure for (6.1), and $\mathcal{F}_-$ be a σ-algebra in $\mathcal{F}$, generated by random variables $\xi(\varpi)$, which depend only on ω_k, $k\le 0$. For the following important result see [Led86, Le 87, Cra91].

Proposition 6.15. *The limit*

$$\mu_\varpi=\lim_{k\to\infty}\phi_k(\theta_{-k}\varpi)\circ\mu$$

[5]for a point $t\in H$ and a set $B\subset H$ we denote $d(z,B)=\quad\inf_{\xi\in B}|x-\xi|$

exists almost surely in the sense of weak convergence of measures. The random measure μ_ω is $\mathcal{F}_-$-measurable in the sense that for any Borel set $Q\subset H$ the random variable $\mu_\varpi(Q)$ is $\mathcal{F}_-$ measurable, and [6]

$$\mu=\mathbf{E}\mu_\varpi. \tag{6.18}$$

The representation (6.18) is called the *Markov disintegration* of a stationary measure μ. It is proved in [Cra01] that for a large class of RDS, which includes the system (6.17), the disintegration μ_ϖ is supported by each random point attractor A'_ϖ, i.e., $\mu_\varpi(A'_\varpi)=1$ a.s. That is to say, $\operatorname{supp}\mu_\varpi\subset A'_\varpi$ a.s. For the specific systems (6.18) which we consider a much stronger result is true:

Theorem 6.16. *Under the assumptions of Theorem 6.4 $\operatorname{supp}\mu_\varpi$ is the minimal random point attractor of the system* (6.1).

In particular, due to the known results on random attractors (see [Deb98]), a.s. the Hausdorff dimension of $\operatorname{supp}\mu_\varpi$ is bounded by a certain deterministic constant.

Theorem 6.16 establishes connection between two important objects, related to randomly perturbed 2D NSE, and provides its minimal random point attractor, written as $\operatorname{supp}\mu_\varpi$, with the natural measure μ_ϖ. Since dimension of an attractor goes to infinity with the Reynolds number, then the measures μ_ϖ may be useful for applications of the random point attractors to 2D flows with high Reynolds numbers as it provides the attractors with a structure which is not sensitive to their dimensions.

The theorem is proved in [KS04a], where it is derived from a general abstract theorem.

For the white–forced NSE (4.3) a natural version of Theorem 6.16 holds true. See [KS04a].

6.7 Appendix: Summary of the proof of Theorem 6.4

Denote $\mu_j(k)=\mathcal{B}_k^*(\mu_j)$, $j=1,2$, $k=1,2,\dots$. To prove the Theorem we construct weak solution of (6.1) such that

$$\mathcal{D}U_j(0)=\mu_j,\quad j=1,2\,. \tag{6.19}$$

Then $\mu_j(k)=\mathcal{D}U_j(k)$, $k\ge 0$, $j=1,2$. We wish to construct them in such a way that

$$d(k):=|U_1(k)-U_2(k)|\ll 1 \ \text{ with high probability, if } k\gg 1 \tag{6.20}$$

This would imply that the measures $\mu_1(k)$, $\mu_2(k)$ are close.

Denote $B=\mathfrak{B}_{\frac{e}{e-1}\sqrt{B_0}}(H)$. We know (see (6.7)) that if $U_j(0)\in B$ for $j=1,2$ a.s., then

$$U_j(k)\in B \text{ for } j=1,2, \text{ and } k\ge 0 \text{ a.s.} \tag{6.21}$$

To simplify the presentation we assume that $\operatorname{supp}\mu_j\subset B$ $(j=1,2)$. Then any coupling $(U_1(k),U_2(k))$ for the measures $(\mu_1(k),\mu_2(k))$ satisfies (6.21). Arguments in the general case are very similar.

[6]The integral in (6.18) is understood in the weak sense. That is, for any $Q\in\mathcal{B}(H)$, $(\mathbf{E}\mu_\varpi)(Q)=\mathbf{E}(\mu_\varpi(Q))$.

From Lemma 6.1 we know the following:
Let $\mu_1 = \delta_{u_1}$, $\mu = \delta_{u_2}$. Denote $d = |u_1 - u_2|$. Then there exist weak solutions $U_1(k)$, $U_2(k)$, $k = 0, 1$, such that with probability $\geq 1 - C_*\, d$ we have

$$d(1) \leq \frac{1}{2}\, d\,.$$

- Let $U_1(0)$, $U_2(0)$ be any coupling for (μ_1, μ_2). These are our weak solution for $k = 0$.
- Assume that we have constructed weak solutions $U_1(t)$, $U_2(t)$ for $t \leq k$. How to construct $U_1(k+1)$, $U_2(k+1)$? The strategy is Markov, i.e. it depends only on $U_1(k)$, $U_2(k)$.

Fix a threshold $d_0 \in (0, 1)$.
Recipe 1. If $d(k) \leq d_0$, then 'we are in coupling', and we construct $U_{1,2}(k+1)$ 'in terms of $U_1(k)$, $U_2(k)$', using Lemma 6.1. Then, with high probability,

$$d(k+1) \leq \frac{1}{2}\, d_0\,.$$

If THIS happens, then 'we are in coupling', and we continue to iterate. We iterate forever with the probability

$$\geq 1 - C_* d - C_*\, \frac{1}{2}\, d - C_*\, \frac{1}{4}\, d - \dots = 1 - C_* d\Big(1 = \frac{1}{2} + \frac{1}{4} + \dots\Big) = 1 - 2C_* d\,.$$

We 'lose coupling' with probability $\leq 2C_* d$.
Recipe 2. If $d(k) > d_0$, then we are not in coupling. This usually happens when $k = 0$. Also, it happens when we lose coupling. Then:

$\exists\, T = T(d_0) \geq 1$, such that with probability $\theta(d_0) > 0$ we have $|U_1(k+T)| \leq \dfrac{1}{2}\, d_0$ and $|U_2(k+T)| \leq \dfrac{1}{2}\, d_0$. Then

$$d(k+T) \leq d_0\,. \tag{6.22}$$

In (6.10) I use a brutal way to achieve (6.22), but I do not know how to do this better!

So, after T steps we switch to Recipe 1 with probability θ. We continue with Recipe 2 with probability $1 - \theta$.

This strategy results in (6.20).

To put this arguments in numbers, the method of Kantorovich functionals is used.

The physical relevance of the results. The convergences $\mathcal{B}_k f(u) \to (f, \mu)$ and $\mathcal{B}_k^* \nu \to \mu$, where μ is the unique stationary measure, are very important for the statistical hydrodynamics, where they are postulated in the form that "statistical properties of the turbulent motion of fluid fast become independent of the initial data", [7] cf. Example 6.14; see [Gal01]. The theorems from this chapter and Section 7 rigorously prove this postulate for the periodic 2D turbulence, driven by a random force.

[7] "...we put our faith in the tendency for dynamical systems with a large number of degrees of freedom, and with coupling between these degrees of freedom, to approach a statistical state which is *independent* (partially, if not wholly) from the initial condition." G. K. Batchelor, [Bat82], pp. 6-7.

The validity of the postulate above for randomly stirred 2D NSE is obvious to physicists. In contrast, its rigorous mathematical proof, given in Theorem 6.4 (kick–forces) and Theorem 7.1 (white–forces), is surprisingly difficult. Moreover, it uses essentially the additional assumption (6.2)=(7.2). If this assumption is violated, the proof has to be modified significantly, and the correct version of Theorem 6.4=Theorem 7.1 is not known yet (see [HM04] and discussion in Section 12). In particular, we do not exclude that in this case the convergence to the unique stationary measure (if it still holds!) is not exponential.

The random attractors, discussed in Section 6.6, and their deterministic counter–parts (see [BV92, CF88]) are well established tools for analytical and numerical study of the 2D NSE (see [Gal01]). Their importance is due to the fact that they reduce the *infinite–dimensional* dynamical system, defined by the NSE in a function space, to *finite–dimensional systems* on attractors. Since dimensions of attractors growth to infinity with the Reynolds number, then it is hard to apply them to study 2D flows with high Reynolds numbers. Theorem 6.16 improves the situation a bit since it provides minimal random point attractors with an additional structure — the natural measure $\mu_{\overline{\omega}}$.

7 Uniqueness of a stationary measure: white-forces

7.1 The main theorem

Let us consider the 2D NSE, perturbed by a white force:

$$\dot{u} + \nu L u + B(u) = \frac{d}{dt}\,\zeta(t,x)\,, \qquad \zeta(t,x) = \sum_{s\in\mathbb{Z}_0^2} b_s\,\beta_s(t)\,e_s(x)\,. \tag{7.1}$$

The random field ζ is as in Section 4. In particular,

$$B_1 = \sum |s|^2\, b_s^2 < \infty\,.$$

By $u(t;u_0)$, $t \geq 0$, we denote a solution, equal u_0 at $t=0$.

Theorem 7.1. *There exists $N \geq 1$, depending on ν and B_1, such that if*

$$b_s \neq 0 \quad \forall\,|s| \leq N\,, \tag{7.2}$$

then there exists a unique stationary measure $\mu \in \mathcal{P}(H)$, and for any solution $u(t)$, satisfying $\mathbf{E}\,|u(0)|^2 =: M_2 < \infty$, we have

$$\big\|\mu - \mathcal{D}\big(u(t)\big)\big\|_L^* \leq C(M_2+1)\,e^{-\kappa t}\,, \tag{7.3}$$

for all $t \geq 0$. Here $\kappa > 0$ and $C \geq 1$ depend on ν, B_1 and $\{b_s, |s| \leq N\}$.

Remark 7.2. *The constant N depends on ν, B_1 and $\{b_s\}$, and goes to ∞ as $\nu \to 0$.*

Amplification 1. If $B_k < \infty$, $k \geq 1$ then supp $\mu \subset H^k$. Moreover, let f be a locally Hölder function on H^{k-1} such that

$$\begin{aligned} |f(u)| &\leq C_1\big(1 + \|u\|_{k-1}^{C_2}\big)\,, \\ |f(u) - f(v)| &\leq \|u - v\|_{k-1}^{\gamma}\, C_1(1 + \|u\|_{k-1}^{C_2} + \|v\|_{k-1}^{C_2}) \end{aligned} \tag{7.4}$$

for all $u, v \in H^{k-1}$, where $\gamma \in (0,1]$ and $C_1, C_2 > 0$. Then for each non-random vector $u_0 \in H$ we have

$$|\mathbf{E}\, f\big(u(t;u_0)\big) - (f,\mu)| \leq C_3(\nu, B_k, f, |u_0|)\,e^{-\kappa_k t}$$

for any $t \geq 0$, where $\kappa_k > 0$. If $k = 1$, then the same is true if in (7.4) $1 + |u|^{C_2}$ is replaced by $\exp \varepsilon |u|^2$, where $\varepsilon > 0$ is a sufficiently small constant.

Amplification 2. Consider the 2D NSE, perturbed by a force with a deterministic component:

$$\dot{u} + \nu L u + B(u) = f(x) + \frac{d}{dt}\,\zeta(t,x)\,,$$

where ζ is as above and $f(x) = \sum f_s e_s(x) \in H$ is a non-random smooth vector-field. If $b_s \neq 0$ for all s, then the assertions of Theorem 7.1 and Amplification 2 hold true for solutions of this equation. Proof remains essentially the same, see [Shi05].

Below we sketch a proof of Theorem 7.1 in comparison with that of Theorem 6.4. For the corresponding complete proof see [KS02a, Shi05], and for a proof of Amplification 1 see [KS03].

	Kick-force	White-force
1.	μ_1, μ_2 – initial measures. We have to compare $\mathcal{B}_k^*\mu_1$ with $\mathcal{B}_k^k\mu_2,\ k \to \infty$.	same
2.	Take $k \in \mathbb{N} \cup \{0\}$. Assume that for $0 \leq t \leq k$ we have constructed 'suitable' weak solutions $U_1(t), U_2(t)$ such that $\mathcal{D}\,U_j(0) = \mu_j$. Set $d(t) = \lvert U_1(t) - U_2(t)\rvert$, $R(t) = \lvert U_1(t)\rvert + \lvert U_2(t)\rvert$.	same
3.	If $d(k) \leq d_0,\ R(k) \leq R_0$ (7.5) then use Lemma 6.1 to construct $U_1(k+1), U_2(k+1)$ such that $d(k+1) \leq \frac{1}{2}\,d(k)$ with high probability.	If $d(t) \leq d_0$ and $R(t) \leq R_0\sqrt{t - T_*}$ (7.6) for $T_* + 1 \leq t \leq k$ (with a suitable $T_* \leq k_1$), then use an 'adjusted Girsanov lemma' to construct $U_1(t)$, $U_2(t)$, $k \leq t \leq k+1$, such that (7.6) holds for $k := k+1$, $d_0 := \frac{1}{2}\,d_0$ with high probability. After this, replace d_0 by $\frac{1}{2}\,d_0$ and k by $k+1$.
4.	If (7.5) violated, then construct $U_1(k+1), U_2(k+1)$ in 'trivial way', i.e. using (6.1) with the same force η_{k+1} for both solutions. Note that in this case with small positive probability, for $k := k+1$ we have (7.5).	same
5.	Iterate the procedure. Construct a suitable Kantorovich functional such that $\mathcal{K}\big(U_1(k+1), U_2(k+1)\big) \leq \kappa\,\mathcal{K}\big(U_1(k), U_2(k)\big)$, $\kappa < 1$.	same

The random forces $\zeta(t,x)$ which are homogeneous in x are the most important for the hydrodynamics. Due to Exercise 4.2, this is the case if $b_s = b_{-s}$ for all s.

Theorem 7.3. *If under the assumptions of Theorem 7.1 $b_s = b_{-s}$ for all $s \in \mathbb{Z}_0^2$, then the measure μ is homogeneous, i.e.*

$$T_h \circ \mu = \mu, \quad \forall h \in \mathbb{T}^2$$

where $T_h\, u(x) = u(x+h)$. Moreover, $S \circ \mu = \mu$, where $S\, u(x) = -u(-x)$. If in addition, $b_s = b_{s^\perp}$ for all $s \in \mathbb{Z}_0^2$, then $R \circ \mu = \mu$, where $R\, u(x) = u^\perp(-x^\perp)$.

Proof. Let $U(t,x)$, $t \geq 0$, be a stationary solution of (7.1). Then $T_h\, U(t,x)$ satisfies (7.1) with ζ replaced by $T_h\zeta$. Since $\mathcal{D}\big(T_h\zeta(\cdot)\big) = \mathcal{D}\big(\zeta(\cdot)\big)$, then $T_h U$ is a stationary process which is a weak solution of (7.1). Hence, $\mathcal{D}\, T_h U(0)$ is a stationary measure and $T_h \circ \mu = \mathcal{D}\, T_h U(0) = \mu$ by the uniqueness.

Since $\mathcal{D}\, S(\zeta) = \mathcal{D}\zeta$, then $S \circ \mu = \mu$ by the same reasons. Proof of the last assertion is the same. □

Let us interpret the torus $\mathbb{T}^2$ as the square $[-\pi,\pi]^2$ with identified opposite edges. Then the rotation by the angle $-\pi/2$,

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \longmapsto \begin{pmatrix} -x^2 \\ x^1 \end{pmatrix} = x^\perp ,$$

maps $\mathbb{T}^2$ to itself. This rotation induces the transformation R of the vector fields $u(x)$. So the last assertion of the theorem means that the rotation by the angle $-\pi/2$ does not change the stationary measure (if $b_s \equiv b_{s^\perp}$). The theory of homogeneous isotropic turbulence deals with measures on the space of vector fields $u(x)$ that are invariant under the translations T_h and the rotations R_θ by any angle θ (see [Bat82, Fri95]). Unfortunately, the periodic vector fields (which we interpret as vector fields on $\mathbb{T}^2$) can be invariant only under rotations by angles proportional to $\pi/2$, since only these rotations preserve the torus.

7.2 Stationary measures for equation, perturbed by high frequency kicks

In Section 4.5 we compared solutions of equation (7.1) with $\nu = 1$ and solutions of the 2D NSE, perturbed by the high–frequency kick-force η_ε,

$$\eta_\varepsilon(t) = \sqrt{\varepsilon} \sum_k \eta_k^\omega \delta(t - \varepsilon k), \quad \eta_k = \sum b_s \xi_{sk} e_s ,$$

where $\mathbf{E}\, \xi_{sk} \equiv 0$ and $B_1 = \sum |s|^2 b_s^2 < \infty$. We proved that

$$\mathcal{D}(u_\varepsilon(T; u_0)) \rightharpoonup \mathcal{D}(u(T; u_0)) \quad \forall\, u_0 ,\ \forall\, T > 0 , \tag{7.7}$$

where u_ε is a solution of the equation, perturbed by the force η_ε, and u is a solution of (7.1). Let μ_ε be a stationary measure of the kicked equation, and μ be a stationary measure of (7.1). It turns out that the measures μ_ε converge to μ:

Theorem 7.4. *Let $\nu = 1$ and assume that $b_s \neq 0$ for all s and $B_1 < \infty$. Then $\mu_\varepsilon \rightharpoonup \mu$ in $\mathcal{P}(H)$ as $\varepsilon \to 0$.*

Sketch of the proof. Let $U_\varepsilon(t)$ be a stationary solution of the kicked equation (so $\mathcal{D}U_\varepsilon(k\varepsilon) = \mu_\varepsilon$ for each k). Then

$$\mathbf{E}\|U_\varepsilon((k+1)\varepsilon)\|_1^2 = \mathbf{E}\|S_\varepsilon(U_\varepsilon(k\varepsilon)) + \eta_{k+1}\|_1^2 \leq e^{-2\varepsilon}\mathbf{E}\|U_\varepsilon(k)\|_1^2 + D_1\, \varepsilon ,$$

where $D_1 = \mathbf{E}\|\eta_{k+1}\|_1^2 \leq B_1$. Therefore,

$$\int \|u\|_1^2 \mu_\varepsilon(du) = \mathbf{E}\|U_\varepsilon(k)\|_1^2 \leq B_1, \tag{7.8}$$

uniformly in $0 < \varepsilon \leq 1$ (cf. (3.17)). Hence, every sequence $\mu_{\tilde\varepsilon_j}$, $\tilde\varepsilon_j \to 0$, contains a converging in $\mathcal{P}(H)$ subsequence $\mu_{\varepsilon_j} \rightharpoonup \mu_0$. It remains to check that $\mu_0 = \mu$ (for every sequence $\{\varepsilon_j\}$ as above).

Let us denote by $\mathcal{B}_t^*$, $t \geq 0$, the semigroup in measures for (7.1), and denote by $\mathcal{B}_k^{\varepsilon_j *}$, $0 \leq k \in \mathbb{Z}$, the semigroup for the kicked equation. Then

$$\mathcal{B}_k^{\varepsilon_j *} \mu_{\varepsilon_j} = \mu_{\varepsilon_j} . \tag{7.9}$$

Denote $\theta_\varepsilon = [\varepsilon^{-1}]$. It is not difficult to derive from (7.9) with $k = \theta_\varepsilon$, (7.8) and (7.7) that $\mathcal{B}_1^* \mu_0 = \mu_0$ (see [KS03] for details). Now by the uniqueness of a stationary measure we have $\mu_0 = \mu$. □

The results, obtained in Sections 6, 7, make very plausible the following

Conjecture. Consider the 2D NSE, forced by the random force

$$\sqrt{\varepsilon} \sum_s b_s \xi_s(t/\varepsilon) e_s(x) , \quad 0 < \varepsilon \leq 1,$$

where $b_s \neq 0$ for all s and ξ_s are i.i.d. stationary processes that are mixing (in an appropriate sense) and satisfy $\mathbf{E}\xi_s(t) \equiv 0$, $\mathbf{E}\xi_s^2(t) \equiv 1$. Then

i) the equation has a unique stationary measure μ_ε, and all solutions converge to μ_ε in distribution as $t \to \infty$;

ii) $\mu_e \rightharpoonup \mu$ as $\varepsilon \to 0$, where μ is the stationary measure for the white–forced equation (7.1).

8 Ergodicity and the strong law of large numbers

The results and their proofs which we discuss in this section are very similar for kick- and white-forced equations, and we restrict ourselves to the kicked-forced NSE (3.3), where the kicks η_j are as in the Section 3.1.

The corresponding random dynamical system

$$u(j+1) = S\big(u(j)\big) + \eta_{j+1} \tag{8.1}$$

defines a Markov process with the transition function $P_k(x,\cdot)$, and has a stationary measure μ. The solution $\{u(j), j \ge 0\}$ such that $\mathcal{D}\,u(0) = \mu$ is a stationary process. Moreover, one of the basic constructions from the theory of Markov processes, provides us with a stationary process $\{U(j),\ j \in \mathbb{Z}\}$, such that $\mathcal{D}\big(U(j)\big) = \mu$ for each j, which is a weak solution of (8.1) (e.g., see [DZ96, KS00]).

Let us denote $\mathcal{A} = H^{\mathbb{Z}}$, equip $\mathcal{A}$ with the Tikhonov topology and the Borel sigma-algebra $\mathcal{B} = \mathcal{B}(\mathcal{A})$. The distribution of the process U, $\mathbf{P} = \mathcal{D}\big(U(\cdot)\big)$, is a measure on $\mathcal{A}$, invariant under the shifts

$$T_\ell : \mathcal{A} \to \mathcal{A}, \qquad (\dots, u_0, u_1, \dots) \mapsto (\dots, u'_0 = u_\ell, u'_1 = u_{\ell+1}, \dots).$$

We have got the probability space $(\mathcal{A}, \mathcal{B}, \mathbf{P})$ and the group of measure-preserving transformations $\{T_\ell\}$. See, e.g. [Ros71], p. 95-96, and [DZ96], Chapter 2.

Definition 8.1. *A set $A \in \mathcal{B}(H)$ is called invariant for (8.1) if*

$$P_1(x, A) = 1 \text{ for a.e. } x \in A\,.$$

A set $\tilde A \in \mathcal{A}$ is called invariant for the group $\{T_\ell\}$, if $\mathbf{P}(\tilde A\,\Delta\,T_\ell \tilde A) = 0$ for each ℓ, where Δ stands for the symmetric difference of sets.

It is obvious that if A is an invariant set for (8.1), then $A^{\mathbb{Z}} \subset \mathcal{A}$ is invariant for the group $\{T_\ell\}$. Conversely, if $\tilde A \in \mathcal{B}$ is an invariant set for $\{T_\ell\}$, then there exists $A \in \mathcal{B}(H)$ such that $\tilde A$ equals $A^{\mathbb{Z}}$ up to a null-set, see [Ros71, DZ96]. Clearly the set A is invariant for (8.1). This relation between invariant sets of (8.1) and the group $\{T_\ell\}$ implies the following classical result:

Theorem 8.2. *If μ is a unique stationary measure for (8.1), then the transformation T_1 is ergodic.*

Proof. Let $\tilde A$ be an invariant set of T_1, and A be the corresponding invariant set for (8.1). Assume that $\mu(A) \neq 0$. Then the conditional measure $\mu(\cdot\,|A)$,

$$\mu(B\,|\,A) = \mu(B \cap A)\,/\,\mu(A)\,,$$

is also stationary for (8.1). So $\mu(A)$ is 0 or 1. Accordingly, $P(\tilde A)$ also is 0 or 1. So T_1 is an ergodic transformation. $\square$

Due to this theorem, the transformation T_1 satisfies the Strong Law of Large Numbers (SLLN), i.e. the Birkhoff–Hinchin ergodic theorem (see [Ros71, DZ96]). In terms of the process $\{U(j)\}$ we have

$$\big\langle f(U)\big\rangle_0^T := \frac{1}{T}\sum_{j=0}^{T-1} f\big(U(j)\big) \to \langle \mu, f\rangle \ \text{ a.e.} \tag{8.2}$$

if $f \in L_1(H, d\mu)$, and in particular, if $f \in C_b(H)$. It turns out that the SLLN holds for all solutions of (8.1) (not only for stationary):

Theorem 8.3. *Let $b_s \neq 0$ for $|s| \le N$, where N is as in Theorem 6.4, and let f be a locally Lipschitz function on H. Then any solution $\{u(j)\}$ of (8.1) such that $\mathcal{D}\,u(0) \in \mathcal{P}_1(H)$ satisfies*

$$\big\langle f(u)\big\rangle_0^T \to (\mu, f) \ \text{ a.e.} \tag{8.3}$$

Proof. We have to show that the convergence (8.3) holds for a.e. $\vec u = (u_0, u_1, \dots) \in \mathcal{A}_+ = H^{\{0\}\cup\mathbb{N}}$, with respect to the measure $\mathcal{D}\big(u(\cdot)\big)$. To check this we can replace $u(\cdot)$ by any weak solution $u_1(\cdot)$ of (8.1), equal $u(0)$ at $t = 0$. Similarly, (8.2) remains true if we replace $U(j)$, $j \ge 0$, by a process $u_2(j)$, $j \ge 0$, distributed as U.

For any real-valued random sequence $\{x(j), j \ge 0\}$ and any a.s. finite random variable $T' \ge 0$, we set

$$\langle x\rangle_{T'}^T = \begin{cases} \dfrac{1}{T}\displaystyle\sum_{j=T'}^{T} x(j), & \text{if } T > T' \\ 0, & \text{if } T \le T'\,. \end{cases}$$

Then a.e. we have

$$\lim_{T\to\infty} \langle x\rangle_{T'}^T = \lim_{T\to\infty} \langle x\rangle_0^T$$

(i.e. if one limit exists, then the second also exists and they are equal). Let us choose for $u_1(k)$, $u_2(k)$ the weak solutions as in Corollary 6.5 (defined on the same probability space). Let $\tilde f \in C_b(H)$ be a Lipschitz function, equal f for $|u| \le r_0 = \frac{e}{e-1}\sqrt{B_0}$, and let T' be the random variable from (6.14). Then we have the following equalities which hold a.e.:

$$\langle \mu, f\rangle = \langle \mu, \tilde f\rangle = \lim \big\langle \tilde f(u_2)\big\rangle_0^T = \lim \big\langle \tilde f(u_2)\big\rangle_{T'}^T$$

(here the first equality holds by Exercise 3.7, and the second by (8.2)).

Next, since $\tilde f$ is a Lipschitz function, then by (6.14) we have

$$\lim \big\langle \tilde f(u_2)\big\rangle_{T'}^T = \lim \big\langle \tilde f(u_1)\big\rangle_{T'}^T = \lim \big\langle f(u_1)\big\rangle_{T'}^T = \lim \big\langle f(u_1)\big\rangle_0^T\,.$$

So (8.3) is proved. $\square$

Now let us discuss the white-forced NSE (7.1). If the assumptions of Theorem 7.1 hold, then any solution $u(t)$ with a non-random initial data $u(0)$ satisfies (8.3), where the sum should be replaced by an integral in the definition of $\langle f(u)\rangle_0^T$:

Theorem 8.4. *Under the assumptions of Theorem 7.1, for any solution $u(t)$ of eq. (7.1) such that $\mathbf{E}\,|u(0)|^2 < \infty$, and any Lipschitz $f \in C_b(H)$ we have*

$$\big\langle f(u)\big\rangle_0^T \to (\mu, f) \;\; a.e. \tag{8.4}$$

Moreover, if $B_k < \infty$ for some $k \geq 1$, then $\langle f(u)\rangle_1^T \to (\mu, f)$ a.e., for any function f as in the Amplification to Theorem 7.1. [8]

Proof remains the same (see [Kuk02a]).

In [Shi06] the rate of convergence in (8.3) and (8.4) was specified, using the technique of martingale approximation (see the next section).

The physical relevance of the results. The SLLN, proved in this section, shows that for the 2D turbulence, driven by a random force, the time average equals the ensemble average. This equality is postulated by the theory of turbulence (e.g., see [Bat82], p. 17, and [Fri95], p. 58), where it is important since it allows to calculate various averaged characteristics of a turbulent flow by running one experiment for a long time (experiments with turbulent flows are expensive, and it is practically impossible to calculate ensemble averages directly).

[8] We replace $\langle f(u)\rangle_0^T$ by $\langle f(u)\rangle_1^T$ since, possibly, $f(u(t)) \to \infty$ as $t \to 0$.

9 The martingale approximation and CLT

Let us consider the white-forced NSE:

$$\dot u + Lu + B(u) = \frac{d}{dt}\,\zeta(t), \quad \zeta = \sum b_s\,\beta_s(t)\,e_s(x), \quad b_s \neq 0 \quad \forall s, \; B_1 < \infty\,, \tag{9.1}$$

(the results remain true for the kicked equation, but the notations become more cumbersome). For simplicity we assume that

$$u(0) = u_0\,,$$

where $u_0 \in H$ is a non-random vector. Let μ be the unique stationary measure and $f \in C_b(H)$ be a Lipschitz functional such that

$$(f, \mu) = 0\,.$$

Denoting

$$S_t = \int_0^t f\big(u(s)\big)\,ds\,,$$

we have $t^{-1}\,S_t \to 0$ a.s. as $t \to \infty$ by the SLLN.

Our goal in this section is to prove that the random variables $t^{-1/2}\,S_t$ satisfy the Central Limit Theorem (CLT):

Theorem 9.1. *Under the assumptions above we have*

$$\mathcal{D}(T^{-1/2}\,S_T) \rightharpoonup \widehat\sigma\,\mathcal{N}(0,1) \;\; as \; T \to \infty\,, \tag{9.2}$$

for some $\widehat\sigma \geq 0$, independent of u_0.

Let $U(t)$, $t \geq 0$, be a stationary solution of (9.1). Then the CLT for stationary processes as in [Dur91] applies to the process $f\big(U(t)\big)$, and the theorem can be proved by comparing the processes $f\big(u(t)\big)$ and $f\big(U(t)\big)$ similar to the proof of Theorem 8.2, see [Kuk02a]. Here we present another proof of the theorem, based on the martingale approximation for the process $f\big(u(t)\big)$, following [Shi06]. This approximation is a powerful technical tool to study Markov processes and random dynamical systems. In particular, it can be used to get for the process $u(t)$ a version of the SLLN with a control of the rate of convergence [Shi06] and to prove the Law of Iterated Logarithm for the process $f\big(u(t)\big)$ [Den04].

For $t \geq 0$ let us set

$$M_t = \int_0^\infty \mathbf{E}\big(f\big(u(s)\big)\,|\mathcal{F}_t\big)\,ds \,-\, I\,, \qquad I = \int_0^\infty \mathbf{E}\,f\big(u(s)\big)\,ds\,, \tag{9.3}$$

where $\{\mathcal{F}_s, s \geq 0\}$ is the filtration of σ-algebras corresponding to the Wiener process $\zeta(t)$. By the Markov property, for $s \geq t$ we have

$$\big|\mathbf{E}\big(f\big(u(s)\big)\,|\mathcal{F}_t\big)\big| = \big|(\mathcal{B}_{s-t}\,f)\big(u(t)\big)\big| \leq C\,e^{-\kappa(s-t)}(1 + |u(t)|^2)$$

(see Theorem 7.1). In particular, the integral in (9.3) converges and $\mathbf{E}|M_t| < \infty$. Since $\mathcal{F}_0$ is the trivial σ-algebra, then

$$M_0 = \int_0^\infty \mathbf{E}\, f(u(s))\, ds \, - I \, = 0\,.$$

Next, for $t_1 > t_2$ we have

$$\mathbf{E}(M_{t_1} \,|\, \mathcal{F}_{t_2}) = \int_0^\infty \Big(\mathbf{E}\big(f(u(s)) \,\big|\, \mathcal{F}_{t_2}\big)\, ds \, - I \, = M_{t_2}\,.$$

So $\{M_t\}$ is a martingale with respect to the filtration $\{\mathcal{F}_s\}$. Since $\mathbf{E}\big(f(u(s))\big|\mathcal{F}_t\big) = f\big(u(s)\big)$ for $s \le t$, then

$$M_t = S_t + \int_t^\infty \mathbf{E}\big(f(u(s)) \,\big|\, \mathcal{F}_t\big)\, ds - I\,. \tag{9.4}$$

The second term in the r.h.s. is a random variable such that the expectation of its norm is bounded uniformly in t. Hence,

$$\|\mathcal{D}(t^{-1/2}M_t) - \mathcal{D}(t^{-1/2}S_t)\|_L^* \le C\, t^{-1/2}\,. \tag{9.5}$$

That is, asymptotical properties of distribution of S_t are the same as of the martingale M_t. The relation (9.4) defines the *martingale approximation of the process S_t*.

Proof of Theorem 9.1. Without lost of generality we assume that $|f| \le 1$.

Since $|f| \le 1$, then it is sufficient to prove the convergence (9.2) for integer t. Moreover, due to (9.5) we can replace S_t by the martingale M_t. Let us consider the function

$$g(v) = \int_0^\infty \mathbf{E}\, f\big(u(s;v)\big)\, ds\,,$$

where $u(s;v)$ is the solution of (9.1) with initial condition v. In particular, $g(u_0) = I$. We have

$$\begin{aligned} |g(v)| &\le \int_0^\infty \big|\mathbf{E}\, f\big(u(s;v)\big)\,\big|\, ds \\ &\le \int_0^\infty \big(C\, e^{-\kappa s}(1+|v|^2) \wedge 1\big)\, ds \le \kappa^{-1}\big(1 + \ln C(1+|u|^2)\big)\,. \end{aligned}$$

Writing (9.4) as

$$M_t = S_t + g\big(u(t)\big) - I \, = \, \int_0^t f(u(s))\, ds + g(u(t)) - I$$

and denoting $X_j = M_j - M_{j-1}$, $j \in \mathbb{N}$, we have

$$X_j = \int_{j-1}^j f\big(u(s)\big)\, ds + g\big(u(j)\big) - g\big(u(j-1)\big)\,.$$

Clearly $\{X_j\}$ is a martingale difference sequence, and $M_n = X_1 + \cdots + X_n$. It is known (see [HH80, LS89, Dur91]) that the sequence $\{M_n\}$ satisfies the CLT (i.e. (9.2) holds with S_t replaced by M_t, $t \in \mathbb{N}$), provided that certain conditions are fulfilled. As $|X_j| \le C_1\big(1 + \ln\big(1+|u(j-1)|^2\big) + \ln(1+|u(j)|^2)\big)$ and $\mathbf{E}|u(t)|^2 \le \frac{1}{2}B_0 + e^{-2t}|u_0|^2$ (see (4.16), then only one of these conditions is non-trivial:

H) $n^{-1}\, V_n \to \widehat{\sigma}^2$ in probability, where $V_n = \sum_{j=1}^n \mathbf{E}(X_j^2 \,|\, \mathcal{F}_{j-1})$.

By the Markov property, $\mathbf{E}(X_j^2 \,|\, \mathcal{F}_{j-1}) = h\big(u(j-1)\big)$, where h is the function

$$h(v) = \mathbf{E}\Big(\int_0^1 f\big(u(s;v)\big)\, ds + f\big(u(1)\big) - f(v)\Big)^2\,.$$

So $n^{-1}V_n = n^{-1}\sum_{j=1}^n h(u(j-1))$. It can be checked (see [Shi06]) that Theorem 8.4 applies to $h\big(u(s)\big)$. Hence, H) follows from the SLLN and Theorem 9.1 is proved. □

The physical relevance of the results. The CLT, proved above, shows that (for randomly forced 2D flows) 'in large time–scales the turbulent flow is a Gaussian process'. This result is well known from experiments. In particular, the PDF of the velocity of turbulent fluid at an arbitrary point, calculated by A. A. Townsend in the middle of the last century using certain mechanical device, surprisingly turned out to be very close to the Gaussian probability density (see in [Bat82], p. 169). [9] We believe that this paradox is explained by the CLT since any mechanical device measures not the instant velocity, but its average over some time–interval.

[9] Townsend's experiments deal with 3D flows, described by the 3D NSE. But numerical investigation of the randomly forced 2D NSE, made in 1980's and 1990's, show for the PDF of their solutions a very similar picture.

10 The Eulerian limit

10.1 White-forces, proportional to the square-root of the viscosity

Let us consider the NSE with the force proportional to $\sqrt{\nu}$ ($\nu > 0$ is the viscosity:

$$\dot u + \nu L u + B(u) = \sqrt{\nu}\, \eta(t,x)\,, \tag{10.1}$$

where

$$\eta = \frac{d}{dt} \sum_s b_s\, \beta_s(t)\, e_s(x)\,. \tag{10.2}$$

We assume that $B_1 < \infty$ and $b_s \neq 0$ for all s. Let μ_ν be the unique stationary measure and $u_\nu(t)$ be the corresponding stationary solution, i.e. $\mathcal{D}\big(u_\nu(t)\big) = \mu_\nu$ for all t. Applying Ito's formula to $\varphi\big(u_\nu(t)\big)$, $\varphi(u) = |u|^2$ (see (4.15)) and taking expectation we get

$$E\,|u_\nu(t)|^2 - \mathbf{E}\,|u_\nu(0)|^2 = -2\nu \int_0^t \mathbf{E}\langle L u_\nu, u_\nu\rangle\, d\tau$$
$$-\,2\int_0^t \mathbf{E}\langle B(u_\nu), u_\nu\rangle\, d\tau + \frac12\,\nu \int_0^t \mathbf{E} \sum_s b_s^2\, d^2\varphi(u_\nu)(e_s, e_s)\, d\tau\,.$$

Noting that the l.h.s. vanishes as well as the second term in the r.h.s., and that

$$\frac12 \sum_s b_s^2\, d^2\,\varphi(u_\nu)\,(e_s, e_s) = \sum b_s^2 = B_0\,,$$

we have $2\int_0^t \mathbf{E}\,\langle L u_\nu, u_\nu\rangle\, d\tau = t B_0$. That is,

$$\mathbf{E}\,\|u_\nu(t)\|_1^2 = \frac12\, B_0 \quad \text{for each } t \ge 0 \text{ and } \nu > 0. \tag{10.3}$$

Similarly, applying Ito's formula to the functional $\varphi_1\big(u_\nu(t)\big) = \|u_\nu(t)\|_1^2$ and using that $\langle B(u), L u\rangle \equiv 0$, we have

$$2\int_0^t \mathbf{E}\langle L^2 u_\nu, u_\nu\rangle\, d\tau = \int_0^t \sum_s b_s\, \|e_s\|_1^2\, d\tau = t\, B_1\,.$$

Therefore

$$\mathbf{E}\,\|u_\nu(t)\|_2^2 = \frac12\, B_1\,. \tag{10.4}$$

Simple analysis of the derivation in Exercise 4.2 the estimate $\mathbf{E}\, e^{\sigma \|u(t;0)\|_1^2} \le C$ (valid for each $t \ge 0$), shows that it holds for solutions of (10.1), uniformly in $\nu > 0$. Since $\mathcal{D}u(t;0) \rightharpoonup \mu_\nu$, then

$$\mathbf{E}\, e^{\sigma \|u_\nu(t)\|_1^2} \le C \quad \forall t \ge 0, \nu > 0 \tag{10.5}$$

(to get (10.5) from the estimate for $\exp(\sigma\|u(t;0)\|_1^2)$ one should use Fatou's lemma as in the proof of Theorem 4.3.)

Since $\|u_\nu\|_1^2 \le |u_\nu| \cdot \|u_\nu\|_2$, then $(\mathbf{E}\,\|u_\nu\|_1^2)^2 \le \mathbf{E}\,|u_\nu|^2 \cdot \mathbf{E}\,\|u_\nu\|_2^2$. Using this inequality and the estimates (10.3), (10.4) we see that

$$\frac12\, \frac{B_0^2}{B_1} \le \mathbf{E}\,|u_\nu(t)|^2 \le \frac12\, B_0 \quad \text{for all } t\,, \;\; \text{all } \nu > 0. \tag{10.6}$$

The Reynolds number $R(u)$ of a velocity-field $u(x)$ was defined in Section 2.7. For a random field u the characteristic scale for u is $(\mathbf{E}|u|^2)^{1/2}$. Therefore, due to (10.6), we have

$$R(u_\nu) \sim \nu^{-1}\,.$$

A traditional definition of a turbulent velocity field is that this is one with a high Reynolds number. Accordingly, the velocity field $u_\nu(t,x)$ becomes turbulent when ν is small.

Since the estimates (10.3)–(10.5) hold uniformly in $\nu > 0$, they allow to pass to a limit as $\nu \to 0$ along sequences:

Theorem 10.1. 1) *Any sequence $\widetilde{\nu}_j \to 0$ contains a subsequence $\nu_j \to 0$ such that*

$$\mathcal{D}\, u_{\nu_j}(\cdot) \rightharpoonup \mathcal{D}\, U(\cdot) \;\; \textit{in}\;\; \mathcal{P}\big(C(0,\infty; H^1)\big)\,. \tag{10.7}$$

The limiting process $U(t) \in H^1$ is stationary and possesses the following properties:

2) *Every trajectory of the process U is such that*

$$U(\cdot) \in L_{2,\,\mathrm{loc}}(0,\infty; H^2)\,,$$
$$\dot U(\cdot) \in L_{1,\,\mathrm{loc}}(0,\infty; H^1) \cap L_{\infty,\,\mathrm{loc}}(0,\infty; L_p)\,,$$

for every $p < 2$. It satisfies the Euler equation:

$$\dot U + B(U) = 0\,, \tag{10.8}$$

and $|U(t)|$ and $\|U(t)\|_1$ are time-independent constants. Besides, if $g : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, then the quantity

$$\int_{\mathbb{T}^2} g(\operatorname{curl} U(t,x))\, dx \tag{10.9}$$

also is time-independent. [10]

3) *For each $t \ge 0$ we have* $\mathbf{E}\,\|U(t)\|_2^2 \le \frac12 B_1$, $\frac12 \frac{B_0^2}{B_1} \le \mathbf{E}\,|U(t)|^2 \le \frac12 B_0$, $\mathbf{E}\, e^{\sigma\|U(t)\|_1^2} \le C$ *and*

$$\mathbf{E}\,\|U(t)\|_1^2 = \frac12\, B_0\,; \tag{10.10}$$

[10] In other words, the push-forward of the Lebesgue measure on the torus by the map $\mathbb{T}^2 \to \mathbb{R}$, $x \mapsto \operatorname{curl} U(t,x)$, is a time-independent measure.

in particular, U is not the zero-process.

4) *If $b_s \equiv b_{-s}$, then the process $U(t,x)$ is homogeneous in x. If, in addition, $B_6 < \infty$, then*

$$\mathbf{E}\, e^{\sigma_1|U(t,x)|} + \mathbf{E}\, e^{\sigma_1|\nabla U(t,x)|^{1/2}} \le C_1 \quad \forall t, x\,, \tag{10.11}$$

where σ_1 and C_1 are suitable positive constants.

Proof. 1) Let us denote

$$Z_n = C(0,n;H^1), \quad Z = C(0,\infty;H^1)$$

(the space Z is endowed with the Tikhonov topology) and set

$$m_\nu^n = \mathcal{D}(u_\nu|_{[0,n]}) \in \mathcal{P}(Z_n), \quad \nu \in \widetilde{\Sigma} = \{\widetilde{\nu}_1, \widetilde{\nu}_2, \dots\}\,.$$

We show first that for each n the sequence $\{m_\nu^n,\, \nu \in \widetilde{\Sigma}\}$ contains a weakly converging subsequence. Let us denote

$$v_\nu(t) = u_\nu(t) - \sqrt{\nu}\,\zeta(t), \quad 0 \le t \le n,\, \nu \in \widetilde{\Sigma}\,.$$

Then

$$\dot{v}_\nu - \nu\, L u_\nu + B(u_\nu) = 0\,. \tag{10.12}$$

Since $B_1 < \infty$, then we get from (10.4) that

$$\mathbf{E}\,|u_\nu|^2_{L_2(0,n;H^2)} \le C_n\,. \tag{10.13}$$

Due to (10.5),

$$\mathbf{E}\,|u_\nu|^4_{L_4(0,n;H^1)} \le C_n\,. \tag{10.14}$$

□

Lemma 10.2. *The nonlinearity B defines continuous quadratic maps $H^2 \to H^1$ and $H^1 \to L_p(\mathbb{T}^2;\mathbb{R}^2)$, for any $p < 2$.*

The proof of these assertions follows from two well known facts. First, the multiplication of functions defines continuous maps $H^2(\mathbb{T}^2) \times H^1(\mathbb{T}^2) \to H^1(\mathbb{T}^2)$ and $H^1(\mathbb{T}^2) \times L_2(\mathbb{T}^2) \to L_p(\mathbb{T}^2)$ ($p < 2$), and second, the Leray projection Π is continuous in the L_p-norm for $1 < p < \infty$.

This lemma and (10.12)–(10.14) imply that

$$\mathbf{E}\,|\dot{v}_\nu|^2_{L_2(0,n;L_{3/2})} \le C_n\,.$$

Let us denote by $\mathcal{C}$ the space

$$\mathcal{C} = \{v \in L_2(0,n;H^2) \mid \dot{v} \in L_2(0,n;L_{3/2})\}\,.$$

Then $\mathcal{C} \Subset Z_n$ (see [Lio69], Section 1.5, cf. Lemma 2.5), and

$$\mathbf{E}\,|v_\nu|^2_{\mathcal{C}} \le C_n'\,.$$

This estimate and Prokhorov's theorem imply that

$$\mathcal{D}\, v_{\nu_j} \rightharpoonup m^n \in \mathcal{P}(Z_n) \tag{10.15}$$

for a suitable subsequence $\{\nu_j\}$. Let f be a bounded Lipschitz functional on Z_n. Since

$$(f, m^n_{\nu_j}) = \mathbf{E}\, f(u_{\nu_j}) = \mathbf{E}\, f(v_{\nu_j} + \sqrt{\nu}_j\,\zeta)\,,$$

we obtain

$$\begin{aligned}|(f,m^n_{\nu_j}) - (f,m^n)| = |\,&\mathbf{E}\big(f(v_{\nu_j} + \sqrt{\nu}_j\,\zeta) - f(v_{\nu_j})\big)\\ &+ \big(\mathbf{E}\, f(v_{\nu_j}) - (f,m^n)\big)| \le C\sqrt{\nu}_j\,\mathbf{E}|\zeta|_{Z_n} + |(f,\mathcal{D}v_{\nu_j}) - (f,m^n)|\,.\end{aligned}$$

The first term in the r.h.s. goes to zero since $\mathbf{E}\,|\zeta|_{Z_n} < \infty$, and the second goes to zero by (10.15). So $m^n_{\nu_j} \rightharpoonup m^n$. Now we can use the diagonalisation procedure to construct a new subsequence $\{\nu_j\}$ such that

$$m^n_{\nu_j} \rightharpoonup m^n \in \mathcal{P}(Z_n) \quad \text{for all } n\,.$$

The measures $\{m^n\}$ form a compatible family and define a measure $m \in \mathcal{P}(Z)$ with the property

$$(\mathcal{D}u_{\nu_j}, f) = (m^n_{\nu_j}, f) \to (m^n, f) = (m, f)$$

for every n and every functional f on Z of the form $f = f_n(u|_{[0,n]})$, $f_n \in C_b(Z_n)$. This relation and Proposition 5.1 imply that

$$\mathcal{D}u_{\nu_j}(\cdot) \rightharpoonup m \quad \text{in} \quad \mathcal{P}(Z).$$

For details see [Kuk04], p. 485.

By Skorokhod's theorem we can find random processes $U_{\nu_j}(t)$, $j \ge 1$, and $U(t)$, defined on the same (new) probability space, such that

$$\mathcal{D}\, U_{\nu_j} = \mathcal{D}\, u_{\nu_j} \;\text{ for all } j,\, \mathcal{D}U = m \;\text{ and }\; U_{\nu_j} \to U \text{ in } Z, \text{ a.s.}. \tag{10.16}$$

Since the processes U_{ν_j} are stationary, then U is stationary as well. The first assertion of the theorem is proved.

2) The estimate (10.13), and Fatou's lemma imply that

$$\mathbf{E}\,|U|^2_{L_2(0,n;H^2)} \le C_n \quad \forall n\,.$$

So the first assertion in 2) follows. Any process $U_{\nu_j}(t)$ is a weak solution of (10.1). So for any n and any $0 \le t_1 \le t_2 \le n$ we have

$$\begin{aligned}&U_\nu(t_2) - U_\nu(t_1) + \int_{t_1}^{t_2} B\big(U_\nu(s)\big)\, ds = -\sqrt{\nu}\int_{t_1}^{t_2} AU_\nu(s)\, ds +\\ &\sqrt{\nu}\big(\zeta_\nu(t_2) - \zeta_\nu(t_1)\big), \qquad \nu \in \Sigma = \{\nu_j\}\,,\end{aligned} \tag{10.17}$$

where $\zeta_\nu(T)$ is a version of the process $\zeta(t)$. Since the map $B : H \to H^{-2}$ is quadratic and continuous (see Section 2.2), then the l.h.s. of (10.17) converges a.s. in H^{-2} to

$$U(t_2) - U(t_1) + \int_{t_1}^{t_2} B\big(U(s)\big)\, ds\,. \tag{10.18}$$

Due to (10.13) and (4.1),

$$\mathbf{E}\Big(\Big|\int_{t_1}^{t_2} AU_\nu\, ds\Big| + |\zeta_\nu(t_2) - \zeta_\nu(t_1)|\Big) < \infty\,.$$

Therefore the r.h.s. of (10.17) converges to zero in H in probability. Hence, (10.18) vanishes a.s. for any fixed $0 \le t_1 \le t_2$. Since the trajectories of the process U are continuous in H^1, then (10.18) holds a.s. for all $0 \le t_1 \le t_2$. That is, (10.8) holds almost surely. Redefining $U(t)$ to be zero on the exceptional null-set, we achieve that every trajectory $U(t)$ belongs to $L_{2,\mathrm{loc}}(0,\infty; H^2)$ and satisfies (10.8). Since $\dot U = -B(U)$, the second assertion in 2) follows now from Lemma 10.2. The established smoothness of U and $\dot U$ easily imply that both $|U(t)|$ and $\|U(t)\|_1$ are integrals of motion, as well as the quantities (10.9) (see [Kuk04]). So 2) is proved.

3) The first and the third inequalities follow from (10.16), the corresponding inequalities for $U_{\nu_j}(t)$ and Fatou's lemma. To prove (10.10) let us consider $U_\nu^N(t) = P_N\, U_\nu(t)$, $\nu \in \Sigma$, where P_N is the Galerkin projection. By (10.16),

$$\|U_\nu^N(t)\|_1^2 \to \|U^N(t)\|_1^2 \ \text{ a.s.}\,.$$

Due to (10.5), the random variables $\|U_\nu^N(t)\|_1^2$ are uniformly integrable, and hence

$$\mathbf{E}\,\|U_\nu^N(t)\|_1^2 \to \mathbf{E}\,\|U^N(t)\|_1^2, \ \ \Sigma \ni \nu \to 0\,.$$

Due to (10.3) and (10.4),

$$\frac12\, B_1 \ge \mathbf{E}\,\|U_\nu^N(t)\|_1^2 \ge \frac12\, B_1 - \varphi(N)\,,$$

where $\varphi(N) \to 0$ when $N \to \infty$. So

$$\frac12\, B_1 \ge \mathbf{E}\,\|U^N(t)\|_1^2 \ge \frac12\, B_1 - \varphi(N)\,,$$

and (10.10) follows from the monotone convergence theorem. Now the second (both–side) inequality follows by the same arguments which prove (10.6).

4) If $b_s \equiv b_{-s}$, then the processes $u_\nu(t,x)$ and $U_\nu(t,x)$ are homogeneous in x by Theorem 7.3. Hence, $U(t,x)$ is homogeneous due to (10.16). Proof of (10.11) is nontrivial and uses obtained below in Section 11.1 explicit algebraical relations, satisfied by the stationary process u_ν. See [Kuk05]. ☐

We call the limiting process $U(t,x)$ the *Eulerian limit.*

Combining assertions of Theorem 7.1 and Theorem 10.1, we obtain that for any $u_0 \in H$ there exists the following double limit:

$$\mathcal{D}\, u_\nu(t;u_0) \underset{t\to\infty}{\rightharpoonup} \mu_\nu \underset{\{\nu_j\}\ni\nu\to 0}{\rightharpoonup} \mu_0\,.$$

Remark 10.3. Since $U(t)$ is a stationary process that satisfies a.s. the Euler equation, then $\mu_0 = \mathcal{D}U(t)$ is a stationary measure for this equation. To prove this statement we need to know that μ_0 is supported by a certain topological space $\mathcal{K}$, where (10.8) defines a measurable dynamical system. Due to the first relation in 3), $\mu_0(H^2) = 1$. Unfortunately, we cannot take $\mathcal{K} = H^2$ since it is unknown if for $U(0) \in H^2$ the Euler equation has a unique solution. Instead we choose for $\mathcal{K}$ the space, formed by all vectors $u(0)$, where $u(t), t \ge 0$, is a continuous in H^1 solution of the Euler equation, satisfying assertions of item 2) of Theorem 10.1. This is a well-defined phase–space, such that $H^3 \subset \mathcal{K} \subset H^1$ (see [Kuk04], Section 3.4). We do not know if the space $\mathcal{K}$ is linear, or not. ☐

Remark 10.4. The space $\mathcal{C}$, defined in the theorem's proof, is compactly embedded in $C^1(0,n;H^\kappa) =: Z_n^\kappa$ for any $\kappa < 2$. If $B_2 < \infty$, then $\mathbf{E}\,|\zeta\ |_{[0,n]}\ |_{Z_n^\kappa} < \infty$. These observations and simple analysis of the theorem's proof show that if $B_2 < \infty$, then for any $\kappa < 2$ the process U can be chosen such that the convergence (10.7) holds in $\mathcal{P}\big(C(0,\infty;H^\kappa)\big)$. Choosing $\kappa > 1$ and using (1.2) we achieve that a.e. realisation $U(t,x)$ is continuous in t and x.

Remark 10.5. By the item 3) of the theorem, $\|U(t)\|_1 \sim |U(t)|$. Therefore the Eulerian limit U is mostly supported by low Fourier modes. From other hand, we know that $|U(t)| \in H^2$, and our attempts to improve the smoothness of U failed. This may indicate that the Fourier coefficients in x of $U(t,x)$, $U_s(t)$, decay with $|s|$ algebraically (e.g., that $|U_s(t)| \lesssim |s|^{-2}$, but $\limsup_{|s|\to\infty} |U_s||s|^{3+\epsilon} = \infty$ for $\epsilon > 0$).

Let us denote by K the set of all Borel measures μ in H^2 such that
i) $\int_{H^2} \Big(\|u\|_2^2 + e^{\sigma\|u\|_1^2}\Big)\, \mu(du) < \infty$,
ii) $\int_{H^2} \|u\|_1^2\, \mu(du) = \frac12 B_0\,, \ \ \frac12\, B_0^2 B_1{}^{-1} \le \int_{H^2} |u|^2\, \mu(du) \le \frac12 B_0$,
iii) $\mu(\mathcal{K}\cap H^2) = 1$ ($\mathcal{K}$ is defined in Remark 10.3), and μ is an invariant measure for the Euler equation.

Corollary 10.6. *The stationary measures μ_ν converge to K in the Lipschitz–dual distance:*

$$\lim_{\nu\to 0}\, \inf_{\rho\in K} \|\mu_\nu - \rho\|_L^* = 0.$$

We stress that the set K is a pre-compact subset of $\mathcal{P}(H^1)$ of infinite codimension, depending only on two characteristics of the force η, which under the double limit "time to infinity, viscosity to zero" attracts distribution of every solution of the NS equation (10.1). [11].

10.2 One negative result

To study further properties of the Eulerian limit $U(t,x)$ and the limiting stationary measure $\mu_0 = \mathcal{D}U(t)$ is a difficult task. A natural idea to resolve it is first to guess the right answer and next to prove that the guess is correct. In particular, O. Penrose and A. Shnirelman explained us that some physical insights on the 2D–turbulence make it plausible that the random field U is a.s. time–independent (that is, the measure μ_0 is supported by time–independent solutions of the Euler equation). We cannot prove or disprove this conjecture. Below we discuss another guess concerning the measure μ_0, coming from the statistical physics, which identifies μ_0 with a certain weighted sum of microcanonical measures (see below for the relevant definitions). Importance of the canonical and microcanonical measures for the statistical hydrodynamics (including the theory of turbulence) is a popular physical idea, e.g. see [KM80], [Fri95], Section 9.7.2 and [Gal01].

To explain how the idea of the microcanonical ensemble enters the problem which we discuss, let us consider the Galerkin approximation to the equation (10.1). That is, the equation (4.13) with the operator L replaced by νL, and with the factor $\sqrt{\nu}$ in front of the force in the r.h.s.. Theorems 7.1 and 10.1 apply to the equation 'uniformly in N'. It means that in Theorem 7.1 the constant C and the exponent κ are independent of N, while in Theorem 10.1 the constants in the estimates for the limiting process $U = U^N$ may be chosen independent of N. The process U^N satisfies the Galerkin approximation to the Euler equation:

$$\dot{u} + \Pi_N B(u) = 0, \quad u \in H_N. \tag{10.19}$$

This is a finite–dimensional Hamiltonian system with two integrals of motion: the energy $E(u) = \frac{1}{2}|u|^2$ and the enstrophy $\Omega(u) = \frac{1}{2}\|u\|_1^2$, see [AK01]. The former is the hamiltonian of the system and the latter is its additional integral of motion. [12]

Exercise 10.7. For $H_N \ni u(t) = \sum_{|s|\le N} u_s(t)$ write (10.19) as an explicit system of differential equations for $\{u_s(t)\}$. Prove that the flow–maps of this system preserve the Euclidean volume in H_N. [13]

In statistical physics the invariant measure Vol$= \prod_{|s|\le N} du_s$ is called the *canonical measure*. For any $0 < a < b$ the surface

$$S^N_{a,b} = \{u \in H_N \mid E(u) = a,\ \Omega(u) = b\}$$

is invariant for the system (10.19). Therefore the normalised δ-measure of this surface

$$\delta^N_{a,b} = \left(Z^N_{a,b}\right)^{-1} \mathrm{Vol}|_{S^N_{a,b}},$$

is invariant for the flow-maps (here $Z^N_{a,b}$ stands for the codimension–two volume of the surface). The measure $\delta^N_{a,b}$ is called the *microcanonical measure for a system with the two energies*, E and Ω. Let us denote $\Sigma^2 = \{(a,b) \mid 0 < a < b\}$. For any Borel measure $\rho \in \mathcal{P}(\Sigma^2)$ we set

$$\mu_\rho = \int_{\Sigma^2} \delta^N_{a,b}\, \rho(da\, db). \tag{10.20}$$

The measures μ_ρ are invariant for (10.19). From the point of view of statistical physics they are the most natural invariant measures. So it is natural to ask the following

Question. Can the measure $\mu_0^N = \mathcal{D}U^N(t)$ be represented in the form (10.20)?

This question becomes interesting when $N \gg 1$, i.e. when the Galerkin system (10.19) approximates well the Euler equation. For such N the answer is negative, if we assume some natural additional regularity of the measures ρ_N. Namely, that $\rho_N = \phi_N(a,b)\, da\, db$, where $|\phi_N| \le C$ and the constant C is independent of N. [14] Indeed, let $\mu_0^N = \mu_{\rho_N}$ for all N large enough. By the recent results of A. Biriuk [Bir06],

$$\int \|u\|_2^2\, \delta^N_{a,b}(du) \to \infty \quad \text{as } N \to \infty, \tag{10.21}$$

for each $(a,b) \in \Sigma^2$, uniformly on compact subsets of Σ^2. Using (10.10) we get that

$$\frac{1}{4} B_0 = \int \frac{1}{2}\|u\|_1^2\, \mu_{\rho_N}(du) = \int_{\Sigma^2} b\, \rho_N(da\, db).$$

This relation and the regularity assumption imply that

$$\rho_N\big(\{(a,b) \in \Sigma^2 \mid \gamma^{-1} \ge b \ge a + \gamma\}\big) \ge \frac{1}{2} \qquad \forall N,$$

for a suitable $\gamma > 0$. Now (10.21) implies that

$$\int \|u\|_2^2\, \mu_{\rho_N}(du) \to \infty,$$

in contradiction with the first estimate in the item 3) of Theorem 10.1 (more precisely, with an analogy of this estimate for the system (10.19)).

[11] If $b_s \equiv b_{-s}$ and $B_6 < \infty$, then the corollary remains true if the set K is modified (i.e., decreased) to satisfy (10.11)

[12] Apart from E and Ω the Euler equation has infinitely many integrals (10.9), but non of them is an integral of motion for (10.19) (apart from (10.9), where g is an affine function, and which equals Ω up to an affine transformation), cf. Exercise 2.12.

[13] The system (10.19) is Hamiltonian with respect to the symplectic structure, given by a two-form with variable coefficients. So this assertion is not obvious.

[14] It suffice to assume that for some $\gamma > 0$ the ρ_N-measure of the set $\{(a,b) \in \Sigma^2 \mid b \ge a + \gamma\}$ is $\ge c > 0$ for all N.

10.3 Other scalings

Now let us scale the 2D NSE differently, and consider the equation

$$\dot u + \nu L u + B(u) = \nu^a\, \eta(t,x)\,, \tag{10.22}$$

where the random force η is as in (10.1) and $a \neq 1/2$. Let μ_ν be the stationary measure for this equation, and $u_\nu(t)$ be the corresponding stationary solution.
Applying Ito's formula to the functions $|u|$ and $\|u\|_1^2$ as at the beginning of Section 10.1, we see that

$$\mathbf{E}\,\|u_\nu(t)\|_1^2 = \nu^{2a-1}\,\tfrac12\,B_0, \quad \mathbf{E}\,\|u_\nu(t)\|_2^2 = \nu^{2a-1}\,\tfrac12\,B_1 \quad \forall\, t\,. \tag{10.23}$$

Theorem 10.8. 1) *If $a > \frac12$, then $\lim \mu_{\nu_j} = \delta_0$.*
2) *If $a < \frac12$, then $\lim \mu_{\nu_j}$ does not exist in $\mathcal{P}(H)$.*

Proof. If φ is a bounded Lipschitz functional on H such that $\mathrm{Lip}(\varphi) \le 1$, then

$$\big|\mathbf{E}\,\big(\varphi(u_\nu(0)) - \varphi(0)\big)\big| \le \mathbf{E}\,|u_\nu(0)| \le \big(\nu^{2a-1}\,\tfrac12\,B_0\big)^{1/2}$$

by (10.23). Therefore,

$$\mathcal{D}\,u_\nu(0) \rightharpoonup \delta_0 \ \text{ if } \ a > 1/2\,, \tag{10.24}$$

and the first assertion follows.

To prove the second, we make in (10.22) the following substitution:

$$u(t) = \nu^b\, v(\tau), \ \ t = \nu^{-b}\tau\,. \tag{10.25}$$

Then $\frac{\partial u}{\partial t} = \nu^b\, v'_\tau$, $\eta = \dfrac{d}{dt}\,\zeta(t) = \nu^b\,\dfrac{d}{d\tau}\,\zeta(\nu^{-b}\tau)$, and (10.22) reads as

$$v'_\tau + \nu^{1-b} L v + B(v) = \nu^{a-\frac32\,b}\,\frac{d}{d\tau}\,\widehat\zeta(\tau)\,,$$

where $\widehat\zeta(\tau) = \nu^{b/2}\,\zeta(\nu^{-b}\tau)$. Clearly $\mathcal{D}\big(\widehat\zeta(\cdot)\big) = \mathcal{D}\big(\zeta(\cdot)\big)$. Choosing $b = a - \frac12$ and denoting $\widetilde\nu = \nu^{1-b} = \nu^{\frac32 - a}$, we see that $\nu^{a-\frac32\,b} = \widetilde\nu^{\frac12}$. Therefore $v(\tau)$ is a weak stationary solution of the equation

$$v'_\tau + \tilde\nu L v + B(v) = \sqrt{\tilde\nu}\,\eta(t,x)\,. \tag{10.26}$$

Now let us assume that the sequence $\mu_{\nu_j} = \mathcal{D}\,u_{\nu_j}(0)$ converges to a limit in $\mathcal{P}(H)$. Then, by Prokhorov's criterion,

$$\mathbf{P}\big(|u_{\nu_j}(0)| > R\big) \underset{R\to\infty}{\longrightarrow} 0$$

uniformly in j. Since $u_{\nu_j}(0) = \nu^{a-1/2}\,v_{\nu_j}(0)$ by (10.25), then the convergence above implies that $\mathbf{P}(|v_{\nu_j}(0)| > \varepsilon) \to 0$ as $\nu_j \to 0$, for each $\varepsilon > 0$. So $\mathcal{D}v_{\nu_j}(0) \rightharpoonup \delta_0$ in $\mathcal{P}(H)$. But by Theorem 10.1, along a suitable subsequence the measures $\mathcal{D}v_{\nu_j}(0)$ converge to a non-zero limit. Contradiction. $\square$

10.4 Discussion

Consider the NSE (7.1), where $0 < \nu \ll 1$. Assume that

$$\frac{B_1}{B_0} \sim 1 \tag{10.27}$$

(that is, the force is essentially supported by low modes). Let $u(t,x)$ be a stationary solution of the equation and assume that its energy is of order one:

$$\mathbf{E}|u(t)|^2 \sim 1. \tag{10.28}$$

Let us find a real number a such that $B_0 = \nu^{2a}$, and set $\eta = \nu^{-a}\frac{d}{dt}\zeta$. Then for η we have $B_0 = 1$, $B_1 \sim 1$. By (10.25) $u(t) = \nu^{a-1/2}v(\tau)$, where $v(\tau)$ is a stationary weak solution of (10.26). By Corollary 10.6, the measure $\mathcal{D}v(0)$ is close to the set K (which depends on the constants B_0, B_1 as above). Since the measures in K satisfy ii), then $\mathbf{E}|v(0)|^2 \sim \frac12\,B_0$, and by (10.28) we must have $a \approx 1/2$.

We have seen that if eq. (7.1) with $0 < \nu \ll 1$ has a stationary solution u with energy of order one, then the force is $\sim \sqrt{\nu}$ and $\mathcal{D}u(t)$ is close to the set K. Arguing as when proving Corollary 10.6 we see that in this case the process $u(\cdot)$ is close in distribution to an Eulerian limit.

In the 3D case the situation, presumably, is very different. Indeed, let us assume (for the purposes of this section) that the 3D NSE, written in the form (10.22), has a unique stationary measure, that the corresponding stationary solution is H^1–smooth in x and its energy stays of order one as $\nu \to 0$. Applying Ito's formula to the energy functional $\frac12\,|u(t)|_{L_2}$ we get, as in the 2D case, that the rate of dissipation of energy $\varepsilon := \mathbf{E}|\nabla u|^2_{L_2}$ equals $\frac12\,\nu^{2a}B_0$. By the Kolmogorov's theory of turbulence (see in [Fri95]), ε converges to a positive limit as $\nu \to 0$. [15] So now we must have $a = 0$.

10.5 Kicked equations

Let us consider the kicked equation

$$\dot u + \nu\, L u + B(u) = \sqrt{\nu}\,\eta(t,x) = \sqrt{\nu}\,\sum_k \eta_k^\omega\,\delta(t-k)\,, \tag{10.29}$$

where $\eta_k = \sum b_s\,\xi_{sk}\,e_s$. Here the random variables ξ_{sk} are as in Section 3, $b_s \neq 0$ for all s and $B_1 = \sum |s|^2\, b_s^2 < \infty$. By Theorem 6.4 and its corollaries, for any $\nu > 0$ the corresponding Markov chain has a unique stationary measure μ_ν. Let $u_\nu(t)$ be the corresponding 1–periodic solution of (10.29), i.e. $\mathcal{D}u_\nu(k) = \mu_\nu$ for each k (see the end of Section 3.3). Due to (3.12),

$$\int_k^{k+1} \mathbf{E}\,\|u_\nu(s)\|_1^2\,ds = \frac12\,D_0, \quad \int_k^{k+1} \mathbf{E}\,\|u_\nu(s)\|_2^2\,ds = \frac12\,D_1 \tag{10.30}$$

[15] We assume that the theory applies to the space–periodic turbulence, which is not at all obvious.

for each $\nu > 0$ and each k, where

$$D_0 = \sum b_s^2 \, \mathbf{E}\, \xi_{s1}^2, \quad D_1 = \sum b_s^2 \, |s|^2 \, \mathbf{E}\, \xi_{s1}^2 \, .$$

The constants D_0 and D_1 were defined in (3.1), and $D_0 \in (0, B_0)$, $D_1 \in (0, B_1)$. Moreover, by Lemma 3.9 the solution $u(t)$ with initial condition $u(0) = 0$ satisfies $\mathbf{E}\, \|u(k)\|_1^{2m} \le C_m$ for all k, uniformly in ν. As at the beginning of Section 10.1, this implies that the periodic solution $u_\nu(t)$ meets the following estimates:

$$\mathbf{E}\, \|u_\nu(k)\|_1^m \le C_m \text{ for all } k, m \in \mathbb{N}, \text{ all } \nu > 0 \, .$$

Using (3.5) and the property (NS1$'$) from Section 2.5 we see that this estimate holds with k replaced by any $t \ge 0$.

Repeating the proof of Theorem 10.1 we get

Theorem 10.9. *Any sequence $\widetilde{\nu}_j \to 0$ contains a subsequence $\nu_j \to 0$ such that*

$$\mathcal{D}\, u_{\nu_j}(\cdot) \rightharpoonup \mathcal{D}\, U(\cdot) \;\; in \;\; \mathcal{P}\big(C(0, \infty; H^1)\big) \, .$$

The limiting process $U(\cdot)$ is 1-periodic and possesses the following properties: every its trajectory $U(t) \in L_{2,\,\mathrm{loc}}(0, \infty; H^2)$ satisfies the Euler equation, and $|U(t)|$, $\|U(t)\|_1$ are time-independent constants, as well as the quantities (10.9)*. Moreover, $\mathbf{E}\, \|U(t)\|_1^2 = \frac{1}{2}\, D_0$ for each t and*

$$\int_k^{k+1} \mathbf{E}\, \|U(s)\|_2^2 \, ds \le \frac{1}{2}\, D_1 \text{ for all } k \, , \quad \mathbf{E}\, \|U(t)\|_1^m \le C_m \quad \text{for all } t \text{ and } m \, .$$

The physical relevance of the results. The results on the Eulerian limit agree with physical heuristics since they show that the 2D Euler equation describes certain classes of the homogeneous 2D turbulence. This claim was often maid by physicists (e.g., by Onsager in [Ons49]). From other hand, the fact that solutions of the 2D NSE, scaled as

$$\dot{u} + \nu L u + B(u) = \sqrt{\nu}\, \eta(t, x) \, , \tag{10.31}$$

under the double limit $t \to \infty$, $\nu \to 0$ converge (in distribution) to solutions of the free Euler equation was not known earlier. Moreover, the results obtained show that the Kraichnan theory of 2D turbulence does not apply to the stationary space–periodic 2D turbulence. Indeed, the theory is based on the assumption that the rate of dissipation of enstrophy $\nu \mathbf{E} |\Delta u_\nu|^2 = \nu \mathbf{E} |\nabla \operatorname{curl} u|^2$ converges (as $\nu \to 0$) to a non-zero limit. But due to the explicit formula for $\mathbf{E}|\Delta u_\nu|^2$ in Section 10.1, this limit vanishes if the NSE is scaled as (10.31). In the same time, the results of Sections 10.3 and 10.4 show that if the randomly forced 2D NSE is scaled differently from (10.31), then its solutions cannot converge to a non-trivial limit with energy of order one. This fact is not surprising since for the 2π–periodic boundary conditions the smallest wave–number is one, so the inverse cascade of energy (to the low frequencies), on which the Kraichnan theory is based, cannot develop properly. It is an exciting open problem to study the limit $t \to \infty$ and the double limit $t \to \infty$, $\nu \to 0$ for the 2D NSE with $x \in \mathbb{R}^2$, perturbed by a space–homogeneous (non-periodic) random force, to find under what scaling of the force the limit is of order one and to compare the result obtained with the Kraichnan theory. [16]

[16] The theory predicts that in this case the double limit of order one exists if the force is independent of the viscosity ν. This is a remarkable conjecture which has to be checked.

11 Balance relations for the white–forced NSE

11.1 The balance relations

Let us consider again the NSE (10.1), where $B_1 < \infty$ and

$$b_s = b_{-s} \neq 0 \quad \forall\, s. \tag{11.1}$$

Now the scaling factor $\sqrt{\nu}$ in the r.h.s. is not important for us (i.e., the force η may depend on ν). We introduce it to make the formulas which we obtain below independent of ν.

Denoting by $\xi(t,x)$ the vorticity $\xi = \operatorname{curl} u$, we get for ξ the equation (see Section 2.7)

$$\dot{\xi} - \nu\Delta\xi + (u\cdot\nabla)\xi = \sqrt{\nu}\,\operatorname{curl}\eta\,, \tag{11.2}$$

where

$$\operatorname{curl}\eta = \frac{d}{dt}\sum_{s\in\mathbb{Z}_0^2} b_s\beta_s(t)\varphi_s(x)$$

with

$$\varphi_s = \frac{|s|}{\sqrt{2}\,\pi}\cos s\cdot x\,, \qquad \varphi_{-s} = -\frac{|s|}{\sqrt{2}\,\pi}\sin s\cdot x,$$

for all $s \in \mathbb{Z}_+^2$.

For a scalar function $g(x)$ we set $\nabla^{\perp} g(x) = (\partial g/\partial x^2, -\partial g/\partial x^1)^t$. Then

$$\operatorname{curl}\nabla^{\perp} g = -\Delta g.$$

For any $l \in \mathbb{R}$ the operator $-\Delta$ defines an isomorphism of the zero–meanvalue Sobolev spaces $H_0^l(\mathbb{T}^2)$ and $H_0^{l-2}(\mathbb{T}^2)$.

Exercise 11.1. Prove that for any $r \in \mathbb{R}$ the differential operator curl defines an isomorphism

$$\operatorname{curl} : H^r \to H_0^{r-1}(\mathbb{T}^2)\,,$$

and that its inverse can be written as $\operatorname{curl}^{-1} = \nabla^{\perp}\circ(-\Delta)^{-1}$.

Accordingly, for any integer $r \geq -1$ eq (11.2) with $u = \operatorname{curl}^{-1}\xi$ is a well defined equation for $\xi(t) \in H_0^r(\mathbb{T}^2)$, equivalent to eq. (10.1).

Let u_ν be a stationary solution of (10.1). Then $\xi_\nu = \operatorname{curl} u_\nu$ is a stationary solution of (11.2). Due to (11.1) and Theorem 7.1 the random fields $u_\nu(t,\, x)$ and $\xi_\nu(t,\, x)$ are homogeneous in x.

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support. Define G to be its second integral, so that

$$G'' = g, \qquad G(0) = G'(0) = 0.$$

Then

$$|G(r)| \leq C|r|, \quad |G'(r)| \leq C, \quad \forall r \in \mathbb{R}. \tag{11.3}$$

Let us consider the stationary process

$$t \mapsto \int_{\mathbb{T}^2} G(\xi_\nu(t,\, x))\,dx.$$

The estimates (11.3) allow us to apply to this process the Ito formula (see [DZ92]). Taking the expectation and arguing as when deriving (10.3), we find that for all $t \geq 0$ the random field $\xi_\nu(t,\, x)$ satisfies

$$\nu\mathbf{E}\int_{\mathbb{T}^2} G'(\xi_\nu(t,\, x))(-\Delta)\xi_\nu(t,\, x)\,dx = \frac{1}{2}\,\nu\mathbf{E}\sum_{s\in\mathbb{Z}_0^2} b_s^2\int_{\mathbb{T}^2} G''(\xi_\nu(t,\, x))\varphi_s^2\,dx.$$

Integrating by parts, we see that the l.h.s. equals

$$\nu\mathbf{E}\int g(\xi_\nu)|\nabla\xi_\nu|^2\,dx.$$

Since $b_s = b_{-s}$ and $\varphi_s^2 + \varphi_{-s}^2 = |s|^2/2\pi^2$, then the r.h.s. is

$$\frac{1}{4}\,\nu\mathbf{E}\int\Big(g(\xi)\sum_{s\in\mathbb{Z}_0^2} b_s^2(\varphi_s^2+\varphi_{-s}^2)\Big)\,dx = \frac{1}{2}(2\pi)^{-2}B_1\mathbf{E}\int g(\xi_\nu)\,dx.$$

Thus we have

$$\mathbf{E}\int g(\xi_\nu)|\nabla\xi_\nu|^2\,dx = \frac{1}{2}(2\pi)^{-2}B_1\mathbf{E}\int g(\xi_\nu)\,dx.$$

Since the random field ξ_ν is homogeneous in x, then

$$\mathbf{E}g(\xi_\nu(t,\, x))|\nabla\xi_\nu(t,\, x)|^2 = \frac{1}{2}(2\pi)^{-2}B_1\mathbf{E}g(\xi_\nu(t,\, x)) \tag{11.4}$$

for any $(t,\, x)$, provided that g is a continuous function with compact support. The equality (11.4) is called a *balance relation.*

Exercise 11.2. Use Exercise 4.12 to show that if $B_3 < \infty$ then (11.4) holds for any continuous function $g(r)$, satisfying

$$|g(r)| \leq C(1+|r|)^c, \tag{11.5}$$

for some $C,\, c > 0$.

Hint: approximate g as above by functions with compact support $g_n(r)$ such that $g_n(r) \to g(r)$ for each r and each g_n meets (11.5). Next pass to the limit in (11.4) with g replaced by g_n. (For a complete proof see [KP05]).

Theorem 11.3. *Let $B_3 < \infty$ and let ξ_ν be a stationary solution of eq.* (11.2)*. Then for any continuous function g, satisfying* (11.5)*, the balance relation* (11.4) *holds.*

Taking in (11.4) $g=1$ (or using (10.4) and Exercise 2.22) we see that

$$\mathbf{E}|\nabla\xi_\nu(t,x)|^2=\frac{1}{2}(2\pi)^{-2}B_1.$$

So another way of stating the assertion of the theorem is that the random variables $g(\xi_\nu(t,\,x))$ and $|\nabla\xi_\nu(t,\,x)|^2$ are uncorrelated for each (t,x) and each function g, satisfying (11.5).

It is straightforward that the relations (11.4), valid for all bounded continuous g, imply that

$$\mathbf{E}(|\nabla\xi_\nu(t,\,x)|^2\mid\mathcal{F}_{\xi_\nu(t,x)})=\frac{1}{2}(2\pi)^{-2}B_1\,, \tag{11.6}$$

where the l.h.s. is a conditional expectation of the random variable $|\nabla\xi_\nu(t,\,x)|^2$ with respect to the σ-algebra, generated by the random variable $\xi_\nu(t,\,x)$. Note that the balance relations in the form (11.6) hold without assuming that $B_3<\infty$.

The balance relations (11.4) and their proof are related to the classical fact that the quantities (10.9) are integrals of motion for the Euler equation. Indeed, to prove the latter we differentiate (10.9) along trajectories of (10.8), and to establish (11.4) we differentiate the same quantity, evaluated for $U:=\xi_\nu$, using Ito's formula. A significant difference between the two results is that (10.9) is an integral of motion for the Euler equation in any closed Riemann surface (as well as in a domain with the boundary condition $u\cdot n=0$ on the boundary). On the contrary, the *balance relations hold only for the periodic boundary conditions* since their proof heavily uses that $\varphi_s^2+\varphi_{-s}^2\equiv|s|^2/2\pi^2$ for each s.

Now let $g(r)$ be a smooth function such that

$$|g(r)|+|g'(r)|\le C(1+|r|^c)\quad\forall\, r \tag{11.7}$$

with some $C,\,c>0$. Since for any $x\in\mathbb{T}^2$ the map

$$H^4\to\mathbb{R}^2,\quad u(\cdot)\mapsto(\operatorname{curl}u(x),|\nabla\operatorname{curl}u(x)|^2),$$

is continuous (see (S3) in Section 1.1), then the map

$$H^4\to\mathbb{R},\quad u(\cdot)\mapsto g(\operatorname{curl}u(x))\,|\nabla\operatorname{curl}u(x)|^2,$$

satisfies (7.4). Therefore, if $B_5<\infty$, then by the Strong Law of Large Numbers (Theorem 8.4), for any solution ξ of (11.2) such that $\mathbf{E}|\xi(0)|^2<\infty$ we have

$$\frac{1}{T}\int_1^{T+1}\big(g(\xi(t,x))\,|\nabla\xi(t,x)|^2\big)\,dt\to\mathbf{E}\big(g(\xi_\nu(0,x))\,|\nabla\xi_\nu(0,x)|^2\big)\quad\text{a.s.}$$

Since a similar relation holds for $g(\xi(t,x))$, then we have

Corollary 11.4. *Let $B_5<\infty$ and $\xi(t,x)$ be a solution of* (11.2) *such that* $\mathbf{E}|\xi(0)|^2<\infty$. *Then for any smooth function g, satisfying* (11.7), *and any $x\in\mathbb{T}^2$ we have*

$$\frac{1}{T}\int_T^{T+1}\big(g(\xi(t,x))\,|\nabla\xi(t,x)|^2-\frac{1}{2}(2\pi)^{-2}B_1g(\xi(t,x))\big)\,dt\to0\quad\textit{as }\;T\to\infty,$$

almost surely.

We repeat that the results of this section also apply to the NSE without the scaling factor $\sqrt{\nu}$ in the r.h.s.:

$$\dot u+\nu Lu+B(u)=\eta(t,x), \tag{11.8}$$

where the force η has the form (10.2) with $B_1<\infty$ and $b_s\equiv b_{-s}\neq0$. Substituting $\eta=\sqrt{\nu}\,\tilde\eta$, we get eq. (10.1) with η replaced by $\tilde\eta$. Hence, the vorticity ξ_ν of a stationary solution of (11.8) satisfies

$$\mathbf{E}g(\xi_\nu(t,\,x))|\nabla\xi_\nu(t,\,x))^2=\frac{1}{2}\nu^{-1}(2\pi)^{-2}B_1\mathbf{E}g(\xi_\nu(t,\,x))$$

for any $(t,\,x)$, if g satisfies (11.5).

11.2 The co-area form of the balance relations

Let us take the balance relation (11.4), where g is a bounded continuous function, $0\le g\le1$. Integrating this relation in dx we get:

$$\mathbf{E}\int g(\xi(x))|\nabla\xi(x)|^2\,dx=\frac{1}{2}(2\pi)^{-2}B_1\mathbf{E}\int g(\xi(x))\,dx. \tag{11.9}$$

Here we abbreviate $\xi_\nu(t,x)=\xi(x)$ ($t\ge0$ is fixed).

Assume that $B_6<\infty$. Then $\xi(x)\in C^3$ a.s. (see Exercise 4.12). Modifying ξ on a null-set we achieve

$$\xi^\omega(x)\in C^3(\mathbb{T}^2)\quad\forall\,\omega.$$

For any $\varepsilon>0$ and $\tau\in\mathbb{R}$ we set

$$K_\varepsilon(\omega)=\{x\in\mathbb{T}^2\mid|\nabla\xi^\omega(x))|\ge\varepsilon\},\quad\Gamma_\varepsilon(\tau,\omega)=\{x\in K_\varepsilon(\omega)\mid\xi^\omega(x)=\tau\}.$$

The random set Γ_ε is a finite union of C^3-smooth curves. We denote by γ points of Γ_ε and denote by $d\gamma$ the length–element. Performing the co-area change of variables

$$K_\varepsilon(\omega)\ni x\mapsto(\tau,\gamma),\quad\tau=\xi^\omega(x),\;\gamma\in\Gamma_\varepsilon(\tau,\omega),$$

we have $d\tau\,d\gamma=|\nabla\xi|\,dx_1\,dx_2$ (e.g., see [Cha84]). So

$$\int_{K_\varepsilon}g(\xi(x))\,|\nabla\xi(x)|^2\,dx=\int_{\mathbb{R}}g(\tau)\int_{\Gamma_e(\tau,\omega)}|\nabla\xi|\,d\gamma\,d\tau\,, \tag{11.10}$$

and

$$\int_{K_\varepsilon}g(\xi(x))\,dx=\int_{\mathbb{R}}g(\tau)\int_{\Gamma_e(\tau,\omega)}|\nabla\xi|^{-1}\,d\gamma\,d\tau\,. \tag{11.11}$$

Now we study the behaviour of expectations of the four integrals in (11.10), (11.11) as $\varepsilon\to0$.

i) Obviously,

$$\mathbf{E}\int_{K_\varepsilon}g(\xi(x))\,|\nabla\xi(x)|^2\,dx\to\mathbf{E}\int_{\mathbb{T}^2}g(\xi(x))\,|\nabla\xi(x)|^2\,dx\,.$$

ii) We have

$$\left|\mathbf{E}\int_{\mathbb{T}^2\setminus K_\varepsilon} g(\xi(x))\,dx\right| \le \mathbf{E}\int_{\mathbb{T}^2} I_{\{|\nabla\xi|<\varepsilon\}}(x,\omega)\,dx\,.$$

When $\varepsilon\to 0$, the r.h.s. converges to

$$\mathbf{E}\int_{\mathbb{T}^2} I_{\{|\nabla\xi|=0\}}(x,\omega)\,dx = (2\pi)^2\mathbf{P}\{\nabla\xi(x_0)=0\}$$

(we use the homogeneity of ξ). It can be shown (see Appendix in [Kuk05]) that $\mathbf{P}\{\nabla\xi(x_0)=0\}=0$. Therefore,

$$\mathbf{E}\int_{K_\varepsilon} g(\xi(x))\,dx \to \mathbf{E}\int_{\mathbb{T}^2} g(\xi(x))\,dx\,.$$

iii) Denote

$$\Gamma(\tau,\omega)=\{x\in\mathbb{T}^2\mid \xi(x)=\tau\}\,. \tag{11.12}$$

By the Sard lemma, [17] for each ω the set $\Gamma(\tau,\omega)$ is C^3-smooth for a.e. $\tau\in\mathbb{R}$. So

$$\int_{\Gamma_\varepsilon(\tau,\omega)}|\nabla\xi|\,d\gamma \nearrow \int_{\Gamma(\tau,\omega)}|\nabla\xi|\,d\gamma<\infty\,,$$

for every ω and a.e. τ. Hence,

$$\mathbf{E}\int_{\mathbb{R}} g(\tau)\int_{\Gamma_\varepsilon(\tau,\omega)}|\nabla\xi|\,d\gamma\,d\tau \to \mathbf{E}\int_{\mathbb{R}} g(\tau)\int_{\Gamma(\tau,\omega)}|\nabla\xi|\,d\gamma\,d\tau$$

by the monotone convergence theorem.

iv) Similar,

$$\int_{\Gamma_\varepsilon(\tau,\omega)}|\nabla\xi|^{-1}\,d\gamma \nearrow \int_{\Gamma(\tau,\omega)}|\nabla\xi|^{-1}\,d\gamma\le\infty\,,$$

for every ω and a.e. τ, where we accepted the following agreement: $0^{-1}=0$. Hence,

$$\mathbf{E}\int_{\mathbb{R}} g(\tau)\int_{\Gamma_\varepsilon(\tau,\omega)}|\nabla\xi|^{-1}\,d\gamma\,d\tau \to \mathbf{E}\int_{\mathbb{R}} g(\tau)\int_{\Gamma(\tau,\omega)}|\nabla\xi|^{-1}\,d\gamma\,d\tau$$

Now (11.9)–(11.11) and i)-iv) imply that

$$\mathbf{E}\int_{\mathbb{R}} g(\tau)\int_{\Gamma(\tau,\omega)}|\nabla\xi|\,d\gamma\,d\tau = \frac12\,(2\pi)^{-2}B_1\,\mathbf{E}\int_{\mathbb{R}} g(\tau)\int_{\Gamma(\tau,\omega)}|\nabla\xi|^{-1}\,d\gamma\,d\tau \tag{11.13}$$

(in particular, the r.h.s. is $<\infty$). Since for each ω the set of critical values τ of ξ has zero measure, then we can arbitrarily define the integrals of $|\nabla\xi|^{-1}$ over critical levels of ξ, without changing the r.h.s.. Below we adopt the following natural convention:

$$\int_{\Gamma(\tau,\omega)}|\nabla\xi|^{-1}\,d\gamma=\infty \ \text{ if } \tau \text{ is a critical value of } \xi\,. \tag{11.14}$$

[17] The lemma applies since $\xi\in C^3$ and $\dim x=2$, $\dim\xi=1$. See [GG73].

The integral $\int_{\Gamma(\tau,\omega)}|\nabla\xi|\,d\gamma\le\infty$ is always well defined as a limit of the integrals over Γ_ε.

The relation (11.13) holds for each continuous g, $0\le g\le 1$. Similar, it holds for continuous g, $-1\le g\le 0$. So it is satisfied by any $g\in C_b(\mathbb{R})$, and we get

Theorem 11.5. *If $B_6<\infty$, then for any $\nu>0$ and $t\ge 0$ we have*

$$\mathbf{E}\int_{\{x|\xi^\omega_\nu(t,x)=\tau\}}|\nabla\xi^\omega_\nu(t,x)|\,d\gamma = \frac12\,(2\pi)^{-2}B_1\mathbf{E}\int_{\{x|\xi^\omega_\nu(t,x)=\tau\}}|\nabla\xi^\omega_\nu(t,x)|^{-1}\,d\gamma\,, \tag{11.15}$$

for a.a. $\tau\in\mathbb{R}$. Here we assume (11.14)*, where $\Gamma=\{x\mid\xi^\omega_\nu(t,x)=\tau\}$.*

Noting that (11.10) and i), iii) with $g=1$ imply

$$\mathbf{E}\int_{\mathbb{T}^2}|\nabla\xi|^2\,dx=\int_{\mathbb{R}}\Big(\mathbf{E}\int_{\Gamma(\tau,\omega)}|\nabla\xi|\,d\gamma\Big)\,d\tau$$

and using (11.9) with $g=1$, we see that the integral in $d\tau$ over $\mathbb{R}$ of the function in (11.15) equals $\frac12 B_1$.

Similar to Section 11.1, the vorticity $\xi(t,x)$ of a stationary solution for equation (11.8) for any $t\ge0$ satisfies

$$\mathbf{E}\int_{\{x|\xi^\omega=\tau\}}|\nabla\xi^\omega(t,x)|\,d\gamma = \frac12\,\nu^{-1}\,(2\pi)^{-2}B_1\mathbf{E}\int_{\{x|\xi^\omega=\tau\}}|\nabla\xi^\omega(t,x)|^{-1}\,d\gamma\,,$$

for a.a. τ.

The physical relevance of the results. The balance relations (11.4) and (11.15) indicate that in some sense 'the periodic 2D turbulence is integrable', if the 2D fluid is stirred by a white in time Gaussian random force. We are certain that these relations make a powerful tool to study the 2D turbulence, but now we know only one their application: in [Kuk05] the balance relations (11.4) are used to prove that a stationary solution of the randomly forced 2D NSE and its vorticity, evaluated at any point, both are random variables with finite exponential moments (see the estimates (10.11)).

12 Comments

Below we comment on the results, presented in the book. Our discussion of the relevant publications is incomplete. Still, we believe that we mention most of the works which contributed significantly to the problems which we consider.

Chapter 1, essentially, is a collections of notations.

Chapter 2 contains well known results on the 2D NSE. We present them here for the sake of the readers who do not belong to the PDE community.

The assertions, presented in the first three sections of **Chapter 3** are basic and well known; Theorem 3.5 and its proof are due to Bogolyubov–Krylov.

The results of Section 3.4 are less known, but rather straightforward. They form a 'poor man's Ito formula' for the kick–forced NSE (their counter–parts for the white–forced equation in Sections 4.3 and 10.1 are obtained by direct application of the Ito formula).

The results of Sections 4.1–4.3 in **Chapter 4** go back to the thesis of Viot [Vio76] and the book [VF88]. Namely, it is proved in these two references that the white forced 2D NSE defines a Markov process in a suitable function space, that the Ito lemma applies to solutions and implies the a priori estimates of Theorem 4.7. A more delicate task is to check these properties for white forces of low smoothness. Since all forces in nature are smooth, we avoid the mentioned difficulty by assuming that $B_1 := \sum |s|^2 b_s^2 < \infty$, i.e. that the forces are 'smooth enough'. For the case of forces of low smoothness see e.g. [Fla94], [FM95], [Fer03]. Also, see [DD02] for a study of the 2D NSE, driven by a 'very rough' force. If the coefficients b_s decay exponentially with $|s|$, then the force is analytic in x, and solutions are analytic with probability one (see [BKL00], [Shi02]). In this case the equation can be studied in the space of analytical functions. It seems that this option gives no serious technical advantages and does not lead to new interesting results. In three space dimensions situation becomes significantly more complicated due to the luck of uniqueness. For results, available for the randomly forced 3D NSE see [VF88, FMRT01, MR03], references in these works, and the review [Fla05].

The estimates of Exercise 4.10 are less traditional. For their proofs see [KS03]. Theorem 4.11 is well known and may be found, e.g. in [VF88]. Theorem 4.13 belongs to the large group of results in modern stochastic PDE (SPDE), motivated by numerical methods, since a popular way to calculate solutions of a white forced equation is to replace it by a kick-forced equation. In numerical analysis this is called the *splitting up method.* For some linear and quasilinear SPDE the rate of convergence for this method can be estimated, see [GK03].

All results of **Chapter 5** are well known and can be found in books (some references are given in the main text). The statement in Section 5.3, which we call the Dobrushin lemma, seems to be a Moscow folklore result from 1960's. Our attempts to trace its inventor failed. Roland Dobrushin was the first who systematically used this result (in a different, but equivalent form) in his celebrated works on the Gibbs systems.

The main result of **Chapter 6**, Theorem 6.4, essentially (i.e., without specifying that the rate of convergence to the stationary measure is exponential) is proved by A. Shirikyan and the author in [KS00] (see also [KS02b]). This paper was written during the year 1999, when its approach was discussed at a number of informal seminars in Heriot-Watt University. At the end of that year the first talk of the results obtained was given at a meeting of Moscow Mathematical Society, and a preprint of the paper appeared. To prove the results we in [KS00] used a Foias–Prodi type reduction of the NSE (3.3) to an N-dimensional system with delay, which is satisfied by the vector, formed by the first N Fourier components of a solution. The new system turned out to be of the Gibbs type (similar systems are considered, say, in [Rue68, Bow75, Sin72]). Since

$$b_s \neq 0 \qquad \forall\, |s| \leq N, \tag{12.1}$$

then the noise, which stirs it, is non-degenerate. Therefore, due to a Ruelle–type theorem (proved in [KS00]) the new system has a unique invariant ("Gibbs") measure, and (3.3) has a unique stationary measure μ. We note that importance of the Gibbs measures for the study of stochastic processes, defined by randomly forced nonlinear PDE, was advocated by Ya. Sinai in 1990's. E.g., see his work [Sin91] on the Burgers equation.

The condition (12.1), which guarantees the non-degeneracy of the reduced system, is crucial for all works on stationary measures for randomly forced PDE's, written after [KS00] up to now (cf. below discussion of the work [HM04]).

In [KS02b], [KS01a] and [KPS02] we developed a coupling-approach to study the NSE and related nonlinear PDE. This approach uses not the Foias–Prodi reduction, but the main lemma the reduction is based upon, which was finally transformed to Lemma 6.1 of this book. It gives a shorter proof of the uniqueness and implies that the convergence to the stationary measure is exponentially fast. The proof was clarified in [Kuk02b], using the technique of the Kantorovich functionals. The proof of Theorem 6.4, presented in Section 6.3, is based on that paper. Theorem 6.4 remains true for the NSE, perturbed by unbounded random kicks, see [KS01b], [Shi04]. A bit later, independently from [KS01a], a similar coupling-approach to study the randomly kicked NSE was developed by Masmoudi–Young [MY02].

Theorem 6.4 and its proof remain valid for a large class of nonlinear PDE, forced by random kicks. In particular, for the equations (0.2)–(0.4) and for the 2D NSE under the Dirichlet boundary conditions. See [KS00], [KS01a], [KPS02] for the corresponding abstract theorems.

Existence of a random attractor for a randomly forced 2D NSE was first established in [CF94]. For more recent results see [Arn98] and references therein.

The main theorem of **Chapter 7**, Theorem 7.1, has a longer history. The first result on the uniqueness of a stationary measure for the white-forced 2D NSE is due to Flandoli and Maslowski [FM95] who established it for equation (7.1), where the force ζ is non-smooth in the space variable x, i.e.

$$C^{-1}|s|^{-1} \leq b_s \leq C|s|^{-\frac{3}{4}-\varepsilon} \quad \forall s, \tag{12.2}$$

with some $\varepsilon > 0$, $C \geq 1$. The result follows by applying a version of the classical Doob theorem to a Markov process, which the equation defines in a Sobolev space of low smoothness. The assumption (12.2) was weakened in [Fer99]. In [BKL01] a similar result was obtained for the case when the estimates (12.2) are replaced by exponential bounds. The mentioned results are not quite satisfactory since it is unnatural to impose a lower bound on the energy of the noise in each Fourier mode.

In [Mat99] the uniqueness of a stationary measure for equation (7.1) was proved in the laminar case $\sqrt{B_0} \ll \nu$.

After the work [KS00] on the kick-forced NSE, E, Mattingly, Sinai [EMS01] and Bricmont, Sprained, Lefevere [BKL02] applied the Foias–Prodi reduction as in [KS00] to study the equation (7.1) for the case when ζ is a finite trigonometrical polynomial,

$$\zeta = \sum_{|s|\le N'} b_s \beta_s(t) e_s(x),$$

satisfying (7.2) (N' is any finite number, bigger than N). In [EMS01] it was proved that the equation has a unique stationary measure, while in [BKL02] it was also established that

$$\mathcal{B}_t f(u) \to (f,\, \mu) \quad \text{exponentially fast}, \tag{12.3}$$

for μ–a.a. $u \in H$ (it was pointed out in [Bri02] that the arguments of [BKL02], in fact, imply the convergence for all u). We note that the uniqueness of a stationary measure is a property of the equation, which is essentially weaker then the exponential mixing (7.3): the former follows from the latter, but does not implies it, as well as it does not imply the consequences from (7.3), discussed in Chapters 8–9.

In [Mat02] the convergence (12.3) was proved for all u. The proof uses the Foias–Prodi reduction, which follows by some coupling arguments. Unfortunately, we found it very difficult to follow the arguments of that work. In the papers [EH01, Hai02] the uniqueness of a stationary measure is obtained for a class of randomly perturbed parabolic problems with strong nonlinear dissipation, including the Ginzburg–Landau equation (0.3).

In [KS02a] it is shown that the ideas, developed earlier in [KS00, KS02a] to study the kicked equations, apply as well in the white–forced case (without the unpleasant restriction that $b_s = 0$ if $|s| > N'$). To implement the ideas some technical lemmas from [EMS01] were used. The scheme of this proof is presented in Section 7.1 (and Theorem 7.1 is the main result of [KS02a]). The results of [KS02a] apply to the 2D NSE both under the periodic and the Dirichlet boundary conditions. A simplified version of the proof see in [Shi05].

A disadvantage of Theorems 7.1 and 6.4 is that the number N of active modes of the force η grows indefinitely as $\nu \to 0$. This assumption is not too restrictive for physical applications since the forces in nature have energy in each Fourier mode, so $b_s \neq 0$ for all s and (12.1) holds for each N. But the assumption is violated in numerical experiments, where usually the force is a final trigonometric polynomial. So it is desirable to have a version of the theorem, where N is a constant, independent of ν. Recently serious progress in this direction was claimed in [HM04]: the main theorem of this work states that eq. (7.1) has a unique stationary measure if $b_s \neq 0$ for all $|s| \le 2$. The proof uses essentially the results of the work [MP04], where it is shown (using the Malliavin calculus) that if $b_s \neq 0$ for all $|s| \le 2$, then any solution $u(t,x)$ of (7.1) is such that the distribution of the vector, formed by its Fourier harmonics $u_s(t)$, $|s| \le N$ (N is the same

as in (12.1)), is absolutely continuous with respect to the Lebesgue measure. For the proof of the uniqueness this result plays a role, similar to that of the assumption (12.1). We did not include the results of [HM04] in the main text since we are waiting till they are properly checked.

Theorem 7.3 is proved in [Kuk02a], and Theorem 7.4 – in [KS03].

Theorem 8.2 in **Chapter 8** is a classical result. Theorems 8.3 and 8.4 were proved in [Kuk02a] and in [Shi06]. We note that the proof of Theorem 8.3, given in Chapter 8, follows from Corollary 6.5. That statement is equivalent to the exponential convergence of distributions of solutions to the stationary measure in the Prokhorov distance (see Exercise 6.7). As it is pointed out at the end of Section 6.3, this convergence, in fact, follows from (6.11).

The CLT from **Chapter 9** was first proved in [Kuk02a] by comparing $f(u(s))$ with $f(U(t))$ where $U(t)$ is a (weak) stationary solution, and evoking the CLT for stationary processes from [Dur91]. The martingale approximation (9.5) is a known tool to study Markov processes and RDS (brought to our attention by Yu. Kifer). The martingale–proof of the theorem is taken from [Shi06], where the rate of convergence to the limit is given. We do not doubt that in (9.2) $\hat\sigma > 0$, but this fact is not proved yet.

The Eulerian limit (Theorem 10.1) from **Chapter 10** was established in [Kuk04]. The results in Section 10.2 are direct consequence from the convergence (10.21), proved in [Bir06]. It is shown in [KS04b] that an analogy of Theorem 10.1 holds for the CGL equation (0.4). More generally, similar results may be established for many other PDEs of the form

$$\langle\,\text{Hamiltonian PDE}\,\rangle + \nu\langle\,\text{damping}\,\rangle = \sqrt{\nu}\,\langle\text{random force}\rangle, \tag{12.4}$$

where the r.h.s. is a kick- or a white–force, and the Hamiltonian PDE has at least two 'good' integrals of motion.

The equation (0.3) has the form (12.4) (without the factor $\sqrt{\nu}$ in its r.h.s.), but the corresponding Hamiltonian PDE is degenerate, as well as its integrals of motion: they contain no partial derivatives of u. The limiting behaviour of solution as $\nu \to 0$ was studies in [Kuk97, Kuk99], using different ideas. In these works lower and upper bounds for high Sobolev norms of stationary solutions were obtained. No lover bounds for $\mathbf{E}\,|u|^2_{L_2}$ are available, and it is not clear how one should scale the force η to keep this quantity ~ 1 as $\nu \to 0$.

The results of **Chapter 11** are very recent. Theorem 11.3 is proved in [KP05] and Theorem 11.5 — in [Kuk05]. The level sets $\Gamma(\tau)$ of the vorticity of a solution for the *deterministic* Navier–Stokes equation in the 2d and 3d cases, and of solutions for equation (11.2) without assuming that $\xi = \operatorname{curl} u$ (but imposing certain a-priori bounds on u and ξ), were studied by P. Constantin and others, e.g see [Con90, CD96]. There the areas of the sets $\Gamma(\tau)$ are estimated (with and without averaging in t and τ), as well as certain integrals over these sets.

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