Anderson Model and Absence of Pure Singular Spectrum *

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Abstract

Absence of singular continuous component, with probability one, in the spectra of random perturbations of multidimensional finite-difference Hamiltonians, is for the first time rigorously established under certain conditions ensuring either absence of point component, or absence of absolutely continuous component in the corresponding regions of spectra. The main technical tool applied is the theory of rank-one perturbations of singular spectra ([AD, STW]).

The respective new result (the non-mixing property) is applied to establish existence and bounds of the (non-empty) pure absolutely continuous component in the spectrum of the Anderson model with bounded random potential in dimension $d = 2$ at low disorder. The same proof holds for $d > 4$.

The new (1999) result implies, via the trace-class perturbation analysis ([SS]), the Anderson model with the unbounded random potential to have only pure point spectrum (complete system of localized wave-functions) with probability one in arbitrary dimension.

The original approximation scheme, based on the resolvent reduction formula, and analogue of the Lippman-Schwinger equations for the generalized eigenfunctions, is applicable in order to establish absence of the singular spectral component (i.e. existence of the non-empty pure absolutely continuous spectral component) in the range of the conductivity spectrum of the arbitrary bounded perturbation in the exactly solvable model, in the surface model, and of some other multidimensional Hamiltonians on $\ell^2(\mathbb{Z}^d)$ and $\mathbb{L}^2(\mathbb{R}^d)$, random and non-random as well.

The new original results imply non-zero value of conductivity in the energy regime corresponding to the high impurity concentration and zero temperature (at low disorder), providing rigorous proof for the Mott conjecture (on existence of localization transition, from the metal-type diffusion, corresponding to the non-empty pure absolutely continuous spectral component at low disorder, to the quantum jump diffusion corresponding to the pure point spectrum at high disorder).

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*personal research of 1999

†no e-mail address, **self alone**, non-slavonic, non-committee's, surname adopted, no relatives, slavonic non-speaking, medical certificate of 1996, never requested any third parties neither to receive, nor to entrust any of my correspondence, nor to communicate on my behalf

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1 Introduction

The purpose of the following paper is to announce the author's new (1999) rigorous proof of absence of pure singular continuous component in the spectra of random perturbations of multidimensional Hamiltonians, and to describe some important applications. The new results imply, in particular, presence of non-empty pure absolutely continuous component in the spectrum of the Anderson tight-binding model at low disorder.

Position of the corresponding mobility edge separating the spectral regions where the eigenstates are localized (Anderson localization of impurity spectrum), and are extended (via delocalization effect of conductivity spectrum), is determined as

$$E_m \sim \inf\left(\sigma(H_A) \cap \sigma(-\Delta)\right) < \infty,$$

depending on the degree of the impurity concentration.

Thus existence of the so-called localization transition, from the metal-type diffusion (corresponding to the non-zero diffusion coefficient at low disorder), to the quantum jump diffusion (corresponding to the zero diffusion coefficient at high disorder), now is rigorously established.

Existence of the corresponding finite critical energy E_m at the high concentration (low disorder) was predicted previously for the Anderson tight-binding Hamiltonian by the physical theory of impurity conduction ([MT, M, A]), and the corresponding hypothesis is known in the physical literature as the Mott conjecture.

In the following paper the new results establish rigorously and for the first time:

A) pure point spectrum in the Anderson model with the unbounded random potential (Appendix B);

B) existence of non-empty pure absolutely continuous component in the spectrum of the Anderson model with random potential with single-site probability distribution having bounded density of compact support (Section 2), in dimension 2 at low disorder.

Similar proofs hold at the same time for $d > 4$.

Some extensions and further generalizations of the new (1999) results are valid for the random and non-random Anderson-type Hamiltonians defined on $\ell^2(\mathbb{Z}^d)$ and $\mathbb{L}^2(\mathbb{R}^d)$ as well, and are supposed to be described by the author's possible forthcoming publication(s). The main technical tool is the approximation scheme based on the generalized eigenfunction's formalism (represented in part by the following paper), and rigorous study of the properties of C.Möller's operator.

Theorem 3 (the new result on absolutely continuous spectrum) is valid, in particular for the disordered surface model ([G6]), disordered exactly solvable model [BG, G4], also in many examples not described by the following paper.

Existence of non-empty pure absolutely continuous component in the spectrum of the Cayley tree ([ICMP]), and of (possibly non-pure) absolutely continuous component in the spectrum of the surface model ([JMP]) had been established by the different methods prior to the results described by the following paper.

The main assumption imposed in order to deduce a.s. absence of the singular continuous component in the spectrum of a random perturbation, is absence either of its point spectrum, or of its absolutely continuous spectrum established á priori. Specifically, the pure absolutely continuous spectrum is rigorously established to exist in the intervals free of the point spectrum of a random (ergodic) Hamiltonian. By the same way, the pure point spectrum is proved to exist in the intervals where the absolutely continuous spectrum is empty.

The main technical tool applied to establish absence of spectra of mixed types is the theory of rank-one perturbations of singular spectra of random Hamiltonians developed by [STW, SW] in order to study the point spectrum in the Anderson tight-binding model, applying some results [AD] had established previously.

In the Anderson tight-binding model, a non-empty pure absolutely continuous spectrum is rigorously established, via proving absence of the mixed point spectrum within its conductivity spectral component, for the bounded random potentials (with single-site probability distribution of compact support and bounded density), satisfying the following condition: if

$$\sup_{q \in \operatorname{supp} dP(q)} (|q| \, + \, |q|^{-1}) \, < \, \infty, \tag{1.1}$$

then

$$\sigma_{pp}(H_A) \, \cap \sigma(H_0) \, = \, \emptyset,$$

where H_A, H_0 are defined by (2.1), (2.2).

As it had been previously established, in the Anderson model, the continuous spectrum vanishes when the disorder parameter increases, or when the degree of the impurity concentration decreases, so when the disorder is sufficiently high (at the low concentration of impurities), the whole spectrum is pure point, with probability 1 ([FS, DLS, STW, SW]).

For the unbounded random (strongly unbounded non-random) potentials (cf. definition in Appendix B), it is rigorously proved (Theorems 7, 8) absence of the absolutely continuous spectrum in arbitrary dimension with probability 1:

$$\sigma_{ac}(H_U) \, = \, \emptyset,$$

which, according to the main new result (Theorems 1, 2), implies that the Anderson tight binding model with unbounded random potential exhibits only pure point spectrum, with probability 1 (Theorem 4).

Consider the random Hamiltonian defined on $\ell^2(\mathbb{Z}^d)$, $d \geq 1$:

$$\overline{H}_\lambda(\omega) \, = \, \overline{H}_0 \, + \, \lambda Q_\omega, \tag{1.2}$$

where the non-perturbed $\overline{H}_0$ is defined by the Laplace operator, and Q_ω is the random perturbation defined by

$$Q(\omega)\psi(x) \, = \, q_\omega(x)\psi(x), \ \ \psi \in \ell^2(\mathbb{Z}^d), \ \ x \in \mathbb{Z}^d, \tag{1.3}$$

$\{q_\omega(x)\}_{x\in \mathbf{Z}^d}$ are independent random variables with identical probability distributions of compact support and bounded density

$$\mathrm{Prob}\{q(0) \in dq\} \;=\; g(q)dq, \;\; g_0^{-1} = \sup_q \, g(q) < \infty. \tag{1.4}$$

The corresponding probability space

$$(\Omega, \mathbb{P}) = \prod_{j\in \mathbf{Z}^d} (\mathbb{Z}_j, dP(q_j)).$$

Theorem 1 (The non-mixing property (A)) *Suppose*

$$\sigma_{pp}(\overline{H}(\omega)) \cap (a, b) \;=\; \emptyset, \tag{1.5}$$

$(a, b) \subset \mathbb{R}$, *with probability 1.*
Then

$$\sigma_{sc}(\overline{H}(\omega)) \cap (a, b) \;=\; \emptyset, \;\; \textit{and} \;\; \sigma(\overline{H}(\omega)) \cap (a, b) \;\subset\; \sigma_{ac}(\overline{H})$$

(i.e. the spectrum of $\overline{H}$ *in* (a, b) *is pure absolutely continuous), with probability 1.*

Theorem 2 (The non-mixing property (B)) *Suppose*

$$\sigma_{ac}(\overline{H}(\omega)) \cap (a, b) \;=\; \emptyset, \tag{1.6}$$

$(a, b) \subset \mathbb{R}$, *with probability 1.*
Then

$$\sigma_{sc}(\overline{H}(\omega)) \cap (a, b) \;=\; \emptyset, \;\; \textit{and} \;\; \sigma(\overline{H}(\omega)) \cap (a, b) \;\subset\; \sigma_{pp}(\overline{H})$$

(i.e. the spectrum of $\overline{H}$ *in* (a, b) *is pure point), with probability 1.*

Remark 1. Theorems 1, 2 hold for arbitrary ergodic self-adjoint $\overline{H}_0 = \overline{H}_0(\overline{\omega})$ on $\ell^2(\mathbb{Z}^d)$, $d \geq 1$.

The main applications of the new general result are described by the following theorems.

Theorem 3 (Absolutely continuous spectrum) *Consider the random operator* $H(\omega)$ *defined on* $\ell^2(\mathbb{Z}^d)$ $(d > 1)$ *by (2.1) -(2.2) and (1.3)-(1.4) (the Anderson model), or by (1.7)-(1.9) (the disordered surface model), with single-site probability distribution having bounded density, satisfying condition (1.1), with probability 1 (Theorem 3 -(2)), or* $\forall \omega \in \Omega$ *(Theorem 3 -(1)). Then:*

(1) *there exists* $\lambda_0 > 0$*, such that if* $0 < \lambda < \lambda_0$*,*

$$\sigma_{ac}(H(\omega)) \;\neq\; \emptyset$$

(i.e. there is non-empty absolutely continuous spectrum at low disorder);

(2)

$$\sigma(H(\omega)) \cap \sigma(H_0) \;\subset\; \sigma_{ac}(H)$$

(i.e. the conductivity component of the spectrum is pure absolutely continuous) with probability 1.

Remark 2. The corresponding "mobility edges" $E_{m\pm}$, separating the (non-random) point and continuous components of the spectrum of $H(\omega)$, are determined by

$$\inf\{\sigma(H(\omega)\} \leq E_{m-} \leq \inf\{\sigma(H_0)\}\} < \sup\{\sigma(H_0)\} \leq E_{m+} \leq \sup\{\sigma(H(\omega)\}.$$

Theorem 4 (Pure point spectrum for unbounded random potential) *Consider Hamiltonian $H(\omega)$ defined on $\ell^2(\mathbb{Z}^d)$ $(d \geq 1)$ by (2.1) -(2.2) and (1.3)-(1.4) (the Anderson model), with random potential (i.i.d.v.). Suppose the single-site probability distribution $dP(q)$ has unbounded support and bounded density, then*

$$\sigma(H(\omega)) = \sigma_{pp}$$

(i.e. the spectrum is pure point), with probability 1.

Remark 3. The condition (1.1) is necessary to ensure a.s. existence of the pure absolutely continuous component in the spectrum of the Anderson model (Section 2) at low disorder, and cannot be weakened, as it is seen via the following examples.

Example 1. Consider the Anderson Hamiltonian with random potential formed by the independent identically distributed random variables (i.i.d.v.) with the Gauss probability distribution:

$$g(q) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(q-m)^2}{2\sigma^2}},$$

$g_0 = \sqrt{2\pi}\sigma$. As it had been previously established, there exist $0 < \overline{g_0} < \infty$, and $0 \leq E_0(g_0) < \infty$, such that $E_0 = 0$, if $g_0 > \overline{g_0}$, and

$$\sigma(H_G) \cap (\pm E_0, \pm\infty) \subset \sigma_{pp}(H_G)$$

(i.e. the impurity spectrum is pure point), with probability 1 ([DLS, SW, STW], 1986). At the other hand, it had been known that the spectrum of H_G is pure point if $d = 1$ ([p1986]).
The new Theorem 4 provides interesting application (strong new result formulated via Example 1), proving that all the spectrum of H_G is pure point for arbitrary value of the disorder parameter g_0, in any dimension $d \geq 1$.

Example 2 (The disordered surface model).
Consider the disordered surface Hamiltonian H_s with the random potential formed by the "subspace" lattice $\mathbb{Z}^\nu$ $(1 \leq \nu \leq d)$ of i.i.d.v.:

$$H_s(\omega) = H_0 + \lambda Q_\omega, \tag{1.7}$$

where H_0 is finite-difference Laplace operator:

$$H_0\Psi(X) = \sum_{\|Y-X\|=1} \Psi(Y), \quad X, Y \in \mathbb{Z}^d, \ \ \Psi \in \ell^2(\mathbb{Z}^d); \tag{1.8}$$

$\sigma(H_0) = [-2d, 2d]$, the potential Q_ω is random multiplication operator on the subspace $\ell^2(\mathbb{Z}^\nu)$, $1 \leq \nu < d$:

$$(Q_\omega\Psi)(x,\xi) \;=\; \delta(x)q_\omega(\xi)\Psi(x,\xi), \tag{1.9}$$

$x \in \mathbb{Z}^{d-\nu}$, $\xi \in \mathbb{Z}^\nu$, $\delta(x) \;=\; \begin{cases} 1, & \text{if } x=0, \\ 0, & \text{otherwise;} \end{cases}$, $\{q_\omega(\xi)\}_{\xi\in\mathbf{Z}^\nu}$ are i.i.d.v. with bounded distribution density of compact support $[-1,1]$.

1. If $\nu < d$, then

$$\sigma_{ac}(H_s(\omega)) \;\neq\; \emptyset \quad \forall\omega\in\Omega.$$

Proof follows via the following expression

$$H_s \;=\; h_0(\mathbb{Z}^{d-\nu})\otimes\mathbf{1}(\mathbb{Z}^\nu) \;+\; \mathbf{1}(\mathbb{Z}^{\mathbf{d}-\nu})\otimes\mathbf{h}_{\mathbf{q}(\omega)} \tag{1.10}$$

(where $h_{q(\omega)} = h_0(\mathbb{Z}^\nu) + \lambda q(\omega)$ denotes operator on $\ell^2(\mathbb{Z}^\nu)$ with the full-space random potential, and $h_0(S)$ denotes the Laplace operator on subspace $\ell^2(S)$, $S\subseteq\mathbb{Z}^d$), since $(\ell^2(\mathbb{Z}^{d-\nu}),\mathbf{0}(\mathbb{Z}^\nu))$ is the invariant subspace of H_s, and

$$(\mathcal{H}_{ac}(h_0),\mathbf{0}(\mathbb{Z}^\nu)) \subset \mathcal{H}_{\mathbf{ac}}(\mathbf{H_s}).$$

2. Suppose $\nu = d$ (i.e. there is considered the Anderson model defined by Hamiltonian H_A with bounded random potential), then there exists $0<\lambda_0<\infty$, such that if $\lambda>\lambda_0$,

$$\sigma(H_0) \;=\; [-2d,2d] \;\subset\; \sigma_{pp}(H_A)$$

(i.e. the spectrum is pure point at high disorder), with probabiliy 1 ([DLS, SW, STW]).

In Example 2, strength of the impurity disorder introduced by the random sources concentrated on a subspace of lower dimension $\nu<d$, is insufficient to ensure localization in the region of the conductivity spectral component.

Remark 4. Theorem 4 holds for arbitrary infinite-order H_0 on $\ell^2(\mathbb{Z}^d)$ satisfying (4.3), if the random potential q_ω satisfies (4.1) (cf. Appendix B).

Remark 5. The results are applicable to study the corresponding spectral properties of the N-particle Hamiltonians, random and non-random as well.

Theorems 1, 2 are proved in Appendix A. Theorem 3 is proved in Section 2 via Theorems 1, 5, 6, except the Statement 1 for the Anderson model with bounded non-random potential, which proof is not presented by the following paper. Theorem 4 is proved in Appendix B via Theorems 2, 7, 8.

The rigorous results presented are in accordance with the physical theory of impurity conduction, and with the previous original research in physics ([A, ET, Il, M, YO]).
These results were new (i.e. were not available in the literature on the subject at the moment), as it had been recognized via private communications by author, and is seen by the corresponding surveys ([S, p1999]).
Remark 6. The wrong statement in [p1999] is nowhere proven.

2 Anderson model and absence of singular spectrum

The Anderson model was initially introduced by P.Anderson [A] in 1958 to model the quantum processes of spin diffusion, impurity conduction, and localization.

The corresponding Hamiltonian is defined by the finite-difference operator

$$H_A(\omega) \;=\; H_0 \;+\; \lambda Q(\omega), \tag{2.1}$$

where H_0 is the Laplace operator

$$H_0\psi(x) \;=\; \sum_{\|x-y\|=1} (\psi(x)-\psi(y)), \;\; \psi \in \ell^2(\mathbb{Z}^d), \; x,y \in \mathbb{Z}^d, \tag{2.2}$$

$\|x\| = \sum\limits_{1\le j\le d} |x_j|$, Q_ω is the random potential defined by (1.3)-(1.4).

Operator H_A is ergodic self-adjoint operator, which spectrum $\sigma(H_A)$, as well as its corresponding point component $\sigma_{pp}(H_A)$, absolutely continuous component $\sigma_{ac}(H_A)$, and singular continuous component $\sigma_{sc}(H_A)$ are non-random subsets of $\mathbb{R}$ ([KS]):

$$\sigma(H_A) = \sigma(H_0)\dot{+}\mathrm{supp}\, dP(\lambda q),$$

where $\dot{+}$ denotes the algebraic sum of subsets of $\mathbb{R}$, and $\sigma(H_0)$ denotes the spectrum of the non-perturbed Laplace operator, which is pure absolutely continuous: $\sigma(H_0) = [0, 4d] = \sigma_{ac}$.

The non-random parameters g_0 (supposing λ is fixed), or λ (supposing g_0 is fixed), are usually used to measure strength of the disorder produced by the impurity sources of random amplitudes, while the value $g_0^{-1}\mathrm{dist}(q_j, q_{j+1})$ is used to denote degree of the concentration of corresponding impurities. In the following there will be convenient to consider g_0 as the disorder parameter, so that "low disorder" means sufficiently small values of g_0, and consequently the "high concentration" (of impurities).

The model considered had been intensively studied in the recent years, and it had been rigorously established via different approximation schemes ([FS, DLS, STW, SW, D, G2, G3, G4]) that the respective spectrum exhibits exponential localization (i.e. it is pure point, and the eigenfunctions decay exponentially at infinity) in the regions of impurity spectrum corresponding to the certain values of disorder parameter (the so-called "high disorder localization"). At the same time the structure of the conductivity spectral component (corresponding to the region of the spectrum of non-perturbed Laplace operator) until recently has not been described in the available literature.

Rigorous study of the corresponding spectral properties was made possible via the new approximation scheme based on the following observation.

The Anderson model may be understood as the limiting case $\nu = d$ of the disordered surface model studied previously by [G3, G6]. This implies, in particular, that all the statements of Theorem 3 are valid, except theorem on existence of non-empty absolutely continuous component in the spectrum of the operator with full-space non-random potential at low disorder, which proof could compose a part of possible separate publication by the author.

Denote by $G(E; x, y) = (H_A - E)^{-1}(x, y)$, $G_0(E; x, y) = (H_0 - E)^{-1}(x, y)$, $x, y \in \mathbb{Z}^d$, the resolvent kernels of operators $H_A(\omega)$, H_0 on $\ell^2(\mathbb{Z}^d)$.

Also denote $\ell^2_{\pm\delta}(\mathbb{Z}^d) = \ell^2(\mathbb{Z}^d, d\mu_{\pm\delta})$, where $d\mu_{\pm\delta}(x) = (1 + |x|)^{\pm 2\delta}dx$, $\delta > \frac{d}{2}$.

The sequence $Q = \{q(\xi) \subset \mathrm{supp}\, dP(q)\}_{\xi\in\mathbf{Z}^d}$ is called an admissible potential. Denote by $\mathcal{A}_Q$ the set of all admissible potentials, and by H_Q operator with the fixed potential $Q = Q(\omega_0) \in \mathcal{A}_Q$ (corresponding to some fixed $\omega = \omega_0 \in \Omega$).

Theorem 5 (General properties of spectrum) *Consider the Anderson model, defined by (2.1)-(2.2), (1.3)-(1.4).*

1) The ergodic property.

$$\sigma(H_A) \;=\; \bigcup_{Q \in \mathcal{A}_Q} \sigma(H_Q).$$

2) The resolvent identity.

A)

Suppose $E \notin \sigma(H_A) \cup \sigma(H_0)$. Then

$$\begin{aligned} G(E;x,y) \;&=\; G_0(E;x,y) \\ &+\; \sum_{\zeta,\mu \in \mathbf{Z}^d} G_0(E;x,\zeta)\,\Gamma_q^{-1}(E;\zeta,\mu)\,G_0(E;\mu,y), \end{aligned}$$

where $x, y \in \mathbb{Z}^d$, and operator $\Gamma_q(E) : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ is defined as follows:

$$\begin{aligned} \Gamma_q(E) \;&=\; -(\lambda q)^{-1} \;-\; h_0(E), \\ h_0(E;\zeta,\mu) \;&=\; G_0(E;\zeta,\mu), \quad \zeta,\mu \in \mathbb{Z}^d. \end{aligned} \tag{2.3}$$

B)

$$\sigma(H_A) \setminus \sigma(H_0) \;=\; \{E \in \mathbb{R} |\, 0 \in \sigma(\Gamma_q(E))\},$$

where $\Gamma_q(E)$ is defined by (2.3)

3) Eigenfunctions.

Suppose

$$\sup_{\xi \in \mathbf{Z}^d} |q(\xi)|^{-1} < \infty.$$

A) *A function $\Psi_E(x)$, $x \in \mathbb{Z}^d$ is the generalized eigenfunction of H_A, corresponding to the generalized eigenvalue $E \in \sigma(H_A)$, if and only if*

$$\Psi_E(x) \;=\; \Phi_E(x) \;+\; \sum_{\zeta \in \mathbf{Z}^d} \varphi_E(\zeta) G_0(E;x,\zeta) \;\in\; \ell^2_{-\delta}(\mathbb{Z}^d),$$

where $\varphi_E \in \ell^2_{-\delta}(\mathbb{Z}^d)$, $x \in \mathbb{Z}^d$, $\delta > \frac{1}{2}$,

$$\Gamma_q(E)\varphi_E(\zeta) \;=\; \Phi_E(\zeta),$$

where $\Gamma_q(E)$ is defined by (2.3), $\Phi_E(x)$ is the distributional solution of the Laplace equation $(H_0 - E)\Phi_E \ = \ 0.$

B) *$E \in \sigma(H_A) \setminus \sigma(H_0)$ is the eigenvalue, and Ψ_E is the corresponding eigenfunction of H_A if and only if*

$$\Psi_E(x) \ = \ \sum_{\zeta \in \mathbf{Z}^d} \varphi_E(\zeta) G_0(E; x, \zeta), \quad x \in \mathbb{Z}^d,$$

where $\Psi_E \in \ell^2(\mathbb{Z}^d)$, and $\varphi_E \in \ell^2(\mathbb{Z}^d)$ is eigenfunction corresponding to the eigenvalue 0 *of* $\Gamma_q(E)$*:*

$$\Gamma_q(E)\varphi_E \ = \ 0.$$

C) *Suppose $E \in \sigma(H_A)$,*

$$\sup_{\xi \in \mathbf{Z}^d} |q(\xi)| < \infty,$$

$$(H_A - E)\Psi_E \ = \ 0. \tag{2.4}$$

Consider

$$\varphi_E \ = \ -\lambda q \Psi_E. \tag{2.5}$$

Then:

I. *$\Psi_E \in \ell^2(\mathbb{Z}^d)$ is a solution of (2.4) (i.e. is the eigenfunction of H_A), if and only if*

$$\varphi_E \ = \ (H_0 - E)\Psi_E \ \in \ \ell^2(\mathbb{Z}^d). \tag{2.6}$$

II. *$\Psi_E \in \ell^2(\mathbb{Z}^d)$ is a solution of (2.4) (the eigenfunction of H_A), if and only if for some $z \notin \sigma(H_A) \cup \sigma(H_0)$,*

$$\Psi_E \ = \ \Phi_z \ + \ (H_0 - z)^{-1}\varphi_E, \tag{2.7}$$

where

$$\Phi_z \ = \ (E - z)(H_0 - z)^{-1}\Psi_E \ \in \ \ell^2(\mathbb{Z}^d). \tag{2.8}$$

4) The point spectrum.

A) Geometrical structure.

$$\sigma_{pp}(H_A) \cap (\sigma(H_A) \setminus \sigma(H_0)) \ = \ \{E \in \mathbb{R} |\ 0 \in \sigma_{pp}(\Gamma_q(E))\},$$

where $\Gamma_q(E)$ is defined by (2.3).

B) Absence of the mixed point spectrum.

Suppose

$$\sup_{x\in \mathbf{Z}^d} (|q(x)| + |q(x)|^{-1}) \ < \ \infty. \tag{2.9}$$

Then

$$\sigma_{pp}(H_q) \cap \sigma(H_0) \ = \ \emptyset.$$

C) Localization at high disorder.

Consider the disordered Anderson model defined by the random Hamiltonian (2.1) -(2.2), (1.3) -(1.4).

Given $\varepsilon > 0$ there exist $\delta_0(\varepsilon) > 0$, and $E_0 = E_0(\delta) > 2d$, $E_0(\delta_0) = 2d + \varepsilon$, such that

$$\sigma(H_A) \ \cap \ (\pm E_0, \pm\infty) \ = \ \sigma_{pp}$$

(i.e. at high disorder impurity spectrum is pure point), and the corresponding eigenfunctions decay exponentially fast at infity, with probability 1. The point spectrum is non-empty (i.e. $E_0 \in (2d, \sup\{\sigma(H_A)\})$, if $\delta > \delta_0(E_0)$.

Theorem 6 (Pure absolutely continuous spectrum)
Suppose (2.9) holds with probability 1. Then

$$\sigma(H_A) \cap \sigma(H_0) \ \subset \ \sigma_{ac}(H_A)$$

(i.e. the spectrum of H_A in $[0, 4d]$ is pure absolutely continuous), with probability 1.

Theorem 6 is a consequence of Theorem 5-4b) and Theorem 1.
Proof of Theorem 5 . Statement 2a) (the resolvent identity) is proved in [G3].
Proofs of 3a), b), 2b), 4a) are analogous to the proofs of Theorems 3a, 3b, and of Corollary 4 of [BG, G4].
Theorem 5-4c) is proved in [G3], and could be seen as some strong version of the theorem on existence of the pure point spectrum in the Anderson model. It implies bounds for the mobility edges and admits certain extension to cover the weak disorder localization ([G8]).
Theorem 5- 1) is proved in [G6].
Proof of Theorem 5-3c)
I. The statement (2.6) is a trivial consequence of (2.4) and (2.5).
II. Relations (2.7) and (2.5) imply

$$\begin{aligned}(H_A - z)\Psi_E \ &= \ (H_A - z)\Phi_z \ + \ (H_A - z)(H_0 - z)^{-1}\varphi_E \\ &= \ (H_0 - z)\Phi_z \ + \ \lambda q(\Phi_z + (H_0 - z)^{-1}\varphi_E) \ + \ \varphi_E \\ &= \ (H_0 - z)\Phi_z \ + \ \lambda q \Psi_E \ + \ \varphi_E \\ &= \ (H_0 - z)\Phi_z.\end{aligned} \tag{2.10}$$

Since (2.8) implies

$$(H_0 - z)\Phi_z = (E - z)\Psi_E$$
$$= (H_A - z)\Psi_E - (H_A - E)\Psi_E,$$

(2.10) holds if and only if (2.4) is valid. Theorem 5-3c) is proved.
Remark 7. Theorem 5-4b) is proved in the following paper in full details for $d = 2$ only (because of the reasons explained by Remark 6). The same approximation scheme (with precision to the values of the dimension-dependent coefficients) step-by-step could be repeated to verify the results for arbitrary $d \geq 5$. The results are valid also if $d \geq 5$.

Proof of Theorem 5-4b), $d = 2$.
Suppose the potential Q satisfies (2.9). Consider $\Psi_E \in \ell^2(\mathbb{Z}^2)$, the eigenfunction of H_A corresponding to eigenvalue $E \in \sigma(H_0)$:

$$(H_A - E)\Psi_E = 0.$$

Theorem 5-3c) implies

$$\Psi_E(x) = \Phi_z(x) + \sum_{\zeta \in \mathbf{Z}^d} G_z^0(x,\zeta)\varphi_E(\zeta)$$
$$= \Phi_z(x) + (H_0 - z)^{-1}\varphi_E(x) \in \ell^2(\mathbb{Z}^2), \quad x \in \mathbb{Z}^2,$$

where

$$\Phi_z(x) = (E - z)(H_0 - z)^{-1}\,\Psi_E \in \ell^2(\mathbb{Z}^2),$$

and

$$\varphi_E = -\lambda q \Psi_E \in \ell^2(\mathbb{Z}^2),$$

$z \in \mathbb{R} \setminus (\sigma(H_A) \cup \sigma(H_0))$.
Denote

$$\begin{aligned}
J_{++} &= \{(j_1, j_2) \in \mathbb{Z}^2 |\, j_1 \geq 0, j_2 \geq 0\},\\
J_{+-} &= \{(j_1, j_2) \in \mathbb{Z}^2 |\, j_1 \geq 1, j_2 \leq -1\},\\
J_{-+} &= \{(j_1, j_2) \in \mathbb{Z}^2 |\, j_1 \leq -1, j_2 \geq 1\},\\
J_{--} &= \{(j_1, j_2) \in \mathbb{Z}^2 |\, j_1 \leq 0, j_2 \leq 0\},
\end{aligned}$$

$$\varphi_{\pm\pm}(j) = \begin{cases} \varphi(j), & \text{if } j \in J_{\pm\pm}, \\ 0, & \text{otherwise} \end{cases},$$

$$\begin{aligned}
\Delta_{++} &= \{(p_1, p_2) \in \mathbb{C}^2 |\, \Im p_1 \leq 0,\ \Im p_2 \leq 0\},\\
\Delta_{+-} &= \{(p_1, p_2) \in \mathbb{C}^2 |\, \Im p_1 \leq 0,\ \Im p_2 \geq 0\},\\
\Delta_{-+} &= \{(p_1, p_2) \in \mathbb{C}^2 |\, \Im p_1 \geq 0,\ \Im p_2 \leq 0\},\\
\Delta_{--} &= \{(p_1, p_2) \in \mathbb{C}^2 |\, \Im p_1 \geq 0,\ \Im p_2 \geq 0\},
\end{aligned}$$

$$E(p,j) \;=\; e^{-i(\Re p_1 j_1 + \Re p_2 j_2)}\, e^{\Im p_1 j_1 + \Im p_2 j_2}.$$

By $\hat\varphi_{\pm\pm}(p)$, $p \in \mathbb{R}^2$, denote the Fourier transform of $\varphi_{\pm\pm}(j) \in \ell^2(\mathbb{Z}^2)$ ($\hat\varphi_{\pm\pm}(p) \in \mathbb{L}^2(\mathbb{R}^2)$ by the Riesz-Fischer theorem [RF]).
Denote by $\mathcal{H}(\Delta)$ the set of functions holomorfic in $\Delta \subset \mathbb{C}^2$, by $\widetilde\Delta_{\pm\pm}$ an open subset of $\overline{\Delta}_{\pm\pm}(\epsilon) = \{p \in \Delta_{\pm\pm} : \; |p| \le \epsilon < \infty\}$.
Since

$$\begin{aligned}
\sum_{j\in J_{\pm\pm}} |E(p,j)\varphi_{\pm\pm}(j)| \;&\le\; (\sum_{(j_1,j_2)\in J_{\pm\pm}} e^{-2(|\Im p_1 j_1| + |\Im p_2 j_2|)})^{\frac{1}{2}} \,\|\varphi_{\pm\pm}\| \\
&< \;\infty, \quad p \in \Delta_{\pm\pm} \subset \mathbb{C}^2,
\end{aligned} \tag{2.11}$$

it follows that $\hat\varphi_{\pm\pm}(p)$ may be continued to $\mathcal{H}(\widetilde\Delta_{\pm\pm})$ by the Osgood lemma, since the series (2.11) converges absolutely and uniformly in each compact $\overline{\Delta}_{\pm\pm} \subset \mathbb{C}^2$ to define continuous in $\widetilde\Delta_{\pm\pm}$ function which is holomorphic in each separate variable $p_j \in \widetilde\Delta^j_{\pm\pm}$, $j = 1, 2$.
Denote

$$\Psi_{\pm\pm}(E;j) \;=\; \begin{cases} \Psi_E(j), & \text{if } j \in J_{\pm\pm}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\Psi_{\pm\pm} \in \ell^2(\mathbb{Z}^2)$ and $\hat\Psi_{\pm\pm}(p) \in \mathcal{H}(\widetilde\Delta_{\pm\pm})$, it follows by Theorem 5-3c) that if q satisfies (2.9) then

$$\begin{aligned}
\varphi_{\pm\pm}(E;j) \;&=\; -\lambda q \Psi_{\pm\pm}(E;j) \\
&=\; (H_0 - E)\Psi(E;j) \\
&=\; (H_0 - E)\Psi_{\pm\pm}(E,j) \\
&+\; \delta(j_1 \mp 1)\Psi_{\pm\pm}(\pm 1, j_2) + \delta(j_2 \mp 1)\Psi_{\pm\pm}(j_1, \pm 1) \\
&+\; \delta(j_1)\,(\Psi_{\pm\pm}(0, j_2+1) \,+\, \Psi_{\pm\pm}(0, j_2-1) \,-\, 3\Psi_{\pm\pm}(0, j_2)) \\
&+\; \delta(j_2)\,(\Psi_{\pm\pm}(j_1+1, 0) \,+\, \Psi_{\pm\pm}(j_1-1, 0) \,-\, 3\Psi_{\pm\pm}(j_1, 0)).
\end{aligned} \tag{2.12}$$

By passing to the Fourier transform in (2.12) ([Hd]),

$$\begin{aligned}
2\pi\hat\varphi_{\pm\pm}(p) \;&=\; 2\pi(|p|^2 - E)\,\hat\Psi_{\pm\pm}(p) \\
&+\; e^{-i(p_1(j_1 \mp 1) + p_2 j_2)}\,\Psi_{\pm\pm}(\pm 1, j_2) \\
&+\; e^{-i(p_1 j_1 + p_2(j_2 \mp 1))}\,\Psi_{\pm\pm}(j_1, \pm 1) \\
&+\; e^{-ip_2(j_2+1)}\,\Psi_{\pm\pm}(0, j_2+1) \\
&+\; e^{-ip_2(j_2-1)}\,\Psi_{\pm\pm}(0, j_2-1) \\
&-\; 3e^{-ip_2 j_2}\,\Psi_{\pm\pm}(0, j_2) \\
&+\; e^{-ip_1(j_1+1)}\,\Psi_{\pm\pm}(j_1+1, 0) \\
&+\; e^{-ip_1(j_1-1)}\,\Psi_{\pm\pm}(j_1-1, 0) \\
&-\; 3e^{-ip_1 j_1}\,\Psi_{\pm\pm}(j_1, 0), \quad (j_1, j_2) \in J_{\pm\pm}.
\end{aligned} \tag{2.13}$$

It follows that

$$\hat{\Psi}_{\pm\pm}(p) \;=\; \frac{\Theta_{\pm\pm}(p)}{|p|^2 - E} \in \mathcal{H}(\widetilde{\Delta}_{\pm\pm}), \tag{2.14}$$

where $\Theta_{\pm\pm}(p)$ is defined by (2.13). Choose $\widetilde{\Delta}_{\pm\pm}$: $E \in \widetilde{\Delta}_{\pm\pm} \subset \Delta_{\pm\pm}$. Since $\hat{\Psi}_{\pm\pm}(p)$ is bounded in $\widetilde{\Delta}_{\pm\pm}$, (2.14) implies

$$\Theta_{\pm\pm}(p)_{|p|^2-E=0;\, p\in\widetilde{\Delta}_{\pm\pm}} \;=\; 0. \tag{2.15}$$

Since $\hat{\varphi}_{\pm\pm}(p) \in \mathcal{H}(\widetilde{\Delta}_{\pm\pm})$, and $e^{-ipj} \in \mathcal{H}(\widetilde{\Delta}_{\pm\pm})$, $j \in \mathbb{Z}^2$, it follows that

$$\Theta_{\pm\pm}(p) \;\in\; \mathcal{H}(\widetilde{\Delta}_{\pm\pm}).$$

Now since

$$\Delta_{\pm\pm} \setminus (\Delta^0_{\pm\pm} \stackrel{\text{def}}{=} \{p \in \Delta_{\pm\pm} :\; |p|^2 - E = 0\})$$

is the non-connected domain, it follows by the Riemann theorem (the unique continuation property for holomorphic functions on C^n, $n > 1$), that $\Delta^0_{\pm\pm}$ could serve as the zero-surface for the holomorhic function $\Theta_{\pm\pm}(p) \in \mathcal{H}(\widetilde{\Delta}_{\pm\pm})$, $\Delta^0_{\pm\pm} \subset \widetilde{\Delta}_{\pm\pm}$, only if this function equals identically to 0 ([GRo] Theorem 2, Part II/E):

$$\Theta_{\pm\pm}(p) \;\equiv\; 0, \;\; p \in \widetilde{\Delta}_{\pm\pm}. \tag{2.16}$$

Relations (2.14) and (2.16) imply

$$\hat{\Psi}_{\pm\pm}(p) \;\equiv\; 0, \;\; p \in \widetilde{\Delta}_{\pm\pm},$$

which implies

$$\Psi_{\pm\pm}(E) \;\equiv\; 0. \tag{2.17}$$

Since (2.17) is established for arbitrary combination of index $\pm\pm$, it follows

$$\Psi_E \;\equiv\; 0.$$

Theorem 5-3c) is proved, if $d = 2$. $\square$

3 Appendix A. Absence of pure singular continuous spectrum

Proof of Theorems 1, 2. Denote by ψ the unit vector in $\mathbb{Z}^d$, by $\mathcal{L}(A)$ is the Lebesgue measure of $\mathcal{L}$-measurable set $A \subset \mathbb{R}$, $d\rho_0 = d\rho(\overline{H}(\omega_0), \psi)$ the spectral measure of $\overline{H}(0) = \overline{H}(\omega_0)$, $\omega_0 \in \Omega$, associated with ψ (e.g. there is considered operator defined by (1.2) with the fixed value of admissible potential $q(\omega_0)$),

$$\begin{aligned} \mathcal{F}_0(z) \;&=\; \int\limits_{\mathbf{R}} \frac{d\rho_0(\lambda)}{\lambda - z} \\ &=\; \langle \psi, (\overline{H}(0) - z)^{-1} \psi \rangle, \end{aligned} \tag{3.1}$$

where $z \notin \sigma(\overline{H})$ ($z = x + i\varepsilon$, $\varepsilon \neq 0$), $(\overline{H}(0) - z)^{-1}$ denotes the resolvent of $\overline{H}(0)$. $\mathcal{F}_0(z)$ is called the Stiltjes transform of the spectral measure $d\rho_0$:

$$\Im \mathcal{F}_0(z) = \int\limits_{\mathbf{R}} \frac{\varepsilon d\rho_0(\lambda)}{(\lambda - x)^2 + \varepsilon^2}, \tag{3.2}$$

$$\Re \mathcal{F}_0(z) = \int\limits_{\mathbf{R}} \frac{(\lambda - x) d\rho_0(\lambda)}{(\lambda - x)^2 + \varepsilon^2}. \tag{3.3}$$

Denote by $d\rho_\gamma = d\rho(\overline{H}(\gamma), \psi)$ the spectral measure of

$$\overline{H}(\gamma) = \overline{H}(0) + \gamma\langle ., \psi\rangle\psi,$$

associated with ψ. If ψ is cyclic for $\overline{H}(0)$ (i.e. the set of finite linear combinations of $\{\overline{H}(0)^n\}_{n=1}^{\infty}$ is dense in $\ell^2(\mathbb{Z}^d)$), then ψ is also cyclic for $\overline{H}(\gamma)$, $\gamma \in \operatorname{supp} dP(q)$.
It follows by (3.1) and by the rank-one perturbation formula for the resolvent the Stiltjes transform of $d\rho_\gamma$ satisfies

$$\mathcal{F}_\gamma(z) = \frac{\mathcal{F}_0(z)}{1 + \gamma \mathcal{F}_0(z)}, \tag{3.4}$$

$\Im z \neq 0$.
Consider also

$$B_0(x) = (\int\limits_{\mathbf{R}} \frac{d\rho_0(\lambda)}{(\lambda - x)^2})^{-1}, \tag{3.5}$$

then $0 \leq B_0(x) < \infty$, $x \in \sigma(\overline{H}(0))$.
Since $\gamma\langle ., \psi\rangle\psi$ is a rank-one perturbation,

$$\operatorname{supp} d\rho^{ac}(\gamma) = \operatorname{supp} d\rho_0^{ac},$$

and

$$\operatorname{supp} d\rho_0^{ac} \cap (a, b) \subseteq \{x \in (a, b) | \Im F_0(x) > 0\}.$$

(A) Choose ω_0 such that

$$\sigma_{pp}(\overline{H}(\omega_0)) \cap (a, b) = \emptyset.$$

Since

$$d\rho_0^{pp}(a, b) = 0,$$

and

$$\mathcal{L}\{\operatorname{supp} d\rho_0^{sc}\} = 0,$$

it follows

$$
\begin{aligned}
& \mathcal{L}\,\{(a,b)\,\cap\,\operatorname{supp} d\rho_0\,\setminus\,S_{reg}\} \\
= \quad & \mathcal{L}\{x\in(a,b)\,\cap\,\operatorname{supp} d\rho_0|\,\Im\mathcal{F}_0(x+i0)\;=\;B_0(x+i0)\;=\;0\} \\
\leq \quad & \mathcal{L}\{\,\operatorname{supp} d\rho_0^{sc}\,\cap\,(a,b)\,\}\;+\;\mathcal{L}\{\,\operatorname{supp} d\rho_0^{pp}\,\cap\,(a,b)\,\} \\
= \quad & 0,
\end{aligned}
\tag{3.6}
$$

where S_{reg} is defined by 3.14.
Proposition 1 (Appendix A) and (3.6) imply

$$
d\rho_\gamma^{sc}(a,b)\;=\;0 \tag{3.7}
$$

for $\mathcal{L}$- a.e. $\gamma\neq 0$.
(3.7) implies

$$
\begin{aligned}
& \mathbb{P}\{\omega|\; d\rho^{sc}(\overline{H}(\omega))\{(a,b)\}\;\neq\;0\} \\
\leq \quad & \int\limits_{\prod\limits_{j\neq j_0}\mathbf{R}_j} g_0^{-1}\,\mathcal{L}\{\gamma\in\mathbb{R}|\,d\rho_\gamma^{sc}\{(a,b)\}\;\neq\;0\}\,dP(q_j) \\
= \quad & 0.
\end{aligned}
\tag{3.8}
$$

Since ψ may be chosen arbitrary over the dense in $\ell^2(\mathbb{Z}^d)$ set of basis vectors $\{e(j)\}_{j\in\mathbf{Z}^d}$, (3.7) and (3.8) imply

$$
\sigma_{sc}(\overline{H}(\omega))\;\cap\;(a,b)\;=\;\emptyset \tag{3.9}
$$

with probability 1. (1.5) and (3.9) imply

$$
\sigma(\overline{H}(\omega))\;\cap\;(a,b)\;\subset\;\sigma_{ac}(\overline{H})
$$

with probability 1. Theorem 1 is proved.
(B) Choose ω_0 such that

$$
\sigma_{ac}(\overline{H}(\omega_0))\cap(a,b)\;=\;\emptyset.
$$

Proposition 2 (Appendix A) implies

$$
\begin{aligned}
& \mathcal{L}\,\{(a,b)\,\cap\,\operatorname{supp} d\rho_0\,\setminus\,\mathcal{B}\} \\
= \quad & \mathcal{L}\{x\in(a,b)\,\cap\,\operatorname{supp} d\rho_0|\,B_0(x+i0)\;=\;0\} \\
\leq \quad & \mathcal{L}\,\{(a,b)\,\cap\,\operatorname{supp} d\rho_\gamma^{ac}\}\;+\;\mathcal{L}\,\{(a,b)\,\cap\,\operatorname{supp} d\rho_\gamma^{sc}\} \\
= \quad & 0,
\end{aligned}
\tag{3.10}
$$

where $\mathcal{B}$ is defined by 3.13, since

$$
\mathcal{L}\{\operatorname{supp} d\rho_\gamma^{sc}\}\;=\;0,
$$

$$
d\rho_\gamma^{ac}(a,b)\;=\;0,
$$

$\gamma \neq 0$.
Proposition 1 (Appendix A) and (3.10) imply

$$d\rho_\gamma(a,b) \;=\; d\rho_\gamma^{pp}(a,b)$$

for $\mathcal{L}$- a.e. $\gamma \neq 0$, hence

$$d\rho_\gamma^{sc}(a,b) \;=\; 0 \tag{3.11}$$

for $\mathcal{L}$- a.e. $\gamma \neq 0$.
Since $dP(q)$ has bounded density, (3.11) implies

$$\sigma_{sc}(\overline{H}(\omega)) \;\cap\; (a,b) \;=\; \emptyset \tag{3.12}$$

with probability 1. Relations (1.6) and (3.12) imply

$$\sigma(\overline{H}(\omega)) \;\cap (a,b) \;\subset\; \sigma_{pp}(\overline{H})$$

with probability 1. Theorem 2 is proved. □

The following propositions are the previously established results of [STW, SW].

Proposition 1. **A**.
Denote

$$\mathcal{B} \;=\; \{x \in (a,b) |\; B_0(x) > 0\} \;=\; 0. \tag{3.13}$$

Then

$$\mathcal{L}\{\mathcal{B}\} \;=\; 0$$

if and only if

$$d\rho_\gamma^{\rm pp}(a,b) \;=\; 0$$

for $\mathcal{L}$- a.e. $\gamma \neq 0$.

B.
Denote

$$S_{\rm reg} \;=\; \{x \in (a,b) |\; \Im\, \mathcal{F}_0(x+i0) + B_0(x) > 0\}. \tag{3.14}$$

Then

$$\mathcal{L}\{\mathrm{supp}\, d\rho_0 \cap (a,b) \setminus S_{\rm reg}\} \;=\; 0,$$

if and only if

$$d\rho_\gamma^{sc}(a,b) \;=\; 0$$

for $\mathcal{L}$- a.e. $\gamma \neq 0$.

A measure $d\rho$ is said to be supported on $A \subset \mathbb{R}$, if $d\rho(\mathbb{R} \setminus A) = 0$ (i.e. $\mathrm{supp}\, d\rho \subseteq A$).

Proposition 2. ([AD, SW]).
Suppose $\gamma \neq 0$, then

(A)

$$\operatorname{supp} d\rho_\gamma^{ac} \cap (a,b) \subseteq \mathcal{A} \overset{\text{def}}{=} \{x \in (a,b) | \Im \mathcal{F}_0(x+i0) > 0\}$$

(i.e. $d\rho_\gamma^{ac}$ is supported on $\mathcal{A}$),

(B)

$$\operatorname{supp} d\rho_\gamma^{pp} \cap (a,b) \subseteq \mathcal{B} \overset{\text{def}}{=} \{x \in (a,b) | B_0(x) > 0\}$$

(i.e. $d\rho_\gamma^{pp}$ is supported on $\mathcal{B}$),

(C)

$$\operatorname{supp} d\rho_\gamma^{sc} \cap (a,b) \subseteq \mathcal{C} \overset{\text{def}}{=} \operatorname{supp} d\rho_\gamma \cap (a,b) \setminus \{\mathcal{A} \cup \mathcal{B}\}$$

(i.e. $d\rho_\gamma^{sc}$ is supported on $\mathcal{C}$).

Proposition 3. ([SW])
Define

$$\eta(\Delta) = \int_{\mathbf{R}} \frac{\rho_\gamma(\Delta)\, d\gamma}{1+\gamma^2},$$

where Δ is ρ_γ - measurable subset of $\mathbb{R}$, $d\rho_0(\Delta) \neq 0$.
Then $d\eta$ is mutually equivalent to the Lebesgue measure.

Propositions 1 - 3 are proved in [SW] (cf. also [G6]).

4 Appendix B. The pure point spectrum for unbounded random potential

While the pure point spectrum had been established to appear in particular in the Anderson model at the edges of its impurity spectral component (the so-called strong disorder localization phenomenon [A, FS, DLS, SW, D, G2, G3, G4], etc.), the pure absolutely continuous spectrum is established to exist within the conductivity component of the corresponding spectrum (the so-called low disorder delocalization effect), in particular in the Anderson model with the bounded random potential (having single-site probability distribution of bounded density and compact support, Theorem 6, Section 2).
At the same time, it is well-known that self-adjoint operators (on $\ell^2(\mathbb{Z}^d)$, or $\mathbb{L}^2(\mathbb{R}^d)$) with growing potential have no absolutely continuous spectrum, for example, if $\lim\limits_{|x|\to\infty} |q(x)| = \infty$, then the spectrum is pure discrete ([Gl]).
In the following section there is proved the analogues theorem for the multidimensional finite-difference operators with random (ergodic) potential (Theorem 4). All the spectrum is established to be pure point with probability 1 assuming that the random potential has unbounded support.
Consider the Anderson tight-binding Hamiltonian H_U defined by (2.1) - (2.2) with the random potential having single-site probability distribution of unbounded support:

$$\begin{aligned} \sup_q \operatorname{supp} dP(q) &= +\infty \quad \text{or} \\ \inf_q \operatorname{supp} dP(q) &= -\infty. \end{aligned} \tag{4.1}$$

Theorem 4 is a consequence of Theorem 7 and of Theorem 2.

Theorem 7 Absence of absolutely continuous spectrum for unbounded random potential

$$\sigma_{ac}(H_U) = \emptyset$$

with probability 1.

The proof follows from the result for the strongly unbounded non-random potentials (Theorem 8). Consider the operator H defined on $\ell^2(\mathbb{Z}^d)$ by (2.1)-(2.2), and assume

$$\limsup_{\Lambda_{L_j} \nearrow \infty} \inf_{x \in \partial \Lambda_{L_j}} |q(x)| = \infty, \tag{4.2}$$

where $\Lambda_{L_j} = \{x \in \mathbb{Z}^d : \|x\| \leq L_j\}$, i.e. the potential $q(x)$ is unbounded over increasing to infinity sequence of concentric spheres $\Lambda_{L_j} \subset \Lambda_{L_{j+1}}$ of radius L_j, $j \in \mathbb{N}$.

Definition. *The potential satisfying (4.2) is referred in the following paper as the strongly unbounded.*

Theorem 8 Absence of absolutely continuous spectrum for the strongly unbounded potential

$$\sigma_{ac}(H) = \emptyset.$$

Remark 8. Theorem 8 holds for arbitrary infinite-order operator $H = H_0 + q$ on $\ell^2(\mathbb{Z}^d)$, with off-diagonal part satisfying

$$H_0(x,y) = H_0(x-y), \quad |H_0(x)| \leq C|x|^{-(d+\varepsilon)}, \tag{4.3}$$

$x, y \in \mathbb{Z}^d$, for some $\varepsilon > 0$, if the non-random potential q is such that that there exist sequences $\{l_n\}_{n \in \mathbf{N}}$, $l_n > 0$, and $\{L_n\}_{n \in \mathbf{N}}$, $L_n > 0$:

$$\limsup_{\Lambda_{L_n} \nearrow \infty} \inf_{x \in \partial^*_{l_n}(\Lambda_{L_n})} |q(x)| = \infty, \tag{4.4}$$

$$\lim_{n \to \infty} l_n = \infty,$$

$$\lim_{n \to \infty} L_n = \infty,$$

$$\partial^*_l(\Lambda_L) = \{x \in \mathbb{Z}^d : \frac{L-l}{2} \leq |x| \leq \frac{L}{2}\}. \tag{4.5}$$

Remark 9. Theorem 7 holds for arbitrary infinite-order operator H on $\ell^2(\mathbb{Z}^d)$ with random potential satisfying (4.1), (4.3).

Theorem 8 (the main technical result involved to establish Theorem 4) is the extension for the case $d > 1$ of the result on absence of the absolutely continuous spectrum of the one-dimensional Jacoby matrix with the unbounded diagonal potential ([SS], 1989), it is explicitly proved in [G6]. The generalization of this result for the case of one-dimensional finite-difference operator of infinite order (Remark 8, $d = 1$) could be found in [G1].
Proof of Theorem 7. Denote as before by $(\Omega, \mathcal{S}, \mathbb{P})$ the probability space of realizations of the random potential (1.3)-(1.4), where $\mathcal{S}$ denotes the σ- algebra of $\mathbb{P}$- measurable subsets of Ω. Consider the sequence $L_{n+1} > L_n > 0$, $n \in \mathbb{N}$, and denote

$$\Omega(b, \Lambda_l) \;=\; \{q \in \mathcal{A}_Q| \inf_{x \in \partial\Lambda_l} |q(x)| \geq b\} \;\in\; \mathcal{S},$$

$$\Omega_n \;=\; \bigcup_{\Lambda_{L_n} \subset \Lambda_{L_{n+1}}} \Omega(b_n, \Lambda_{L_n}) \;\in\; \mathcal{S}.$$

Then

$$\mathbb{P}\{\Omega_n\} \;\geq\; \frac{L_{n+1}}{L_n}\, \mathbb{P}\{\Omega(b_n, \Lambda_{L_n})\}. \tag{4.6}$$

So choose $\{L_n\}_{n \in \mathbf{N}}$:

$$L_{n+1} \;\geq\; \frac{1}{|n| dP\{(\pm b_n, \pm\infty)\}^{C_d L_n^{d-1}}}\, L_n.$$

Condition (4.1) implies that it is possible to choose $\{b_n\}_{n \in \mathbf{N}}$:

$$\lim_{n \to \infty} |b_n| \;=\; \infty,$$
$$dP\{(\pm b_n, \pm\infty)\} \;\neq\; 0. \tag{4.7}$$

Then (4.6), (4.7) imply

$$\sum_{n \to \infty} \mathbb{P}\{\Omega_n\} \;=\; \infty. \tag{4.8}$$

Now Borel-Kantelli lemma via (4.8) implies

$$\mathbb{P}\{\overline{\Omega} = \bigcap_{n \geq 0} \bigcup_{k \geq n} \Omega_n\} \;=\; 1,$$

i.e. the potential Q_ω satisfying (4.1), is strongly unbounded with probability 1.
Theorem 8 implies

$$\sigma_{ac}(H_U) \;=\; \emptyset$$

holds with probability 1. Theorem 7 is proved. $\square$

5 Appendix C. Authorship's proofs

A.

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Received: from uran.kharkiv.net (relay.kharkiv.net [194.44.156.30])
by burda.kharkiv.net (8.9.3/8.9.1/burda) with ESMTP id BAA20721
for ¡vgr@burda.kharkiv.net¿; Wed, 1 Mar 2000 01:04:56 +0200 (EET)
Received: from dune.kharkiv.net (dune.kharkiv.net [194.44.156.50])
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CONFERENCE PROGRAM AS OF FEB 29, 2000

International Conference on
Differential Geometry and Quantum Physics
Berlin, March 6–10, 2000.

Organized by Sonderforschungsbereich 288,
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Scientific Program

****

Grinshpun, Vadim Fri Dec 31 12:28:44 MET 1999

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Absolutely Continuous Spectrum in the Tight Binding Anderson Model

It is proved that the operator $H = H_0 + \lambda Q$, acting on $\ell^2(Z^d)$, $d \geq 2$, where H_0 is the finite-difference Laplace operator, and Q is the multiplication operator (independent random variables with identical distribution P of compact support), exhibits pure absolutely continuous spectrum at weak disorder: $\sigma_{ac}(H) \neq \emptyset$ with probablity 1, if $0 < \lambda \leq \lambda_1(d, P)$.

It is proved, by applying the analogue of the multiscale analysis scheme, that $\sigma(H) \setminus \sigma(H_0 \pm \varepsilon) \subset \sigma_{pp}(H)$ with probablity 1 for $0 < \lambda \leq \lambda_2(d, P)$, if P is Hölder continuous of finite order $\rho > 0$.

It was established (1984) that $\sigma(H) = \sigma_{pp}(H)$ if $\lambda \geq \bar{\lambda}(d, P)$, and $\sigma(H) \setminus (-\bar{E}, \bar{E}) \subset \sigma_{pp}(H)$ for $\bar{E} = \bar{E}(d, P)$.

Thus it is proved that there occurs the transition from the absolutely continuous spectrum to the pure point spectrum, when the disorder increases, and the bounds on the respective mobility edges are found. The rate of decay of the localized states at the weak disorder is estimated, and the representation of the localized and extended states is obtained.

Remark. More extended abstracts were sent to the official representatives of the conference (Profs. V.Bach and J.Brüning) on December 10, 1999 via e-mail (computer room N 205, Math. Dept., ILTP). Entrance to eu was not permitted in german embassy on Febr 29, 2000. Prepublication was refused because of "absence of presentation" via e-mail on March 14-15, 2000.

FROM: ¡client@..ua¿
TO: najanas@cyf-kr.edu.pl, stolz@math.uab.edu,
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DATE: 23 December2003, 21:06:15
SUBJECT: From: Dr. V.Grinshpun

Official application for personal presentation at Conference OTAMP-2004.
Possible titles (Generalized Eigenfunctions and Absence of Singular Spectrum).
(no relation to ILTP, "INTAS" since 1999, no e-mail address, undelivery, postal address is to be changed).

Conference "Operator Theory & Applications in Mathematical Physics"
Mathematical Research & Conference Center
July 6-11, 2004
Bedlewo, Poland

Chairman Scientific Programme
Fourth ECM-2004, Stockholm, Sweden

This is my application for personal presentation of my new (1999) research results (1)-(2):

(1) "Generalized Eigenfunctions and Absence of Singular Spectrum of Some Multidimensional Operators",
and/or
(2) "On Properties of Essential Spectrum of Schroedinger Operator with Zero-range (Surface) Random Potential in Dimensions Two and Three".

I suppose to provide my new postal address (required for letter of invitation) with my official registration form later. I suppose to pay my conference fees via onsite registration, if possible.

Yours sincerely,

Dr. V.Grinshpun

Remark. Abstracts were sent (officially submitted) on May 4, 2004. There had not been received any correspondence neither from organizers of ECM-2004 (sweden), nor from organizers of OTAMP (satellite conference, poland).

B.
From vgr@online.kharkiv.com Wed Jan 5 16:52:53 2000 +0200
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Date: Wed, 5 Jan 2000 16:52:01 +0200 (EET)
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To: pastur@ilt.kharkov.ua, lpastur@ilt.kharkov.ua
cc: grinshpun@ilt.kharkov.ua
Subject: absence of permition
Message-ID: ¡Pine.BSF.4.10.10001051616001.22065-100000@tiny.office.kharkiv.com¿

Dear Prof. L.Pastur,
as I wrote in the e-mail letter of November 23, I proved existence of the continuous (pure absolutely continuous) spectrum in the tight binding Anderson model, and also for some other random operators. I promised you the abstract of the corresponding results. Please, let me know where and when it could be done. I am enclosing a copy of my request for permition to enter the institute, in order to be able to finish the research (i.e. to submit the papers for publication). If you agree in principle to sign it, then I will have the possibility to follow further official procedure. If you do not accept it, please, let me know your official reason. Please, let me know if it is required to offer the Curriculum Vitae, list of publications, a copy of the PhD thesis, letter from INTAS, research report, research project, abstract of my recent reasults concerning absolutely continuous spectrum of some random operators, etc.

Yours sincerely,
Dr. V.Grinshpun

Professor L.Pastur
Chief of Department N24
Institute of Low Temperature Physics

I would like to request to issue me a permition to enter the Institute, from January 4, till September 30, 2000, to finish the research concerning spectral properties of some random operators.

Dr. V.Grinshpun

From lpastur@fint.ilt.kharkov.ua Thu Jan 6 12:24:38 2000
for ¡vgr@burda.kharkiv.net¿; Thu, 6 Jan 2000 12:24:38 +0200 (EET)
for ¡vgr@online.kharkiv.com¿; Thu, 6 Jan 2000 12:24:31 +0200 (EET)
Date: Thu, 6 Jan 2000 12:24:21 +0200 (EET)
From: "Leonid A. Pastur" ¡lpastur@fint.ilt.kharkov.ua¿
To: "V.Grinshpun" ¡vgr@online.kharkiv.com¿

Subject: Re: absence of permition
In-Reply-To: ¡Pine.BSF.4.10.10001051616001.22065-100000@tiny.office.kharkiv.com¿
Message-ID: ¡Pine.BSI.4.05L.10001061218210.23504-100000@fint.ilt.kharkov.ua¿
Status: RO

Dear Dima,
I believe that the question concerning your pass to the Institute for January 2000 is settled in the sense that Khruslov and myself made all necessary official moves. As I have explained you while giving you the "zayavka", signed by Khruslov and myself, the only thing that you have to do now is to give me your passport and the "zayavka" and I ask somebody from our Department to contact respective person, issuing passes.

Best regards
L.Pastur

C.
Date:Thu, 30 Mar 2006 21:10:14 +0100 (BST)
From:"V. Grinshpun" ¡v gn n@yahoo.com¿
Subject: paper submission
To:CommMathPhys@caltech.edu
CC:CommMathPhys@caltech.edu
The Editor
"Communications in Mathematical Physics"
Springer
Dear Editor,
the following is my letter of submission of my paper
"On Absence of Pure Singular Spectrum of Multidimensional Random Hamiltonians at Low Disorder".

I announced my new results via e-mail on November 24, 1999, and via telephone call on March 2, 2001 (Prof. E.Lieb, IAMP).

I have had no opportunity to submit for publication my mentioned research results since 1999: when my (enclosed) paper had been under preparation for submission, my e-mail box at Institute for Low Temperature Physics (Kharkov, Ukraine, Warsaw Pact) was suddenly closed, and permit for entrance was denied (not prolonged). Later my invited exit to the international conference (Berlin, EU, March 2000) was not permitted without any reason explained. I also had no financial opportunity to present my research at ICMP-2000 (UK), the corresponding proceedings contain the wrong reference to my attendance (XIII ICMP proceedings, UK (International Press), p.490).

This is why I would be grateful if You could confirm receipt of my letter of submission via current (temporary) e-mail address.

The paper is enclosed as file-attachments (LaTeX 73.571 Kb, dvi 97.68 Kb) The printed copy of the paper should be received via air-mail in April.
Sincerely yours,
Dr V.Grinshpun

Remark. The printed copy of paper [G6] with very few corrections in the text (supplied by all required detailed original proofs) is supposed to be sent via air-mail upon receipt of respective inquire of the referee.

Remark.

IAMP dues were paid in euro in 2005.
ICMP-2006 fees were paid at the minimal rate ($65) officially permitting author's participation. The only possible transfer via private author's account was made on May 22, 2006 to the official bank address.
Comments in russian were demanded to make the transfers from the private usd-accounts in local banks (karaganda, kazakhstan).

Acknowledgements. Personal research (1999, kharkov ("ukraine", kyev-russe, former-soviet union, warsaw pact since 1955)), typesetted in December 2005 in former-soviet union (pact of May 9-11, 2005) on private PC (intel pentium II (korea), OS Windows XP Certificate Authenticity (Microsoft corp) 00049-120-546-750, N09-01178, X10-60277, no internet access), with possible unauthorized illegal external access by united former-soviet ko-gb, was supported only by research grant "AMS-1995" ($150).
United former-soviet **KO-gb (former-"cossacks", slavonic ([Ph]) military forces in XV-XX centuries)**,
"INTAS", and hannover have no rights in any use of the following research because of support of slavonic-based national communism and "political" repressions against minoritie(s) of non-cyrillic alphabet(s) of standard orientation.
The author's education within USSR was made possible in part thanks to the efforts of the Communist Party of Ukrainian Soviet Socialist Republic, which had been employed in the USSR.
VG would like to request excuse for not answering to the e-mail correspondence could had arrived to his previous e-mail address (grinshpun@ilt.kharkov.ua): mentioned e-mail box was closed, and permit for entrance to the host-keeping institution was denied (not prolonged) by institute for low temperature physics, kharkov ("ukraine") on December 31 (1999), when the following paper had been under preparation for submission for publication by author.
He had had no opportunity to present his described in part personal research results at ICMP XIII, the wrong reference in [ICMP2000].
Exit to the international conference (berlin, "eu", March 2000) was not permitted in "eu" embassy, kyev ("ukraine"), on Febr 29, 2000.

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