Cluster expansion for abstract polymer models. New bounds from an old approach

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Abstract

We revisit the classical approach to cluster expansions, based on tree graphs, and establish a new convergence condition that improves those by Kotecký-Preiss and Dobrushin, as we show in some examples. The strategy is to better exploit a well known tree-graph expression, due to Penrose.

1. Introduction

Cluster expansions, originally developed to express thermodynamic potentials as power series in activities, are at the heart of important perturbative arguments in statistical mechanics and other branches of mathematical physics. The classical approach to obtain convergence conditions was based on combinatorial considerations [9, 17], which were greatly simplified through the use of tree-graph bounds [4, 2]. A completely new inductive approach originated in the work of Kotecký and Preiss [8], later refined by Dobrushin [5, 6] and many others [11, 1, 10, 18, 16]. This later approach is mathematically very appealing and, in its more elegant version [5, 16], it even disposes of any reference to power series, becoming, in Dobrushin's words, a "no-cluster-expansion" approach. The combinatorial approach, however, kept its adepts who reformulated it in a very clear and compact way [12] and showed how it can lead to bounds at least as good as those given by Kotecký and Preiss [14].

In this paper, we revisit the classical combinatorial approach and point out that it can be used, in a rather simple and natural way, to produce improved bounds on the convergence region (and the sum of the expansion). Our main tool is an *identity*, due to Oliver Penrose [13], relating the coefficients of the expansion to a family of trees determined by compatibility constraints. (In fact, we learnt this identity from the nice exposition in [12, Section 3].) Succesive approximations are obtained by considering larger families of trees that neglect some of the constraints. If only the very basic constraint is kept (links in the tree must relate incompatible objects), the Kotecky-Preiss condition emerges. To the next order of precision (branches must end in different objects) Dobrushin's condition is found. By refining this last constraint (branches' ends must be mutually compatible rather than just different) we obtain a new convergence condition which leads to improvements in several well-studied cases. All these conditions are obtained for general systems under minimal hypotheses. The same approach should yield further improvements for more specific systems through a better description of the resulting constraints on tree diagrams.

2. The set up and the problem

We adopt the abstract polymer setting of [8]. Its starting point is an unoriented graph $\mathcal{G} = (\mathcal{P}, \mathcal{E})$ —the *interaction graph*— on a countable vertex set. The vertices $\gamma \in \mathcal{P}$ are called *polymers* for historical reasons [7] (the name is misleading; Dobrushin [6] proposes to call them *animals*, but the traditional name holds on). The edge set corresponds to an *incompatibility relation*: Two polymers γ, γ' are incompatible if $\{\gamma, \gamma'\} \in \mathcal{E}$, in which case we write $\gamma \nsim \gamma'$. Otherwise they are *compatible* and we write $\gamma \sim \gamma'$. The set of edges is arbitrary, except for the assumption that it contains all pairs of the form $\{\gamma, \gamma\}$, that is, *every polymer is assumed to be incompatible with itself.* In in particular vertices can be of infinite degree (each polymer can be incompatible with infinitely many other polymers). This happens, for instance, for graphs associated to gases of low-temperature contours or "defects".

The physical information of each polymer model is given by the incompatibility relation and a family of *activities* $\underline{z} = \{z_\gamma\}_{\gamma \in \mathcal{P}} \in \mathbb{C}^{\mathcal{P}}$. For each *finite* family $\Lambda \subset \mathcal{P}$, these ingredients define probability weights on the set of subsets of Λ:

$$\mathrm{Prob}_\Lambda(\{\gamma_1, \gamma_2, \ldots, \gamma_n\}) \;=\; \frac{1}{\Xi_\Lambda(\underline{z})}\, z_{\gamma_1} z_{\gamma_2} \cdots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j \sim \gamma_k\}} \tag{2.1}$$

for $n \geq 1$ and $\mathrm{Prob}_\Lambda(\emptyset) = 1/\Xi_\Lambda$, where

$$\Xi_\Lambda(\underline{z}) = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{(\gamma_1, \ldots, \gamma_n) \in \Lambda^n} z_{\gamma_1} z_{\gamma_2} \ldots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j \sim \gamma_k\}} \;. \tag{2.2}$$

In physical terms, the measure (2.1) corresponds to the grand-canonical ensemble of a polymer gas with activities $\underline{z}$ and hard-core interaction defined by the incompatibility relation. The abstract formalism makes it equivalent to a lattice gas on the graph $\mathcal{G}$ with self- and nearest-neighbor hard-core repulsion. The normalization constant (2.2) is the grand-canonical partition function in the "volume" Λ. Cluster expansions allow the control of the measures (2.1) uniformly in Λ and absolutely in the activities. [Thus, the control extends to the unphysical region of non-positive (complex) activities, where the expressions on the right-hand side of (2.1) do not define probability measures.] This is achieved by determining the radius of convergence of the logarithm of the partition function, whose well-known formal power series (Mayer expansion, see e.g. [15]) is

$$\log \Xi_\Lambda(\underline{z}) \;=\; \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1, \ldots, \gamma_n) \subset \Lambda^n} \phi^T(\gamma_1, \ldots, \gamma_n)\, z_{\gamma_1} \ldots z_{\gamma_n} \tag{2.3}$$

with

$$\phi^T(\gamma_1, \ldots, \gamma_n) \;=\; \begin{cases} 1 & n = 1 \\[2mm] 0 & n \geq 2\,,\ \mathcal{G}_{\{\gamma_1, \ldots, \gamma_n\}} \text{ not connected} \\[2mm] \displaystyle\sum_{\substack{G \subset \mathcal{G}_{\{\gamma_1, \ldots, \gamma_n\}} \\ G \text{ connected}}} (-1)^{|E(G)|} & n \geq 2\,,\ \mathcal{G}_{\{\gamma_1, \ldots, \gamma_n\}} \text{ connected} \end{cases} \tag{2.4}$$

where $\mathcal{G}_{\{\gamma_1, \ldots, \gamma_n\}}$ is the restriction of $\mathcal{G}$ to the vertex set $\{\gamma_1, \ldots, \gamma_n\}$ and G ranges over all its connected subgraphs, each of which has edge set $E(G)$. The functions ϕ^T are the *truncated functions* of order n (also called *Ursell functions*).

Simple arguments show that the uniform convergence of (2.3) is determined by that of the "pinned" (formal) expansions

$$\Sigma_{\gamma_0}(\underline{\rho}) \;:=\; \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(\gamma_1,\ldots,\gamma_n)\subset\mathcal{P}^n \\ \exists i:\, \gamma_i=\gamma_0}} \bigl|\phi^T(\gamma_1,\ldots,\gamma_n)\bigr|\; \rho_{\gamma_1}\ldots\rho_{\gamma_n} \tag{2.5}$$

for $\underline{\rho} \in [0,\infty)^{\mathcal{P}}$. The convergence of this positive-term series for a certain $\underline{\rho}$ implies the absolute convergence of (2.3) —and the control of the measures (2.1) [6]— uniformly for $|z_\gamma| \le \rho_\gamma$.

A slightly more efficient alternative is to consider, instead of (2.5), the formal series

$$\Pi_{\gamma_0}(\underline{\rho}) \;:=\; 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\subset\mathcal{P}^n} \bigl|\phi^T(\gamma_0,\gamma_1,\ldots,\gamma_n)\bigr|\; \rho_{\gamma_1}\ldots\rho_{\gamma_n}\;. \tag{2.6}$$

The convergence of (2.6) implies that of (2.5) via the relation

$$\Sigma_{\gamma_0}(\underline{\rho}) \;=\; \rho_{\gamma_0} \int_0^1 \Pi_{\gamma_0}(\underline{\rho}^{\alpha})\, d\alpha \tag{2.7}$$

with

$$\rho^{\alpha}_{\gamma} \;=\; \begin{cases} \rho_\gamma & \text{if } \gamma \ne \gamma_0 \\[1ex] \alpha\rho_\gamma & \text{if } \gamma = \gamma_0 \end{cases}$$

and, thus,

$$\Sigma_{\gamma_0}(\underline{\rho}) \;\le\; \rho_{\gamma_0}\, \Pi_{\gamma_0}(\underline{\rho})\;. \tag{2.8}$$

In the sequel we focus on the convergence of the series (2.6) for positive activities.

3. New convergence criterion

Our new criterion involves the grand-canonical partition functions $\Xi_{\mathcal{P}_{\gamma_0}}$, associated to the polymer families $\mathcal{P}_{\gamma_0} = \{\gamma \in \mathcal{P} : \gamma \nsim \gamma_0\}$, $\gamma_0 \in \mathcal{P}$. These functions, defined in (2.2), can also be written in the form

$$\Xi_{\mathcal{P}_{\gamma_0}}(\underline{\mu}) = 1 + \sum_{n\ge 1} \frac{1}{n!} \sum_{\substack{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n \\ \gamma_0\nsim\gamma_i\,,\, \gamma_i\sim\gamma_j\,,\, 1\le i,j\le n}} \mu_{\gamma_1}\mu_{\gamma_2}\ldots\mu_{\gamma_n} \tag{3.1}$$

because compatible polymers are different.

Theorem 1 *Let $\underline{\rho} \in [0,\infty)^{\mathcal{P}}$. If there exists a $\underline{\mu} \in [0,\infty)^{\mathcal{P}}$ such that*

$$\rho_{\gamma_0}\, \Xi_{\mathcal{P}_{\gamma_0}}(\underline{\mu}) \;\le\; \mu_{\gamma_0}\;, \quad \forall \gamma_0 \in \mathcal{P}\;, \tag{3.2}$$

then the series $\Pi_{\gamma_0}(\rho)$ [defined in (2.6)] converges for such $\underline{\rho}$ and satisfies $\rho_{\gamma_0}\,\Pi_{\gamma_0}(\rho) \le \mu_{\gamma_0}$.

Remark 2 Kotecký-Preiss and Dobrushin criteria produce similar results, but starting from the conditions

$$\rho_{\gamma_0}\; \exp\Bigl[\sum_{\gamma\nsim\gamma_0} \mu_\gamma\Bigr] \;\le\; \mu_{\gamma_0} \quad \text{(Kotecký-Preiss)} \tag{3.3}$$

and

$$\rho_{\gamma_0} \prod_{\gamma \nsim \gamma_0} (1+\mu_\gamma) \;\le\; \mu_{\gamma_0} \quad \text{(Dobrushin)}\,. \tag{3.4}$$

The inequalities

$$\Xi_{\mathcal{P}_{\gamma_0}}(\underline{\mu}) \;\le\; \prod_{\gamma \nsim \gamma_0} (1+\mu_\gamma) \;\le\; \exp\Big[\sum_{\gamma \nsim \gamma_0} \mu_\gamma\Big] \tag{3.5}$$

show that condition (3.2) is an improvement over Dobrushin's condition which in turns is an improvement over Kotecký-Preiss' condition. The examples below show that the successive improvements can be noticeable.

Remark 3 Condition (3.2) can be exploited in two complementary directions: (i) To find the radius of convergence, that is the largest values ρ_γ for which there is a $\underline{\mu}$ satisfying (3.2). (ii) To find the sharpest bound on Π_{γ_0}, that is, the smallest values μ_γ yielding (3.2) for a fixed $\underline{\rho}$. In the following examples we focus in direction (i). Regarding direction (ii), Corollary 5 below implies that bounds on Π_{γ_0} can be improved up to limit functions $\Pi^*(\underline{\rho},\underline{\mu})$ and $\Pi^*(\underline{\rho})$ constructed as follows. Let $T : [0,\infty)^{\mathcal{P}} \to [0,\infty]^{\mathcal{P}}$ defined by

$$T_{\gamma_0}(\underline{\mu}) \;=\; \underline{\rho}\, \Xi_{\mathcal{P}_{\gamma_0}}(\underline{\mu}) \tag{3.6}$$

for each $\gamma_0 \in \mathcal{P}$. Then,

$$\mu_{\gamma_0} \ge T_{\gamma_0}(\underline{\mu}) \ge \cdots \ge T^n_{\gamma_0}(\underline{\mu}) \underset{n\to\infty}{\searrow} T^\infty_{\gamma_0}(\underline{\mu}) =: \rho_{\gamma_0}\, \Pi^*_{\gamma_0}(\underline{\rho},\underline{\mu}) \ge T^\infty_{\gamma_0}(\underline{\rho}) =: \rho_{\gamma_0}\, \Pi^*_{\gamma_0}(\underline{\rho}) \ge \rho_{\gamma_0}\, \Pi_{\gamma_0}(\underline{\rho})\,. \tag{3.7}$$

Similar chains of inequalities hold within the Kotecký-Preiss and Dobrushin criteria, for operators T defined as in (3.6) but with $\Xi_{\mathcal{P}_{\gamma_0}}(\underline{\mu})$ substituted, respectively, by $\exp[\sum_{\gamma\nsim\gamma_0}\mu_\gamma]$ and $\prod_{\gamma\nsim\gamma_0}(1+\mu_\gamma)$. Of course, $\Pi^{*\mathrm{KP}} \ge \Pi^{*\mathrm{Dob}} \ge \Pi^*$.

To test our criterion we compare the criteria (3.2)–(3.4) for two families of benchmark examples.

Polymers with finitely many incompatibilities These are examples where $\mathcal{G}$ has finite coordination numbers. We shall suppose that all polymers have equal activity $\rho_\gamma \equiv \rho$ for all $\gamma \in \mathcal{G}$, and therefore we search for equally constant functions $\mu_\gamma \equiv \mu$. The preceding criteria take the form $\rho \le \mu/\varphi(\mu)$ for appropriate functions φ, and the maximization of the right-hand side with respect to μ yields the best lower bounds of the radius of convergence of (2.6) [and hence of (2.3)]. We study two systems. First, the domino model in $\mathbb{Z}^2$, in which polymers are pairs of nearest-neighbor sites and incompatibility is just overlapping. Second, the hard-core lattice gas on graphs of bounded degree (=number of neighbors). To highlight differences in criteria, we consider two graphs, which we denote $\mathcal{G}_\Delta$ and $\mathcal{G}_{\mathrm{Triang}}$. The first one is any graph of maximum degree Δ where *all neighbors to a vertex are compatible between themselves* (e.g. $\mathbb{Z}^d$, trees). The second graph is the bidimensional triangular lattice ($\Delta = 6$). Calculations and bounds on the radius of convergence are summarized in Tables 1 and 2.

We see that Dobrushin and Kotecký-Preiss criteria are only sensitive to the coordination number, that is, the number of polymers that are incompatible to a given one. In the new condition (3.2), incompatibilities between these neighbors are also accounted for. This is not the only difference, because there is also a (less important) improvement for the graph $\mathcal{G}_\Delta$, where neighbors are mutually

Model	Kotecký-Preiss	Dobrushin	(3.2)
Hard-core in $\mathcal{G}_\Delta$	$\rho \le \mu\, e^{-(\Delta+1)\mu}$	$\rho \le \dfrac{\mu}{(1+\mu)^{\Delta+1}}$	$\rho \le \dfrac{\mu}{\mu + (1+\mu)^{\Delta}}$
Hard-core in $\mathcal{G}_{\rm Triang}$	$\rho \le \mu\, e^{-7\mu}$	$\rho \le \dfrac{\mu}{(1+\mu)^7}$	$\rho \le \dfrac{\mu}{1+7\mu+8\mu^2+2\mu^3}$
Domino in $\mathbb{Z}^2$	Same as above	Same as above	$\rho \le \dfrac{\mu}{1+7\mu+9\mu^2}$

Table 1: Comparison of convergence conditions for some polymer systems with finitely many incompatibilities

Model	Kotecký-Preiss	Dobrushin	(3.2)
Hard-core in $\mathcal{G}_\Delta$	$\dfrac{1}{(\Delta+1)\, e}$	$\dfrac{\Delta^\Delta}{(\Delta+1)^{\Delta+1}}$	$\dfrac{1}{1+\frac{\Delta^\Delta}{(\Delta-1)^{\Delta-1}}}$
Hard-core in $\mathcal{G}_{\rm Triang}$	$\dfrac{1}{7e} \approx 0.0525$	$\dfrac{\frac{1}{6}}{(1+\frac{1}{6})^7} \approx 0.0566$	$4R^3+8R^2=1\ ,\ R\approx 0,078$
Domino in $\mathbb{Z}^2$	Same as above	Same as above	$\dfrac{1}{13} \approx 0.0769$

Table 2: Lower bounds of the radius of convergence for cluster expansions of some polymer systems with finitely many incompatibilities

compatible. The best bound available for this last graph, however, is due to Scott and Sokal [16]. Their estimation of the radius of convergence is

$$R \;=\; \frac{(\Delta-1)^{(\Delta-1)}}{\Delta^\Delta}\;, \tag{3.8}$$

which, as they show, is optimal for trees. While our estimate approaches (3.8) for large Δ, the differences for small Δ are important (1/4 vs. 1/5 for $\Delta = 2$). On the other hand, our estimate for the triangular graph is better than that coming from (3.8) (0.078 vs. 0.067), precisely because of the presence of incompatibilities between neighbors of a site. The estimate (3.8) was derived through a sequence of volume-dependent Dobrushin conditions. It would be interesting to see whether a similar strategy could be developed within our approach.

Random sets on a graph This is the general example of cluster expansions for graphs with vertices of infinite degree. Applications include contour ensembles of low-temperature phases, geometrical objects of high-temperature expansions, random sets of the Fortuin-Kasteleyn representation of the Potts model, . . . The general setup for these models is a polymer family formed by the finite parts of a given set $\mathbb{V}$ with incompatibility defined by overlapping. (Usually, $\mathbb{V}$ is formed by the vertices of a graph with respect to which polymers form connected sets.)

For these systems it is useful and traditional to pass to exponential weight functions $a(\gamma)$ defined by $\mu_\gamma = \rho_\gamma\, e^{a(\gamma)}$. Condition (3.2) becomes

$$1 + \sum_{n\geq 1} \sum_{\substack{\{\gamma_1,\dots,\gamma_n\}\subset\mathcal{P} \\ \gamma_0\cap\gamma_i\neq\emptyset\,,\,\gamma_i\cap\gamma_j=\emptyset\,,\,1\leq i,j\leq n}} \prod_{i=1}^{n} \rho_{\gamma_i}\, e^{a(\gamma_i)} \;\leq\; e^{a(\gamma_0)} \tag{3.9}$$

From the constraint in the sum we only keep the fact that each of the polymers $\gamma_1,\dots,\gamma_n$ must intersect *different* points in γ_0 (otherwise they would overlap). This implies: (i) $n \leq |\gamma_0|$, and (ii) there are n different points in γ_0 touched by $\gamma_1 \cup \cdots \cup \gamma_n$. These points can be chosen in $\binom{|\gamma_0|}{n}$ ways. Hence, the left-hand side of (3.9) is less or equal than

$$1 + \sum_{n=1}^{|\gamma_0|} \binom{|\gamma_0|}{n} \Biggl[\sup_{x\in\gamma_0} \sum_{\substack{\gamma\in\mathcal{P}\\ \gamma\ni x}} \rho_\gamma\, e^{a(\gamma)}\Biggr]^n \;=\; \Biggl[1 + \sup_{x\in\gamma_0} \sum_{\substack{\gamma\in\mathcal{P}\\ \gamma\ni x}} \rho_\gamma\, e^{a(\gamma)}\Biggr]^{|\gamma_0|} ,$$

which leads us to the following sufficient condition for (3.9):

$$\sup_{x\in\gamma_0} \sum_{\substack{\gamma\in\mathcal{P}\\ \gamma\ni x}} \rho_\gamma\, e^{a(\gamma)} \;\leq\; e^{a(\gamma_0)/|\gamma_0|} - 1 \tag{3.10}$$

The corresponding convergence conditions obtained from our reference criteria are:

$$\sup_{x\in\gamma_0} \sum_{\substack{\gamma\in\mathcal{P}\\ \gamma\ni x}} \rho_\gamma\, e^{a(\gamma)} \;\leq\; \frac{a(\gamma_0)}{|\gamma_0|} \quad \text{(Kotecký-Preiss)} \tag{3.11}$$

and

$$\sup_{x\in\gamma_0} \prod_{\substack{\gamma\in\mathcal{P}\\ \gamma\ni x}} \Bigl[1 + \rho_\gamma\, e^{a(\gamma)}\Bigr] \;\leq\; e^{a(\gamma_0)/|\gamma_0|} \quad \text{(Dobrushin)}\,. \tag{3.12}$$

In all practical cases the function $a(\gamma)$ has been chosen of the form $a(\gamma) = a\,|\gamma|$, with a a positive constant. Also, the criteria are usually presented in the slightly weaker form obtained by replacing the supremum over $x \in \gamma_0$ by a supremum over $x \in \mathbb{V}$. In this form, a condition like (3.10) is, in fact, present in the seminal paper by Gruber and Kunz [7] [formula (42) with normalization $\phi(x) = 1$ and parametrization $\xi_0 = e^a$.]

4. The proof and the approach

The argument has two distinct parts. The first one is a rather elementary convergence argument for positive series; the second one is the Penrose tree identity for the truncated functions.

4.1 Trees and convergence

Let us fix some $\underline{\rho} \in [0,\infty)^{\mathcal{P}}$ and consider maps of the form

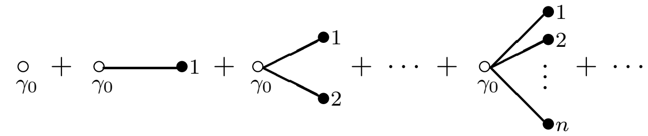
$$\begin{aligned} T : [0,\infty)^{\mathcal{P}} &\longrightarrow [0,\infty]^{\mathcal{P}} \\ \underline{\mu} &\longmapsto T_\gamma(\underline{\mu}) = \rho_\gamma\, \varphi_\gamma(\underline{\mu}) \end{aligned} \tag{4.1}$$

with

$$\varphi_{\gamma_0}(\underline{\mu}) \;=\; 1+\sum_{n\geq 1}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} b_n(\gamma_0,\gamma_1,\ldots,\gamma_n)\,\mu_{\gamma_1}\ldots\mu_{\gamma_n} \tag{4.2}$$

for certain functions $b_n : \mathcal{P}^{n+1} \to [0,\infty)$. We find it useful to visualize these maps in the diagrammatic form

$T_{\gamma_0}(\underline{\mu}) \;=$



The sum is over all single-generation rooted trees. In each tree, open circles represents a factor ρ, bullets a factor μ and vertices other than the root must be summed over all possible polymers γ. At each vertex with n descendants, a "vertex function" b_n acts, having as arguments the ordered $n+1$-tuple formed by the polymer at the vertex, the polymer at the top offspring, the polymer at the next offspring from the top,..., in that order. With this representation, the iteration $T^2(\underline{\mu})$ corresponds to replacing each of the bullets by each one of the diagrams of the expansion for T. This leads to rooted trees of up to two generations, with open circles at first-generation vertices and bullets at second-generation ones. In particular, all single-generation trees have only open circles. Notice that the two drawings of Figure 1 appear in two different terms of the expansion, and hence should be counted as *different* diagrams. More generally, the k-th iteration of T involves all possible rooted tree diagrams, counting as different those obtained by permutations of non-identical branches. We shall call these diagrams *planar rooted trees*. In each term of the expansion, vertices of generation k are occupied by bullets and all the others by open circles.

Formally, the definition of planar rooted trees is determined by a labeling choice which we fix as follows. There is a special vertex, labeled 0 (the root), placed, say, at the leftmost position of the drawing. From it there emerge s_0 branches ending at the first-generation vertices. The value $s_0 = 0$ describes the trivial tree with the root as its only vertex. Otherwise these vertices are drawn along a vertical line at the right of the root and labeled $(0,1),\ldots(0,s_0)$ with the second subscript increasing from the top to the bottom of the line. The construction continues rightwards: Each of the vertices $(0,i)$, gives rise to a family of second-generation vertices $(0,i,1),\ldots(0,i,s_{(0,i)})$ and so on. The vertex v is of generation ℓ if its label has the form $v=(0,i_1,\ldots,i_\ell)$ with $1\leq i_j\leq s_{(0,i_1,\ldots,i_{j-1})}$, $1\leq j\leq\ell$ ($i_0\equiv 0$). A planar rooted tree of maximal generation number k is defined by a sequence

$$\Bigl(s_{(0,i_1,\ldots,i_\ell)}\in\mathbb{N}\cup\{0\} : 1\leq i_j\leq s_{(0,i_1,\ldots,i_{j-1})}\,,\;1\leq j\leq\ell\Bigr)_{1\leq\ell\leq k} \tag{4.3}$$

with $s_{(0,i_1,\ldots,i_k)}=0$. Let us denote $\mathbb{T}^{0,k}$ the set of trees with maximal generation number k; $\mathbb{T}^{0,0}$ being the trivial tree. Figure 1 shows two different trees of $\mathbb{T}^{0,2}$. The labels define a natural total order among the vertices, with $(0,i_1,\ldots,i_\ell)<(0,i'_1,\ldots,i'_{\ell'})$ if $\ell<\ell'$ or $\ell=\ell'$ and $i_j\leq i'_j$ for all $j=1,\cdots,\ell$ with some $i_j<i'_j$. We use this order to enumerate the set $V_t=\{v_1,v_2,\ldots,v_{|V_t|}\}$ formed by the vertex, other than the root, of a planar tree t; $v_i<v_j$ if $i<j$.

A straightforward inductive argument shows that

$$T^k_{\gamma_0}(\underline{\mu}) \;=\; \rho_{\gamma_0}\Bigl[\sum_{\ell=0}^{k-1}R^{(\ell)}_{\gamma_0}(\underline{\rho})+R^{(k)}_{\gamma_0}(\underline{\rho},\underline{\mu})\Bigr] \tag{4.4}$$

(a) (b)

Figure 1: Planar rooted trees defined by (a) $s_0 = s_{(0,1)} = 2$ and $s_{(0,2)} = s_{(0,1,1)} = s_{(0,1,2)} = 0$; (b) $s_0 = s_{(0,2)} = 2$ and $s_{(0,1)} = s_{(0,2,1)} = s_{(0,2,2)} = 0$

with

$$R^{(\ell)}_{\gamma_0}(\underline{\rho}) \;=\; \sum_{t\in\mathbb{T}^{0,\ell}} \; \sum_{(\gamma_{v_1},\ldots,\gamma_{v_{|V_t|}})\in\mathcal{P}^{|V_t|}} \; \prod_{i=0}^{|V_t|} b_{s_{v_i}}(\gamma_{v_i},\gamma_{(v_i,1)},\ldots,\gamma_{(v_i,s_{v_i})})\,\rho_{\gamma_{(v_i,1)}}\cdots\rho_{\gamma_{(v_i,s_{v_i})}} \tag{4.5}$$

and $R^{(k)}_{\gamma_0}(\underline{\rho},\underline{\mu})$ has a similar expression but with the activities of the vertex of the k-th generation weighted by $\underline{\mu}$. Here we agree that $b_0(\gamma_v) \equiv 1$ and $\prod_\emptyset \equiv 1$. We are interested in the $k\to\infty$ limit of (4.4). Let us denote $\mathbb{T}^0 = \cup_\ell \mathbb{T}^{0,\ell}$.

Proposition 4 *Let T be a map of the form (4.1)–(4.2) and assume there exists $\underline{\mu}\in[0,\infty)^{\mathcal{P}}$ such that*

$$T_\gamma(\underline{\mu}) \;\le\; \mu_\gamma \quad \forall\gamma\in\mathcal{P}\,. \tag{4.6}$$

Then the series

$$\rho^*_{\gamma_0} \;:=\; \rho_{\gamma_0} \sum_{t\in\mathbb{T}^{0}} \; \sum_{(\gamma_{v_1},\ldots,\gamma_{v_{|V_t|}})\in\mathcal{P}^{|V_t|}} \; \prod_{i=0}^{|V_t|} b_{s_{v_i}}(\gamma_{v_i},\gamma_{(v_i,1)},\ldots,\gamma_{(v_i,s_{v_i})})\,\rho_{\gamma_{(v_i,1)}}\cdots\rho_{\gamma_{(v_i,s_{v_i})}} \tag{4.7}$$

is finite for each $\gamma_0\in\mathcal{P}$. Furthermore, the limits $T^\infty(\underline{\mu}) = \lim_n T^n(\underline{\mu})$ and $T^\infty(\underline{\rho}) = \lim_n T^n(\underline{\rho})$ belong to $[0,\infty)^{\mathcal{P}}$ and are such that

$$\mu_{\gamma_0} \ge T_{\gamma_0}(\underline{\mu}) \ge \cdots \ge T^n_{\gamma_0}(\underline{\mu}) \underset{n\to\infty}{\searrow} T^\infty_{\gamma_0}(\underline{\mu}) \ge T^\infty_{\gamma_0}(\underline{\rho}) = \rho^*_{\gamma_0} \underset{n\to\infty}{\nwarrow} T^n_{\gamma_0}(\underline{\rho}) \ge \cdots \ge T_{\gamma_0}(\underline{\rho}) \ge \rho_{\gamma_0} \tag{4.8}$$

for each $\gamma_0\in\mathcal{P}$.

Proof. Let us consider the coordinatewise partial order of $[0,\infty]^{\mathcal{P}}$ defined by $\underline{\rho}\le\underline{\mu} \iff \rho_\gamma \le \mu_\gamma\ \forall\gamma\in\mathcal{P}$. With respect to it T is an increasing map and $\underline{\mu}\ge T(\underline{\mu})\ge\underline{\rho}$. By induction

$$\underline{\mu} \;\ge\; T^n(\underline{\mu}) \;\ge\; T^{n+1}(\underline{\mu}) \;\ge\; T^{n+\ell+1}(\underline{\mu}) \;\ge\; T^{n+\ell+1}(\underline{\rho}) \;\ge\; T^{\ell+1}(\underline{\rho}) \;\ge\; T^{\ell}(\underline{\rho}) \;\ge\; \underline{\rho} \tag{4.9}$$

for all $\ell,n\in\mathbb{N}$. This shows that, for each $\gamma\in\mathcal{G}$, the series $[T^\ell_\gamma(\underline{\rho})]_\ell$ is increasing and bounded above while $[T^n_\gamma(\underline{\mu})]_n$ is decreasing and bounded below. Thus, the limits $T^\infty(\underline{\rho}) := \sup_\ell T^\ell(\underline{\rho})$ and $T^\infty(\underline{\mu}) := \inf_\ell T^\ell(\underline{\mu})$ exist and are finite and, by letting alternatingly $\ell\to\infty$ and $n\to\infty$ in (4.8), we obtain the inequalities

$$\underline{\mu} \;\ge\; T^n(\underline{\mu}) \;\ge\; T^{n+1}(\underline{\mu}) \;\ge\; T^\infty(\underline{\mu}) \;\ge\; T^\infty(\underline{\rho}) \;\ge\; T^{\ell+1}(\underline{\rho}) \;\ge\; T^{\ell}(\underline{\rho}) \;\ge\; \underline{\rho}\,. \tag{4.10}$$

The fact that $T^\infty(\underline{\rho}) = \underline{\rho}^*$ is immediate from expression (4.4). □

We notice that $T^\infty_{\gamma_0}(\underline{\mu}) = \rho^*_{\gamma_0} + \lim_k R^{(k)}_{\gamma_0}(\underline{\mu})$. The last limit is in fact an infimum because $R^{(k)}(\underline{\rho},\underline{\mu}) \le R^{(k-1)}(\underline{\rho}, T(\underline{\mu})) \le R^{(k-1)}(\underline{\rho},\underline{\mu})$.

To apply this proposition to the series (2.6), it is convenient to permute the sums in (4.7) (this is allowed because it is a series of positive terms). For each $n \in \mathbb{N}$ let us denote $\mathbb{T}^0_n$ the set of planar rooted trees with n vertices besides the root. We have

$$\rho^*_{\gamma_0} \;=\; \rho_{\gamma_0} \sum_{n\ge 0} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} \Big[\sum_{t\in\mathbb{T}^0_n} w_t \prod_{i=1}^n \mathbb{1}_{\{\gamma_{v_i}=\gamma_i\}}\Big]\, \rho_{\gamma_1}\cdots\rho_{\gamma_n} \tag{4.11}$$

with

$$w_t \;=\; \prod_{i=0}^n b_{s_{v_i}}(\gamma_{v_i}, \gamma_{(v_i,1)}, \ldots, \gamma_{(v_i,s_{v_i})})\;. \tag{4.12}$$

We are particularly interested in vertex functions $b_n(\gamma_0,\gamma_1,\ldots,g_n)$ that are invariant under permutations of the last n arguments. They are associated to weights w_t that are invariant under permutations of the branches of the planar tree t. That is, they depend only on the underlying tree τ obtained by neglecting the order of the vertices. Formally, if $\mathcal{T}^0_n$ is the set of rooted trees on $\{0,1,\ldots,n\}$ (=labelled trees of $n+1$ vertices), there is a map $\mathcal{T}^0_n \ni \tau \mapsto t_\tau \in \mathbb{T}^0_n$ where t_τ is the planar tree obtained by drawing branches starting on the root according to the order given by the labels of the first offspring, and continuing in this way for branches within branches. This map is many-to-one, in fact, the cardinality of the preimage of a tree t (=number of ways of labelling the n non-root vertices of a planar rooted tree with n distinct labels consistently with the rule "from high to low") is given by

$$\beta_n(t) \;=\; \frac{n!}{\prod_{i=0}^n s_{v_i}!} \tag{4.13}$$

(see e.g. theorem 145B in [3]). Therefore, for permutation-invariant functions b_n, we can pass to a sum over trees $\tau\in\mathcal{T}^0_n$ by changing $w_t \to w_{t_\tau}/\beta_n(t_\tau)$ and we obtain

$$\rho^*_{\gamma_0} \;=\; \rho_{\gamma_0} \sum_{n\ge 0} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} \Big[\sum_{\tau\in\mathcal{T}^0_n} \prod_{i=0}^n s_i!\, b_{s_i}(\gamma_i,\gamma_{i_1},\ldots,\gamma_{i_{s_i}})\Big]\, \rho_{\gamma_1}\cdots\rho_{\gamma_n} \tag{4.14}$$

where i_j denotes the descendants of the vertex i and s_i its coordination number.

Comparing (4.14) with (2.6) we immediately see that Proposition 4 yields the following criterion for the convergence of cluster expansions.

Corollary 5 *Let $\mathcal{G} = (\mathcal{P},\mathcal{E})$ be a polymer system and assume there exist functions $c_n : \mathcal{P}^{n+1} \to [0,\infty)$, for $n\in\mathbb{N}$, invariant under permutations of the last n arguments and satisfying*

$$\big|\phi^T(\gamma_0,\gamma_1,\ldots,\gamma_n)\big| \;\le\; \sum_{\tau\in\mathcal{T}^0_n} \prod_{i=0}^n c_{s_i}(\gamma_i,\gamma_{i_1},\ldots,\gamma_{i_{s_i}})\;. \tag{4.15}$$

Let $\underline{\rho}, \underline{\mu} \in [0,\infty)$ and

$$\varphi_{\gamma_0}(\underline{\mu}) \;=\; 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} c_n(\gamma_0,\gamma_1,\ldots,\gamma_n)\,\mu_{\gamma_1}\ldots\mu_{\gamma_n}\;, \tag{4.16}$$

$\gamma_0 \in \mathcal{P}$. Then, the existence of $\underline{\mu} \in [0,\infty)^{\mathcal{P}}$ such that

$$\rho_{\gamma_0}\,\varphi_{\gamma_0}(\underline{\mu}) \;\leq\; \mu_{\gamma_0} \tag{4.17}$$

for each $\gamma_0 \in \mathcal{P}$, entails the absolute convergence of the cluster expansion (2.3) for the system $\mathcal{G}$, uniformly in Λ and in the activities $\underline{z}$ with $|z_\gamma| \leq \rho_\gamma$, $\gamma \in \mathcal{P}$. Furthermore, if $T(\underline{\mu})$ and $T^\infty(\underline{\mu})$ are defined as in the previous proposition for φ defined in (4.16),

$$\mu_{\gamma_0} \;\geq\; T_{\gamma_0}(\underline{\mu}) \;\geq\; \cdots \;\geq\; T^n_{\gamma_0}(\underline{\mu}) \underset{n\to\infty}{\searrow} T^\infty_{\gamma_0}(\underline{\mu}) \;\geq\; \rho_{\gamma_0}\,\Pi_{\gamma_0}(\underline{\rho}) \tag{4.18}$$

for each $\gamma_0 \in \mathcal{P}$.

4.2 Penrose bound and the proof of Theorem 1

The core of our approach is the deep tree identity obtained by Oliver Penrose [13] in 1967. It relies on a map established between graphs and trees defined in such a way that each connected graph has a unique associated tree, and each tree defines a unique maximal graph among those mapped into it. The latter is the direction needed for our purposes. Consider a finite graph $G = (\{v_0, v_1, \ldots, v_n\}, E_G)$ and let $\mathbb{T}_G$ be the set of subgraphs that are trees (thought of as trees rooted in v_0). Given a tree $\tau \in \mathbb{T}_G$ and $i = 1, \ldots, n$, let $d(i)$ denote the tree distance of the vertex v_i to v_0. Let p be the map that to each tree τ associates the graph $p(\tau) \subset G$ formed by adding (only once) to τ all edges $\{v_i, v_j\} \in E_G \setminus E_\tau$ such that either:

(p1) $d(i) = d(j)$ (edges between vertices of the same generation), or

(p2) $d(i) = d(j) - 1$ and $i < j$ (edges connecting to predecessors with smaller index).

The set of *Penrose trees* of G is

$$\mathbb{P}_G \;=\; \{\tau \in \mathbb{T}_G : p(\tau) = \tau\}\;. \tag{4.19}$$

Thus, a Penrose tree must be such that if two vertices v_i and v_j

(t1) are descendants of the same predecessor, or

(t2) have the same generation number, or

(t3) their generation number differs by one and the smaller number corresponds to $v_{i\wedge j}$,

then $\{v_i, v_j\} \notin E_G$.

The remarkable *Penrose identity* says that

$$\phi^T(\gamma_0,\gamma_1,\ldots,\gamma_n) \;=\; (-1)^n \left|\mathbb{P}_{\mathcal{G}_{(\gamma_0,\gamma_1,\ldots,\gamma_n)}}\right| \tag{4.20}$$

where $\mathcal{G}_{(\gamma_0,\gamma_1,\dots,\gamma_n)}$ is the graph of vertices $\{0,1,\dots,n\}$ and edges $\{\{i.j\}:\gamma_i\nsim\gamma_j, 0\le i,j\le n\}$. To obtain our new criterion, we keep only condition (t1), which refers to individual vertices. That is, rather than Penrose trees we consider the larger family of trees τ such that

$$\text{If } \{i,i_1\} \text{ and } \{i,i_2\} \text{ are edges of } \tau, \text{ then } \gamma_{i_1}\sim\gamma_{i_2} \tag{4.21}$$

In this way we obtain bounds of the form (4.15) with

$$c_n(\gamma_0,\gamma_1,\dots,\gamma_n) \;=\; \prod_{i=1}^n \mathbb{1}_{\{\gamma_0\nsim\gamma_i\}} \prod_{j=1}^n \mathbb{1}_{\{\gamma_i\sim\gamma_j\}}\;, \tag{4.22}$$

and Corollary 5 applies with

$$\varphi_{\gamma_0}(\underline{\mu}) \;=\; 1+\sum_{n\ge1}\frac{1}{n!}\sum_{\substack{(\gamma_1,\dots,\gamma_n)\in\mathcal{P}^n\\ \gamma_0\nsim\gamma_i\,,\,\gamma_i\sim\gamma_j\,,\,1\le i,j\le n}} \mu_{\gamma_1}\dots\mu_{\gamma_n}\;, =\; \Xi_{\mathcal{P}_{\gamma_0}}(\underline{\mu})\;.\;\square \tag{4.23}$$

If we replace in (4.21) the condition $\gamma_i\nsim\gamma_j$ by the weaker requirement $\gamma_i\neq\gamma_j$ we obtain

$$c_n^{\rm Dob}(\gamma_0,\gamma_1,\dots,\gamma_n) \;=\; \prod_{i=1}^n \mathbb{1}_{\{\gamma_0\nsim\gamma_i\}} \prod_{j=1}^n \mathbb{1}_{\{\gamma_i\neq\gamma_j\}}\;, \tag{4.24}$$

and

$$\varphi^{\rm Dob}_{\gamma_0}(\underline{\mu}) \;=\; 1+\sum_{n\ge1}\frac{1}{n!}\sum_{\substack{(\gamma_1,\dots,\gamma_n)\in\mathcal{P}^n\\ \gamma_0\nsim\gamma_i\,,\,\gamma_i\neq\gamma_j\,,\,1\le i,j\le n}} \mu_{\gamma_1}\dots\mu_{\gamma_n} \;=\; \prod_{\gamma\nsim\gamma_0}(1+\mu_\gamma)\;, \tag{4.25}$$

which corresponds to Dobrushin condition. Finally, if requirement (4.21) is ignored altogether,

$$c_n^{\rm KP}(\gamma_0,\gamma_1,\dots,\gamma_n) \;=\; \prod_{i=1}^n \mathbb{1}_{\{\gamma_0\nsim\gamma_i\}}\;, \tag{4.26}$$

and

$$\varphi^{\rm KP}_{\gamma_0}(\underline{\mu}) \;=\; 1+\sum_{n\ge1}\frac{1}{n!}\sum_{\substack{(\gamma_1,\dots,\gamma_n)\in\mathcal{P}^n\\ \gamma_0\nsim\gamma_i\,,\,1\le i\le n}} \mu_{\gamma_1}\dots\mu_{\gamma_n} \;=\; \exp\Big[\sum_{\gamma\nsim\gamma_0}\mu_\gamma\Big] \tag{4.27}$$

yields the criterion of Kotecký and Preiss.

Acknowledgements

It is a pleasure to thank Aernout van Enter and Alan Sokal for useful comments and for a critical reading of the manuscript (AvE), and Charles-Edouard Pfister and Daniel Ueltschi for encouragement. The work of AP was supported by a visitor grant of CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, Brasil). He also thanks the Mathematics Laboratory Raphael Salem of the University of Rouen for the invitation that started the project and for hospitality during its realization.

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