May 14, 2006

Conditional Proof of the Boltzmann-Sinai Ergodic Hypothesis (Assuming the Hyperbolicity of Typical Singular Orbits)

Nándor Simányi[1]

University of Alabama at Birmingham
Department of Mathematics
Campbell Hall, Birmingham, AL 35294 U.S.A.
E-mail: simanyi@math.uab.edu

Dedicated to Yakov G. Sinai and Domokos Szász

Abstract. We consider the system of N (≥ 2) elastically colliding hard balls of masses $m_1, \dots, m_N$ and radius r on the flat unit torus $\mathbb{T}^\nu$, $\nu \geq 2$. We prove the so called Boltzmann-Sinai Ergodic Hypothesis, i. e. the full hyperbolicity and ergodicity of such systems for every selection $(m_1, \dots, m_N; r)$ of the external geometric parameters, under the assumption that almost every singular trajectory is geometrically hyperbolic (sufficient), i. e. the so called Chernov-Sinai Ansatz holds true for the model. The present proof does not use at all the formerly developed, rather involved algebraic techniques, instead it employs exclusively dynamical methods and tools from geometric analysis.

Primary subject classification: 37D50

Secondary subject classification: 34D05

[1]Research supported by the National Science Foundation, grant DMS-0457168.

§1. Introduction

In this paper we prove the Boltzmann–Sinai Ergodic Hypothesis under the condition of the Chernov-Sinai Ansatz (see §2). In a loose form, as attributed to L. Boltzmann back in the 1880's, this hypothesis asserts that gases of hard balls are ergodic. In a precise form, which is due to Ya. G. Sinai in 1963 [Sin(1963)], it states that the gas of $N \geq 2$ identical hard balls (of "not too big" radius) on a torus $\mathbb{T}^\nu$, $\nu \geq 2$, (a ν-dimensional box with periodic boundary conditions) is ergodic, provided that certain necessary reductions have been made. The latter means that one fixes the total energy, sets the total momentum to zero, and restricts the center of mass to a certain discrete lattice within the torus. The assumption of a not too big radius is necessary to have the interior of the configuration space connected.

Sinai himself pioneered rigorous mathematical studies of hard ball gases by proving the hyperbolicity and ergodicity for the case $N = 2$ and $\nu = 2$ in his seminal paper [Sin(1970)], where he laid down the foundations of the modern theory of chaotic billiards. Then Chernov and Sinai extended this result to ($N = 2$, $\nu \geq 2$), as well as proved a general theorem on "local" ergodicity applicable to systems of $N > 2$ balls [S-Ch(1987)]; the latter became instrumental in the subsequent studies. The case $N > 2$ is substantially more difficult than that of $N = 2$ because, while the system of two balls reduces to a billiard with strictly convex (spherical) boundary, which guarantees strong hyperbolicity, the gases of $N > 2$ balls reduce to billiards with convex, but not strictly convex, boundary (the latter is a finite union of cylinders) – and those are characterized by very weak hyperbolicity.

Further development has been mostly due to A. Krámli, D. Szász, and the present author. We proved hyperbolicity and ergodicity for $N = 3$ balls in any dimension [K-S-Sz(1991)] by exploiting the "local" ergodic theorem of Chernov and Sinai [S-Ch(1987)], and carefully analyzing all possible degeneracies in the dynamics to obtain "global" ergodicity. We extended our results to $N = 4$ balls in dimension $\nu \geq 3$ next year [K-S-Sz(1992)], and then I proved the ergodicity whenever $N \leq \nu$ [Sim(1992)-I-II] (this covers systems with an arbitrary number of balls, but only in spaces of high enough dimension, which is a restrictive condition). At this point, the existing methods could no longer handle any new cases, because the analysis of the degeneracies became overly complicated. It was clear that further progress should involve novel ideas.

A breakthrough was made by Szász and myself, when we used the methods of algebraic geometry [S-Sz(1999)]. We assumed that the balls had arbitrary masses $m_1, \dots, m_N$ (but the same radius r). Now by taking the limit $m_N \to 0$, we were able to reduce the dynamics of N balls to the motion of $N - 1$ balls, thus utilizing a natural induction on N. Then algebro-geometric methods allowed us to effectively analyze all possible degeneracies, but only for typical (generic) $(N + 1)$-tuples of "external" parameters $(m_1, \dots, m_N, r)$; the latter needed to avoid some exceptional submanifolds of codimension one, which remained unknown. This approach led to a proof of full hyperbolicity (but not yet ergodicity) for all $N \geq 2$ and $\nu \geq 2$, and for generic $(m_1, \dots, m_N, r)$, see [S-Sz(1999)]. Later the present author simplified the arguments and made them more "dynamical", which allowed me to obtain full hyperbolicity for hard balls with any set of external geometric parameters $(m_1, \dots, m_N, r)$ [Sim(2002)]. Thus, the hyperbolicity has been fully established for all systems of hard balls on tori.

To upgrade the full hyperbolicity to ergodicity, one needs to refine the analysis of the aforementioned degeneracies. For hyperbolicity, it was enough that the degeneracies made a subset of codimension ≥ 1 in the phase space. For ergodicity, one has to show that its codimension is ≥ 2, or to find some other ways to prove that the (possibly) arising codimension-one manifolds of non-sufficiency are incapable of separating distinct ergodic components. The latter approach will be pursued in this paper. In the paper [Sim(2003)] I took the first step in the direction of proving that the codimension of exceptional manifolds is at least two: I proved that the systems of $N \geq 2$ balls on a 2D torus (i.e., $\nu = 2$) are ergodic for typical (generic) $(N+1)$-tuples of external parameters $(m_1, \dots, m_N, r)$. The proof again involves some algebro-geometric techniques, thus the result is restricted to generic parameters $(m_1, \dots, m_N; r)$. But there was a good reason to believe that systems in $\nu \geq 3$ dimensions would be somewhat easier to handle, at least that was indeed the case in early studies.

Finally, in my recent paper [Sim(2004)] I was able to further improve the algebro-geometric methods of [S-Sz(1999)], and proved that for any $N \geq 2$, $\nu \geq 2$ and for almost every selection $(m_1, \dots, m_N; r)$ of the external geometric parameters the corresponding system of N hard balls on $\mathbb{T}^\nu$ is (fully hyperbolic and) ergodic.

In this paper I will prove the following result.

Theorem. For any integer values $N \geq 2$, $\nu \geq 2$, and for every $(N+1)$-tuple $(m_1, \dots, m_N, r)$ of the external geometric parameters the standard hard ball system $\left(\mathbf{M}_{\vec{m},r}, \left\{S^t_{\vec{m},r}\right\}, \mu_{\vec{m},r}\right)$ is (fully hyperbolic and) ergodic, provided that the so Chernov-Sinai Ansatz (see the closing part of §2 below) is true for $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ and for all of its subsystems.

Remark 1.1. The novelty of the theorem (as compared to the result in [Sim(2004)]) is that it applies to each $(N+1)$-tuple of external parameters (provided that the interior of the phase space is connected), without an exceptional zero-measure set.

Remark 1.2. The present result speaks about exactly the same models as the result of [Sim(2002)], but the assertion of this new theorem is obviously stronger than that of the theorem in [Sim(2002)]: It has been known for a long time that, for the family of semi-dispersive billiards, ergodicity cannot be obtained without also proving full hyperbolicity.

Remark 1.3. As it follows from the results of [C-H(1996)] and [O-W(1998)], all standard hard ball systems $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ (the models covered by the theorem) are not only ergodic, but they enjoy the Bernoulli mixing property, as long as they are known to be ergodic.

Remark 1.4. The reason for assuming the Ansatz not only for the considered model $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ but also for all of its subsystems is the inductive nature of the proof, see §5.

The Organization of the Paper. In the subsequent section we overview the necessary technical prerequisites of the proof, along with the needed references to the literature. The fundamental objects of this paper are the so called "exceptional J-manifolds": they are codimension-one submanifolds of the phase space that are separating distinct, open ergodic components of the billiard flow. In §3 we obtain the necessary linear estimations for the contraction coefficients of some specific tangent vectors in small, tubular neighborhoods of exceptional manifolds. By using the results of §3, in §4 we prove that at least one phase point of an exceptional J-manifold is actually sufficient (Main Lemma 4.5).

Finally, in the closing section we complete the inductive proof of ergodicity (with respect to the number of balls N) by utilizing Main Lemma 4.5 and earlier results from the literature. Actually, a consequence of Main Lemma 4.5 will be that exceptional J-manifolds do not exist, and this will imply the the fact that no distinct, open ergodic components can coexist.

Finally, a short appendix of this paper serves the purpose of making the reading of the proof of §4 easier, by providing a chart of the hierarchy of the selection of several constants playing a role in the proof of Main Lemma 4.5.

§2. Prerequisites

Consider the ν-dimensional ($\nu \geq 2$), standard, flat torus $\mathbb{T}^\nu = \mathbb{R}^\nu/\mathbb{Z}^\nu$ as the vessel containing N (≥ 2) hard balls (spheres) $B_1, \dots, B_N$ with positive masses $m_1, \dots, m_N$ and (just for simplicity) common radius $r > 0$. We always assume that the radius $r > 0$ is not too big, so that even the interior of the arising configuration space $\mathbf{Q}$ (or, equivalently, the phase space) is connected. Denote the center of the ball B_i by $q_i \in \mathbb{T}^\nu$, and let $v_i = \dot{q}_i$ be the velocity of the i-th particle. We investigate the uniform motion of the balls $B_1, \dots, B_N$ inside the container $\mathbb{T}^\nu$ with half a unit of total kinetic energy: $E = \dfrac{1}{2}\sum_{i=1}^N m_i||v_i||^2 = \dfrac{1}{2}$. We assume that the collisions between balls are perfectly elastic. Since — beside the kinetic energy E — the total momentum $I = \sum_{i=1}^N m_i v_i \in \mathbb{R}^\nu$ is also a trivial first integral of the motion, we make the standard reduction $I = 0$. Due to the apparent translation invariance of the arising dynamical system, we factorize the configuration space with respect to uniform spatial translations as follows: $(q_1, \dots, q_N) \sim (q_1 + a, \dots, q_N + a)$ for all translation vectors $a \in \mathbb{T}^\nu$. The configuration space $\mathbf{Q}$ of the arising flow is then the factor torus $\left((\mathbb{T}^\nu)^N / \sim\right) \cong \mathbb{T}^{\nu(N-1)}$ minus the cylinders

$$C_{i,j} = \left\{(q_1, \dots, q_N) \in \mathbb{T}^{\nu(N-1)} \colon \operatorname{dist}(q_i, q_j) < 2r\right\}$$

$(1 \le i < j \le N)$ corresponding to the forbidden overlap between the i-th and j-th spheres. Then it is easy to see that the compound configuration point

$$q = (q_1, \dots, q_N) \in \mathbf{Q} = \mathbb{T}^{\nu(N-1)} \setminus \bigcup_{1 \le i < j \le N} C_{i,j}$$

moves in $\mathbf{Q}$ uniformly with unit speed and bounces back from the boundaries $\partial C_{i,j}$ of the cylinders $C_{i,j}$ according to the classical law of geometric optics: the angle of reflection equals the angle of incidence. More precisely: the post-collision velocity v^+ can be obtained from the pre-collision velocity v^- by the orthogonal reflection across the tangent hyperplane of the boundary $\partial\mathbf{Q}$ at the point of collision. Here we must emphasize that the phrase "orthogonal" should be understood with respect to the natural Riemannian metric (the kinetic energy) $||dq||^2 = \sum_{i=1}^N m_i ||dq_i||^2$ in the configuration space $\mathbf{Q}$. For the normalized Liouville measure μ of the arising flow $\{S^t\}$ we obviously have $d\mu = \text{const} \cdot dq \cdot dv$, where dq is the Riemannian volume in $\mathbf{Q}$ induced by the above metric, and dv is the surface measure (determined by the restriction of the Riemannian metric above) on the unit sphere of compound velocities

$$\mathbb{S}^{\nu(N-1)-1} = \left\{(v_1, \dots, v_N) \in (\mathbb{R}^\nu)^N \colon \sum_{i=1}^N m_i v_i = 0 \text{ and } \sum_{i=1}^N m_i ||v_i||^2 = 1\right\}.$$

The phase space $\mathbf{M}$ of the flow $\{S^t\}$ is the unit tangent bundle $\mathbf{Q} \times \mathbb{S}^{d-1}$ of the configuration space $\mathbf{Q}$. (We will always use the shorthand notation $d = \nu(N-1)$ for the dimension of the billiard table $\mathbf{Q}$.) We must, however, note here that at the boundary $\partial\mathbf{Q}$ of $\mathbf{Q}$ one has to glue together the pre-collision and post-collision velocities in order to form the phase space $\mathbf{M}$, so $\mathbf{M}$ is equal to the unit tangent bundle $\mathbf{Q} \times \mathbb{S}^{d-1}$ modulo this identification.

A bit more detailed definition of hard ball systems with arbitrary masses, as well as their role in the family of cylindric billiards, can be found in §4 of [S-Sz(2000)] and in §1 of [S-Sz(1999)]. We denote the arising flow by $(\mathbf{M}, \{S^t\}_{t \in \mathbb{R}}, \mu)$.

In the series of articles [K-S-Sz(1989)], [K-S-Sz(1991)], [K-S-Sz(1992)], [Sim(1992-I)], and [Sim(1992-II)] the authors developed a powerful, three-step strategy for proving the (hyperbolic) ergodicity of hard ball systems. First of all, these proofs are inductions on the number N of balls involved in the problem. Secondly, the induction step itself consists of the following three major steps:

Step I. To prove that every non-singular (i. e. smooth) trajectory segment $S^{[a,b]}x_0$ with a "combinatorially rich" (in a well defined sense) symbolic collision sequence is automatically sufficient (or, in other words, "geometrically hyperbolic", see below in this section), provided that the phase point x_0 does not belong to a countable union J of smooth sub-manifolds with codimension at least two. (Containing the exceptional phase points.)

The exceptional set J featuring this result is negligible in our dynamical considerations — it is a so called slim set. For the basic properties of slim sets, see again below in this section.

Step II. Assume the induction hypothesis, i. e. that all hard ball systems with N' balls ($2 \leq N' < N$) are (hyperbolic and) ergodic. Prove that there exists a slim set $S \subset \mathbf{M}$ with the following property: For every phase point $x_0 \in \mathbf{M} \setminus S$ the entire trajectory $S^{\mathbb{R}}x_0$ contains at most one singularity and its symbolic collision sequence is combinatorially rich, just as required by the result of Step I.

Step III. By using again the induction hypothesis, prove that almost every singular trajectory is sufficient in the time interval $(t_0, +\infty)$, where t_0 is the time moment of the singular reflection. (Here the phrase "almost every" refers to the volume defined by the induced Riemannian metric on the singularity manifolds.)

We note here that the almost sure sufficiency of the singular trajectories (featuring Step III) is an essential condition for the proof of the celebrated Theorem on Local Ergodicity for semi-dispersive billiards proved by Chernov and Sinai [S-Ch(1987)]. Under this assumption, the result of Chernov and Sinai states that in any semi-dispersive billiard system a suitable, open neighborhood U_0 of any sufficient phase point $x_0 \in \mathbf{M}$ (with at most one singularity on its trajectory) belongs to a single ergodic component of the billiard flow $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$.

In an inductive proof of ergodicity, steps I and II together ensure that there exists an arc-wise connected set $C \subset \mathbf{M}$ with full measure, such that every phase point $x_0 \in C$ is sufficient with at most one singularity on its trajectory. Then the cited Theorem on Local Ergodicity (now taking advantage of the result of Step III) states that for every phase point $x_0 \in C$ an open neighborhood U_0 of x_0 belongs to one ergodic component of the flow. Finally, the connectedness of the set C and $\mu(\mathbf{M} \setminus C) = 0$ imply that the flow $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ (now with N balls) is indeed ergodic, and actually fully hyperbolic, as well.

The subsets $\mathbf{M}^0$ and $\mathbf{M}^{\#}$. Denote by $\mathbf{M}^{\#}$ the set of all phase points $x \in \mathbf{M}$ for which the trajectory of x encounters infinitely many non-tangential collisions in both time directions. The trajectories of the points $x \in \mathbf{M} \setminus \mathbf{M}^{\#}$ are lines: the motion is linear and uniform, see the appendix of [Sz(1994)]. It is proven in lemmas A.2.1 and A.2.2 of [Sz(1994)] that the closed set $\mathbf{M} \setminus \mathbf{M}^{\#}$ is a finite union of hyperplanes. It is also proven in [Sz(1994)] that, locally, the two sides of a hyper-planar component of $\mathbf{M} \setminus \mathbf{M}^{\#}$ can be connected by a positively measured beam of trajectories, hence, from the point of view of ergodicity, in this paper it is enough to show that the connected components of $\mathbf{M}^\#$ entirely belong to one ergodic component. This is what we are going to do in this paper.

Denote by $\mathbf{M}^0$ the set of all phase points $x \in \mathbf{M}^\#$ the trajectory of which does not hit any singularity, and use the notation $\mathbf{M}^1$ for the set of all phase points $x \in \mathbf{M}^\#$ whose orbit contains exactly one, simple singularity. According to Lemma 4.1 of [K-S-Sz(1990)-I], the set $\mathbf{M}^\# \setminus (\mathbf{M}^0 \cup \mathbf{M}^1)$ is a countable union of smooth, codimension-two (≥ 2) submanifolds of $\mathbf{M}$, and, therefore, this set may be discarded in our study of ergodicity, please see also the properties of slim sets above. Thus, we will restrict our attention to the phase points $x \in \mathbf{M}^0 \cup \mathbf{M}^1$.

The "Chernov-Sinai Ansatz". An essential precondition for the Theorem on Local Ergodicity by Chernov and Sinai [S-Ch(1987)] is the so called "Chernov-Sinai Ansatz" which we are going to formulate below. Denote by $\mathcal{SR}^+ \subset \partial\mathbf{M}$ the set of all phase points $x_0 = (q_0, v_0) \in \partial\mathbf{M}$ corresponding to singular reflections (a tangential or a double collision at time zero) supplied with the post-collision (outgoing) velocity v_0. It is well known that $\mathcal{SR}^+$ is a compact cell complex with dimension $2d-3 = \dim\mathbf{M}-2$. It is also known (see Lemma 4.1 in [K-S-Sz(1990)-I]) that for ν_1-almost every phase point $x_0 \in \mathcal{SR}^+$ the forward orbit $S^{(0,\infty)}x_0$ does not hit any further singularity. (Here ν_1 is the Riemannian volume of $\mathcal{SR}^+$ induced by the restriction of the natural Riemannian metric of $\mathbf{M}$.) The Chernov-Sinai Ansatz postulates that for ν_1-almost every $x_0 \in \mathcal{SR}^+$ the forward orbit $S^{(0,\infty)}x_0$ is sufficient (geometrically hyperbolic).

The Theorem on Local Ergodicity. The Theorem on Local Ergodicity for semi-dispersive billiards (Theorem 5 of [S-Ch(1987)]) claims the following: Let $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ be a semi-dispersive billiard flow with the property that the smooth components of the boundary $\partial\mathbf{Q}$ of the configuration space are algebraic hypersurfaces. (The cylindric billiards automatically fulfill this algebraicity condition.) Assume – further – that the Chernov-Sinai Ansatz holds true, and a phase point $x_0 \in \left(\mathbf{M}^0 \cup \mathbf{M}^1\right) \setminus \partial\mathbf{M}$ is sufficient.

Then some open neighborhood $U_0 \subset \mathbf{M}$ of x_0 belongs to a single ergodic component of the flow $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$. (Modulo the zero sets, of course.)

A few years ago Bálint, Chernov, Szász, and Tóth [B-Ch-Sz-T(2002)] discovered that, in addition, the algebraic nature of the scatterers needs to be assumed, in order for the proof of this result to work. Fortunately, systems of hard balls are, by nature, automatically algebraic.

§3. Expansion and Contraction Rate Estimates

We would like to get a useful lower estimate for the expansion of a tangent vector $(\delta q_0, \delta v_0) \in \mathcal{T}_{x_0}\mathbf{M}$ with positive infinitesimal Lyapunov function $Q(\delta q_0, \delta v_0) =$

$\langle \delta q_0, \delta v_0 \rangle$. The expression $\langle \delta q_0, \delta v_0 \rangle$ is the scalar product in $\mathbb{R}^d$ defined via the mass (or kinetic energy) metric, see §2. It is also called the infinitesimal Lyapunov function associated with the tangent vector $(\delta q_0, \delta v_0)$, see [K-B(1994)], or part A.4 of the Appendix in [Ch(1994)], or §7 of [Sim(2003)]. For a detailed exposition of the relationship between the quadratic form $Q(\,.\,)$, the relevant symplectic geometry of the Hamiltonian system and the dynamics, please also see [L-W(1995)].

Note. The original idea of using infinitesimal Lyapunov exponents to measure the expansion rate of codimension-one submanifolds in the phase space of semi-dispersive billiards came from N. Chernov back in the late '80s. These ideas have been explored in detail and further developed by him and myself in recent personal communications, so that we obtained at least linear (but uniform!) expansion rates for such submanifolds with negative infinitesimal Lyapunov exponents for their normal vector. These results will be presented in our forthcoming joint paper [Ch-Sim(2006)], and they will be used later in this article in the proof of Lemma 5.2. Also, closely related to the above said, the following ideas (to estimate the expansion rates of tangent vectors from below) are derived from the thoughts being published in [Ch-Sim(2006)].

Denote by $(\delta q_t, \delta v_t) = (DS^t)(\delta q_0, \delta v_0)$ the image of the tangent vector $(\delta q_0, \delta v_0)$ under the linearization DS^t of the map S^t, $t \geq 0$. (We assume that the base phase point x_0 — for which $(\delta q_0, \delta v_0) \in \mathcal{T}_{x_0}\mathbf{M}$ — has a non-singular forward orbit.) The time-evolution $(\delta q_{t_1}, \delta v_{t_1}) \mapsto (\delta q_{t_2}, \delta v_{t_2})$ $(0 \leq t_1 < t_2)$ on a collision free segment $S^{[t_1,t_2]}x_0$ is described by the equations

$$\begin{aligned} \delta v_{t_2} &= \delta v_{t_1}, \\ \delta q_{t_2} &= \delta q_{t_1} + (t_2 - t_1)\delta v_{t_1}. \end{aligned} \tag{3.1}$$

Correspondingly, the change $Q(\delta q_{t_1}, \delta v_{t_1}) \mapsto Q(\delta q_{t_2}, \delta v_{t_2})$ in the infinitesimal Lyapunov function $Q(\,.\,)$ on the collision free orbit segment $S^{[t_1,t_2]}x_0$ is

$$Q(\delta q_{t_2}, \delta v_{t_2}) = Q(\delta q_{t_1}, \delta v_{t_1}) + (t_2 - t_1)||\delta v_{t_1}||^2, \tag{3.2}$$

thus $Q(\,.\,)$ steadily increases between collisions.

The passage $(\delta q_t^-, \delta v_t^-) \mapsto (\delta q_t^+, \delta v_t^+)$ through a reflection (i. e. when $x_t = S^t x_0 \in \partial\mathbf{M}$) is given by Lemma 2 of [Sin(1979)] or formula (2) in §3 of [S-Ch(1987)]:

$$\begin{aligned} \delta q_t^+ &= R\delta q_t^-, \\ \delta v_t^+ &= R\delta v_t^- + 2\cos\phi R V^* K V \delta q_t^-, \end{aligned} \tag{3.3}$$

where the operator $R : \mathcal{T}\mathbf{Q} \to \mathcal{T}\mathbf{Q}$ is the orthogonal reflection (with respect to the mass metric) across the tangent hyperplane $\mathcal{T}_{q_t}\partial\mathbf{Q}$ of the boundary $\partial\mathbf{Q}$ at the configuration component q_t of $x_t=(q_t,v_t^{\pm})$, $V:(v_t^-)^{\perp}\to\mathcal{T}_{q_t}\partial\mathbf{Q}$ is the v_t^--parallel projection of the orthocomplement hyperplane $(v_t^-)^{\perp}$ onto $\mathcal{T}_{q_t}\partial\mathbf{Q}$, $V^*:\mathcal{T}_{q_t}\partial\mathbf{Q}\to(v_t^-)^{\perp}$ is the adjoint of V (i. e. the $\nu(q_t)$-parallel projection of $\mathcal{T}_{q_t}\partial\mathbf{Q}$ onto $(v_t^-)^{\perp}$, where $\nu(q_t)$ is the inner normal vector of $\partial\mathbf{Q}$ at $q_t\in\partial\mathbf{Q}$), $K:\mathcal{T}_{q_t}\partial\mathbf{Q}\to\mathcal{T}_{q_t}\partial\mathbf{Q}$ is the second fundamental form of the boundary $\partial\mathbf{Q}$ at q_t (with respect to the field $\nu(q)$ of inner unit normal vectors of $\partial\mathbf{Q}$) and, finally, $\cos\phi=\langle\nu(q_t),\,v_t^+\rangle>0$ is the cosine of the angle ϕ ($0\le\phi<\pi/2$) subtended by v_t^+ and $\nu(q_t)$. Regarding formulas (3.3), please see the last displayed formula in §1 of [S-Ch(1987)] or (i)–(ii) in Proposition 2.3 of [K-S-Sz(1990)-I]. The instantaneous change in the infinitesimal Lyapunov function $Q(\delta q_t,\delta v_t)$ caused by the reflection at time $t>0$ is easily derived from (3.3):

$$\begin{aligned}Q(\delta q_t^+,\delta v_t^+)&=Q(\delta q_t^-,\delta v_t^-)+2\cos\phi\langle V\delta q_t^-,\,KV\delta q_t^-\rangle\\&\ge Q(\delta q_t^-,\delta v_t^-).\end{aligned}\tag{3.4}$$

In the last inequality we used the fact that the operator K is positive semi-definite, i. e. the billiard is semi-dispersive.

We are primarily interested in getting useful lower estimates for the expansion rate $||\delta q_t||/||\delta q_0||$. The needed result is

Proposition 3.5. Use all the notations above, and assume that

$$\langle\delta q_0,\,\delta v_0\rangle/||\delta q_0||^2\ge c_0>0.$$

We claim that $||\delta q_t||/||\delta q_0||\ge 1+c_0t$ for all $t\ge0$.

Proof. Clearly, the function $||\delta q_t||$ of t is continuous for all $t\ge0$ and continuously differentiable between collisions. According to (3.1), $\frac{d}{dt}\delta q_t=\delta v_t$, so

$$\frac{d}{dt}||\delta q_t||^2=2\langle\delta q_t,\delta v_t\rangle.\tag{3.6}$$

Observe that not only the positive valued function $Q(\delta q_t,\delta v_t)=\langle\delta q_t,\delta v_t\rangle$ is nondecreasing in t by (3.2) and (3.4), but the quantity $\langle\delta q_t,\delta v_t\rangle/||\delta q_t||$ is nondecreasing in t, as well. The reason is that $\langle\delta q_t,\delta v_t\rangle/||\delta q_t||=||\delta v_t||\cos\alpha_t$ (α_t being the acute angle subtended by δq_t and δv_t), and between collisions the quantity $||\delta v_t||$ is unchanged, while the acute angle α_t decreases, according to the time-evolution equations (3.1). Finally, we should keep in mind that at a collision the norm $||\delta q_t||$ does not change, while $\langle\delta q_t,\delta v_t\rangle$ cannot decrease, see (3.4). Thus we obtain the inequalities

$$\langle\delta q_t,\delta v_t\rangle/||\delta q_t||\ge\langle\delta q_0,\delta v_0\rangle/||\delta q_0||\ge c_0||\delta q_0||,$$

so

$$\frac{d}{dt}||\delta q_t||^2 = 2||\delta q_t||\frac{d}{dt}||\delta q_t|| = 2\langle \delta q_t, \delta v_t\rangle \geq 2c_0||\delta q_0|| \cdot ||\delta q_t||$$

by (3.6). This means that $\frac{d}{dt}||\delta q_t|| \geq c_0||\delta q_0||$, so $||\delta q_t|| \geq ||\delta q_0||(1 + c_0 t)$, proving the proposition. □

Next we need an effective lower estimation c_0 for the curvature $\langle \delta q_0,\, \delta v_0\rangle/||\delta q_0||^2$ of the trajectory bundle:

Lemma 3.7. Assume that the perturbation $(\delta q_0^-,\, \delta v_0^-) \in \mathcal{T}_{x_0}\mathbf{M}$ (as in Proposition 3.5) is being performed at time zero right before a collision, say, $\sigma_0 = (1,\, 2)$ taking place at that time. Select the tangent vector $(\delta q_0^-,\, \delta v_0^-)$ in such a specific way that $\delta v_0^- = 0$, $\delta q_0^- = (m_2 w, -m_1 w, 0, 0, \dots, 0)$ with a nonzero vector $w \in \mathbb{R}^\nu$, $\langle w, v_1^- - v_2^-\rangle = 0$. This scalar product equation is exactly the condition that guarantees that δq_0^- be orthogonal to the velocity component $v^- = (v_1^-, v_2^-, \dots, v_N^-)$ of $x_0 = (q, v^-)$. The last, though crucial requirement is that w should be selected from the two-dimensional plane spanned by $v_1^- - v_2^-$ and $q_1 - q_2$ (with $||q_1 - q_2|| = 2r$) in $\mathbb{R}^\nu$. The purpose of this condition is to avoid the unwanted phenomenon of "astigmatism" in our billiard system, discovered first by Bunimovich and Rehacek in [B-R(1997)] and [B-R(1998)]. Later on the phenomenon of astigmatism gathered further prominence in the paper [B-Ch-Sz-T(2002)] as the main driving mechanism behind the wild non-differentiability of the singularity manifolds (at their boundaries) in hard ball systems in dimensions bigger than 2. We claim that

$$\frac{\langle \delta q_0^+, \delta v_0^+\rangle}{||\delta q_0||^2} = \frac{||v_1 - v_2||}{r\cos\phi_0} \geq \frac{||v_1 - v_2||}{r} \tag{3.8}$$

for the post-collision tangent vector $(\delta q_0^+, \delta v_0^+)$, where ϕ_0 is the acute angle subtended by $v_1^+ - v_2^+$ and the outer normal vector of the sphere $\left\{y \in \mathbb{R}^\nu \middle|\, ||y|| = 2r\right\}$ at the point $y = q_1 - q_2$. Note that in (3.8) there is no need to use $+$ or $-$ in $||\delta q_0||^2$ or $||v_1 - v_2||$, for $||\delta q_0^-|| = ||\delta q_0^+||$, $||v_1^- - v_2^-|| = ||v_1^+ - v_2^+||$.

Proof. The proof of the equation in (3.8) is a simple, elementary geometric argument in the plane spanned by $v_1^- - v_2^-$ and $q_1 - q_2$, so we omit it. We only note that the outgoing relative velocity $v_1^+ - v_2^+$ is obtained from the pre-collision relative velocity $v_1^- - v_2^-$ by reflecting the latter one across the tangent hyperplane of the sphere $\left\{y \in \mathbb{R}^\nu \middle|\, ||y|| = 2r\right\}$ at the point $y = q_1 - q_2$. □

The previous lemma shows that, in order to get useful lower estimations for the "curvature" $\langle \delta q, \delta v\rangle/||\delta q||^2$ of the trajectory bundle, it is necessary (and sufficient) to find collisions $\sigma = (i, j)$ on the orbit of a given point $x_0 \in \mathbf{M}$ with a "relatively big" value of $||v_i - v_j||$. Finding such collisions will be based upon the following result:

Proposition 3.9. Consider orbit segments $S^{[0,T]}x_0$ of N-ball systems with masses $m_1, m_2, \dots, m_N$ in $\mathbb{T}^\nu$ (or in $\mathbb{R}^\nu$) with collision sequences $\Sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ corresponding to connected collision graphs. (Now the kinetic energy is not necessarily normalized, and the total momentum $\sum_{i=1}^N m_i v_i$ may be different from zero.) We claim that there exists a positive-valued function $f(a; m_1, m_2, \dots, m_N)$ ($a > 0$, f is independent of the orbit segments $S^{[0,T]}x_0$) with the following two properties:

(1) If $||v_i(t_l) - v_j(t_l)|| \le a$ for all collisions $\sigma_l = (i, j)$ ($1 \le l \le n$, t_l is the time of σ_l) of some trajectory segment $S^{[0,T]}x_0$ with a symbolic collision sequence $\Sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ corresponding to a connected collision graph, then the norm $||v_{i'}(t) - v_{j'}(t)||$ of any relative velocity at any time $t \in \mathbb{R}$ is at most $f(a; m_1, \dots, m_N)$;

(2) $\lim_{a \to 0} f(a; m_1, \dots, m_N) = 0$ for any $(m_1, \dots, m_N)$.

Proof. We begin with

Lemma 3.10. Consider an N-ball system with masses $m_1, \dots, m_N$ (an $(m_1, \dots, m_N)$-system, for short) in $\mathbb{T}^\nu$ (or in $\mathbb{R}^\nu$). Assume that the inequalities $||v_i(0) - v_j(0)|| \le a$ hold true ($1 \le i < j \le N$) for all relative velocities at time zero. We claim that

$$||v_i(t) - v_j(t)|| \le 2a\sqrt{\frac{M}{m}} \tag{3.11}$$

for any pair (i, j) and any time $t \in \mathbb{R}$, where $M = \sum_{i=1}^N m_i$,

$$m = \min\{m_i |\ 1 \le i \le N\}.$$

Note. The estimate (3.11) is far from the optimal one, however, it will be sufficient for our purposes.

Proof. The assumed inequalities directly imply that $||v_i'(0)|| \le a$ ($1 \le i \le N$) for the velocities $v_i'(0)$ measured at time zero in the baricentric reference system. Therefore, for the total kinetic energy E_0 (measured in the baricentric system) we get the upper estimation $E_0 \le \frac{1}{2}Ma^2$, and this inequality remains true at any time t. This means that all the inequalities $||v_i'(t)||^2 \le \frac{M}{m_i}a^2$ hold true for the baricentric velocities $v_i'(t)$ at any time t, so

$$||v_i'(t) - v_j'(t)|| \le a\sqrt{M}\left(m_i^{-1/2} + m_j^{-1/2}\right) \le 2a\sqrt{\frac{M}{m}},$$

thus the inequalities

$$||v_i(t) - v_j(t)|| \le 2a\sqrt{\frac{M}{m}}$$

hold true, as well. $\square$

Proof of the proposition by induction on the number N.

For $N = 1$ we can take $f(a; m_1) = 0$, and for $N = 2$ the function $f(a; m_1, m_2) = a$ is obviously a good choice for f. Let $N \geq 3$, and assume that the orbit segment $S^{[0,T]}x_0$ of an $(m_1, \dots, m_N)$-system fulfills the conditions of the proposition. Let $\sigma_k = (i, j)$ be the collision in the symbolic sequence $\Sigma_n = (\sigma_1, \dots, \sigma_n)$ of $S^{[0,T]}x_0$ with the property that the collision graph of $\Sigma_k = (\sigma_1, \dots, \sigma_k)$ is connected, while the collision graph of $\Sigma_{k-1} = (\sigma_1, \dots, \sigma_{k-1})$ is still disconnected. Denote the two connected components (as vertex sets) of Σ_{k-1} by C_1 and C_2, so that $i \in C_1$, $j \in C_2$, $C_1 \cup C_2 = \{1, 2, \dots, N\}$, and $C_1 \cap C_2 = \emptyset$. By the induction hypothesis and the condition of the proposition, the norm of any relative velocity $v_{i'}(t_k - 0) - v_{j'}(t_k - 0)$ right before the collision σ_k (taking place at time t_k) is at most $a + f(a; \overline{C}_1) + f(a; \overline{C}_2)$, where $\overline{C}_l$ stands for the collection of the masses of all particles in the component C_l, $l = 1, 2$. Let $g(a; m_1, \dots, m_N)$ be the maximum of all possible sums

$$a + f(a; \overline{D}_1) + f(a; \overline{D}_2),$$

taken for all two-class partitions (D_1, D_2) of the vertex set $\{1, 2, \dots, N\}$. According to the previous lemma, the function

$$f(a; m_1, \dots, m_N) := 2\sqrt{\frac{M}{m}}g(a; m_1, \dots, m_N)$$

fulfills both requirements (1) and (2) of the proposition. $\square$

Corollary 3.12. Consider the original $(m_1, \dots, m_N)$-system with the standard normalizations $\sum_{i=1}^N m_i v_i = 0$, $\frac{1}{2}\sum_{i=1}^N m_i ||v_i||^2 = \frac{1}{2}$. We claim that there exists a threshold $G = G(m_1, \dots, m_N) > 0$ (depending only on N, $m_1, \dots, m_N$) with the following property:

In any orbit segment $S^{[0,T]}x_0$ of the $(m_1, \dots, m_N)$-system with the standard normalizations and with a connected collision graph, one can always find a collision $\sigma = (i, j)$, taking place at time t, so that $||v_i(t) - v_j(t)|| \geq G(m_1, \dots, m_N)$.

Proof. Indeed, we choose $G = G(m_1, \dots, m_N) > 0$ so small that $f(G; m_1, \dots, m_N) < M^{-1/2}$. Assume, contrary to 3.12, that the norm of any relative velocity $v_i - v_j$ of any collision of $S^{[0,T]}x_0$ is less than the above selected value of G. By the proposition, we have the inequalities $||v_i(0) - v_j(0)|| \leq f(G; m_1, \dots, m_N)$ at time zero. The normalization $\sum_{i=1}^N m_i v_i(0) = 0$, with a simple convexity argument, implies that $||v_i(0)|| \leq f(G; m_1, \dots, m_N)$ for all i, $1 \leq i \leq N$, so the total kinetic energy is at most $\frac{1}{2}M\left[f(G; m_1, \dots, m_N)\right]^2 < \frac{1}{2}$, a contradiction. $\square$

Corollary 3.13. For any phase point x_0 with a non-singular backward trajectory $S^{(-\infty,0]}x_0$ and with infinitely many consecutive, connected collision graphs on $S^{(-\infty,0]}x_0$, and for any number $L > 0$ one can always find a time $-t < 0$

and a tangent vector $(\delta q_0, \delta v_0) \in \mathcal{T}_{x_{-t}}\mathbf{M}$ $(x_{-t} = S^{-t}x_0)$ with $\langle \delta q_0, \delta v_0 \rangle > 0$ and $||\delta q_t||/||\delta q_0|| > L$, where $(\delta q_t, \delta v_t) = DS^t(\delta q_0, \delta v_0)$.

Proof. Indeed, select a number $t > 0$ so big that $1+\frac{t}{r}G(m_1, \dots, m_N) > L$ and $-t$ is the time of a collision (on the orbit of x_0) with the relative velocity $v_i^-(-t) - v_j^-(-t)$, for which $||v_i^-(-t) - v_j^-(-t)|| \ge G(m_1, \dots, m_N)$. By Lemma 3.7 we can choose a tangent vector $(\delta q_0^-, 0)$ right before the collision at time $-t$ in such a way that the lower estimation

$$\frac{\langle \delta q_0^+, \, \delta v_0^+ \rangle}{||\delta q_0^+||^2} \ge \frac{1}{r} G(m_1, \dots, m_N)$$

holds true for the "curvature" $\langle \delta q_0^+, \, \delta v_0^+ \rangle / ||\delta q_0^+||^2$ associated with the post-collision tangent vector $(\delta q_0^+, \, \delta v_0^+)$. According to Proposition 3.5 we have then the lower estimation

$$\frac{||\delta q_t||}{||\delta q_0||} \ge 1 + \frac{t}{r} G(m_1, \dots, m_N) > L$$

for the δq-expansion rate between $(\delta q_0^-, \, 0)$ and $(\delta q_t, \, \delta v_t) = DS^t(\delta q_0^-, \, 0)$. $\square$

§4. The Exceptional J-Manifolds (The asymptotic measure estimates)

First of all, we define the fundamental object for the proof of our theorem.

Definition 4.1. A smooth submanifold $J \subset \mathrm{int}\mathbf{M}$ of the interior of the phase space $\mathbf{M}$ is called an *exceptional J-manifold* (or simply an exceptional manifold) with a negative Lyapunov function Q if

(1) $\dim J = 2d - 2$ $(= \dim\mathbf{M} - 1)$;

(2) the pair of manifolds $(\overline{J}, \, \partial J)$ is diffeomorphic to the standard pair

$$\left(B^{2d-2}, \, \mathbb{S}^{2d-3}\right) = \left(B^{2d-2}, \, \partial B^{2d-2}\right),$$

where B^{2d-2} is the closed unit ball of $\mathbb{R}^{2d-2}$;

(3) J is locally flow-invariant, i. e. $\forall x \in J \; \exists \, a(x), \, b(x), \, a(x) < 0 < b(x)$, such that $S^t x \in J$ for all t with $a(x) < t < b(x)$, and $S^{a(x)}x \in \partial J$, $S^{b(x)}x \in \partial J$;

(4) the manifold J has some thin, open, tubular neighborhood $\tilde{U}_0$ in $\mathrm{int}\mathbf{M}$, and there exists a number $T > 0$ such that

(i) $S^T\left(\tilde{U}_0\right) \cap \partial\mathbf{M} = \emptyset$, and all orbit segments $S^{[0,T]}x$ ($x \in \tilde{U}_0$) are non-singular, hence they share the same symbolic collision sequence Σ;

(ii) $\forall x \in \tilde{U}_0$ the orbit segment $S^{[0,T]}x$ is sufficient if and only if $x \notin J$;

(5) $\forall x \in J$ we have $Q(n(x)) := \langle z(x),\, w(x)\rangle \le -c_1 < 0$ for a unit normal vector field $n(x) = (z(x),\, w(x))$ of J with a fixed constant $c_1 > 0$;

(6) the set W of phase points $x \in J$ never again returning to J (After first leaving it, of course. Keep in mind that J is locally flow-invariant!) has relative measure greater than $1 - 10^{-8}$ in J, i. e. $\dfrac{\mu_1(W)}{\mu_1(J)} > 1 - 10^{-8}$, where μ_1 is the hypersurface measure of the smooth manifold J.

We begin with an important proposition on the structure of forward orbits $S^{[0,\infty)}x$ for $x \in J$.

Proposition 4.2. For μ_1-almost every $x \in J$ the forward orbit $S^{[0,\infty)}x$ is non-singular.

Proof. According to Proposition 7.12 of [Sim(2003)], the set

$$J \cap \left[\bigcup_{t>0} S^{-t}\left(\mathcal{SR}^-\right)\right]$$

of forward singular points $x \in J$ is a countable union of smooth, proper submanifolds of J, hence it has μ_1-measure zero. $\square$

In the future we will need

Lemma 4.3. The concave, local orthogonal manifolds $\Sigma(y)$ passing through points $y \in J$ are uniformly transversal to J.

Note. A local orthogonal manifold $\Sigma \subset \mathrm{int}\mathbf{M}$ is obtained from a codimension-one, smooth submanifold Σ_1 of $\mathrm{int}\mathbf{Q}$ by supplying Σ_1 with a selected field of unit normal vectors as velocities. Σ is said to be concave if the second fundamental form of Σ_1 (with respect to the selected orientation) is negative semi-definite at every point of Σ_1. Similarly, the convexity of Σ requires positive semi-definiteness here, see also §2 of [K-S-Sz(1990)-I].

Proof. We will only prove the transversality. It will be clear from the uniformity of the estimations used in the proof that the claimed transversalities are actually uniform across J.

Assume, to the contrary of the transversality, that a concave, local orthogonal manifold $\Sigma(y)$ is tangent to J at some $y \in J$. Let $(\delta q,\, B\delta q)$ be any vector of $\mathcal{T}_y\mathbf{M}$ tangent to $\Sigma(y)$ at y. Here $B \le 0$ is the second fundamental form of the projection $q\left(\Sigma(y)\right) = \Sigma_1(y)$ of $\Sigma(y)$ at the point $q = q(y)$. The assumed tangency means that $\langle \delta q,\, z\rangle + \langle B\delta q,\, w\rangle = 0$, where $n(y) = (z(y),\, w(y)) = (z, w)$ is the unit normal vector of J at y. We get that $\langle \delta q,\, z+Bw\rangle = 0$ for any vector $\delta q \in v(y)^{\perp}$. We note that the components z and w of n are necessarily orthogonal to the velocity $v(y)$, because the manifold J is locally flow-invariant. The last equation means that $z = -Bw$, thus $Q(n(y)) = \langle z,\, w\rangle = \langle -Bw,\, w\rangle \geq 0$, contradicting to the assumption $Q(n(y)) \leq -c_1$ of (5) in 4.1. This finishes the proof of the lemma. $\square$

In order to formulate the main result of this section, we need to define two important subsets of J.

Definition 4.4. Let

$$A = \left\{ x \in J \middle|\, S^{[0,\infty)}x \text{ is nonsingular and } \dim \mathcal{N}_0 \left(S^{[0,\infty)}x \right) = 1 \right\},$$

$$B = \left\{ x \in J \middle|\, S^{[0,\infty)}x \text{ is nonsingular and } \dim \mathcal{N}_0 \left(S^{[0,\infty)}x \right) > 1 \right\}.$$

The two Borel subsets A and B of J are disjoint and, according to Proposition 4.2 above, their union $A \cup B$ has full μ_1-measure in J.

The anticipated main result of this section is

Main Lemma 4.5. Use all of the above definitions and notations. We claim that $A \neq \emptyset$.

Proof. The proof will be a proof by contradiction, and it will be subdivided into several lemmas. Thus, from now on, we assume that $A = \emptyset$.

First, select and fix a non-periodic point (a "base point") $x_0 \in B$. The following step is to use Corollary 3.13 in such a way that the role of x_0 in 3.13 is now played by the time-inverted version $-x_0 = (q_0,\, -v_0)$ of our fixed base point $x_0 \in J$. Thus, for a large constant $L_0 >> 1$ (to be specified later) select a big enough time $c_3 >> 1$ (playing the role of t in 3.13) and a tangent vector $(\delta q_0,\, -\delta v_0) \in \mathcal{T}_{-x_{c_3}}\mathbf{M}$ ($x_{c_3} = S^{c_3}x_0$) with $\langle \delta q_0,\, -\delta v_0\rangle > 0$ and

$$\text{(4.6)} \qquad \frac{||\delta q_{c_3}||}{||\delta q_0||} > L_0,$$

where

$$\text{(4.7)} \qquad (\delta q_{c_3},\, \delta v_{c_3}) = \left(DS^{-c_3}\right)(\delta q_0,\, \delta v_0).$$

The normalized tangent vector

$$\text{(4.8)} \qquad (\delta \tilde{q}_0,\, \delta \tilde{v}_0) := \left(||\delta q_{c_3}||^2 + ||\delta v_{c_3}||^2\right)^{-1/2} \cdot (\delta q_{c_3},\, \delta v_{c_3}) \in \mathcal{T}_{x_0}\mathbf{M}$$

will play a crucial role in the proof.

By slightly perturbing the tangent vector $(\delta q_0,\, -\delta v_0)$, we can always achieve that

$$
\tag{4.9}
\begin{cases}
\text{the unit tangent vector } (\delta\tilde{q}_0,\, \delta\tilde{v}_0) \text{ of (4.8)} \\
\text{is transversal to } J \text{ and, corresponding to (4.6),} \\
\frac{||\delta\tilde{q}_{c_3}||}{||\delta\tilde{q}_0||} < L_0^{-1},
\end{cases}
$$

where $(\delta\tilde{q}_{c_3},\, \delta\tilde{v}_{c_3}) = (DS^{c_3})\,(\delta\tilde{q}_0,\, \delta\tilde{v}_0)$.

We choose the orientation of the unit normal field $n(x)$ $(x \in J)$ of J in such a way that $\langle n(x_0),\, (\delta\tilde{q}_0,\, \delta\tilde{v}_0)\rangle < 0$, and define the one-sided tubular neighborhood U_δ of radius $\delta > 0$ as the set of all phase points $\gamma_x(s)$, where $x \in J$, $0 \le s < \delta$. Here $\gamma_x(\,.\,)$ is the geodesic line passing through x (at time zero) with the initial velocity $n(x)$, $x \in J$. The radius (thickness) $\delta > 0$ here is a variable, which will eventually tend to zero. We are interested in getting useful asymptotic estimates for certain subsets of U_δ, as $\delta \to 0$.

Our main working domain will be the set

$$
\tag{4.10}
\begin{aligned}
D_0 = \Big\{ y \in U_{\delta_0} \setminus J \big|\; & y \notin \bigcup_{t>0} S^{-t}\left(\mathcal{SR}^-\right),\ \exists \text{ a sequence} \\
& t_n \nearrow \infty \text{ such that } S^{t_n} y \in U_{\delta_0} \setminus J, \quad n = 1, 2, \dots \Big\},
\end{aligned}
$$

a set of full μ-measure in U_{δ_0}. We will use the shorthand notation $U_0 = U_{\delta_0}$ for a fixed, small value δ_0 of δ. For any $y \in \mathbf{M}$ we use the traditional notations

$$
\tag{4.11}
\begin{aligned}
&\tau(y) = \min\left\{ t > 0 \big|\; S^t y \in \partial\mathbf{M} \right\}, \\
&T(y) = S^{\tau(y)} y
\end{aligned}
$$

for the first hitting of the collision space $\partial\mathbf{M}$. The first return map (Poincaré section, collision map) $T : \partial\mathbf{M} \to \partial\mathbf{M}$ (the restriction of the above T to $\partial\mathbf{M}$) is known to preserve the finite measure ν that can be obtained from the Liouville measure μ by projecting the latter one onto $\partial\mathbf{M}$ along the flow. Following 4. of [K-S-Sz(1990)-II], for any point $y \in \mathrm{int}\mathbf{M}$ (with $\tau(y) < \infty$, $\tau(-y) < \infty$, where $-y = (q, -v)$ for $y = (q, v)$) we denote by $z_{tub}(y)$ the supremum of all radii $\rho > 0$ of tubular neighborhoods V_ρ of the projected segment

$$
q\left(\left\{ S^t y \big|\; -\tau(-y) \le t \le \tau(y) \right\}\right) \subset \mathbf{Q}
$$

for which even the closure of the set

$$
\left\{ (q,\, v(y)) \in \mathbf{M} \big|\; q \in V_\rho \right\}
$$

does not intersect the set $\mathcal{SR}$ of singular reflections.

We remind the reader that both Lemma 2 of [S-Ch(1987)] and Lemma 4.10 of [K-S-Sz(1990)-I] use this tubular distance function $z_{tub}(\,.\,)$ (despite the notation $z(\,.\,)$ in those papers), see the important note 4. on [K-S-Sz(1990)-II].

Following the fundamental construction of local stable invariant manifolds [S-Ch(1987)] (see also §5 of [K-S-Sz(1990)-I]), for any $y \in D_0$ we define the concave, local orthogonal manifolds

$$\begin{aligned} &\Sigma_t^t(y) = SC_{y_t}\left(\left\{(q,\, v(y_t)) \in \mathbf{M}\middle|\; q - q(y_t) \perp v(y_t)\right\} \setminus (\mathcal{S}_1 \cup \mathcal{S}_{-1})\right), \\ &\Sigma_0^t(y) = SC_y\left[S^{-t}\Sigma_t^t(y)\right], \end{aligned} \tag{4.12}$$

where $\mathcal{S}_1 := \left\{x \in \mathbf{M}\middle|\; Tx \in \mathcal{SR}^-\right\}$ (the set of phase points on singularities of order 1), $\mathcal{S}_{-1} := \left\{x \in \mathbf{M}\middle|\; -x \in \mathcal{S}_1\right\}$ (the set of phase points on singularities of order -1), $y_t = S^t y$, and $SC_y(\,.\,)$ stands for taking the smooth component of the given set that contains the point y. The local, stable invariant manifold $\gamma^{(s)}(y)$ of y is known to be a superset of the C^2-limiting manifold $\lim_{t\to\infty}\Sigma_0^t(y)$.

On all these local orthogonal manifolds, appearing in the proof, we will always use the so called δq-metric to measure distances. The length of a smooth curve with respect to this metric is the integral of $||\delta q||$ along the curve. The proof of the Theorem on Local Ergodicity [S-Ch(1987)] shows that the δq-metric is the relevant notion of distance on the local orthogonal manifolds Σ, also being in good harmony with the tubular distance function $z_{tub}(\,.\,)$ defined earlier.

On any manifold $\Sigma_0^t(y) \cap U_0$ $(y \in D_0)$ we define the smooth field $\mathcal{X}_{y,t}(y')$ $(y' \in \Sigma_0^t(y) \cap U_0)$ of unit tangent vectors of $\Sigma_0^t(y) \cap U_0$ as follows:

$$\mathcal{X}_{y,t}(y') = \frac{\Pi_{y,t,y'}\left((\delta\tilde{q}_0,\, \delta\tilde{v}_0)\right)}{\|\Pi_{y,t,y'}\left((\delta\tilde{q}_0,\, \delta\tilde{v}_0)\right)\|}, \tag{4.13}$$

where $\Pi_{y,t,y'}$ denotes the orthogonal projection of $\mathbb{R}^d \oplus \mathbb{R}^d$ onto the tangent space of $\Sigma_0^t(y)$ at the point $y' \in \Sigma_0^t(y) \cap U_0$. Recall that $(\delta\tilde{q}_0,\, \delta\tilde{v}_0)$ is the unit tangent vector of $\mathbf{M}$ at the base point x_0 from (4.8) and (4.9). We also remind the reader that $(\delta\tilde{q}_0,\, \delta\tilde{v}_0)$ points toward the side of J opposite to the side where the one-sided neighborhoods U_δ reside.

Note 4.14. By the construction of $(\delta\tilde{q}_0, \delta\tilde{v}_0)$ in (4.6)–(4.8), if the threshold c_3 is big enough, then the vector $(\delta\tilde{q}_0, \delta\tilde{v}_0)$ is close to the tangent space $\mathcal{T}\gamma^s(x_0)$ of the local stable manifold of x_0. On the other hand, for large enough t the tangent space of $\Sigma_0^t(y) \cap U_0$ at y' makes a small angle with $\mathcal{T}\gamma^s(x_0)$. All the necessary upper estimations for the mentioned angles follow from the well known result stating that the difference (in norm) between the second fundamental forms of the S^t-images $(t > 0)$ of two local, convex orthogonal manifolds is at most $1/t$, see, for instance, inequality (4) in [Ch(1982)]. These facts imply that the vector in the numerator of (4.13) is actually very close to $(\delta\tilde{q}_0, \delta\tilde{v}_0)$, in particular its magnitude is almost one.

For any $y \in D_0$ let $t_k = t_k(y)$ $(0 < t_1 < t_2 < \dots)$ be the time of the k-th collision σ_k on the forward orbit $S^{[0,\infty)}y$ of y. Assume that the time t in the construction of $\Sigma_0^t(y)$ and $\mathcal{X}_{y,t}$ is between σ_{k-1} and σ_k, i. e. $t_{k-1}(y) < t < t_k(y)$. We define the smooth curve $\rho_{y,t} = \rho_{y,t}(s)$ (with the arc length parametrization s, $0 \le s \le h(y,t)$) as the maximal integral curve of the vector field $\mathcal{X}_{y,t}$ emanating from y and not intersecting any forward singularity of order $\le k$, i. e.

$$
(4.15) \qquad \begin{cases} \rho_{y,t}(0) = y, \\ \frac{d}{ds}\rho_{y,t}(s) = \mathcal{X}_{y,t}\left(\rho_{y,t}(s)\right), \\ \rho_{y,t}(\,.\,) \text{ does not intersect any singularity of order } \le k, \\ \rho_{y,t} \text{ is maximal among all curves with the above properties.} \end{cases}
$$

We remind the reader that a phase point x lies on a singularity of order k $(k \in \mathbb{N})$ if and only if the k-th collision on the forward orbit $S^{(0,\infty)}x$ is a singular one. It is also worth noting here that, as it immediately follows from (4.15), the curve $\rho_{y,t}$ can only terminate at a boundary point of the manifold $\Sigma_0^t(y) \cap U_0$.

Note 4.16. From now on, we will use the notations $\Sigma_0^k(y)$, $\mathcal{X}_{y,k}$, and $\rho_{y,k}$ for $\Sigma_0^{t_k^*}(y)$, $\mathcal{X}_{y,t_k^*}$, and ρ_{y,t_k^*}, respectively, where $t_k^* = t_k^*(y) = \frac{1}{2}\left(t_{k-1}(y) + t_k(y)\right)$.

Due to these circumstances, the curves $\rho_{y,t_k^*} = \rho_{y,k}$ can now terminate at a point z such that z is not on any singularity of order at most k and $S^{t_k^*}z$ is a boundary point of $\Sigma_{t_k^*}^{t_k^*}(y)$, so that at the point $S^{t_k^*}z$ the manifold $\Sigma_{t_k^*}^{t_k^*}(y)$ touches the boundary of the phase space in a nonsingular way. This means that, when we continuously move the points $\rho_{y,k}(s)$ by varying the parameter s between 0 and $h(y,k)$, either the time $t_k\left(\rho_{y,k}(s)\right)$ or the time $t_{k-1}\left(\rho_{y,k}(s)\right)$ becomes equal to $t_k^* = t_k^*(y)$ when the parameter value s reaches its maximum value $h(y,k)$. The length of the curve $\rho_{y,k}$ is at most δ_0, and an elementary geometric argument shows that the time of collision $t_k(\rho_{y,k}(s))$ (or $t_{k-1}(\rho_{y,k}(s))$) can only change by at most the amount of $c^*\sqrt{\delta_0}$, as s varies between 0 and $h(y,k)$. (Here c^* is an absolute constant.) Thus, we get that the unpleasant situation mentioned above can only occur when the difference $t_k(y) - t_{k-1}(y)$ is at most $c^*\sqrt{\delta_0}$. These collisions have to be excluded from the limiting process of Lemma 4.28 below. Still, everything works by the main result of [B-F-K(1998)], which guarantees that the indices k of the collisions (on the forward orbit $S^{(0,\infty)}y$) with $t_k(y) - t_{k-1}(y) > c^*\sqrt{\delta_0}$ have a positive density amongst the natural numbers.

As far as the terminal point $\rho_{y,k}(h(y,k))$ of $\rho_{y,k}$ is concerned, there are exactly three, mutually exclusive possiblities for this point:

(A) $\rho_{y,k}(h(y,k)) \in J$ and this terminal point does not belong to any forward singularity of order $\le k$,

(B) $\rho_{y,k}(h(y,k))$ lies on a forward singularity of order $\le k$,

(C) the terminal point $\rho_{y,k}(h(y,k))$ does not lie on any singularity of order $\le k$ but lies on the part of the boundary ∂U_0 of U_0 different from J.

Note 4.17. Under the canonical identification $U_0 \cong J \times [0, \delta_0)$ of U_0 via the geodesic lines perpendicular to J, the above mentioned part of ∂U_0 (the "side" of U_0) corresponds to $\partial J \times [0, \delta_0)$. Therefore, the set of points with property (C) inside a layer U_δ ($\delta \le \delta_0$) will have μ-measure small ordo of δ (actually, of order δ^2), thus this set will be negligible in our asymptotic measure estimations, as $\delta \to 0$. That is why in the future we will not be dealing with any phase point with property (C).

Should (B) occur for some value of k ($k \ge 2$), the minimum of all such integers k will be denoted by $\overline{k} = \overline{k}(y)$. The exact order of the forward singularity on which the terminal point $\rho_{y,\overline{k}}\left(h(y,\overline{k})\right)$ lies is denoted by $\overline{k}_1 = \overline{k}_1(y)$. If (B) does not occur for any value of k, then we take $\overline{k}(y) = \overline{k}_1(y) = \infty$.

We can assume that the manifold J and its one-sided tubular neighborhood $U_0 = U_{\delta_0}$ are already so small that for any $y \in U_0$ no singularity of $S^{(0,\infty)}y$ can take place at the first collision, so the indices $\overline{k}$ and $\overline{k}_1$ above are automatically at least 2. For our purposes the important index will be $\overline{k}_1 = \overline{k}_1(y)$ for phase points $y \in D_0$.

Note 4.18. Refinement of the construction. Instead of selecting a single contracting unit vector $(\delta\tilde{q}_0, \delta\tilde{v}_0)$ in (4.8), we should do the following: Choose a compact set $K_0 \subset B$ with the following properties:

(i) $\frac{\mu_1(K_0)}{\mu_1(J)} > 1 - 10^{-6}$,
(ii) every point $x \in K_0$ has a non-singular forward orbit.

Now the running point $x \in K_0$ will play the role of x_0 in the construction of the contracting unit tangent vector $u(x) := (\delta\tilde{q}_0, \delta\tilde{v}_0) \in \mathcal{T}_x\mathbf{M}$ on the left-hand-side of (4.8). For every $x \in K_0$ there is a small, open ball neighborhood $B(x)$ of x and a big threshold $c_3(x) >> 1$ such that (4.9) holds true for $u(x)$ and $c_3 = c_3(x)$ for all $x \in K_0$.

By the continuity of the contraction/expansion factor, one can also achieve that the contraction estimation L_0^{-1} of (4.9) holds true not only for $u(x)$, but also for any projected copy of it appearing in (4.13), provided that $y' \in B(x)$, i. e. y' is close enough to x.

Now select a finite subcovering $\bigcup_{i=1}^n B(x_i)$ of K_0, and replace J by $J_1 = J \cap \bigcup_{i=1}^n B(x_i)$, U_δ by $U'_\delta = U_\delta \cap \bigcup_{i=1}^n B(x_i)$ (for $\delta \le \delta_0$) and, finally, choose the threshold c_3 to be the maximum of all thresholds $c_3(x_i)$ for $i = 1, 2, \dots, n$. In this way the assertion of Corollary 4.20 will be indeed true.

We note that the new exceptional manifold J_1 is no longer so nicely "round shaped" as J, but it is still pretty well behaved, being a domain in J with a piecewise smooth boundary.

The reason why we cannot switch completely to a round and much smaller manifold $B(x) \cap J$ is that the measure $\mu_1(J)$ should be kept bounded from below after having fixed L_0, see 4. in the Appendix.

In addition, it should be noted that, when constructing the vector field in (4.13) and the curves $\rho_{y,t}$, an appropriate directing vector $u(x_i)$ needs to be chosen for (4.13). To be definite and not arbitrary, a convenient choice is the first index $i \in \{1, 2, \dots, n\}$ for which $y \in B(x_i)$. In that way the whole curve $\rho_{y,t}$ will stay in the slightly enlarged ball $B'(x_i)$ with double the radius of $B(x_i)$, and one can organize things so that the required contraction estimates of (4.9) be still valid even in these enlarged balls.

In the future, a bit sloppily, J_1 will be denoted by J, and U'_δ by U_δ.

Note 4.19. When defining the returns of a forward orbit to U_δ, we used to say that "before every new return the orbit must first leave the set U_δ". Since the newly obtained J is no longer round shaped as it used to be, the above phrase is not satisfactory any longer. Instead, one should say that the orbit leaves even the κ-neighborhood of U_δ, where κ is two times the diameter of the original J. This guarantees that not only the new U_δ, but also the original U_δ will be left by the orbit, so we indeed are dealing with a genuine return. This note also applies to two more slight shrinkings of J that will take place later in the proof.

As an immediate corollary of (4.9) and the above note, we get

Corollary 4.20. For the given sets J, U_0, and the large constant L_0 we can select the threshold $c_3 > 0$ large enough so that for any point $y \in D_0$ any time t with $c_3 \le t < t_{\overline{k}_1(y)}(y)$ the δq-expansion rate of S^t between the curves $\rho_{y,\overline{k}(y)}$ and $S^t\left(\rho_{y,\overline{k}(y)}\right)$ is less than L_0^{-1}, i. e. for any tangent vector $(\delta q_0,\, \delta v_0)$ of $\rho_{y,\overline{k}(y)}$ we have

$$\frac{||\delta q_t||}{||\delta q_0||} < L_0^{-1},$$

where $(\delta q_t,\, \delta v_t) = (DS^t)(\delta q_0,\, \delta v_0)$.

A further immediate consequence of the previous result is

Corollary 4.21. For any $y \in D_0$ with $\overline{k}(y) < \infty$ and $t_{\overline{k}_1(y)-1}(y) \ge c_3$, and for any t with $t_{\overline{k}_1(y)-1}(y) < t < t_{\overline{k}_1(y)}(y)$, we have

$$z_{tub}\left(S^t y\right) < L_0^{-1} l_q\left(\rho_{y,\overline{k}(y)}\right) < \frac{c_4}{L_0} \operatorname{dist}(y,\, J), \tag{4.22}$$

where $l_q\left(\rho_{y,\overline{k}(y)}\right)$ denotes the δq-length of the curve $\rho_{y,\overline{k}(y)}$, and $c_4 > 0$ is a constant, independent of L_0, depending only on the (asymptotic) angles between the curves $\rho_{y,\overline{k}(y)}$ and J.

By further shrinking the exceptional manifold J a little bit, and by selecting a suitably thin, one-sided neighborhood $U_1 = U_{\delta_1}$ of J, we can achieve that the open $2\delta_1$-neighborhood of U_1 (on the same side of J as U_0 and U_1) is a subset of U_0.

For a varying δ, $0 < \delta \le \delta_1$, we introduce the layer

$$\begin{aligned} \overline{U}_\delta = \Big\{ y \in (U_\delta \setminus U_{\delta/2}) \cap D_0 \Big| \ \exists \text{ a sequence } t_n \nearrow \infty \\ \text{such that } S^{t_n} y \in (U_\delta \setminus U_{\delta/2}) \text{ for all } n \Big\}. \end{aligned} \tag{4.23}$$

Since almost every point of the layer $(U_\delta \setminus U_{\delta/2}) \cap D_0$ returns infinitely often to this set and the asymptotic equation

$$\mu\left((U_\delta \setminus U_{\delta/2}) \cap D_0 \right) \sim \frac{\delta}{2} \mu_1(J)$$

holds true, we get the asymptotic equation

$$\mu\left(\overline{U}_\delta\right) \sim \frac{\delta}{2} \mu_1(J). \tag{4.24}$$

We will need the following subsets of $\overline{U}_\delta$:

$$\begin{aligned} \overline{U}_\delta(c_3) &= \left\{ y \in \overline{U}_\delta \big| \ t_{\overline{k}_1(y)-1}(y) \ge c_3 \right\}, \\ \overline{U}_\delta(\infty) &= \left\{ y \in \overline{U}_\delta \big| \ \overline{k}_1(y) = \infty \right\}. \end{aligned} \tag{4.25}$$

Here c_3 is the constant from Corollary 4.20, the exact value of which will be specified later, at the end of the proof of Main Lemma 4.5. By selecting the pair of sets $(U_1, \ J)$ small enough, we can assume that

$$z_{tub}(y) > c_4 \delta_1 \quad \forall y \in U_1. \tag{4.26}$$

This inequality guarantees that the collision time $t_{\overline{k}_1(y)}(y)$ ($y \in \overline{U}_\delta$) cannot be near any return time of y to the layer $(U_\delta \setminus U_{\delta/2})$, for $\delta \le \delta_1$, provided that $y \in \overline{U}_\delta(c_3)$. More precisely, the whole orbit segment $S^{[-\tau(-z),\, \tau(z)]} z$ will be disjoint from U_1, where $z = S^t y$, $t_{\overline{k}_1(y)-1}(y) < t < t_{\overline{k}_1(y)}(y)$.

Lemma 4.27. $\mu\left(\overline{U}_\delta \setminus \overline{U}_\delta(c_3)\right) = o(\delta)$ (small ordo of δ), as $\delta \to 0$.

Proof. The points y of the set $\overline{U}_\delta \setminus \overline{U}_\delta(c_3)$ have the property $t_{\overline{k}_1(y)-1}(y) < c_3$. By doing another slight shrinking to J, the same way as in Note 4.18, we can achieve that $t_{\overline{k}_1(y)}(y) < 2c_3$ for all $y \in \overline{U}_\delta \setminus \overline{U}_\delta(c_3)$, $0 < \delta \le \delta_1$. This means that all points of the set $\overline{U}_\delta \setminus \overline{U}_\delta(c_3)$ are at most at the distance of δ from the singularity set

$$\bigcup_{0\le t\le 2c_3} S^{-t}\left(\mathcal{SR}^-\right).$$

This singularity set is a compact collection of codimension-one, smooth submanifolds (with boundaries), each of which is uniformly transversal to the manifold J. This uniform transversality follows from Lemma 4.3 above, and from the fact that the inverse images $S^{-t}(\mathcal{SR}^-)$ $(t > 0)$ of singularities can be smoothly foliated with local, concave orthogonal manifolds. Thus, the δ-neighborhood of this singularity set inside $\overline{U}_\delta$ clearly has μ-measure small ordo of δ, actually, of order $\le \text{const}\cdot\delta^2$. $\square$

Lemma 4.28. For any point $y \in \overline{U}_\delta(\infty)$ the curves $\rho_{y,k}(s)$ $(0 \le s \le h(y,k))$ have a C^2-limiting curve $\rho_{y,\infty}(s)$ $(0 \le s \le h(y,\infty))$, with $h(y,\, k) \to h(y,\, \infty)$, as $k \to \infty$.

Proof. Besides the concave, local orthogonal manifolds $\Sigma_0^k(y) = \Sigma_0^{t_k^*}(y)$ of (4.12) (where $t_k^* = t_k^*(y) = \frac{1}{2}\left(t_{k-1}(y) + t_k(y)\right)$), let us also consider another type of concave, local orthogonal manifolds defined by the formula

$$\tilde{\Sigma}_0^k(y) = \tilde{\Sigma}_0^{t_k^*}(y) = SC_y\left(S^{-t_k^*}\left(SC_{y_{t_k^*}}\left\{y' \in \mathbf{M}\middle|\; q(y') = q(y_{t_k^*})\right\}\right)\right), \tag{4.29}$$

the so called "candle manifolds", containing the phase point $y \in \overline{U}_\delta(\infty)$ in their interior. It was proved in §3 of [Ch(1982)] that the second fundamental forms $B\left(\Sigma_0^k(y),\, y\right) \le 0$ are monotone non-increasing in k, while the second fundamental forms $B\left(\tilde{\Sigma}_0^k(y),\, y\right) < 0$ are monotone increasing in k, so that

$$B\left(\tilde{\Sigma}_0^k(y),\, y\right) < B\left(\Sigma_0^k(y),\, y\right)$$

is always true. It is also proved in §3 of [Ch(1982)] that

$$\lim_{t\to\infty} B\left(\tilde{\Sigma}_0^k(y),\, y\right) = \lim_{t\to\infty} B\left(\Sigma_0^k(y),\, y\right) := B_\infty(y) < 0$$

uniformly in y, and these two-sided, monotone curvature limits give rise to uniform C^2-convergences

$$\lim_{t\to\infty} \Sigma_0^k(y) = \Sigma_0^\infty(y), \quad \lim_{t\to\infty} \tilde{\Sigma}_0^k(y) = \Sigma_0^\infty(y),$$

and the limiting manifold $\Sigma_0^\infty(y)$ is the local stable invariant manifold $\gamma^{(s)}(y)$ of y, once it contains y in its smooth part. These monotone, two-sided limit relations, together with the definition of the curves $\rho_{y,t_k^*} = \rho_{y,k}$ prove the existence of the C^2-limiting curve $\rho_{y,\infty} = \lim_{k\to\infty} \rho_{y,k}$, $h(y,k) \to h(y,\infty)$, as $k \to \infty$. They also prove the inclusion $\rho_{y,\infty}\left([0,\, h(y,\infty)]\right) \subset \gamma^{(s)}(y)$. $\square$

Lemma 4.30. For any point $y \in \overline{U}_\delta(\infty)$ the projection $\Pi(y) := \rho_{y,\infty}\left(h(y,\infty)\right)$ $(\in J)$ is either an element of the set A (of 4.4) or it is a forward singular point of J.

Proof. Assume that the forward orbit of $\Pi(y)$ is non-singular. Select a return time $t > c_3$ for which $S^t y \in \overline{U}_\delta \subset \left(U_\delta \setminus U_{\delta/2}\right) \cap D_0$. The distance $\mathrm{dist}(S^t y,\, J)$ between $S^t y$ and J is bigger than $\delta/2$. According to the contraction result 4.20, if the contraction factor L_0^{-1} is chosen small enough, the distance between $S^t\left(\Pi(y)\right)$ and J stays bigger than $\delta/4$, so $S^t\left(\Pi(y)\right) \in U_0 \setminus J$ will be true. This means, however, that the forward orbit of $\Pi(y)$ is sufficient, according to (4)/(ii) of Definition 4.1. $\square$

Corollary 4.31. $\mu\left(\overline{U}_\delta(\infty)\right) = 0$.

Proof. By our indirect assumption $A = \emptyset$, so the previous lemma says that

$$\Pi(y) \in \bigcup_{t>0} S^{-t}\left(\mathcal{SR}^-\right)$$

for all $y \in \overline{U}_\delta(\infty)$. Due to the transversality result 4.3 above, the singularity set $\bigcup_{t>0} S^{-t}\left(\mathcal{SR}^-\right)$ is a countable union of smooth, proper submanifolds of J. Since the curves $\rho_{y,\infty}$ belong to (at least) the C^2 smoothness class, we get that $\overline{U}_\delta(\infty)$ is a countable union of proper, C^2-smooth submanifolds of U_δ, hence $\mu\left(\overline{U}_\delta(\infty)\right) = 0$. $\square$

Next we need a useful upper estimation for the μ-measure of the set $\overline{U}_\delta(c_3) \setminus \overline{U}_\delta(\infty)$ as $\delta \to 0$. We will classify the points $y \in \overline{U}_\delta(c_3) \setminus \overline{U}_\delta(\infty)$ according to whether $S^t y$ returns to the layer $\left(U_\delta \setminus U_{\delta/2}\right) \cap D_0$ (after first leaving it, of course) before the time $t_{\overline{k}_1(y)-1}(y)$ or not. Thus, we define the sets

$$\begin{aligned} E_\delta(c_3) =& \Big\{ y \in \overline{U}_\delta(c_3) \setminus \overline{U}_\delta(\infty) \Big|\, \exists\, 0 < t_1 < t_2 < t_{\overline{k}_1(y)-1}(y) \\ & \quad \text{such that } S^{t_1} y \notin \tilde{U}_0,\ S^{t_2} y \in \left(U_\delta \setminus U_{\delta/2}\right) \cap D_0 \Big\}, \\ F_\delta(c_3) =& \overline{U}_\delta(c_3) \setminus \left[\overline{U}_\delta(\infty) \cup E_\delta(c_3)\right]. \end{aligned} \tag{4.32}$$

Recall that the threshold $t_{\overline{k}_1(y)-1}(y)$, being a collision time, is far from any possible return time t_2 to the layer $\left(U_\delta \setminus U_{\delta/2}\right) \cap D_0$, see the remark right after (4.26).

Now we will be doing the "slight shrinking" trick of Note 4.18 the third (and last) time. We slightly further decrease J to obtain a smaller J_1 with almost the same μ_1-measure. Indeed, by using property (6) of 4.1, inside the set $J \cap B$ we choose a compact set K_1 for which

$$\frac{\mu_1(K_1)}{\mu_1(J)} > 1 - 10^{-6},$$

and no point of K_1 ever returns to J. For each point $x \in K_1$ the distance between the orbit segment $S^{[a_0,c_3]}x$ and J is at least $\epsilon(x) > 0$. Here a_0 is needed to guarantee that we certainly drop the initial part of the orbit, which still stays near J, and c_3 was chosen earlier. By the non-singularity of the orbit segment $S^{[a_0,c_3]}x$ and by continuity, the point $x \in K_1$ has an open ball neighborhood $B(x)$ of radius $r(x) > 0$ such that for every $y \in B(x)$ the orbit segment $S^{[a_0,c_3]}y$ is non-singular and stays away from J by at least $\epsilon(x)/2$. Choose a finite covering $\bigcup_{i=1}^{n} B(x_i) \supset K_1$ of K_1, replace J and U_δ by their intersections with the above union (the same way as it was done in Note 4.18), and fix threshold value of δ_0 so that

$$\delta_0 < \frac{1}{2}\min\{\epsilon(x_i) \,|\, i = 1, 2, \dots, n\}.$$

In the future we again keep the old notations J and U_δ for these intersections.

In this way we achieve that the following statement is true:

$$\text{(4.33)} \qquad \begin{cases} \text{Any return time } t_2 \text{ of any point } y \in \left(U_\delta \setminus U_{\delta/2}\right) \cap D_0 \text{ to} \\ \left(U_\delta \setminus U_{\delta/2}\right) \cap D_0 \text{ is always greater than } c_3 \text{ for } 0 < \delta \le \delta_1. \end{cases}$$

For any phase point $y \in E_\delta(c_3)$ we define the first return time $\bar{t}_2 = \bar{t}_2(y)$ as the infimum of all the return times t_2 of y featuring (4.32). By using this definition of $\bar{t}_2(y)$, formulas (4.32)–(4.33), and the contraction result 4.20, we easily get

Lemma 4.34. If the contraction coefficient L_0^{-1} in 4.20 is chosen suitably small, then for any point $y \in E_\delta(c_3)$ the projected point

$$\text{(4.35)} \qquad \Pi(y) := \rho_{y,\bar{t}_2(y)}\left(h(y, \bar{t}_2(y))\right) \in J$$

is either a forward singular point of J, or $\Pi(y)$ belongs to the set A of regular sufficient points of J, defined in 4.4.

Proof. Since $\bar{t}_2(y) < t_{\bar{k}_1(y)-1}(y)$, we get that, indeed, $\Pi(y) \in J$. Assume that the forward orbit of $\Pi(y)$ is non-singular.

Since $S^{\bar{t}_2(y)}y \in \overline{\left(U_\delta \setminus U_{\delta/2}\right) \cap D_0}$, we get that $\operatorname{dist}\left(S^{\bar{t}_2(y)}y,\, J\right) \ge \delta/2$. On the other hand, by using (4.33) and Corollary 4.20, we get that for a small enough contraction coefficient L_0^{-1} the distance between $S^{\bar{t}_2(y)}y$ and $S^{\bar{t}_2(y)}(\Pi(y))$ is less than $\delta/4$. (The argument is the same as in the proof of Lemma 4.30.) In this way we obtain that $S^{\bar{t}_2(y)}(\Pi(y)) \in U_0 \setminus J$, so $\Pi(y) \in A$, according to condition (4)/(ii) in 4.1. $\square$

Corollary 4.36. $\mu\left(E_\delta(c_3)\right)=0$.

Proof. The proof will be analogous with the proof of Corollary 4.31 above. Indeed, $A=\emptyset$ (according to our indirect assumption), and the exceptional set of Lemma 4.2 (i. e. the set of all forward singular points of J) is a countable union of smooth, proper submanifolds of J. Since the curves $\rho_{y,\bar{t}_2(y)}$ belong to (at least) the C^2 smoothness class, we get that $E_\delta(c_3)$ is a countable union of proper, C^2-smooth submanifolds of U_δ, hence $\mu\left(E_\delta(c_3)\right)=0$. $\square$

For the points $y\in F_\delta(c_3)$ we define the projection $\Pi(y)$ by the formula

$$\Pi(y):=S^{t_{\overline{k}_1(y)-1}(y)}y\in\partial\mathbf{M}. \tag{4.37}$$

Now we prove

Lemma 4.38. For the measure $\nu\left(\Pi\left(F_\delta(c_3)\right)\right)$ of the projected set $\Pi\left(F_\delta(c_3)\right)\subset\partial\mathbf{M}$ we have the upper estimate

$$\nu\left(\Pi\left(F_\delta(c_3)\right)\right)\le c_2c_4L_0^{-1}\delta,$$

where $c_2>0$ is the geometric constant (also denoted by c_2) in Lemma 2 of [S-Ch(1987)] or in Lemma 4.10 of [K-S-Sz(1990)-I], c_4 is the constant in (4.22) above, and ν is the natural T-invariant measure on $\partial\mathbf{M}$ that can be obtained by projecting the Liouville measure μ onto $\partial\mathbf{M}$ along the billiard flow.

Proof. Let $y\in F_\delta(c_3)$. From the inequality $t_{\overline{k}_1(y)-1}(y)\ge c_3$ and from Corollary 4.21 we conclude that $z_{tub}\left(\Pi(y)\right)<c_4L_0^{-1}\delta$. This inequality, along with the fundamental measure estimate of Lemma 2 of [S-Ch(1987)] (see also Lemma 4.10 in [K-S-Sz(1990)-I]) yield the required upper estimate for $\nu\left(\Pi\left(F_\delta(c_3)\right)\right)$. $\square$

The next lemma claims that the projection $\Pi:\ F_\delta(c_3)\to\partial\mathbf{M}$ (considered here only on the set $F_\delta(c_3)$) is "essentially one-to-one", from the point of view of the Poincaré section.

Lemma 4.39. Suppose that $y_1,\,y_2\in F_\delta(c_3)$ are non-periodic points ($\delta\le\delta_1$), and $\Pi(y_1)=\Pi(y_2)$. We claim that y_1 and y_2 belong to an orbit segment S of the billiard flow lying entirely in the one-sided neighborhood $U_0=U_{\delta_0}$ of J and, consequently, the length of the segment S is at most $1.1\mathrm{diam}(J)$.

Remark. We note that, obviously, in the length estimate $1.1\mathrm{diam}(J)$ above, the coefficient 1.1 could be replaced by any number bigger than 1, provided that the parameter $\delta>0$ is small enough.

Proof. The relation $\Pi(y_1) = \Pi(y_2)$ implies that y_1 and y_2 belong to the same orbit, so we can assume, for example, that $y_2 = S^a y_1$ with some $a > 0$. We need to prove that $S^{[0,a]} y_1 \subset U_0$. Assume the opposite, i. e. that there is a number t_1, $0 < t_1 < a$, such that $S^{t_1} y_1 \notin U_0$. This, and the relation $S^a y_1 \in \left(U_\delta \setminus U_{\delta/2}\right) \cap D_0$ mean that the first return of y_1 to $\left(U_\delta \setminus U_{\delta/2}\right) \cap D_0$ occurs not later than at time $t = a$. On the other hand, since $\Pi(y_1) = \Pi(S^a y_1)$ and y_1 is non-periodic, we get that $t_{\overline{k}_1(y_1)-1}(y_1) > a$, see (4.37). The obtained inequality $t_{\overline{k}_1(y_1)-1}(y_1) > a \geq \overline{t}_2(y)$, however, contradicts to the definition of the set $F_\delta(c_3)$, to which y_1 belongs as an element, see (4.32). The upper estimate $1.1\text{diam}(J)$ for the length of S is an immediate corollary of the containment $S \subset U_0$. $\square$

As a direct consequence of lemmas 4.38 and 4.39, we obtain

Corollary 4.40. For all small enough $\delta > 0$ the inequality

$$\mu\left(F_\delta(c_3)\right) \leq 1.1 c_2 c_4 L_0^{-1} \delta \text{diam}(J)$$

holds true.

Finishing the Indirect Proof of Main Lemma 4.5.

It follows immediately from Lemma 4.27 and corollaries 4.31, 4.36, and 4.40 that

$$\mu\left(\overline{U}_\delta\right) \leq 1.2 c_2 c_4 \text{diam}(J) L_0^{-1} \delta$$

for all small enough $\delta > 0$. This fact, however, contradicts to (4.24) if L_0^{-1} is selected so small that

$$1.2 c_2 c_4 \text{diam}(J) L_0^{-1} < \frac{1}{4} \mu_1(J^*),$$

where J^* stands for the original exceptional manifold before the three slight shrinkings in the style of Note 4.18. Clearly, $\mu_1(J) > (1 - 10^{-5})\mu_1(J^*)$. The obtained contradiction finishes the indirect proof of Main Lemma 4.5. $\square$

§5. Proof of Ergodicity
The Induction on N

By using several results of Sinai [Sin(1970)], Chernov-Sinai [S-Ch(1987)], and Krámli-Simányi-Szász, in this section we finally prove the ergodicity (hence also the Bernoulli property, see Chernov-Haskell [C-H(1996)] or Ornstein-Weiss [O-W(1998)]) for every hard ball system $(\mathbf{M}, \{S^t\}, \mu)$, under the assumption of the Ansatz for the considered hard ball system $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ and for all of its subsystems, by carrying out an induction on the number N (≥ 2) of interacting balls.

The base of the induction (i. e. the ergodicity of any two-ball system on a flat torus) was proved in [Sin(1970)] and [S-Ch(1987)].

Assume now that ($\mathbf{M}$, $\{S^t\}$, μ) is a given system of N (≥ 3) hard spheres with masses $m_1, m_2, \dots, m_N$ and radius $r > 0$ on the flat unit torus $\mathbb{T}^\nu = \mathbb{R}^\nu/\mathbb{Z}^\nu$ ($\nu \geq 2$), as defined in §2. Assume further that the ergodicity of every such system is already proved to be true for any number of balls N' with $2 \leq N' < N$, and the Chernov-Sinai Ansatz is true for the considered N-ball system $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$. We will carry out the induction step by following the strategy for the proof laid down in the series of papers [K-S-Sz(1989)], [K-S-Sz(1990)-I], [K-S-Sz(1991)], and [K-S-Sz(1992)].

By using the induction hypothesis, Theorem 5.1 of [Sim(1992)-I], together with the slimness of the set Δ_2 of doubly singular phase points, shows that there exists a slim subset $S_1 \subset \mathbf{M}$ of the phase space such that for every $x \in \mathbf{M} \setminus S_1$ the point x has at most one singularity on its entire orbit $S^{(-\infty,\infty)}x$, and each branch of $S^{(-\infty,\infty)}x$ is not eventually splitting in any of the time directions. By Corollary 3.26 and Lemma 4.2 of [Sim(2002)] there exists a locally finite (hence countable) family of codimension-one, smooth, exceptional submanifolds $J_i \subset \mathbf{M}$ such that for every point $x \notin (\bigcup_i J_i) \cup S_1$ the orbit of x is sufficient (geometrically hyperbolic). This means, in particular, that the considered hard ball system $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ is fully hyperbolic.

By the assumed Ansatz (the ultimate global hypothesis of the Theorem on Local Ergodicity by Chernov and Sinai, Theorem 5 in [S-Ch(1987)], see also Corollary 3.12 in [K-S-Sz(1990)-I] and the main result of [B-Ch-Sz-T(2002)]) an open neighborhood $U_x \ni x$ of any phase point $x \notin (\bigcup_i J_i) \cup S_1$ belongs to a single ergodic component of the billiard flow. (Modulo the zero sets, of course.) Therefore, the billiard flow $\{S^t\}$ has at most countably many, open ergodic components C_1, C_2, $\dots$.

Remark. Note that theorem 5.1 of [Sim(1992)-I] (used above) requires the induction hypothesis as an assumption.

Assume that, contrary to the statement of our theorem, the number of ergodic components C_1, C_2, $\dots$ is more than one. The above argument shows that, in this case, there exists a codimension-one, smooth (actually analytic) submanifold $J \subset \mathbf{M} \setminus \partial\mathbf{M}$ separating two different ergodic components C_1 and C_2, lying on the two sides of J. By the Theorem on Local Ergodicity for semi-dispersive billiards, no point of J has a sufficient orbit. (Recall that sufficiency is clearly an open property, so the existence of a sufficient point $y \in J$ would imply the existence of a sufficient point $y' \in J$ with a non-singular orbit.) By shrinking J, if necessary, we can achieve that the infinitesimal Lyapunov function $Q(n)$ be separated from zero on J, where n is a unit normal field of J. By replacing J with its time-reversed copy

$$-J = \left\{(q, v) \in \mathbf{M} \middle|\ (q, -v) \in J\right\},$$

if necessary, we can always achieve that $Q(n) \leq -c_1 < 0$ uniformly across J.

There could be, however, a little difficulty in achieving the inequality $Q(n) < 0$ across J. Namely, it may happen that $Q(n_t) = 0$ for every $t \in \mathbb{R}$. This is, however, shown to be impossible in Remark 7.9 of [Sim(2003)].

To make sure that the submanifold J is neatly shaped (i. e. it fulfills (2) of 4.1) is a triviality. Condition (3) of 4.1 clearly holds true. We can achieve (4) as follows: Select a base point $x_0 \in J$ with a non-singular and not eventually splitting forward orbit $S^{(0,\infty)}x_0$. This can be done according to the transversality result Lemma 4.3 (see also 7.12 in [Sim(2003)]), and by using the fact that the points with an eventually splitting forward orbit form a slim set in $\mathbf{M}$ (Theorem 5.1 of [Sim(1992)-I]), henceforth a set of first category in J. After this, choose a large enough time $T > 0$ so that $S^T x_0 \notin \partial\mathbf{M}$, and the symbolic collision sequence $\Sigma_0 = \Sigma\left(S^{[0,T]}x_0\right)$ is combinatorially rich in the sense of Definition 3.28 of [Sim(2002)]. By further shrinking J, if necessary, we can assume that $S^T(J) \cap \partial\mathbf{M} = \emptyset$ and S^T is smooth on J. Choose a thin, tubular neighborhood $\tilde{U}_0$ of J in $\mathbf{M}$ in such a way that S^T be still smooth across $\tilde{U}_0$, and define the set

$$NS\left(\tilde{U}_0,\, \Sigma_0\right) = \left\{x \in \tilde{U}_0 \middle|\, \dim\mathcal{N}_0\left(S^{[0,T]}x\right) > 1\right\} \tag{5.1}$$

of not Σ_0-sufficient phase points in $\tilde{U}_0$. Clearly, $J \subset NS\left(\tilde{U}_0,\, \Sigma_0\right)$. We can assume that the selected (generic) base point $x_0 \in J$ belongs to the smooth part of the closed algebraic set $NS\left(\tilde{U}_0,\, \Sigma_0\right)$. This guarantees that actually $J = NS\left(\tilde{U}_0,\, \Sigma_0\right)$, as long as the manifold J and its tubular neighborhood are selected small enough, thus achieving property (4) of 4.1.

Proof of why property (6) of Definition 4.1 can be assumed.

We recall that J is a codimension-one, smooth manifold of non-sufficient phase points separating two open ergodic components, as described in (0)–(3) at the end of §3 of [Sim(2003)].

Let P be the subset of J containing all points with non-singular forward orbit and recurring to J infinitely many times.

Lemma 5.2. $\mu_1(P) = 0$.

Proof. Assume that $\mu_1(P) > 0$. Take a suitable Poincaré section to make the time discrete, and consider the on-to-one first return map $T:\ P \to P$ of P. According to the measure expansion theorem for hypersurfaces J (with negative infinitesimal Lyapunov function $Q(n)$ for their normal field n), proved in [Ch-Sim(2006)], the measure $\mu_1\left(T(P)\right)$ is strictly larger than $\mu_1(P)$, though $T(P) \subset P$. The obtained contradiction proves the lemma. $\square$

Next, we claim that the above lemma is enough for our purposes to prove (6) of 4.1. Indeed, the set $W \subset J$ consisting of all points $x \in J$ never again returning to

J (after leaving it first, of course) has positive μ_1-measure by Lemma 5.2. Select a Lebesgue density base point $x_0 \in W$ for W with a non-singular forward orbit, and shrink J at the very beginning to such a small size around x_0 that the relative measure of W in J be bigger than $1 - 10^{-8}$.

Finally, Main Lemma 4.5 asserts that $A \neq \emptyset$, contradicting to our earlier statement that no point of J is sufficient. The obtained contradiction completes the inductive step of the proof of the Theorem. $\square$

Appendix. The Constants of §3–4

In order to make the reading of sections 3–4 easier, here we briefly describe the hierarchy of the constants used in those sections.

1. The geometric constant $c_0 > 0$ of Proposition 3.5 is a lower estimation for the "curvature" $\langle \delta q_0, \delta v_0 \rangle / ||\delta q_0||^2$ of an expanding tangent vector $(\delta q_0, \delta v_0)$.

2. The geometric constant $-c_1 < 0$ provides an upper estimation for the infinitesimal Lyapunov function $Q(n)$ of J in (5) of Definition 4.1. It cannot be freely chosen in the proof of Main Lemma 4.5.

3. The constant $c_2 > 0$ is present in the upper measure estimation of Lemma 2 of [S-Ch(1987)], or Lemma 4.10 in [K-S-Sz(1990)-I]. It cannot be changed in the course of the proof of Main Lemma 4.5.

4. The contraction coefficient $0 < L_0^{-1} << 1$ plays a role all over §4. It must be chosen suitably small by selecting the time threshold $c_3 >> 1$ large enough (see Corollary 4.20), after having fixed U_0, δ_0, and J. The phrase "suitably small" for L_0^{-1} means that the inequality

$$L_0^{-1} < \frac{0.25\mu_1(J^*)}{1.2c_2c_4\mathrm{diam}(J)}$$

should be true, see the end of §4.

5. The geometric constant $c_4 > 0$ of (4.22) bridges the gap between two distances: the distance $\mathrm{dist}(y, J)$ between a point $y \in U_\delta$ and J, and the arc length $l_q\left(\rho_{y,\overline{k}(y)}\right)$. It cannot be freely chosen during the proof of Main Lemma 4.5.

Acknowledgement. The author expresses his sincere gratitude to N. I. Chernov for his numerous, very valuable questions, remarks, and suggestions.

References

[B-Ch-Sz-T(2002)] P. Bálint, N. Chernov, D. Szász, I. P. Tóth, *Multidimensional semidispersing billiards: singularities and the fundamental theorem*, Ann. Henri Poincaré **3, No. 3 (2002)**, 451–482.

[B-F-K(1998)] D. Burago, S. Ferleger, A. Kononenko, *Uniform estimates on the number of collisions in semi-dispersing billiards*, Annals of Mathematics **147 (1998)**, 695-708.

[B-R(1997)] L. A. Bunimovich, J. Rehacek, *Nowhere Dispersing 3D Billiards with Non-vanishing Lyapunov Exponents*, Commun. Math. Phys. **189 (1997), no. 3**, 729–757.

[B-R(1998)] L. A. Bunimovich, J. Rehacek, *How High-Dimensional Stadia Look Like*, Commun. Math. Phys. **197 (1998), no. 2**, 277–301.

[Ch(1982)] N. I. Chernov, *Construction of transverse fiberings in multidimensional semi-dispersed billiards*, Functional Anal. Appl. **16 (1982), no. 4**, 270–280.

[Ch(1994)] N. I. Chernov, *Statistical Properties of the Periodic Lorentz Gas. Multidimensional Case*, Journal of Statistical Physics **74, Nos. 1/2**, 11-54.

[C-H(1996)] N. I. Chernov, C. Haskell, *Nonuniformly hyperbolic K-systems are Bernoulli*, Ergod. Th. & Dynam. Sys. **16 (1996)**, 19-44.

[Ch-Sim(2006)] N. I. Chernov, N. Simányi, *Flow-invariant hypersurfaces in hard ball systems*, Manuscript, under preparation.

[E(1978)] R. Engelking, *Dimension Theory*, North Holland.

[G(1981)] G. Galperin, *On systems of locally interacting and repelling particles moving in space*, Trudy MMO **43 (1981)**, 142-196.

[K-B(1994)] A. Katok, K. Burns, *Infinitesimal Lyapunov functions, invariant cone families and stochastic properties of smooth dynamical systems*, Ergodic Theory Dyn. Syst. **14, No. 4**, 757-785.

[K-S-Sz(1989)] A. Krámli, N. Simányi, D. Szász, *Ergodic Properties of Semi–Dispersing Billiards I. Two Cylindric Scatterers in the 3–D Torus*, Nonlinearity **2 (1989)**, 311–326.

[K-S-Sz(1990)-I] A. Krámli, N. Simányi, D. Szász, *A "Transversal" Fundamental Theorem for Semi-Dispersing Billiards*, Commun. Math. Phys. **129 (1990)**, 535–560.

[K-S-Sz(1990)-II] A. Krámli, N. Simányi, D. Szász, *Erratum. A "Transversal" Fundamental Theorem for Semi-Dispersing Billiards*, Commun. Math. Phys. **138 (1991)**, 207–208.

[K-S-Sz(1991)] A. Krámli, N. Simányi, D. Szász, *The K–Property of Three Billiard Balls*, Annals of Mathematics **133 (1991)**, 37–72.

[K-S-Sz(1992)] A. Krámli, N. Simányi, D. Szász, *The K–Property of Four Billiard Balls*, Commun. Math. Phys. **144 (1992)**, 107-148.

[L-W(1995)] C. Liverani, M. Wojtkowski, *Ergodicity in Hamiltonian systems*, Dynamics Reported **4**, 130-202, arXiv:math.DS/9210229..

[O-W(1998)] D. Ornstein, B. Weiss, *On the Bernoulli Nature of Systems with Some Hyperbolic Structure*, Ergod. Th. & Dynam. Sys. **18 (1998)**, 441-456.

[Sim(1992)-I] N. Simányi, *The K-property of N billiard balls I*, Invent. Math. **108 (1992)**, 521-548.

[Sim(1992)-II] N. Simányi, *The K-property of N billiard balls II*, Invent. Math. **110 (1992)**, 151-172.

[Sim(2002)] N. Simányi, *The Complete Hyperbolicity of Cylindric Billiards*, Ergodic Th. & Dyn. Sys. **22 (2002)**, 281-302.

[Sim(2003)] N. Simányi, *Proof of the Boltzmann-Sinai Ergodic Hypothesis for Typical Hard Disk Systems*, Inventiones Mathematicae **154, No. 1 (2003)**, 123-178.

[Sim(2004)] N. Simányi, *Proof of the Ergodic Hypothesis for Typical Hard Ball Systems*, Annales Henri Poincaré **5 (2004)**, 203-233.

[Sin(1963)] Ya. G. Sinai, *On the Foundation of the Ergodic Hypothesis for a Dynamical System of Statistical Mechanics*, Soviet Math. Dokl. **4 (1963)**, 1818-1822.

[Sin(1970)] Ya. G. Sinai, *Dynamical Systems with Elastic Reflections*, Russian Math. Surveys **25:2 (1970)**, 137-189.

[Sin(1979)] Ya. G. Sinai, *Development of Krylov's ideas. Afterword to N. S. Krylov's "Works on the foundations of statistical physics", see reference [K(1979)]*, Princeton University Press.

[S-Ch(1987)] Ya. G. Sinai, N.I. Chernov, *Ergodic properties of certain systems of 2–D discs and 3–D balls*, Russian Math. Surveys **42, No. 3 (1987)**, 181-207.

[S-Sz(1994)] N. Simányi, D. Szász, *The K-property of 4-D Billiards with Non-Orthogonal Cylindric Scatterers*, J. Stat. Phys. **76, Nos. 1/2**, 587-604.

[S-Sz(1999)] N. Simányi, D. Szász, *Hard ball systems are completely hyperbolic*, Annals of Mathematics **149 (1999)**, 35-96.

[S-Sz(2000)] N. Simányi, D. Szász, *Non-integrability of Cylindric Billiards and Transitive Lie Group Actions*, Ergod. Th. & Dynam. Sys. **20 (2000)**, 593-610.

[Sz(1994)] D. Szász, *The K-property of 'Orthogonal' Cylindric Billiards*, Commun. Math. Phys. **160 (1994)**, 581-597.

[V(1979)] L. N. Vaserstein, *On Systems of Particles with Finite Range and/or Repulsive Interactions*, Commun. Math. Phys. **69 (1979)**, 31-56.