Green's Functions of Generalized Laplacians

Philippe Poulin *

Department of Mathematics and Statistics
McGill University
805, rue Sherbrooke Ouest
Montréal, Québec, H3A-2K6, Canada

Abstract

The Green's function of any discretization, Δ, of the Laplacian in dimension d is given by

$$G(n-m,z) = \int_{\mathbb{T}^d} \frac{e^{\mathrm{i}(n-m)\cdot x}}{\Phi(x)-z} \mathrm{d}x,$$

where $m,n \in \mathbb{Z}^d$, $z \in \mathbb{C}_+$, and $\Phi(x)$ is the symbol of Δ. Using the stationary phase method we study the decay of $G(n, e+\mathrm{i}0)$ when $|n| \to \infty$ for values of energy, e, inside the range of $\Phi(x)$, where $\Phi(x)$ is analytic. We focus on two specific examples: the standard Laplacian and the Molchanov–Vainberg Laplacian.

1 Introduction

The present text, which constitutes the first part of my doctoral thesis and details my contribution to the *MolchanovFest*, is based on [P]; however, it presents more general results.

There has been recent interest in establishing decay properties of Green's functions of discrete Laplacians for values of energy inside their associated spectra [ESY, MV, SV]. Such *a priori* estimates are useful in several contexts, for instance in scattering theory of the Anderson model.[1] In concrete examples the problem consists of investigating the decay of $G(n, e+\mathrm{i}0)$ when $|n| \to \infty$, where

$$G(n,z) = \int_{\mathbb{T}^d} \frac{\mathrm{e}^{\mathrm{i}n\cdot x}}{\Phi(x)-z} \, \mathrm{d}x,$$

$n \in \mathbb{Z}^d$, $e \in \operatorname{ran} \Phi$, and Φ is analytic. Such a decay is calculated in [P] for the Molchanov–Vainberg Laplacian. The mentioned calculation takes its origin in

*This research was supported in part by NSERC.

[1] For a list of papers using such estimates, see the introduction in [P].

[SV] and works for strictly convex constant-energy surfaces,

$$\Gamma(e) = \{x \in \mathbb{T}^d \ ; \ \Phi(x) = e\}.$$

We generalize this study to non convex surfaces. To this end, we take special care of a supplemental parameter when developing the stationary phase method. This development is substantially based on [St, V] and on the observation in [Li] that the method also gives a decay for Fourier transforms over non convex surfaces.[2] In the sequel we prove in detail this last observation, which leads to the treatment of a large class of generalized Laplacians.

Results Consider firstly the standard Laplacian. An elementary computation shows that except for $e \in \{-2d, -2d+4, \ldots, 2d-4, 2d\} \cup \{0\}$, $\Gamma(e)$ is a regular surface without planar point. The stationary phase method then yields

$$G(n, e + i0) = O(|n|^{-\frac{1}{2}})$$

in any dimension. Without pretending that this result is optimal, it improves a naive estimate based on Bessel functions, which is presented in the appendix.

Secondly, consider the Molchanov–Vainberg Laplacian, which is defined in section 6. Then, except for $e \in \{-2^d, 0, 2^d\}$, $\Gamma(e)$ is a regular, strictly convex surface. The classical stationary phase method thus yields the optimal decay,

$$G(n, e + \mathrm{i}0) = O(|n|^{-\frac{d-1}{2}}),$$

valid in any dimension. This is why Molchanov and Vainberg introduced this Laplacian in the literature in their work about scattering theory of the Anderson model [MV].

Acknowledgments I would like to thank Vojkan Jakšić for the invitation to the *MolchanovFest*; Boris Vainberg for useful discussions when writing [P]; and my dear friend and colleague Marco Merkli, who helped me to enhance the introduction and took interest in this text.

* * *

An *oscillatory integral* is an integral of the form

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\varphi(x)} f(x)\, \mathrm{d}x,$$

where $r \in \mathbb{R}$ and $\varphi(x)$ is real valued. Its *amplitude* and *phase* are $f(x)$ and $\varphi(x)$, respectively. The phase is *stationary at* x_0 if $\nabla\varphi(x_0) = 0$. Such a stationary phase point, x_0, is *non degenerate* if in addition $\det \mathrm{D}_x^2 \varphi(x_0) \neq 0$.

[2] New developments and results are to appear in [ES].

Our plan is to establish *1)* the rapid decay of an oscillatory integral whose phase is nowhere stationary, for a given compactly supported amplitude; *2)* the polynomial decay of an oscillatory integral whose stationary phase points are not degenerate; *3)* similar results for Cauchy principal values of oscillatory integrals. Given an analytic function, $\Phi(x)$, on $\mathbb{T}^d$, we then consider Fourier transforms of functions over level surfaces of the form

$$\Gamma(e) = \{x \in \mathbb{T}^d \ ; \ \Phi(x) = e\}$$

and investigate their decay when the $\Gamma(e)$'s are regular, compact, and admit at every point at least $\kappa > 0$ non vanishing principal curvatures. Then, we do a similar study for Cauchy principal values of such Fourier transforms. Finally, we deduce the decay of Green's functions of generalized Laplacians for energies inside their associated spectra.

We adopt the following conventions: most of our theorems establish the existence of a neighborhood on which a certain phenomenon occurs. For sake of simplicity (and without loss of generality), we consider only non empty, bounded, open, cubic neighborhoods and we call them *cubes*.[3]

Given a real valued phase, $\varphi(x,t)$, we define

$$I_f(r,t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\varphi(x,t)} f(x,t)\,\mathrm{d}x,$$

where $r > 0$, $t \in \mathbb{R}^m$, and $f(x,t)$ is a complex valued function—provided that this integral makes sense.

Smooth is used in this text for *infinitely differentiable*. The vector space of all (complex valued) smooth functions on $\mathbb{R}^d$ is denoted by $C^\infty(\mathbb{R}^d)$. It contains two important subspaces: $C^\omega(\mathbb{R}^d)$, consisting of all analytic functions on $\mathbb{R}^d$, and $C_c^\infty(\mathbb{R}^d)$, consisting of all compactly supported smooth functions on $\mathbb{R}^d$.

The *transpose* of a linear transformation on $C_c^\infty(\mathbb{R}^d)$ is defined by duality with respect to the following bracket:

$$(f(x)|g(x)) = \int_{\mathbb{R}^d} f(x)g(x)\,\mathrm{d}x.$$

For instance, denoting by $\partial_{x^{(j)}}$ the differentiation with respect to $x^{(j)}$, integration by parts gives ${\partial_{x^{(j)}}}^{\mathrm{t}} = -\partial_{x^{(j)}}$. Moreover, denoting by $F(x)$ the multiplication by a smooth function of the same name, $F(x)^{\mathrm{t}} = F(x)$.

Our notations regarding asymptotic behavior are standard: for instance,

$$f(r) = O(r^{-\alpha}) \ \text{ when } r \to \infty$$

means the existence of a positive constant, C, such that

$$|f(r)| \leqslant C r^{-\alpha}$$

[3]Recall that a *cubic neighborhood* in $\mathbb{R}^d$ is a subset of the form $I_1 \times \cdots \times I_d$, where each I_d is an interval.

when r is sufficiently large. If such a constant (depending on α) exists for any $\alpha \geqslant 0$, we write

$$f(r) = O(r^{-\infty}) \quad \text{when } r \to \infty.$$

Finally,

$$f(r) \sim \sum_{j=0}^{\infty} a_j r^{-j} \quad \text{when } r \to \infty$$

means that for any $N \geqslant 0$

$$f(r) - \sum_{j=0}^{N} a_j r^{-j} = O(r^{-N-1}).$$

If the function f also depends on a parameter, t, we say that the previous estimates/asymptotics are *uniform* in t if the constants C may be chosen independently of t.

2 Oscillatory Integral without Stationary Phase Point

The following theorems are stated in the way they are used when studying Fourier transforms over surfaces. They establish the existence of neighborhoods on which a certain phenomenon occurs, given a fixed phase. The given phase, $\varphi(x, t)$, is supposed to be smooth in $(x, t) \in \mathbb{R}^d \times \mathbb{R}^m$.

Lemma *Let $d = 1$. Suppose $\partial_x \varphi \neq 0$ at a given $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}^m$. Then, there exists an arbitrarily small cube, $U \times B$, containing (x_0, t_0) such that the following holds: if the amplitude, $f(x, t)$, is smooth in the neighborhood of $\mathbb{R} \times \overline{B}$ and vanishes on $U^{\mathrm{c}} \times \overline{B}$, then*

$$|I_f(r, t)| = O(r^{-\infty})$$

uniformly in $t \in \overline{B}$.[4]

Proof: By continuity, $\partial_x \varphi(x, t) \neq 0$ on a certain cube $U' \times B'$ containing (x_0, t_0). The operator $D = \frac{1}{\partial_x \varphi(x,t)} \circ \partial_x$ is thus well defined on this cube, where it satisfies $D\mathrm{e}^{\mathrm{i}r\varphi(x,t)} = \mathrm{i}r\mathrm{e}^{\mathrm{i}r\varphi(x,t)}$ and $D^{\mathrm{t}} = -\partial_x \circ \frac{1}{\partial_x \varphi(x,t)}$. Let $U \times B \ni (x_0, t_0)$ be a cube whose closure is in $U' \times B'$. If $f(x, t)$ satisfies the asserted properties,

[4] U^{c} denotes the complementary of U with respect to a set determined by the context; here, $U^{\mathrm{c}} = \mathbb{R} \setminus U$.

then for any $N \geqslant 0$ and $t \in \overline{B}$

$$\begin{aligned} |I_f(r,t)| &= \frac{1}{r^N}\left|\int_U (D^N \mathrm{e}^{\mathrm{i}r\varphi(x,t)}) f(x,t)\,\mathrm{d}x\right| \\ &= \frac{1}{r^N}\left|\int_U \mathrm{e}^{\mathrm{i}r\varphi(x,t)} (D^{\mathrm{t}})^N f(x,t)\,\mathrm{d}x\right| \\ &\leqslant \frac{C_N}{r^N}, \end{aligned}$$

where the constant C_N does not depend on $t \in \overline{B}$. □

The multidimensional analogue follows:

Theorem 1 *Suppose $\nabla_x \varphi(x_0,t_0) \neq 0$ for a given $(x_0,t_0) \in \mathbb{R}^d \times \mathbb{R}^m$. Then, there exists an arbitrarily small cube, $U \times B$, containing (x_0,t_0) such that the following holds: if the amplitude, $f(x,t)$, is smooth in the neighborhood of $\mathbb{R}^d \times \overline{B}$ and vanishes on $U^{\mathrm{c}} \times \overline{B}$, then*

$$|I_f(r,t)| = O(r^{-\infty})$$

uniformly in $t \in \overline{B}$.

Proof: By assumption, $\partial_{x^{(k)}} \varphi(x_0,t_0) \neq 0$ for a certain $1 \leqslant k \leqslant d$, say, for $k=1$. Interpreting $(x^{(2)},\ldots,x^{(d)},t)$ as a parameter, there exists an arbitrarily small cube, $U_1 \times (U_2 \times \cdots \times U_d \times B)$, containing $(x_0^{(1)};x_0^{(2)},\ldots,x_0^{(d)},t_0)$ on which the previous lemma applies. Hence, if $f(x,t)$ satisfies the asserted properties, then

$$\left|\int_{U_1} \mathrm{e}^{\mathrm{i}r\varphi(x,t)} f(x,t)\,\mathrm{d}x^{(1)}\right| \leqslant \frac{C_N}{r^N}$$

uniformly in $(x^{(2)},\ldots,x^{(d)},t) \in \overline{U_2} \times \ldots \times \overline{U_d} \times \overline{B}$. In particular,

$$\begin{aligned} |I_f(r,t)| &= \left|\int_{U_2\times\ldots\times U_d}\int_{U_1} \mathrm{e}^{\mathrm{i}r\varphi(x,t)} f(x,t)\,\mathrm{d}x^{(1)}\ \mathrm{d}x^{(2)}\ldots\ \mathrm{d}x^{(d)}\right| \\ &\leqslant |U_2 \times \ldots \times U_d| \frac{C_N}{r^N} \end{aligned}$$

uniformly in $t \in \overline{B}$. □

Scholium *A similar result holds when considering oscillatory integrals over the torus,*

$$\int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}r\varphi(x,t)} f(x,t)\,\mathrm{d}x,$$

without stationary phase point. One then assumes $f(\cdot,t)$ is periodic for every t (instead of being compactly supported). The same proof works, due to the absence of boundary term when integrating by parts a smooth periodic function.

Cauchy Principal Value We now turn our attention to Cauchy principal values of oscillatory integrals without stationary phase point. To this end, we now consider a parameter $(e, t) \in \mathbb{R} \times \mathbb{R}^m$ and a phase, $\varphi(x; e, t)$, smooth in $(x; e, t) \in \mathbb{R}^d \times (\mathbb{R} \times \mathbb{R}^m)$. We thus let

$$I_f(r; e, t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\varphi(x;e,t)} f(x; e, t)\, \mathrm{d}x$$

and study

$$\text{p.v.} \int_{|\eta - e| < \delta} \frac{1}{\eta - e} I_f(r; \eta, t)\, \mathrm{d}\eta = \lim_{\varepsilon \downarrow 0} \int_{\varepsilon < |\eta - e| < \delta} \frac{1}{\eta - e} I_f(r; \eta, t)\, \mathrm{d}\eta$$

for a given $\delta > 0$. Our results are based on the following elementary estimate:

Lemma *Suppose $f(h)$ is a continuously differentiable function in the neighborhood of $[-\delta, \delta]$, where $\delta > 0$ is given. Then, for any $\varepsilon \in]0, \delta[$*

$$\left| \int_{\varepsilon \leqslant |h| \leqslant \delta} \frac{f(h)}{h}\, \mathrm{d}h \right| \leqslant 2\delta \max_{|h| \leqslant \delta} |f'(h)|.$$

In particular,

$$\left| \text{p.v.} \int_{-\delta}^{\delta} \frac{f(h)}{h}\, \mathrm{d}h \right| \leqslant 2\delta \max_{|h| \leqslant \delta} |f'(h)|.$$

The proof, based on the Mean Value Theorem, is left to the reader.

Theorem 2 *Suppose $\nabla_x \varphi(x_0, 0, t_0) \neq 0$ for a given $(x_0, t_0) \in \mathbb{R}^d \times \mathbb{R}^m$. Then, there exists an arbitrarily small cube, $U \times B$, containing (x_0, t_0) and an arbitrarily small $\delta > 0$ such that the following holds: if the amplitude, $f(x, h, t)$, is smooth in the neighborhood of $\mathbb{R}^d \times [-\delta, \delta] \times \overline{B}$ and vanishes on $U^{\mathrm{c}} \times [-\delta, \delta] \times \overline{B}$, then*

$$\left| \text{p.v.} \int_{-\delta}^{\delta} \frac{I_f(r, h, t)}{h}\, \mathrm{d}h \right| = O(r^{-\infty})$$

uniformly in $t \in \overline{B}$.

Proof: By the lemma it suffices to estimate $\partial_h I_f(r, h, t)$. In fact, the dominated convergence theorem implies

$$\partial_h I_f(r, h, t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\varphi(x,h,t)} (\partial_h f(x, h, t) + \mathrm{i}r f(x, h, t) \partial_h \varphi(x, h, t))\, \mathrm{d}x.$$

Hence, by Theorem 1 there exist an arbitrarily small cube, $U \times B$, containing (x_0, t_0) and an arbitrarily small $\delta > 0$ such that for $f(x, h, t)$ as stipulated

$$|\partial_h I_f(r, h, t)| \leqslant \frac{C_N}{r^N}$$

uniformly in $(h,t) \in [-\delta,\delta] \times \overline{B}$. The result follows. $\square$

Applying the above to the phase $\tilde{\varphi}(x,h;e,t) = \varphi(x,h+e,t)$ and the amplitude $\tilde{f}(x,h;e,t) = f(x,h+e,t)$, one obtains:

Corollary *Suppose $\nabla_x \varphi(x_0,e_0,t_0) \neq 0$ for a given $(x_0,e_0,t_0) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^m$. Then, there exist an arbitrarily small cube, $U \times B$, containing (x_0,t_0) and an arbitrarily small $\delta > 0$ such that the following holds: if the amplitude, $f(x,e,t)$, is smooth in the neighborhood of $\mathbb{R}^d \times [e_0-\delta, e_0+\delta] \times \overline{B}$ and vanishes on $U^{\mathrm{c}} \times [e_0-\delta,e_0+\delta] \times \overline{B}$, then*

$$\left| \mathrm{p.v.} \int_{|\eta-e|<\delta} \frac{I_f(r,\eta,t)}{\eta - e} \, \mathrm{d}\eta \right| = O(r^{-\infty})$$

uniformly in $(e,t) \in [e_0-\delta,e_0+\delta] \times \overline{B}$.

3 Oscillatory Integral with Non Degenerate Stationary Phase Points

Quadratic Phase We now investigate the decay of oscillatory integrals admitting non degenerate stationary phase points. Via the *Morse lemma*, which is proven below, our study reduces to oscillatory integrals whose phases are canonical, not degenerate quadratic forms:

$$Q(x) = \sum_{j=1}^{s} (x^{(j)})^2 - \sum_{k=s+1}^{d} (x^{(k)})^2,$$

where $0 \leqslant s \leqslant d$.

Exceptionally, in the next three lemmas the variable $r \in \mathbb{R}$ is allowed to be negative. Moreover $d = 1$ (so x varies in $\mathbb{R}$).

Lemma *Suppose $|r| > 1$. For all $l \in \mathbb{N}$ there exist constants $c_j^{(l)} \in \mathbb{C}$ independent of $|r|$ such that*

$$\int_{-\infty}^{\infty} \mathrm{e}^{-(1-\mathrm{i}r)x^2} x^l \, \mathrm{d}x = |r|^{-\frac{l+1}{2}} \sum_{j=0}^{\infty} c_j^{(l)} |r|^{-j}.$$

Proof: Let us denote by $(\cdot)^{\frac{1}{2}}$ the branch of the square root whose singular cut is the positive imaginary axis. Let $z = (1-\mathrm{i}r)^{\frac{1}{2}} x$. Then,

$$\int_{\mathbb{R}} \mathrm{e}^{-(1-\mathrm{i}r)x^2} x^l \, \mathrm{d}x = \left(\int_{\gamma} \mathrm{e}^{-z^2} z^l \, \mathrm{d}z \right) [(1-\mathrm{i}r)^{\frac{1}{2}}]^{-l-1}, \tag{1}$$

where γ is the oriented path $(1-\mathrm{i}r)^{\frac{1}{2}}\mathbb{R}$. Observe that

$$(1-\mathrm{i}r)^{\frac{1}{2}} = \sigma_r |r|^{\frac{1}{2}} \left(\frac{1}{r} - \mathrm{i}\right)^{\frac{1}{2}}, \tag{2}$$

where

$$\sigma_r = \begin{cases} 1 & \text{if} \quad r > 1 \\ \mathrm{i} & \text{if} \quad r < -1 \end{cases}$$

(so σ_r does not depend on $|r|$). Moreover the Taylor expansion

$$[(w-\mathrm{i})^{\frac{1}{2}}]^{-l-1} = \sum_{j=0}^{\infty} a_j^{(l)} w^j \tag{3}$$

is valid for all $|w| < 1$, by choice of the branch of $z^{\frac{1}{2}}$. Thus, substituting $w = \mathrm{sgn}(r)\frac{1}{|r|}$ in the above, the result follows from (1), (2), and (3). $\square$

Lemma *Suppose $f(x,t)$ is smooth in the neighborhood of $\mathbb{R}\times\overline{B}$ and vanishes on $U^{\mathrm{c}}\times\overline{B}$, where $U\subset\mathbb{R}$ and $B\subset\mathbb{R}^m$ are cubes. For any $l\in\mathbb{N}$ there exists a $C_l>0$ independent of $|r|$ satisfying*

$$\left|\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}rx^2} x^l f(x,t)\,\mathrm{d}x\right| \leqslant C_l |r|^{-\frac{l+1}{2}}$$

uniformly in $t\in\overline{B}$.

Proof: Let $\chi(x)$ be a compactly supported smooth function on $\mathbb{R}$ such that $0\leqslant\chi(x)\leqslant 1$ everywhere and

$$\chi(x) = \begin{cases} 0 & \text{if} \quad |x| \geqslant 2 \\ 1 & \text{if} \quad |x| \leqslant 1. \end{cases}$$

For an arbitrarily fixed $\varepsilon>0$, $\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}rx^2} x^l f(x,t)\,\mathrm{d}x = I + II$, where

$$\begin{aligned} I &= \int_{|x|<2\varepsilon} \mathrm{e}^{\mathrm{i}rx^2} x^l f(x,t)\chi\left(\frac{x}{\varepsilon}\right)\,\mathrm{d}x, \\ II &= \int_{|x|>\varepsilon} \mathrm{e}^{\mathrm{i}rx^2} x^l f(x,t)\left(1-\chi\left(\frac{x}{\varepsilon}\right)\right)\,\mathrm{d}x. \end{aligned}$$

Concerning I, there exist constants (generically denoted by *Const*) independent of $t\in\overline{B}$, but depending on l, satisfying

$$|I| \leqslant Const \int_{-2\varepsilon}^{2\varepsilon} |x|^l\,\mathrm{d}x = Const\,\varepsilon^{l+1}.$$

Concerning II, let $D = \frac{1}{x} \circ \partial_x$, which is well defined on the support of the integrand. Since $\frac{1}{2\mathrm{i}r} D$ fixes $\mathrm{e}^{\mathrm{i}rx^2}$,

$$II = \left(\frac{1}{2\mathrm{i}r}\right)^N \int_{|x|>\varepsilon} \mathrm{e}^{\mathrm{i}rx^2} (D^{\mathrm{t}})^N x^l f(x,t) \left(1 - \chi\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x.$$

In general, if $g(x)$ is a compactly supported smooth function, then

$$D^{\mathrm{t}} x^l g(x) = \left(-\partial_x \circ \frac{1}{x}\right) x^l g(x)$$

is of the form $x^{l-2}h(x)$, where $h(x)$ is also a compactly supported smooth function. It follows by induction that there exist constants independent of t and r, but depending on N and l, which satisfy

$$\begin{aligned} |II| &\leqslant Const\,|r|^{-N} \int_{|x|>\varepsilon} |x|^{l-2N}\,\mathrm{d}x \\ &= Const\,|r|^{-N} \varepsilon^{l-2N+1}, \end{aligned}$$

provided that $N > \frac{l+1}{2}$.

In total,

$$|I + II| \leqslant Const\,(\varepsilon^{l+1} + |r|^{-N}\varepsilon^{l-2N+1})$$

for such an N. Choosing $\varepsilon = |r|^{-\frac{1}{2}}$ then completes the proof. □

Lemma *Consider a function, $f(x,t)$, smooth in the neighborhood of $\mathbb{R} \times \overline{B}$ (where $B \subset \mathbb{R}^m$ is a cube) and vanishing when $|x| < \varepsilon$ (where $\varepsilon > 0$). Suppose that for all $N \geqslant 0$ there exist a $D_N > 0$ and an $\alpha_N > 0$ such that*

$$|\partial_x^N f(x,t)| \leqslant D_N(1 + |x|^{\alpha_N})$$

uniformly in $t \in \overline{B}$. Then,

$$\int_{-\infty}^{\infty} \mathrm{e}^{(\mathrm{i}r-1)x^2} f(x,t)\,\mathrm{d}x = O(r^{-\infty})$$

uniformly in $t \in \overline{B}$.

Proof: The derivation $D = \frac{1}{x} \circ \partial_x$ is well defined on the support of the integrand. Notice that $\frac{1}{2\mathrm{i}r} D$ fixes $\mathrm{e}^{\mathrm{i}rx^2}$ and $D^{\mathrm{t}} = -\partial_x \circ \frac{1}{x}$. Then,

$$\left|\int_{\mathbb{R}} \mathrm{e}^{(\mathrm{i}r-1)x^2} f(x,t)\,\mathrm{d}x\right| = \left(\frac{1}{2|r|}\right)^N \left|\int_{|x|\geqslant\varepsilon} \mathrm{e}^{\mathrm{i}rx^2} (D^{\mathrm{t}})^N (\mathrm{e}^{-x^2} f(x,t))\,\mathrm{d}x\right|.$$

Our assumption on the derivatives of $f(x,t)$ makes the integral on the right side of the previous equation uniformly bounded in $t \in \overline{B}$, where N is arbitrarily fixed. The result follows. $\square$

In dimension one we are interested in the phase x^2, so let

$$I_f(r,t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}rx^2} f(x,t)\,\mathrm{d}x.$$

According to our convention $r > 0$. However, since we are also interested in the phase $-x^2$, we allow r to be negative. We are ready to compute the asymptotic expansion of $I_f(r,t)$ (resp. $I_f(-r,t)$) when $|r| \to \infty$:

Theorem 3 *Consider an amplitude, $f(x,t)$, smooth in the neighborhood of $\mathbb{R} \times \overline{B}$ and vanishing on $U^{\mathrm{c}} \times \overline{B}$, where $U \subset \mathbb{R}$ and $B \subset \mathbb{R}^m$ are cubes. There exist constants, $a_j(t)$, depending smoothly on $t \in \overline{B}$ such that*

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}rx^2} f(x,t)\,\mathrm{d}x \sim |r|^{-\frac{1}{2}} \sum_{j=0}^{\infty} a_j(t) |r|^{-\frac{j}{2}}$$

uniformly in $t \in \overline{B}$ when $r \to \infty$. The same result holds when $r \to -\infty$, with different constants $a_j(t)$.

Proof: Let $\chi(x)$ be a smooth, bounded, compactly supported function such that $\chi(x) = 1$ on an interval containing $\overline{U} \cup \{0\}$. Then, for $f(x,t)$ as stated

$$I_f(r,t) = \int_{\mathbb{R}} \mathrm{e}^{(\mathrm{i}r-1)x^2} (\,\mathrm{e}^{x^2} f(x,t)\,) \chi(x)\,\mathrm{d}x.$$

By Taylor's theorem (in dimension 1), for any fixed N there exists a polynomial,

$$P_t(x) = \sum_{l=0}^{N} b_l(t) x^l,$$

and a smooth remainder, $R_t(x)$, both depending smoothly on $(x,t) \in \mathbb{R} \times \overline{B}$, such that

$$\mathrm{e}^{x^2} f(x,t) = P_t(x) + x^{N+1} R_t(x).$$

Thus, $I_f(r,t)$ decomposes into $I + II + III$, where

$$\begin{aligned}
I &= \sum_{l=0}^{N} b_l(t) \int_{\mathbb{R}} \mathrm{e}^{(\mathrm{i}r-1)x^2} x^l\,\mathrm{d}x,\\
II &= \int_{\mathbb{R}} \mathrm{e}^{(\mathrm{i}r-1)x^2} x^{N+1} R_t(x) \chi(x)\,\mathrm{d}x,\\
III &= \int_{\mathbb{R}} \mathrm{e}^{(\mathrm{i}r-1)x^2} P_t(x)(\chi(x) - 1)\,\mathrm{d}x.
\end{aligned}$$

Concerning I the first lemma gives the existence of $c_j^{(l)} \in \mathbb{C}$ such that for any $|r| > 1$

$$\begin{aligned} I &= \sum_{l=0}^{N} b_l(t)|r|^{-\frac{l+1}{2}} \sum_{j=0}^{\infty} c_j^{(l)} |r|^{-j} \\ &= |r|^{-\frac{1}{2}} \sum_{l=0}^{N} b_l(t)|r|^{-\frac{l}{2}} \sum_{j \leqslant \frac{N-l}{2}} c_j^{(l)} |r|^{-j} + \sum_{l=0}^{N} b_l(t)|r|^{-\frac{l+1}{2}} \sum_{j > \frac{N-l}{2}} c_j^{(l)} |r|^{-j}. \end{aligned}$$

Notice that

$$\sum_{l=0}^{N} b_l(t)|r|^{-\frac{l+1}{2}} \sum_{j > \frac{N-l}{2}} c_j^{(l)} |r|^{-j} = O(|r|^{-\frac{N}{2}-1})$$

uniformly in $t \in \overline{B}$ when $r \to \infty$ (resp. $r \to -\infty$). Consequently, we have found coefficients, $a_k(t)$, smooth in $t \in \overline{B}$, satisfying

$$I = |r|^{-\frac{1}{2}} \sum_{k=0}^{N} a_k(t)|r|^{-\frac{k}{2}} + O(|r|^{-\frac{N}{2}-1})$$

uniformly in $t \in \overline{B}$.

Moreover, the last two lemmas give respectively

$$II = O(|r|^{-\frac{N}{2}-1}) \quad \text{and} \quad III = O(|r|^{-\infty})$$

uniformly in $t \in \overline{B}$. In total,

$$I_f(r,t) = |r|^{-\frac{1}{2}} \sum_{k=0}^{N} a_k(t)|r|^{-\frac{k}{2}} + O(|r|^{-\frac{N}{2}-1})$$

uniformly in $t \in \overline{B}$ when $r \to \infty$ (resp. $r \to -\infty$), as desired. □

Our treatment of a parameter $t \in \mathbb{R}^m$ permits to deduce from the previous theorem its multidimensional analogue by induction. Indeed, let us consider the phase

$$Q(x) = \sum_{j=1}^{s} (x^{(j)})^2 - \sum_{k=s+1}^{d} (x^{(k)})^2,$$

where $x \in \mathbb{R}^d$ and $0 \leqslant s \leqslant d$. Returning to the convention that $r > 0$, we let

$$I_f(r,t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}rQ(x)} f(x,t)\,\mathrm{d}x.$$

Then,

Corollary *Consider an amplitude, $f(x,t)$, smooth in the neighborhood of* $\mathbb{R}^d \times$

$\overline{B}$ and vanishing on $U^c \times \overline{B}$, where $U \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^m$ are cubes. There exist constants $a_j(t)$ depending smoothly on $t \in \overline{B}$ such that

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}rQ(x)} f(x,t)\,\mathrm{d}x \sim r^{-\frac{d}{2}} \sum_{j=0}^{\infty} a_j(t) r^{-\frac{j}{2}}$$

uniformly in $t \in \overline{B}$, when $r \to \infty$.

Proof: Suppose the result holds for a certain $d-1$. Let $x_1 = (x^{(1)}, \ldots, x^{(d-1)})$ and $Q_1(x_1)$ be defined by

$$Q(x) = Q_1(x_1) \pm (x^{(d)})^2.$$

Recall that U may be written $U_1 \times \cdots \times U_d$. Then, by the inductive hypothesis, for an arbitrary $N \geqslant 0$

$$\begin{aligned} I_f(r,t) &= \int_{U_d} \mathrm{e}^{\pm \mathrm{i}r(x^{(d)})^2} \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}rQ_1(x_1)} f(x_1; x^{(d)}, t)\,\mathrm{d}x_1\,\mathrm{d}x^{(d)} \\ &= \int_{U_d} \mathrm{e}^{\pm \mathrm{i}r(x^{(d)})^2} \left(r^{-\frac{d-1}{2}} \sum_{k=0}^{N+1} b_k(x^{(d)}, t) r^{-\frac{k}{2}} + O(r^{-\frac{d+N+1}{2}}) \right) \mathrm{d}x^{(d)} \end{aligned}$$

uniformly in $(x^{(d)}, t) \in \overline{U_d} \times \overline{B}$. The following estimates then hold when $r \to \infty$, uniformly in $t \in \overline{B}$:

$$\begin{aligned} I_f(r,t) &= \sum_{k=0}^{N+1} r^{-\frac{d+k-1}{2}} \int_{U_d} \mathrm{e}^{\pm \mathrm{i}r(x^{(d)})^2} b_k(x^{(d)}, t)\,\mathrm{d}x^{(d)} + O(r^{-\frac{d+N+1}{2}}) \\ &= \sum_{k=0}^{N+1} r^{-\frac{d+k-1}{2}} \left(r^{-\frac{1}{2}} \sum_{l=0}^{N} c_{k,l}(t) r^{-\frac{l}{2}} \right) + O(r^{-\frac{d+N+1}{2}}) \\ &= r^{-\frac{d}{2}} \sum_{j=0}^{N} a_j(t) r^{-\frac{j}{2}} + O(r^{-\frac{d+N+1}{2}}). \end{aligned}$$

Since N is arbitrary, this completes the proof. $\square$

Morse Lemma The investigation of oscillatory integrals with non degenerate stationary phase points reduces to the above case by means of *Morse's lemma*, which we now prove.

Let the anticipated phase, $\phi(h,t)$, be a smooth function in $(h,t) \in \mathbb{R}^d \times \mathbb{R}^m$ satisfying

$$\phi(0,t) = 0, \quad \nabla_h \phi(0,t) = 0 \quad \text{and} \quad \det \mathrm{D}_h^2 \phi(0,t) \neq 0$$

for *all* $t \in B$, where $B \subset \mathbb{R}^m$ is a given cube containing a given $t_0 \in \mathbb{R}^m$. Such a function can be expressed *like* a quadratic form, but with coefficients varying smoothly in (h,t):

Lemma *Under the above circumstances there exist functions, $\phi_{jk}(h,t)$, smooth in $(h,t) \in \mathbb{R}^d \times \mathbb{R}^m$ and satisfying*

$$\phi(h,t) = \sum_{j=1}^{d} \sum_{k=1}^{d} \phi_{jk}(h,t) h^{(j)} h^{(k)},$$

where $\phi_{jk}(h,t) = \phi_{kj}(h,t)$.

Proof: Using our hypotheses on $\phi(h,t)$, the fundamental theorem of calculus and integration by parts give for any $t \in B$

$$\phi(h,t) = \int_0^1 \partial_s(\phi(sh,t))\,\mathrm{d}s = \int_0^1 (1-s)\partial_s^2(\phi(sh,t))\,\mathrm{d}s.$$

Expanding $\partial_s^2(\phi(sh,t))$ in the above gives the result. □

The next step consists of applying *Lagrange's algorithm*, which is better understood using matrices. Using the standard basis on $\mathbb{R}^d$, h is represented by a column (also denoted by h), while the "quadratic form" given by the previous lemma is represented by a $d \times d$ matrix, denoted by $\Phi(h,t)$. The (j,k)-th element of $\Phi(h,t)$ is then given by the function $\phi_{jk}(h,t)$, so the previous lemma gives

$$\phi(h,t) = h^{\mathrm{t}} \Phi(h,t) h.$$

For any given $t \in B$ the rank of $\Phi(0,t)$ is read through the Hessian of $\phi(h,t)$ using the following, straightforward relation, which is left to the reader:

Lemma *Under the above circumstances, for* any $t \in B$, $\mathrm{D}_h^2 \phi(0,t) = 2\Phi(0,t)$.

In order to perform Lagrange's algorithm one uses the following elementary line/column operations:

- Given a scalar $c \neq 0$, to multiply the j-th row and then the j-th column by c, which is denoted by $CL_j(c)$;
- Given a scalar $c \in \mathbb{C}$, to add c times the k-th row to the j-th row and then c times the k-th column to the j-th column, which is denoted by $CL_{jk}(c)$;
- To interchange the j-th row with the k-th row and then the j-th column with the k-th column, which is denoted by CL_{jk}.

By the previous lemma, since $\det \mathrm{D}_h^2 \phi(0,t) \neq 0$ and since $\Phi(0,t)$ consists of smooth elements, there exists a cube $V_0 \times B_0 \subset \mathbb{R}^d \times B$ containing $(0,t_0)$ such that $\Phi(h,t)$ is invertible for all $(h,t) \in V_0 \times B_0$.

Without loss of generality $\phi_{11}(h,t) \neq 0$ on a certain cube $V_1 \times B_1 \subseteq V_0 \times B_0$ containing $(0,t_0)$. Otherwise, $\phi_{11}(h,t)$ vanishes at $(0,t_0)$. However, considering the Laplace expansion of the above determinant along the first row, there exist a cube, $V_1' \times B_1' \subseteq V_0 \times B_0$, containing $(0, t_0)$ and an index $1 \leqslant k \leqslant d$ such that $\phi_{1k}(h,t) \neq 0$ for all $(h,t) \in V_1' \times B_1'$. Applying $CL_{1k}(1)$ to $\Phi(h,t)$, the resulting upper left element does not vanish on $V_1' \times B_1'$ (which then replaces $V_1 \times B_1$), as desired.

Then, one may reduce the upper left element to ± 1 (depending on the sign of $\phi_{11}(h,t)$, which does not change on $V_1 \times B_1$) by applying $CL_1(|\phi_{11}(h,t)|^{-\frac{1}{2}})$ to (the possibly refreshed) $\Phi(h,t)$. Finally, this resulting constant on the upper left corner permits to cancel the rest of the first line and column, by applying $CL_{k1}(f(h,t))$ for $k = 2, \ldots, d$ successively—where $f(h,t)$ is equal to the element to cancel (up to the sign). All these operations are represented by matrices having smooth elements in $(h,t) \in V_1 \times B_1$. They transform $\Phi(h,t)$ in a block diagonal matrix, having ± 1 as its first block and a square $(d-1) \times (d-1)$ matrix as its second block.

Repeating this procedure for the second block, $\Phi(h,t)$ is transformed in a block diagonal matrix having ± 1 as its first two blocks and a square $(d-2) \times (d-2)$ matrix as its third block. All the operations used for this second step are represented by matrices having smooth elements in $(h,t) \in V_2 \times B_2$, where

$$(0, t_0) \in V_2 \times B_2 \subseteq V_1 \times B_1 \subseteq V_0 \times B_0.$$

So on and so forth one transforms $\Phi(h,t)$ into a diagonal matrix having elements ± 1 only. All the required operations are smooth (in the previous sense) for (h,t) varying in a cube $V_d \times B_d$ containing $(0, t_0)$. One then applies permutations CL_{jk}, so the resulting matrix becomes $\operatorname{diag}(1, \ldots, 1, -1, \ldots, -1)$, where the element 1 is repeated, say, s times. Since s does not depend on $(h,t) \in V_d \times B_d$, the previous lemma and the Sylvester Inertia Theorem permit to recover s from the signature of $\mathrm{D}_h^2 \phi(0, t_0)$, which is then $(s, d-s, 0)$.

In summary, we have proven:

Lemma *Under the above circumstances there exist a cube, $V_d \times B_d$, containing $(0, t_0)$ and linear map, $Q(h,t)$, whose matrix elements are smooth in $(h,t) \in V_d \times B_d$ such that*

$$Q(h,t)^{\mathrm{t}} \Phi(h,t) Q(h,t) = \operatorname{diag}(1, \ldots, 1, -1, \ldots, -1).$$

In the above the element 1 is repeated s times, where $(s, d-s, 0)$ is the signature of $\mathrm{D}_h^2 \phi(0, t_0)$.

It is thus tempting to consider the *non linear* mapping $h \mapsto Q(h,t)^{-1} h$ defined on V_d as a potential change of variables given a fixed $t \in B_d$. Considering $t \in B_d$ as a parameter, $\{Q(h,t)^{-1} h\}_{t \in B_d}$ is indeed a family of smooth mappings depending smoothly on t in the strong sense that $(h,t) \mapsto Q(h,t)^{-1} h$ is jointly smooth on $V_d \times B_d$. Moreover, all these mappings map 0 to 0. They really consist of invertible changes of variables when restricting suitably the ranges of h and t, as shown below:

Lemma *Under the above circumstances there exists a cube, $V' \times B'$, such that*

$$(0, t_0) \in V' \times B' \subset V_d \times B_d,$$

on which

$$(h, t) \mapsto (Q(h, t)^{-1} h, t)$$

is a smooth diffeomorphism. In addition, this diffeomorphism maps $V' \times B'$ onto an open bounded set, hence contained in a certain cube $\tilde{V'} \times B'$.

Proof: Let $F(h, t) = (Q(h, t)^{-1} h, t)$ for $(h, t) \in V_d \times B_d$. A direct computation shows that

$$\det \mathrm{D}_{(h,t)} F(0, t_0) = \det Q(0, t_0)^{-1},$$

which is not zero. Thus, $F(h, t)$ is a local diffeomorphism in a neighborhood of $(0, t_0)$. Choosing a cube, $V' \times B' \ni (0, t_0)$, whose closure is included in this last neighborhood (and in $V_d \times B_d$) then yields the result. $\square$

The previous diffeomorphism maps $V' \times B'$ onto a bounded open set $\tilde{\mathcal{D}} \subseteq \tilde{V'} \times B'$. We want to fix $t \in B'$, so let $\tilde{\mathcal{D}}_t = \{h \in \mathbb{R}^d \ ; \ (h, t) \in \tilde{\mathcal{D}}\}$. Then, $\tilde{\mathcal{D}}_t \subseteq \tilde{V'}$ is an open set in $\mathbb{R}^d$. Let us consider the restriction

$$V' \to \tilde{\mathcal{D}}_t, \ h \mapsto Q(h, t)^{-1} h,$$

which is also a smooth diffeomorphism. It may be used as a smooth invertible change of variables by setting

$$\tilde{h} = Q(h, t)^{-1} h$$

for $h \in V'$. Notice that 0 is then mapped to 0. Let P_t be the inverse change of variables, so

$$h = P_t(\tilde{h})$$

for $\tilde{h} \in \tilde{\mathcal{D}}_t$. Then, $\tilde{h} = P_t^{-1}(h) = Q(h, t)^{-1} h$, which implies $h = Q(h, t)\tilde{h}$ and hence

$$\begin{aligned} \phi(P_t(\tilde{h}), t) &= \phi(h, t) \\ &= \phi(Q(h, t)\tilde{h}, t) \\ &= \tilde{h}^{\mathrm{t}} Q(h, t)^{\mathrm{t}} \Phi(h, t) Q(h, t) \tilde{h} \\ &= \sum_{j=1}^{s} (\tilde{h}^{(j)})^2 - \sum_{k=s+1}^{d} (\tilde{h}^{(k)})^2. \end{aligned}$$

We have thus proven the *Morse lemma* with special care of the parameter t:

Theorem 4 *Given a cube $B \subset \mathbb{R}^m$ containing a fixed t_0, suppose $\phi(h, t)$ is smooth in $(h, t) \in \mathbb{R}^d \times B$ and satisfies*

$$\phi(0, t) = 0, \quad \nabla_h \phi(0, t) = 0, \quad \textit{and} \quad \det \mathrm{D}_h^2 \phi(0, t) \neq 0$$

for all $t \in B$. *Then, there exists a cube* $V' \times B' \subset \mathbb{R}^d \times B$ *containing* $(0, t_0)$ *such that the following holds: for all* $t \in B'$ *there exists an invertible change of variables*

$$\tilde{h} = P_t^{-1}(h)$$

on V'*, smooth and with smooth inverse, mapping* 0 *to* 0*, which satisfies*

$$\phi(P_t(\tilde{h}), t) = \sum_{j=1}^{s} (\tilde{h}^{(j)})^2 - \sum_{k=s+1}^{d} (\tilde{h}^{(k)})^2.$$

The resulting family of changes of variables, $\{P_t(\tilde{h})\}_{t \in B'}$*, depends diffeomorphically on* $t \in B'$ *in the following sense: setting* $\tilde{\mathcal{D}}_t = P_t^{-1}(V')$*,*

$$\tilde{\mathcal{D}} = \bigcup_{t \in B'} \tilde{\mathcal{D}}_t \times \{t\}$$

is an open set in $\mathbb{R}^d \times \mathbb{R}^m$ *(contained in a cube* $\tilde{V}' \times B'$*) on which*

$$\tilde{\mathcal{D}} \to V' \times B', \quad (\tilde{h}, t) \mapsto (P_t(\tilde{h}), t)$$

is a diffeomorphism.

Continuation The corollary of Theorem 3 joined with the Morse lemma finally yield:

Theorem 5 *Suppose* $\varphi(x, t)$ *is smooth in* $(x, t) \in \mathbb{R}^d \times \mathbb{R}^m$ *and satisfies*

$$\nabla_x \varphi(x_0, t_0) = 0 \quad \textit{and} \quad \det \mathrm{D}_x^2 \varphi(x_0, t_0) \neq 0.$$

Then, there exists an arbitrarily small cube, $U \times B$*, containing* (x_0, t_0) *such that the following holds: if* $f(x, t)$ *is smooth in the neighborhood of* $\mathbb{R}^d \times \overline{B}$ *and vanishes on* $U^{\mathrm{c}} \times \overline{B}$*, then*

$$I_f(r, t) \sim \mathrm{e}^{\mathrm{i} r \theta(t)} r^{-\frac{d}{2}} \sum_{j=0}^{\infty} a_j(t) r^{-\frac{j}{2}},$$

where $\theta(t)$ *is real valued, the* $a_j(t)$*'s are complex valued, and all these functions are smooth in* $t \in \overline{B}$*. Moreover, these estimates are uniform in* $t \in \overline{B}$*.*

Proof: By the Implicit Function Theorem there exists a smooth function

$$B_0 \subset \mathbb{R}^m \to \mathbb{R}^d, \quad t \mapsto \mathrm{x}(t)$$

defined on a cube $B_0 \ni t_0$ such that $(\nabla_x \varphi)(\mathrm{x}(t), t) = 0$ for all $t \in B_0$ and $\mathrm{x}(t_0) = x_0$. Without loss of generality we also suppose

$$\det \mathrm{D}_x^2 \varphi(\mathrm{x}(t), t) \neq 0$$

for all $t \in B_0$. Let

$$\phi(h,t) = \varphi(\mathrm{x}(t)+h,t) - \varphi(\mathrm{x}(t),t),$$

which is smooth in $(h,t) \in \mathbb{R}^d \times B_0$. Then, $\phi(h,t)$ satisfies the hypotheses of Morse's lemma. Hence, there exists a cube, $V' \times B'$, containing $(0,t_0)$ and whose closure is in $\mathbb{R}^d \times B_0$, and a family of diffeomorphisms

$$P_t^{-1}\colon\, V' \to \tilde{\mathcal{D}}_t \subseteq \tilde{V}' \quad \text{(where } t \in B'\text{)}$$

such that, letting $\tilde{h} = P_t^{-1}(h)$, one obtains

$$\phi(P_t(\tilde{h}),t) = \sum_{j=1}^{s} (\tilde{h}^{(j)})^2 - \sum_{k=s+1}^{d} (\tilde{h}^{(k)})^2.$$

Notice that $0 = x_0 - \mathrm{x}(t_0) \in V'$. Consequently, there exists an arbitrarily small cube, $U \times B \ni (x_0,t_0)$, such that

$$(x - \mathrm{x}(t),t) \in V' \times B'$$

for all $(x,t) \in \overline{U} \times \overline{B}$. In other words, $U \times B$ is mapped onto a region whose closure is in $V' \times B'$ via the change of variables $(h,t) = (x - \mathrm{x}(t),t)$.

Let us consider an amplitude, $f(x,t)$, satisfying the asserted properties. Since for a fixed $t \in \overline{B}$ the integrand in

$$I_f(r,t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\varphi(x,t)} f(x,t)\,\mathrm{d}x$$

is supported in U, it follows that the right-hand side in

$$I_f(r,t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\varphi(h+\mathrm{x}(t),t)} f(h+\mathrm{x}(t),t)\mathrm{d}h = \mathrm{e}^{\mathrm{i}r\varphi(\mathrm{x}(t),t)} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\phi(h,t)} f(h+\mathrm{x}(t),t)\mathrm{d}h$$

is supported in V'. Then, by Morse's lemma the change of variables $\tilde{h} = P_t^{-1}(h)$ is available, yielding

$$I_f(r,t) = \mathrm{e}^{\mathrm{i}r\varphi(\mathrm{x}(t),t)} \int_{\tilde{\mathcal{D}}_t} \mathrm{e}^{\mathrm{i}rQ(\tilde{h})} f(P_t(\tilde{h}) + \mathrm{x}(t),t) J_t(\tilde{h})\,\mathrm{d}\tilde{h},$$

where $J_t(\tilde{h})$ is the Jacobian and

$$Q(\tilde{h}) = \sum_{j=1}^{s} (\tilde{h}^{(j)})^2 - \sum_{j=s+1}^{d} (\tilde{h}^{(j)})^2.$$

Since the amplitude in the above extends smoothly on $\mathbb{R}^d \times \overline{B}$ and vanishes for $\tilde{h} \notin \tilde{V}'$, the corollary of Theorem 3 then completes the proof. $\square$

Finally, the following result is an interesting application of our treatment of a parameter:

Theorem 6 *Suppose $\varphi(x,t)$ is smooth in $(x,t) \in \mathbb{R}^d \times \mathbb{R}^m$ and satisfies*

$$\nabla_x \varphi(x_0,t_0) = 0 \quad \textit{and} \quad \operatorname{rank} \mathrm{D}_x^2 \varphi(x_0,t_0) = \kappa,$$

where $\kappa \geqslant 1$. Then, there exists an arbitrarily small cube, $U \times B$, containing (x_0,t_0) such that the following holds: if $f(x,t)$ is smooth in a neighborhood of $\mathbb{R}^d \times \overline{B}$ and vanishes on $U^{\mathrm{c}} \times \overline{B}$, then

$$I_f(r,t) = O(r^{-\frac{\kappa}{2}})$$

uniformly in $t \in \overline{B}$.

Proof: Since the rank of $\mathrm{D}_x^2\varphi(x_0,t_0)$ is equal to the order of its largest non zero principal minor, there exist indices, $j_1,\ldots,j_\kappa$, such that, letting

$$\xi = (x^{(j_1)},\ldots,x^{(j_\kappa)}),$$

$\det \mathrm{D}_\xi^2(x_0,t_0) \neq 0$. After permuting the variables we write

$$x = (\xi,\chi), \quad \text{and} \quad x_0 = (\xi_0,\chi_0),$$

with the obvious definitions of χ, ξ_0, and χ_0. Interpreting (χ,t) as a parameter the previous theorem gives the result. $\square$

Remark *Using the previous decomposition one cannot derive the complete asymptotics of the considered oscillatory integral, since the resulting coefficients would be oscillatory integrals themselves! Their decay is not known* a priori.

Cauchy Principal Value We now derive similar results for Cauchy principal values of oscillatory integrals with non degenerate stationary phase points. To this end let us consider first

$$\text{p.v.} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}rh} \frac{f(h)}{h}\,\mathrm{d}h = \lim_{\varepsilon \downarrow 0} \int_{|h|>\varepsilon} \mathrm{e}^{\mathrm{i}rh} \frac{f(h)}{h}\,\mathrm{d}h,$$

where the amplitude, $f(h)$, is smooth in $h \in \mathbb{R}$ and compactly supported.

Notice that the lemma of Theorem 1 generalizes to a complex valued phase provided that the path of integration remains in $\overline{\mathbb{C}_+}$, explicitly: *Given a smooth regular*[5] *path, $\gamma(t)$, lying in the closure of $\mathbb{C}_+$, and an amplitude, $f(z) = f(x,y)$, smooth in x and compactly supported along this path,*

$$\int_\gamma \mathrm{e}^{\mathrm{i}rz} f(z)\,\mathrm{d}z = O(r^{-\infty})$$

when $r \to \infty$. The proof is left to the reader.

[5]A smooth path, $\gamma(t)$, is *regular* if $\gamma'(t) \neq 0$ for any t.

Lemma *Suppose $f(h)$ is smooth in $h \in \mathbb{R}$, compactly supported, and analytic at* 0*. Then,*

$$\mathrm{p.v.} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}rh} \frac{f(h)}{h} \, \mathrm{d}h = \pi \mathrm{i} f(0) + O(r^{-\infty})$$

when $r \to \infty$.

Proof: For $\varepsilon > 0$, let C_ε be a smooth regular path starting at -2ε, going through $[-2\varepsilon, -\varepsilon]$, then avoiding the origin, but staying in

$$\{z \in \mathbb{C} \,;\, |\mathrm{Re}\,(z)| < \varepsilon \text{ and } 0 < \mathrm{Im}\, z < \varepsilon\},$$

and finally going through $[\varepsilon, 2\varepsilon]$. Since

$$\mathrm{e}^{\mathrm{i}rz} \frac{f(z)}{z} = \frac{f(0)}{z} + \text{ \textit{analytic}}$$

in a punctured neighborhood of the origin,

$$\lim_{\varepsilon \downarrow 0} \int_{C_\varepsilon} \mathrm{e}^{\mathrm{i}rz} \frac{f(z)}{z} \, \mathrm{d}z = -\pi \mathrm{i} f(0).$$

Thus, letting $\gamma_\varepsilon = \left]-\infty, -2\varepsilon\right] * C_\varepsilon * \left[2\varepsilon, \infty\right[$ (where $*$ denotes the concatenation), which is smooth and regular,

$$\begin{aligned} \mathrm{p.v.} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}rh} \frac{f(h)}{h} \, \mathrm{d}h - \pi \mathrm{i} f(0) &= \lim_{\varepsilon \downarrow 0} \int_{\gamma_\varepsilon} \mathrm{e}^{\mathrm{i}rz} \frac{f(z)}{z} \, \mathrm{d}z \\ &= \int_{\gamma} \mathrm{e}^{\mathrm{i}rz} \frac{f(z)}{z} \, \mathrm{d}z, \end{aligned}$$

where $\gamma = \gamma_{\varepsilon_0}$ for a fixed, but small enough $\varepsilon_0 > 0$. The result follows from the statement preceding the lemma. $\square$

The analyticity assumption may be removed in the following way—where a parameter $t \in \mathbb{R}^m$ is also introduced for later purpose:

Theorem 7 *Given $\delta > 0$ and a cube $B \subset \mathbb{R}^m$, consider a function, $f(h, t)$, smooth in a neighborhood of $\mathbb{R} \times \overline{B}$ and vanishing on $\left]-\delta, \delta\right[^{\mathrm{c}} \times \overline{B}$. Then,*

$$\mathrm{p.v.} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}rh} \frac{f(h,t)}{h} \, \mathrm{d}h = \pi \mathrm{i} f(0, t) + O(r^{-\infty})$$

uniformly in $t \in \overline{B}$, when $r \to \infty$.

Proof: Let $\chi(x)$ be a smooth function in $x \in \mathbb{R}$, compactly supported, such that $0 \leqslant \chi(x) \leqslant 1$ on $\mathbb{R}$ and $\chi(x) \equiv 1$ in a neighborhood of 0. The considered principal value then decomposes into $I + II + III$, where

$$\begin{aligned} I &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}rh} \frac{f(h,t)}{h}(1-\chi(h))\,\mathrm{d}h, \\ II &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}rh} \frac{f(h,t)-f(0,t)}{h}\chi(h)\,\mathrm{d}h, \quad \text{and} \\ III &= f(0,t)\mathrm{p.v.}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}rh} \frac{\chi(h)}{h}\,\mathrm{d}h. \end{aligned}$$

The lemma of Theorem 1 implies $I = O(r^{-\infty})$ and $II = O(r^{-\infty})$ uniformly in $t \in \overline{B}$.[6] Since $\chi(x)$ is analytic at 0 and $\chi(0) = 1$, the result follows from the lemma. □

We now turn our attention to a phase, $\varphi(h,t)$, smooth in $(h,t) \in \mathbb{R} \times \mathbb{R}^m$, such that $\partial_h\varphi(0,t_0) \neq 0$. Our study reduces to the previous theorem by a change of variables, regarding which the following elementary result is helpful: *Given a cube $B \subset \mathbb{R}^m$, suppose $f(h,t)$ is smooth in a neighborhood of $\{0\} \times \overline{B}$, where it satisfies*

$$f(0,t) = 0 \quad \textit{and} \quad \partial_h f(0,t) \neq 0.$$

Then, $\frac{h\partial_h f(h,t)}{f(h,t)}$ extends smoothly to a neighborhood of $\{0\} \times \overline{B}$, the extension being equal to 1 *when $h = 0$.* The proof is left to the reader.

Lemma *Suppose $\partial_h\varphi(0,t_0) \neq 0$ for a certain $t_0 \in \mathbb{R}^m$. There exist a $\delta > 0$ and an arbitrarily small cube, B, containing t_0 such that the following holds: if $f(h,t)$ is smooth in a neighborhood of $\mathbb{R} \times \overline{B}$ and vanishes on $]-\delta,\delta[^{\mathrm{c}} \times \overline{B}$, then*

$$\mathrm{p.v.}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}r\varphi(h,t)} \frac{f(h,t)}{h}\,\mathrm{d}h = \sigma\pi\mathrm{i} f(0,t)\mathrm{e}^{\mathrm{i}r\varphi(0,t)} + O(r^{-\infty})$$

uniformly in $t \in \overline{B}$, where $\sigma = \mathrm{sgn}(\partial_h\varphi(0,t_0))$.

Proof: By continuity there exists a cube, $]-\delta,\delta[\, \times B' \ni (0,t_0)$, on which $\partial_h\varphi(h,t) \neq 0$. Let $\varphi_t(h) = \varphi(h,t) - \varphi(0,t)$ and $F(h,t) = (\varphi_t(h), t)$, which is smooth in $(h,t) \in \mathbb{R} \times \mathbb{R}^m$. Then,

$$\det \mathrm{D}_{(h,t)} F(h,t) = \partial_h\varphi(h,t) \neq 0$$

on $]-\delta,\delta[\, \times B'$. Hence, $F(h,t)$ admits a smooth inverse, $G(\tilde{h},t) = (\psi_t(\tilde{h}), t)$, defined on $F(]-\delta,\delta[\, \times B')$. Notice that $\varphi_t(0) = 0 = \psi_t(0)$ and

$$\psi_t'(0) = \frac{1}{\varphi_t'(0)} \neq 0.$$

In particular, $\sigma = \mathrm{sgn}(\psi_t'(0))$. Moreover, by the statement preceding the lemma $\tilde{h}\psi_t'(\tilde{h})/\psi_t(\tilde{h})$ has a smooth extension to a neighborhood of $\{0\} \times \overline{B}$ (where

[6]More precisely, the result follows from the proof of this lemma, by noticing that $U' \times B'$ may be chosen to be equal to $\mathbb{R} \times \mathbb{R}^m$—since the derivative of the phase is $1 \neq 0$ everywhere.

$\overline{B} \subset B'$ is arbitrarily chosen), which is equal to 1 when $\tilde{h} = 0$. For $f(h,t)$ of the stipulated form and $t \in \overline{B}$, the change of variables $\tilde{h} = \varphi_t(h)$ then gives

$$\begin{aligned}
\text{p.v.} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}r\varphi(h,t)} \frac{f(h,t)}{h} \,\mathrm{d}h &= \text{p.v.} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}r\tilde{h}} \mathrm{e}^{\mathrm{i}r\varphi(0,t)} \frac{f(\psi_t(\tilde{h}),t)}{\psi_t(\tilde{h})} |\psi_t'(\tilde{h})| \,\mathrm{d}\tilde{h} \\
&= \sigma \mathrm{e}^{\mathrm{i}r\varphi(0,t)} \text{p.v.} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}r\tilde{h}} \frac{g(\tilde{h},t)}{\tilde{h}} \,\mathrm{d}\tilde{h},
\end{aligned}$$

where

$$g(\tilde{h},t) = f(\psi_t(\tilde{h}),t) \frac{\tilde{h}\psi_t'(\tilde{h})}{\psi_t(\tilde{h})}.$$

Notice that $g(0,t) = f(0,t)$. Moreover, $g(\tilde{h},t)$ vanishes when $t \in \overline{B}$ and $\tilde{h} \notin]-\tilde{\delta},\tilde{\delta}[$ for a certain $\tilde{\delta} > 0$. The previous theorem thus applies, which completes the proof. □

Let us now consider a phase, $\varphi(x,h,t)$, smooth in $(x,h,t) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^m$, and

$$I_f(r,h,t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\varphi(x,h,t)} f(x,h,t) \,\mathrm{d}x.$$

Theorem 8 *Suppose* $\nabla_x \varphi(x_0,0,t_0) = 0$, $\det \mathrm{D}_x^2 \varphi(x_0,0,t_0) \neq 0$, *and*

$$\partial_h \varphi(x_0,0,t_0) \neq 0.$$

Then, there exist a $\delta > 0$ *and an arbitrarily small cube,* $U \times B$, *containing* (x_0,t_0) *such that the following holds: if* $f(x,h,t)$ *is smooth in the neighborhood of* $\mathbb{R}^d \times \mathbb{R} \times \overline{B}$ *and vanishes on* $(U \times]-\delta,\delta[)^{\mathrm{c}} \times \overline{B}$, *then*

$$\text{p.v.} \int_{\mathbb{R}} \frac{1}{h} I_f(r,h,t) \,\mathrm{d}h \sim \mathrm{e}^{\mathrm{i}r\theta(t)} r^{-\frac{d}{2}} \sum_{j=0}^{\infty} a_j(t) r^{-\frac{j}{2}}$$

uniformly in $t \in \overline{B}$, *where* $\theta(t)$ *and* $a_j(t)$ *are smooth in the neighborhood of* $\overline{B}$.

Proof: Interpreting $(x,t) \in \mathbb{R}^d \times \mathbb{R}^m$ as a parameter, the previous lemma holds, since $\partial_h \varphi(x_0,0,t_0) \neq 0$. Moreover, interpreting $(h,t) \in \mathbb{R} \times \mathbb{R}^m$ as a parameter, the theorem 5 also holds, since $\nabla_x \varphi(x_0,0,t_0) = 0$ and $\det \mathrm{D}_x^2 \varphi(x_0,0,t_0) \neq 0$. Consequently, there exist a $\delta > 0$ and an arbitrarily small cube, $U \times B$, containing (x_0,t_0) such that for $f(x,h,t)$ of the stipulated form

$$\text{p.v.} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}r\varphi(x,h,t)} \frac{f(x,h,t)}{h} \,\mathrm{d}h = \sigma\pi\mathrm{i} f(x,0,t) \mathrm{e}^{\mathrm{i}r\varphi(x,0,t)} + O(r^{-\infty}) \tag{4}$$

uniformly in $(x,t) \in \overline{U} \times \overline{B}$ and

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\varphi(x,0,t)} f(x,0,t) \,\mathrm{d}x \sim \mathrm{e}^{\mathrm{i}r\theta(t)} r^{-\frac{d}{2}} \sum_{j=0}^{\infty} a_j(t) r^{-\frac{j}{2}} \tag{5}$$

uniformly and smoothly in $t \in \overline{B}$, both when $r \to \infty$.

By Fubini's and the dominated convergence theorems

$$\text{p.v.} \int_{\mathbb{R}} \frac{1}{h} I_f(r,h,t)\,\mathrm{d}h = \int_{\mathbb{R}^d} \text{p.v.} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}r\varphi(x,h,t)} \frac{f(x,h,t)}{h}\,\mathrm{d}h\,\mathrm{d}x,$$

where the dominator is given by the lemma of Theorem 2. By the equation (4), the above is equal to

$$\int_{\mathbb{R}^d} \sigma\pi\mathrm{i} f(x,0,t)\mathrm{e}^{\mathrm{i}r\varphi(x,0,t)}\,\mathrm{d}x + O(r^{-\infty})$$

uniformly in $t \in \overline{B}$. The equation (5) then yields the result. □

In the same way as we derived from Theorem 2 its corollary,

Corollary *Suppose* $\nabla_x \varphi(x_0, e_0, t_0) = 0$, $\det \mathrm{D}_x^2 \varphi(x_0, e_0, t_0) \neq 0$, *and*

$$\partial_e \varphi(x_0, e_0, t_0) \neq 0.$$

Then, there exist a $\delta > 0$ *and an arbitrarily small cube* $U \times B$ *containing* (x_0, t_0) *such that the following holds: if* $f(x,e,t)$ *is smooth in the neighborhood of* $\mathbb{R}^d \times \mathbb{R} \times \overline{B}$ *and vanishes on* $(U \times]e_0 - \delta, e_0 + \delta[)^{\mathrm{c}} \times \overline{B}$, *then*

$$\text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} I_f(r,\eta,t)\,\mathrm{d}\eta \sim \mathrm{e}^{\mathrm{i}r\theta(e,t)} r^{-\frac{d}{2}} \sum_{j=0}^{\infty} a_j(e,t) r^{-\frac{j}{2}}$$

uniformly in $(e,t) \in [e_0 - \delta, e_0 + \delta] \times \overline{B}$, *where* $\theta(e,t)$ *and* $a_j(e,t)$ *are smooth in the neighborhood of* $[e_0 - \delta, e_0 + \delta] \times \overline{B}$.

More generally, by the argument used in Theorem 6 (which we repeat!),

Theorem 9 *Suppose* $\nabla_x \varphi(x_0, e_0, t_0) = 0$, $\operatorname{rank} \mathrm{D}_x^2 \varphi(x_0, e_0, t_0) = \kappa$, *and*

$$\partial_e \varphi(x_0, e_0, t_0) \neq 0.$$

Then, there exist a $\delta > 0$ *and an arbitrarily small cube,* $U \times B$, *containing* (x_0, t_0) *such that the following holds: if* $f(x,e,t)$ *is smooth in the neighborhood of* $\mathbb{R}^d \times \mathbb{R} \times \overline{B}$ *and vanishes on* $(U \times]e_0 - \delta, e_0 + \delta[)^{\mathrm{c}} \times \overline{B}$, *then*

$$\text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} I_f(r,\eta,t)\,\mathrm{d}\eta = O(r^{-\frac{\kappa}{2}})$$

uniformly in $(e,t) \in [e_0 - \delta, e_0 + \delta] \times \overline{B}$.

Proof: There exist indices, $j_1, \cdots, j_\kappa$, such that, for

$$\xi = (x^{(j_1)}, \cdots, x^{(j_\kappa)}),$$

$\det \mathrm{D}_\xi^2(x_0, t_0) \neq 0$. After permuting the variables we write

$$x = (\xi, \chi) \quad \text{and } x_0 = (\xi_0, \chi_0)$$

with the obvious definitions of χ, ξ_0, and χ_0. Interpreting (χ, t) as a parameter, the above corollary gives the existence of a $\delta > 0$ and an arbitrarily small cube, $U' \times (U'' \times B) \ni (\xi_0; \chi_0, t_0)$, such that, letting $U = U' \times U'' \ni x_0$, for $f(x,t)$ of the stipulated form

$$\text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} \int_{\mathbb{R}^\kappa} \mathrm{e}^{\mathrm{i}r\varphi(\xi;\chi,\eta,t)} f(\xi; \chi, \eta, t)\, \mathrm{d}\xi\, \mathrm{d}\eta = O(r^{-\frac{\kappa}{2}})$$

uniformly in $(\chi, e, t) \in \overline{U''} \times [e_0 - \delta, e_0 + \delta] \times \overline{B}$. Therefore,

$$\int_{\mathbb{R}^{d-\kappa}} \text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} \int_{\mathbb{R}^\kappa} \mathrm{e}^{\mathrm{i}r\varphi(\xi;\chi,\eta,t)} f(\xi; \chi, \eta, t)\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\chi = O(r^{-\frac{\kappa}{2}})$$

uniformly in $(e,t) \in [e_0 - \delta, e_0 + \delta] \times \overline{B}$. By Fubini's and the dominated convergence theorems this last integral is equal to

$$\int_{\mathbb{R}^d} \text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} \mathrm{e}^{\mathrm{i}r\varphi(x;\eta,t)} f(x; \eta, t)\, \mathrm{d}\eta\, \mathrm{d}x,$$

and hence (for the same reasons) to

$$\text{p.v.} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{1}{\eta - e} \mathrm{e}^{\mathrm{i}r\varphi(x;\eta,t)} f(x; \eta, t)\, \mathrm{d}x\, \mathrm{d}\eta,$$

where both times the dominator is given by the lemma of Theorem 2. The proof is thus complete. $\square$

4 Fourier Transforms over Level Surfaces of Analytic Functions

We now consider a real valued function, $\Phi(x)$, analytic in $x \in \mathbb{R}^d$. Usually, the level surfaces of $\Phi(x)$ consist of several connected components; let us focus on some of them, say, the connected components whose reunion is given by

$$\Gamma(e) = \{x \in \mathcal{R}\ ;\ \Phi(x) = e\}$$

for an appropriate domain $\mathcal{R} \subseteq \mathbb{R}^d$.

Given a function, $f(x)$, summable on $\Gamma(e)$, its Fourier transform over $\Gamma(e)$ is defined as

$$\mathcal{F}(\Gamma(e), f)(n) = \int_{\Gamma(e)} \mathrm{e}^{\mathrm{i}n\cdot x} f(x)\, \mathrm{dS}(x)$$

for $n \in \mathbb{Z}^d$, where $\mathrm{dS}(x)$ denotes the element of surface on $\Gamma(e)$. In this section we derive the decay of $\mathcal{F}(\Gamma(e), f)(n)$ when $|n| \to \infty$ and show its uniformity when e varies on an appropriate interval. We then derive an analogous result for the Cauchy principal value of such a Fourier transform.

Let $a' < b'$ and $\mathcal{S}' = \bigcup_{e \in]a',b'[} \Gamma(e)$ be given. We assume:

Assumption A

- $\Phi(x)$ *is real valued and analytic in* $\mathbb{R}^d$*;*
- $\nabla\Phi(x) \neq 0$ *for all* $x \in \mathcal{S}'$*;*
- $\Gamma(e)$ *is compact for all* $e \in]a', b'[$.

The second statement in Assumption A and the Implicit Function Theorem ensure that $\Gamma(e)$ is a *regular smooth surface* for any $e \in]a', b'[$. In particular, $\Gamma(e)$ may be covered by real-analytic local parameterizations, $U \xrightarrow{\sigma} \Gamma(e)$, where each $U \subset \mathbb{R}^{d-1}$ is open, each $\sigma(u)$ is a smooth homeomorphism between U and $\sigma(U)$ in the topology of $\Gamma(e)$, and

$$\bigcup_{(U,\sigma)} \sigma(U) = \Gamma(e).$$

Moreover, if two of the previous local parameterizations, (U, σ) and (V, τ), have a non empty overlap, $\mathcal{D} = \sigma(U) \cap \tau(V) \subseteq \Gamma(e)$, then the change of parameterizations $\tau^{-1} \circ \sigma(u)$ is a real-analytic diffeomorphism from $\sigma^{-1}(\mathcal{D})$ to $\tau^{-1}(\mathcal{D})$.

Given a local parameterization, (U, σ), the restriction of the Fourier transform to $\sigma(U)$ gives

$$\begin{aligned}\mathcal{F}(\sigma(U), f)(n) &= \int_{\sigma(U)} \mathrm{e}^{\mathrm{i}n\cdot x} f(x)\, \mathrm{dS}(x) \\ &= \int_U \mathrm{e}^{\mathrm{i}n\cdot\sigma(u)} f(\sigma(u)) J(u)\, \mathrm{d}u\end{aligned}$$

for any $f(x)$ summable on $\sigma(U)$, where $J(u)\,\mathrm{d}u$ is the element of surface. Of course the decay of such an integral when $|n| \to \infty$ is studied by means of the stationary phase method.

Let $n = r\omega$ be the polar form of $n \in \mathbb{Z}^d$, so $r = |n|$ and $\omega \in \mathrm{S}^{d-1}$. The *phase* in the previous integral then becomes $\varphi(u, \omega) = \omega\cdot\sigma(u)$. Let us consider a point, $x_0 = \sigma(u_0)$, in $\sigma(U)$. Remarkably, the fact that u_0 is or is not a stationary phase point depends on *intrinsic properties of* $\Gamma(e)$ *only*; if u_0 is stationary, the rank of $\mathrm{D}_u^2\varphi(u_0, \omega)$ is also intrinsic. Indeed,

Theorem 10 *In the above circumstances* u_0 *is stationary if, and only if* ω *is perpendicular to* $\Gamma(e)$ *at* $x_0 = \sigma(u_0)$*. Then, the rank of the Hessian of* $\varphi(u, \omega)$ *at* u_0 *is equal to the number of non vanishing principal curvatures of* $\Gamma(e)$ *at* x_0*.*

Proof: Notice that u_0 is stationary iff

$$\nabla_u \omega \cdot \sigma(u_0) = (\omega \cdot \mathrm{d}_{u^{(1)}}\sigma(u_0), \ldots, \omega \cdot \mathrm{d}_{u^{(d-1)}}\sigma(u_0)) = 0.$$

Since the tangent plane of $\Gamma(e)$ at x_0 is generated by

$$\{\mathrm{d}_{u^{(j)}}\sigma(u_0)\ ;\ j = 1, \ldots, d-1\},$$

the first statement follows.

Suppose u_0 is a stationary phase point and consider any other local parameterization, (V, τ), of a neighborhood of $x_0 = \sigma(u_0)$. Let $\mathcal{D} = \sigma(U) \cap \tau(V)$ and $F(v) = \sigma^{-1} \circ \tau(v)$. Then, $F(v)$ is a smooth diffeomorphism from $\tau^{-1}(\mathcal{D})$ to $\sigma^{-1}(\mathcal{D})$ satisfying

$$\omega \cdot \tau(v) = \omega \cdot \sigma(F(v)). \tag{6}$$

Let $v_0 = \tau^{-1}(x_0)$, so $(\nabla_u(\omega \cdot \sigma))(F(v_0)) = \nabla_u(\omega \cdot \sigma)(u_0) = 0$. Then, the chain rule applied to (6) gives

$$\mathrm{D}_v^2(\omega \cdot \tau)(v_0) = \mathrm{D}_u^2(\omega \cdot \sigma)(u_0)(\mathrm{D}_v F(v_0))^2.$$

Since $F(v)$ is a diffeomorphism, $\mathrm{D}_v F(v_0)$ is invertible and hence

$$\operatorname{rank} \mathrm{D}_v^2(\omega \cdot \tau)(v_0) = \operatorname{rank} \mathrm{D}_u^2(\omega \cdot \sigma)(u_0),$$

which shows that this last rank is intrinsic.

Finally, since u_0 is a stationary phase point, ω is perpendicular to $\Gamma(e)$ at x_0, so

$$\omega = \frac{\pm\nabla_x \Phi(x_0)}{\|\nabla_x \Phi(x_0)\|}.$$

Moreover, there exists a $j \in \{1, \ldots, d\}$ such that $\omega^{(j)} \neq 0$. Suppose without loss of generality $\omega^{(d)} \neq 0$, and hence $\partial_{x^{(d)}}\Phi(x_0) \neq 0$. Let $w = (x^{(1)}, \ldots, x^{(d-1)})$ and $w_0 = (x_0^{(1)}, \ldots, x_0^{(d-1)})$. By the Implicit Function Theorem there exists a function, $h(w)$, smooth in the neighborhood of w_0, such that $\Phi(w, h(w)) = e$ and $h(w_0) = x_0^{(d)}$. Hence, $\gamma(w) = (w, h(w))$ gives a smooth local parameterization of a neighborhood of x_0 as a graph of a smooth function. Differential geometry then shows that $\operatorname{rank} \mathrm{D}_w^2 h(w)$ is equal to the number of non vanishing principal curvatures at $x_0 \in \Gamma(e)$. Since

$$\mathrm{D}_w^2(\omega \cdot \gamma)(w) = \omega^{(d)} \mathrm{D}_w^2 h(w),$$

the proof is complete. $\square$

Joint system of parameterizations Given a fixed $e_0 \in\ ]a', b'[$, we now construct a system of parameterizations for $\Gamma(e_0)$ compatible with all $\Gamma(e)$'s for e varying in a small neighborhood of e_0 (so the derived estimate for Fourier transforms will be uniform in e). We make the following hypothesis:

Assumption B *For every $e \in\ ]a', b'[$, $\Gamma(e)$ admits at least κ non vanishing principal curvatures at any point, where $\kappa \geqslant 1$ is a fixed integer.*

The plan is the following: starting from an arbitrary system of real-analytic parameterizations for $\Gamma(e_0)$, we will parametrize $\Gamma(e)$ using the local coordinates of $\Gamma(e_0)$, by lifting them orthogonally to $\Gamma(e_0)$ (for e very close to e_0).

Let $\{(\mathcal{U}_\beta, \gamma_\beta)\}_{\beta=1}^{M}$ be a system of real-analytic parameterizations covering $\Gamma(e_0)$. Since $\Gamma(e_0)$ is compact, we assume M to be finite. Without loss of generality, we also assume $\gamma_\beta(u)$ is analytic in a neighborhood of $\overline{\mathcal{U}_\beta}$, so the expression $\gamma_\beta(u)$ for $u \in \partial\mathcal{U}_\beta$ makes sense.

Let $x_0 \in \Gamma(e_0)$, say, $x_0 = \gamma_\beta(u_0)$ for a given $u_0 \in \mathcal{U}_\beta$ and a given $1 \leqslant \beta \leqslant M$. Since $\nabla_x\Phi(x)$ is perpendicular to $\Gamma(e_0)$ at any $x \in \Gamma(e_0)$, we need to solve

$$\Phi(\, \gamma_\beta(u) + \lambda\nabla_x\Phi(\gamma_\beta(u))\,) - e = 0 \tag{7}$$

in the neighborhood of $(u, e, \lambda) = (u_0, e_0, 0)$. The derivative at 0 with respect to λ of the left-hand side in (7) is $\|\nabla_x\Phi(\gamma_\beta(u))\|^2$, which is strictly positive. Hence, there exists an analytic function, $\lambda(u, e)$, defined on an arbitrarily small cube, $\mathcal{U}' \times \,]e_0 - \Delta', e_0 + \Delta'[\, \ni (u_0, e_0)$, satisfying

$$\gamma_\beta(u) + \lambda(u, e)\nabla_x\Phi(\gamma_\beta(u)) \in \Gamma(e), \;\; \lambda(u_0, e_0) = 0. \tag{8}$$

Remarkably, $\lambda(u, e)$ depends on $\gamma_\beta(u)$ only, not on its local coordinates u:

Lemma *Suppose $\gamma_{\beta'}(u'_0) = x_0$ and let $u' = \gamma_{\beta'}^{-1} \circ \gamma_\beta(u)$, where u varies in*

$$\gamma_\beta^{-1}(\gamma_\beta(\mathcal{U}_\beta) \cap \gamma_{\beta'}(\mathcal{U}_{\beta'})).$$

Define $\lambda'(u', e)$ as above with respect to $\gamma_{\beta'}(u')$—while $\lambda(u, e)$ was defined with respect to $\gamma_\beta(u)$. Then, $\lambda(u, e) = \lambda'(u', e)$ when e is close enough to e_0.

The proof, based on the uniqueness part of the Implicit Function Theorem,[7] is left to the reader. Since the considered surfaces are compact, *this last lemma ensures the existence of an analytic function, $\Lambda(x, e)$, defined on $\Gamma(e_0) \times \,]e_0 - \Delta', e_0 + \Delta'[$ (where $\Gamma(e_0)$ is endowed with its surface structure), such that*

$$x + \Lambda(x, e)\nabla_x\Phi(x) \in \Gamma(e) \quad \textit{and} \quad \Lambda(x_0, e_0) = 0. \tag{9}$$

Incidentally, the function $\lambda(u, e)$ has the following, interesting properties:

Lemma *For every $u \in \mathcal{U}'$, $\lambda(u, e_0) = 0$. In particular, $\partial_{u^{(j)}}\lambda(u, e_0) = 0$ for any $j = 1, \ldots, d - 1$.*

The proof is left to the reader.

Lemma $\partial_e\lambda(u_0, e_0) > 0$.

[7] Of course, this uniqueness holds *locally*.

Proof: In fact, since $\Phi(x_0 + \lambda(u_0, e)\nabla_x \Phi(x_0)) = e$, the chain rule gives

$$\partial_e \lambda(u_0, e_0) = \frac{1}{\|\nabla_x \Phi(x_0)\|^2}.$$

□

The equation (8) permits to define the following local parameterization of $\Gamma(e)$:

$$\sigma(u, e) = \gamma_\beta(u) + \lambda(u, e)\nabla_x \Phi(\gamma_\beta(u)),$$

where $u \in \mathcal{U}'$ and $|e - e_0| < \Delta'$. Notice that $\sigma(u, e)$ is real-analytic. Indeed,

Lemma *$\sigma(u, e)$ is a real-analytic diffeomorphism in a neighborhood of (u_0, e_0).*

Proof: Notice that the ambient space, $\mathbb{R}^d$, is generated by the tangent vectors of $\Gamma(e_0)$ at x_0 (namely, $\mathrm{d}_{u^{(j)}} \gamma_\beta(u_0)$ for $1 \leqslant j \leqslant d - 1$) and $\nabla_x \Phi(x_0)$. The penultimate lemma and a direct computation show that the columns of the matrix $\mathrm{D}_{(u,e)} \sigma(u_0, e_0)$ in canonical basis are

$$\begin{array}{rcl} \mathrm{d}_{u^{(j)}} \gamma_\beta(u_0) & \text{for} & 1 \leqslant j \leqslant d-1 \\ \partial_e \lambda(u_0, e_0)\nabla_x \Phi(x_0) & \text{for} & j = d, \end{array}$$

which are linearly independent by the last lemma. The Inverse Mapping Theorem then completes the proof. □

We thus select an arbitrarily small cube, $\mathcal{U} \times]e_0 - \Delta, e_0 + \Delta[\ni (u_0, e_0)$, contained in $\mathcal{U}' \times]e_0 - \Delta', e_0 + \Delta'[$, such that $\sigma(u, e)$ is a real-analytic diffeomorphism in the neighborhood of $\overline{\mathcal{U}} \times [e_0 - \Delta, e_0 + \Delta]$.

Let $\varphi(u; e, \omega) = \omega \cdot \sigma(u, e)$ be the anticipated phase, where $u \in \mathcal{U}$, $|e - e_0| < \Delta$, and $\omega \in \mathrm{S}^{d-1}$. By Theorem 10, the associated stationary phase points and $\operatorname{rank} \mathrm{D}^2_u \varphi$ at these points are intrinsic properties of $\Gamma(e)$. Let $\omega_0 \in \mathrm{S}^{d-1}$ be arbitrarily fixed. It appears that if x_0 is a stationary phase point, then $\partial_e \varphi(u_0; e_0, \omega_0)$ is also "intrinsic" in the following sense:

Theorem 11 *Suppose $x_0 = \sigma(u_0, e_0)$ is a stationary phase point. Then,*

$$\partial_e \varphi(u_0; e_0, \omega_0) = \frac{\pm 1}{\|\nabla_x \Phi(x_0)\|}.$$

Proof: By Theorem 10, since x_0 is stationary, $\omega_0 = \pm \frac{\nabla_x \Phi(x_0)}{\|\nabla_x \Phi(x_0)\|}$. In particular,

$$\begin{aligned} \partial_e \varphi(u_0; e_0, \omega_0) &= \omega_0 \cdot \partial_e \sigma(u_0, e_0) \\ &= \frac{\pm 1}{\|\nabla_x \Phi(x_0)\|} \nabla_x \Phi(x_0) \cdot \partial_e \sigma(u_0, e_0). \end{aligned}$$

Since $\Phi(\sigma(u, e)) = e$, the chain rule gives $\nabla_x \Phi(x_0) \cdot \partial_e \sigma(u_0, e_0) = 1$, which completes the proof. □

Consequently, $\partial_e\varphi(u_0;e_0,\omega_0)\neq 0$ if x_0 is a stationary phase point. Hence, whether $\nabla_u\varphi(u_0;e_0,\omega_0)\neq 0$ or

$$\begin{aligned}\nabla_u\varphi(u_0;e_0,\omega_0) &= 0,\\ \operatorname{rank}\mathrm{D}_u^2\varphi(u_0;e_0,\omega_0) &\geqslant \kappa,\\ \partial_e\varphi(u_0;e_0,\omega_0) &\neq 0.\end{aligned}$$

By Theorems 1, 6, the corollary of Theorem 2, and Theorem 9 there exists an arbitrarily small cube, $U\times B\subset\mathcal{U}\times\mathbb{R}^d$, containing (u_0,ω_0) and an arbitrarily small $0<\delta<\Delta$ such that the following holds:

> If $f(u;e,\omega)$ is smooth in the neighborhood of $\mathbb{R}^{d-1}\times\mathbb{R}\times\overline{B}$ and vanishes on $(U\times]e_0-\delta,e_0+\delta[)^{\mathrm{c}}\times\overline{B}$, then
>
> $$I_f(r;e,\omega)=O(r^{-\frac{\kappa}{2}})\ \text{ and}$$
> $$\text{p.v.}\int_{\mathbb{R}}\frac{1}{\eta-e}I_f(r;\eta,\omega)\,\mathrm{d}\eta=O(r^{-\frac{\kappa}{2}}),$$
>
> where both estimates are uniform in $(e,\omega)\in[e_0-\delta,e_0+\delta]\times\overline{B}$.

Of course, the previous cube and associated constructions depend on the fixed e_0, ω_0, and $x_0\in\Gamma(e_0)$. Repeating this procedure for all $x\in\Gamma(e_0)$, where e_0 and ω_0 are still fixed, one obtains a system of real-analytic diffeomorphisms, $\{(U_x\times]e_0-\delta_x,e_0+\delta_x[\,,\sigma_x)\}_{x\in\Gamma(e_0)}$, and a family of cubes, $\{B_\alpha\}_{\alpha=1}^N$, satisfying the above properties, from which one extracts a finite subsystem, $\{(U_\alpha\times]e_0-\delta_\alpha,e_0+\delta_\alpha[\,,\sigma_\alpha)\}_{\alpha=1}^N$, which covers $\Gamma(e_0)$:

$$\Gamma(e_0)=\bigcup_{\alpha=1}^N\sigma_\alpha(U_\alpha,e_0).$$

We limit our considerations to an arbitrarily small cube, B, whose closure is inside $\bigcap_{\alpha=1}^N B_\alpha$, and to $\{(U_\alpha\times]e_0-\delta',e_0+\delta'[\,,\sigma_\alpha)\}_{\alpha=1}^N$, where $0<\delta'<\min_{\alpha=1}^N\delta_\alpha$ is arbitrarily small.

By the equation (9) the local parameterizations

$$\sigma_\alpha(u,e)=\gamma_{\beta_\alpha}(u)+\lambda_\alpha(u,e)\nabla_x\Phi(\gamma_{\beta_\alpha}(u))$$

yield the following, arguably global parameterization of $\Gamma(e)$:

$$\Sigma(x,e)=x+\Lambda(x,e)\nabla_x\Phi(x),\tag{10}$$

where $x\in\Gamma(e_0)$ and $|e-e_0|<\delta'$. In particular, when x varies over the entire $\Gamma(e_0)$, $\Sigma(x,e)$ describes a whole "closed" surface included in $\Gamma(e)$ (which should therefore corresponds to $\Gamma(e)$, as long as $\Gamma(e_0)$ and $\Gamma(e)$ have the same number of connected components). In other words, one expects

$$\Gamma(e)=\bigcup_{\alpha=1}^N\sigma_\alpha(U_\alpha,e).\tag{11}$$

However, it is easier to shorten δ' in order to *prove* the previous relation.

This may be done in the following way: suppose by contradiction there does not exist a δ such that the relation (11) holds for all $e \in\,]e_0-\delta, e_0+\delta[$, where $0<\delta\leqslant\delta'$. Then there exist a sequence, $\{e_n\}$, converging to e_0, and points, x_n, on $\Gamma(e_n)$, such that $x_n \notin \bigcup_{\alpha=1}^N \sigma_\alpha(U_\alpha, e_n)$. Letting $\mathcal{S} = \bigcup_{|e-e_0|<\delta} \Gamma(e)$, the x_n's lie in the compact set $\overline{\mathcal{S}}$, so they accumulate towards a certain $x^* \in \overline{\mathcal{S}}$. Going to a subsequence, again denoted by $x_n \to x^*$, one finds

$$\Phi(x^*) = \Phi(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} \Phi(x_n) = \lim_{n\to\infty} e_n = e.$$

Therefore, x^* lies in $\bigcup_{\alpha=1}^N \sigma_\alpha(U_\alpha \times\,]e_0-\delta', e_0+\delta'[)$, while the x_n's do not, contradicting the fact that this last region is open in $\mathbb{R}^d$.

Indeed,

Theorem 12 *There exists a $\delta \in\,]0,\delta'[$ such that for any $e\in\,]e_0-\delta, e_0+\delta[$, $\Sigma(x,e)$ is a real-analytic diffeomorphism between $\Gamma(e_0)$ and $\Gamma(e)$ (endowed with their surface structures).*

Proof: The previous paragraph shows that $\Sigma(x,e)$ is surjective for e close enough to e_0. By a similar argument suppose there does not exist a $\delta>0$ such that $\Sigma(x,e)$ is injective for every $e\in\,]e_0-\delta, e_0+\delta[$. Then, there exist a sequence, $\{e_n\}$, converging to e_0, and points, x_n, on $\Gamma(e_n)$, such that

$$x_n = \Sigma(y_n, e_n) = \Sigma(z_n, e_n)$$

for distinct points y_n and z_n on $\Gamma(e_0)$. Since the x_n's accumulate towards a certain $x^*\in\Gamma(e_0)$, y_n and z_n are eventually in the same coordinates neighborhood, contradicting the fact that each σ_α is a diffeomorphism. $\square$

We limit our considerations to the system $\{(U_\alpha \times\,]e_0-\delta, e_0+\delta[\,, \sigma_\alpha)\}_{\alpha=1}^N$, where δ is specified by the previous theorem. We have proven:

Theorem 13 *Let $\omega_0\in \mathrm{S}^{d-1}$, $e_0\in\,]a',b'[$, and $\varepsilon>0$ be arbitrarily fixed. Under the assumptions A and B, there exists a finite family of cubes of diameters less than ε,*

$$\{U_\alpha \times\,]e_0-\delta, e_0+\delta[\, \times B\}_{\alpha=1}^N,$$

where $[e_0-\delta, e_0+\delta] \subset\,]a',b'[$ and $\omega_0 \in B\subset \mathbb{R}^{d-1}$, and functions,

$$\sigma_\alpha\colon\, U_\alpha \times\,]e_0-\delta, e_0+\delta[\, \to \bigcup_{|e-e_0|<\delta} \Gamma(e)$$

such that the following holds:

1. *For every α, $\sigma_\alpha(u,e)$ is a real-analytic diffeomorphism from a neighborhood of $\overline{U_\alpha}\times [e_0-\delta, e_0+\delta]$ to its image.*

2. *For all* $e \in \,]e_0-\delta, e_0+\delta[$, $\Gamma(e) = \bigcup_{\alpha=1}^{N} \sigma_\alpha(U_\alpha, e)$. [8]

3. *Let us denote by* $I_f^{(\alpha)}(r; e, \omega)$ *the oscillatory integral of amplitude* $f(u; e, \omega)$ *with respect to the phase* $\varphi_\alpha(u; e, \omega) = \omega \cdot \sigma_\alpha(u, e)$. *If* $f(u; e, \omega)$ *is smooth in the neighborhood of* $\mathbb{R}^{d-1} \times \mathbb{R} \times \overline{B}$ *and vanishes on* $(U_\alpha \times \,]e_0-\delta, e_0+\delta[)^{\mathrm{c}} \times \overline{B}$*, then*

$$I_f^{(\alpha)}(r; e, \omega) = O(r^{-\frac{\kappa}{2}}) \quad \textit{and}$$

$$\text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} I_f(r; \eta, \omega)\, \mathrm{d}\eta = O(r^{-\frac{\kappa}{2}}),$$

where both estimates are uniform in $(e, \omega) \in [e_0-\delta, e_0+\delta] \times \overline{B}$.

We now construct a smooth, joint "partition of unity" subordinated to the system of neighborhoods $\{\sigma_\alpha(U_\alpha, e)\}_{\alpha=1}^{N}$.

Theorem 14 *Let* $0 < \tilde{\delta} < \delta$ *and* $\mathcal{S} = \bigcup_{|e-e_0|<\tilde{\delta}} \Gamma(e)$. *There exists a family of functions,* $\{\chi_\alpha(x)\}_{\alpha=1}^{N}$*, smooth in* $x \in \mathbb{R}^d$*, satisfying:*

- $0 \leqslant \chi_\alpha \leqslant 1$,
- $\operatorname{supp} \chi_\alpha \subset \sigma_\alpha(U_\alpha \times \,]e_0-\delta, e_0+\delta[)$,
- $\sum_{\alpha=1}^{N} \chi_\alpha(x) = 1$ *for all* $x \in \overline{\mathcal{S}}$.

Proof: Notice that $\overline{\mathcal{S}} \subset \bigcup_{\alpha=1}^{N} \sigma_\alpha(U_\alpha \times \,]e_0-\delta, e_0+\delta[)$, so in particular

$$\{\sigma_1(U_1 \times \,]e_0-\delta, e_0+\delta[), \ldots, \sigma_N(U_N \times \,]e_0-\delta, e_0+\delta[), \mathbb{R}^d \setminus \overline{\mathcal{S}}\}$$

is a finite open covering of $\mathbb{R}^d$. After discarding $\chi_{N+1}(x)$, any partition of unity, $\{\chi_\alpha(x)\}_{\alpha=1}^{N+1}$, subordinated to the previous covering satisfies the stated properties. □

Fourier Transforms We are ready to compute (uniform!) decays of Fourier transforms, $\mathcal{F}(\Gamma(e), f)$, for suitable amplitudes and derive an analogous result for Cauchy principal values.

Theorem 15 *Let* $n = r\omega$ *be the polar form of* $n \in \mathbb{Z}^d$*, where* $n \neq 0$*, and let the amplitude,* $f(x)$*, be smooth in* $x \in \mathbb{R}^d$. *Consider any interval* $[a, b] \subset \,]a', b'[$. *Under Assumptions A and B,*

$$\mathcal{F}(\Gamma(e), f)(n) = O(r^{-\frac{\kappa}{2}})$$

uniformly in $(e, \omega) \in [a, b] \times \mathrm{S}^{d-1}$.

[8] In other words, $\{(U_\alpha, \sigma_\alpha(\cdot, e))\}_{\alpha=1}^{N}$ is a system of real-analytic parameterizations of $\Gamma(e)$ for *any* $e \in \,]e_0-\delta, e_0+\delta[$.

Proof: Let $e_0 \in [a,b]$ and $\omega_0 \in \mathrm{S}^{d-1}$ be fixed. By Theorems 13 and 14, there exist a $\delta > 0$, a cube, $B \ni \omega_0$, a joint system of parameterizations, $\{(U_\alpha \times]e_0 - 2\delta, e_0 + 2\delta[\,, \sigma_\alpha)\}_{\alpha=1}^N$, and a joint partition of unity, $\{\chi_\alpha\}_{\alpha=1}^N$, summing at 1 on $\bigcup_{|e-e_0|<\delta} \Gamma(e)$, such that the following holds: Let

$$I^{(\alpha)}(r; e, \omega) = \int_{U_\alpha} \mathrm{e}^{\mathrm{i}r\omega\cdot\sigma_\alpha(u,e)} f(\sigma_\alpha(u,e))\chi_\alpha(\sigma_\alpha(u,e))J_\alpha(u,e)\,\mathrm{d}u,$$

where $J_\alpha(u,e)\,\mathrm{d}u$ is the element of surface of $\sigma_\alpha(U_\alpha, e)$. Then,

$$\mathcal{F}(\Gamma(e), f)(r\omega) = \sum_{\alpha=1}^{N} I^{(\alpha)}(r; e, \omega)$$

for every $(e,\omega) \in [e_0 - \delta, e_0 + \delta] \times \overline{B}$. Since $\chi_\alpha \circ \sigma_\alpha$, and hence the integrand of $I^{(\alpha)}(r; e, \omega)$ vanish outside $U_\alpha \times]e_0 - 2\delta, e_0 + 2\delta[$, by Theorem 13, $I^{(\alpha)}(r; e, \omega) = O(r^{-\frac{\kappa}{2}})$ uniformly in $(e,\omega) \in [e_0 - \delta, e_0 + \delta] \times \overline{B}$. Thus,

$$\mathcal{F}(\Gamma(e), f)(r\omega) = O(r^{-\frac{\kappa}{2}})$$

uniformly in $[e_0 - \delta, e_0 + \delta] \times \overline{B}$. Since $e_0 \in [a,b]$ and $\omega_0 \in \mathrm{S}^{d-1}$ are arbitrary, the result follows from the compactness of $[a,b] \times \mathrm{S}^{d-1}$. $\square$

Cauchy Principal Value An analogous result for principal values may be derived when affecting the amplitude by a cutoff function.

We establish first that the element of surface on any local part of $\Gamma(e)$ varies smoothly in (u,e). Having in hand a common system of parameterizations, $\sigma_\alpha(u,e)$, for $u \in U_\alpha$ and $|e - e_0| < \delta$ (where $e_0 \in]a', b'[$ is arbitrarily fixed), let us compute this element of surface. To this end, one considers the determinants, $M_\alpha^{(l)}$, of the submatrices of format $(d-1) \times (d-1)$ obtained from $[\partial_{u^{(k)}} \sigma_\alpha^{(l)}(u,e)]_{k,l}$ by removing its l^{th} column, where $\sigma_\alpha = (\sigma_\alpha^{(1)}, \ldots, \sigma_\alpha^{(d)})$. Let

$$J_\alpha(u,e) = (M_\alpha^{(1)}(u,e), -M_\alpha^{(2)}(u,e), \cdots, (-1)^{d-1} M_\alpha^{(d)}(u,e)).$$

By definition the element of surface is $\|J_\alpha(u,e)\|\,\mathrm{d}u$.

Theorem 16 $\|J_\alpha(u,e)\| = \|\nabla_x \Phi(\sigma_\alpha(u,e))\|\,|\det \mathrm{D}_{(u,e)}\sigma_\alpha(u,e)|$.

Proof: The chain rule applied to $\Phi(\sigma_\alpha(u,e)) = e$ gives

$$\nabla_x \Phi(\sigma_\alpha(u,e))\mathrm{D}_{(u,e)}\sigma_\alpha(u,e) = [0 \,\cdots\, 0\; 1],$$

which we abbreviate $\nabla\Phi \mathrm{D}\sigma_\alpha = {e_d}^{\mathrm{t}}$. Since $\sigma_\alpha(u,e)$ is a diffeomorphism, $\mathrm{D}\sigma_\alpha$ is invertible. Thus,

$$\nabla\Phi = {e_d}^{\mathrm{t}}(\mathrm{D}\sigma_\alpha)^{-1} = \frac{1}{\det \mathrm{D}\sigma_\alpha} {e_d}^{\mathrm{t}}(\mathrm{adj}\mathrm{D}\sigma_\alpha),$$

where adj stands for the classical adjoint. The result follows from

$$e_d{}^{\mathrm{t}}(\mathrm{adj}\mathrm{D}\sigma_\alpha) = (-1)^{d-1} J_\alpha(u,e).$$

$\square$

Corollary *$\|J_\alpha(u,e)\|$ is smooth in $(u,e) \in U_\alpha \times]e_0-\delta, e_0+\delta[$.*

For fixed $\omega_0 \in \mathrm{S}^{d-1}$ and $e_0 \in]a', b'[$, let $\{(U_\alpha \times]e_0 - 3\delta, e_0 + 3\delta[\,, \sigma_\alpha)\}_{\alpha=1}^N$, $B \ni \omega_0$, and $\{\chi_\alpha\}_{\alpha=1}^N$, summing at 1 on $\bigcup_{|e-e_0|<2\delta} \Gamma(e)$, be given by Theorems 13 and 14. Let us define a cutoff function, $0 \leqslant \chi_{e_0}(x) \leqslant 1$, smooth in $x \in \mathbb{R}^d$, such that [9]

$$\chi_{e_0}(x) = \begin{cases} 1 & \text{if} \quad |\Phi(x) - e_0| < \delta, \\ 0 & \text{if} \quad |\Phi(x) - e_0| > 2\delta. \end{cases}$$

Notice that $\chi_{e_0}(x)$ is analytic in the neighborhood of $\Gamma(e_0)$—which will become important later. Under Assumptions A and B,

Theorem 17 *For $f(x)$ smooth in $x \in \mathbb{R}^d$ and $r > 0$*

$$\mathrm{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} \mathcal{F}(\Gamma(\eta), \chi_{e_0} f)(r\omega)\, \mathrm{d}\eta = O(r^{-\frac{\kappa}{2}})$$

uniformly in $(e,\omega) \in [e_0 - \delta, e_0 + \delta] \times \mathrm{S}^{d-1}$.

Proof: The considered principal value is equal to

$$\mathrm{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} \sum_{\alpha=1}^{N} I_{F_\alpha}^{(\alpha)}(r;\eta,\omega)\, \mathrm{d}\eta,$$

where $F_\alpha(u,\eta) = (\chi_{e_0} f \chi_\alpha)(\sigma_\alpha(u,\eta)) \|J_\alpha(u,\eta)\|$. Notice that

$$\chi_{e_0}(\sigma_\alpha(u,\eta)) = \begin{cases} 1 & \text{if} \quad |\eta - e_0| < \delta, \\ 0 & \text{if} \quad |\eta - e_0| > 2\delta. \end{cases}$$

Hence, $F_\alpha(u,\eta)$ is smooth on $\mathbb{R}^{d-1} \times \mathbb{R}$ and vanishes when

$$(u,\eta) \notin U_\alpha \times]e_0 - 2\delta, e_0 + 2\delta[\,.$$

By Theorem 13 the principal value under consideration is thus $O(r^{-\frac{\kappa}{2}})$ uniformly in $(e,\omega) \in [e_0 - \delta, e_0 + \delta] \times \overline{B}$. Since $\omega_0 \in \mathrm{S}^{d-1}$ is arbitrarily fixed, the result follows from compactness of S^{d-1}. $\square$

[9] Since the amplitude we will use does not depend on ω, the following cutoff function does not depend on ω_0; in other circumstances if it does, the following argument still holds.

Analyticity of Fourier Transform We close this section by showing that for any fixed $n \in \mathbb{Z}^d$ the following Fourier transform, $\mathcal{F}(\Gamma(e), f)(n)$, is analytic at e_0 when the amplitude, $f(x)$, is analytic in the neighborhood of $\Gamma(e_0) \subset \mathbb{R}^d$ (where $e_0 \in]a', b'[$ is arbitrarily fixed). To this end the diffeomorphism between $\Gamma(e_0)$ and $\Gamma(e)$, $\Sigma(x, e)$, defined in the relation (10) is helpful, because, given a system of *disjoint* open neighborhoods on $\Gamma(e_0)$ covering all $\Gamma(e_0)$ except a set of area zero, its lifting to $\Gamma(e)$ via $\Sigma(x, e)$ also covers the whole $\Gamma(e)$ except a set of area zero.

In details, let $\{(U_\alpha \times]e_0 - \delta, e_0 + \delta[\,, \sigma_\alpha)\}_{\alpha=1}^N$ be given by Theorem 13.

Theorem 18 *There exists a finite, joint system of local parameterizations,*

$$\{(V_\beta \times]e_0 - \delta, e_0 + \delta[\,, \sigma_\beta)\}_{\beta=1}^M,$$

such that:

- *For all* $\beta \in \{1, \ldots, M\}$ *there exists an* $\alpha_\beta \in \{1, \ldots, N\}$ *such that* $V_\beta \subseteq U_{\alpha_\beta}$ *and* $\sigma_\beta = \sigma_{\alpha_\beta} \restriction V_\beta$*;*
- *For all* $e \in]e_0 - \delta, e_0 + \delta[$ *the coordinates neighborhoods* $\{\sigma_\beta(V_\beta, e)\}_{\beta=1}^M$ *are mutually disjoints;*
- *For all* $e \in]e_0 - \delta, e_0 + \delta[$ *the area of* $\Gamma(e) \setminus \bigcup_{\beta=1}^M \sigma_\beta(V_\beta, e)$ *is zero.*[10]

Proof: One may construct a system of disjoint, open neighborhoods of full area on $\Gamma(e_0)$ by considering all non empty cells of the form[11]

$$\sigma_{\alpha_1}(U_{\alpha_1}, e_0) \cap \cdots \cap \sigma_{\alpha_l}(U_{\alpha_l}, e_0) \cap \sigma_{\alpha_{l+1}}(\overline{U_{\alpha_{l+1}}}, e_0)^c \cap \cdots \cap \sigma_{\alpha_N}(\overline{U_{\alpha_N}}, e_0)^c,$$

where $\{\alpha_1, \cdots, \alpha_N\} = \{1, \cdots, N\}$. These cells are open, disjoint, and cover all $\Gamma(e_0)$ except the set $\sigma_1(\partial U_1, e_0) \cup \cdots \cup \sigma_N(\partial U_N, e_0)$, whose area is zero. The diffeomorphism $\Sigma(x, e)$ lifts this exceptional set onto an exceptional set in $\Gamma(e)$ (which is not a surprise). The peculiarity of $\Sigma(x, e)$ is that *all points on* $\Gamma(e)$ *which are not the image of an exceptional* $x \in \Gamma(e_0)$ *are covered.* The result follows. $\square$

Theorem 19 *Suppose* $f(x)$ *is analytic in a neighborhood of* $\Gamma(e_0) \subset \mathbb{R}^d$*. Then, for any fixed* $n \in \mathbb{Z}^d$ *the Fourier transform* $\mathcal{F}(\Gamma(e), f)(n)$ *is analytic at* $e_0 \in]a', b'[$*.*

Proof: By the previous theorem, for any $e \in]e_0 - \delta, e_0 + \delta[$

$$\mathcal{F}(\Gamma(e), f)(n) = \sum_{\beta=1}^M \int_{V_\beta} \mathrm{e}^{\mathrm{i}n \cdot \sigma_\beta(u,e)} f(\sigma_\beta(u, e)) J_\beta(u, e)\, \mathrm{d}u,$$

where $J_\beta(u, e)\, \mathrm{d}u$ is the element of surface of $\sigma_\beta(V_\beta, e)$. Since the integrand in this last expression analytic in the neighborhood of $\overline{V_\beta} \times [e_0 - \delta/2, e_0 + \delta/2]$ (that is, converges to its Taylor's series), the result follows. $\square$

[10] In other words, the integral of 1 over this surface is zero.

[11] The complements are taken with respect to $\Gamma(e_0)$.

5 Green's Functions

We turn our attention to

$$G(n,z) = \int_{\mathbb{T}^d} \frac{\mathrm{e}^{\mathrm{i}n\cdot x}}{\Phi(x) - z}\,\mathrm{d}x, \tag{12}$$

where $z \in \mathbb{C}_+$, $n \in \mathbb{Z}^d$, and $\Phi(x)$ is real valued, analytic, and *periodic* on $\mathbb{T}^d$. In this context,

$$\{x \in \mathbb{T}^d\ ;\ \Phi(x) = e\} = \{x \in \mathcal{R}\ ;\ \Phi(x) = e\} = \Gamma(e)$$

for an appropriate domain, $\mathcal{R} \subset \mathbb{R}^d$, which does not depend on $e \in\,]a', b'[$ *when this last interval is sufficiently small.* However, this restriction on $]a', b'[$ may be removed by a compactness argument (as long as e varies in an arbitrary cube, $]a, b[$, whose *closure* is in $]a', b'[$). We still assume A and B.

For $e \in \mathbb{R}$ let

$$G(n,e) = \lim_{\substack{z\to e\\ z\in\mathbb{C}_+}} G(n,z),$$

where of course the existence of such a limit has to be established. In this section we compute the decay of $G(n,e)$ as $|n| \to \infty$. We are especially interested in the case where e is in the range of $\Phi(x)$; otherwise, since by assumption $\Phi(x)$ admits a holomorphic extension, one may slightly change the domain of integration in (12) and deduce that $G(n,e)$ decays exponentially.

The proof of the following elementary lemma, based on Cauchy's theorem, is left to the reader.

Lemma *Consider a function, $f(\eta)$, continuous in $\eta \in [a,b]$ and analytic at $e \in\,]a,b[$. Then,*

$$\lim_{\substack{z\to e\\ z\in\mathbb{C}_+}} \int_a^b \frac{f(\eta)}{\eta - z}\,\mathrm{d}\eta = \pi\mathrm{i} f(e) + \mathrm{p.v.} \int_a^b \frac{f(\eta)}{\eta - e}\,\mathrm{d}\eta.$$

For fixed $\omega_0 \in \mathrm{S}^{d-1}$ and $e_0 \in\,]a', b'[$, let $\{(U_\alpha \times\,]e_0 - 3\delta, e_0 + 3\delta[\,, \sigma_\alpha)\}_{\alpha=1}^N$, $B \ni \omega_0$, $\{\chi_\alpha\}_{\alpha=1}^N$, summing to 1 on $\bigcup_{|e-e_0|<2\delta} \Gamma(e)$, and χ_{e_0} be given by Theorems 13, 14, and 17.

Theorem 20 *Suppose $|e - e_0| < \delta$ and $\omega \in B$. Then, $G(n,e)$ exists and is equal to*

$$\pi\mathrm{i}\mathcal{F}(\Gamma(e), P)(n) + \mathrm{p.v.} \int_{-\infty}^{\infty} \frac{1}{\eta - e} \mathcal{F}(\Gamma(\eta), \chi_{e_0} P)(n)\,\mathrm{d}\eta + \int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}n\cdot x} \frac{1 - \chi_{e_0}(x)}{\Phi(x) - e}\,\mathrm{d}x,$$

where $P(x) = \frac{1}{\|\nabla_x \Phi(x)\|}$.

Proof: Notice that

$$\int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}n\cdot x} \frac{\chi_{e_0}(x)}{\Phi(x)-z}\,\mathrm{d}x = \sum_{\alpha=1}^{N} \int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}n\cdot x} \frac{\chi_{e_0}(x)\chi_\alpha(x)}{\Phi(x)-z}\,\mathrm{d}x.$$

The change of variables $x=\sigma_\alpha(u,\eta)$ applied to the above gives

$$\sum_{\alpha=1}^{N} \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}n\cdot\sigma_\alpha(u,\eta)} \frac{\chi_{e_0}(\sigma_\alpha(u,\eta))\chi_\alpha(\sigma_\alpha(u,\eta))}{\eta-z} |\det \mathrm{D}_{(u,\eta)}\sigma_\alpha(u,\eta)|\,\mathrm{d}u\,\mathrm{d}\eta,$$

which, by Theorem 16, is equal to

$$\sum_{\alpha=1}^{N} \int_{\mathbb{R}} \frac{1}{\eta-z} \int_{\Gamma(\eta)} \mathrm{e}^{\mathrm{i}n\cdot x} \chi_{e_0}(x)\chi_\alpha(x)P(x)\,\mathrm{dS}(x)\,\mathrm{d}\eta,$$

where $\mathrm{dS}(x)$ denotes the element of surface. Hence,

$$\int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}n\cdot x} \frac{\chi_{e_0}(x)}{\Phi(x)-z}\,\mathrm{d}x = \int_{\mathbb{R}} \frac{1}{\eta-z} \mathcal{F}(\Gamma(\eta),\chi_{e_0}P)\,\mathrm{d}\eta.$$

Notice that $\chi_{e_0}P$ is analytic in the neighborhood of $\Gamma(e)$ when $|e-e_0|<\delta$. Hence, by Theorem 19, $\mathcal{F}(\Gamma(\eta),\chi_{e_0}P)$ is analytic at $\eta=e$. The result follows from the lemma and the dominated convergence theorem. □

The desired decay follows:

Theorem 21 *Let $n=r\omega$ be the polar form of $n\in\mathbb{Z}^d$, where $n\neq 0$, and consider any interval $[a,b]\subset\,]a',b'[$. Under Assumptions A and B*

$$G(n,e)=O(r^{-\frac{\kappa}{2}})$$

uniformly in $(e,\omega)\in[a,b]\times \mathrm{S}^{d-1}$.

Proof: For ω_0, e_0, etc. as above, by Theorems 15 and 17

$$\pi\mathrm{i}\mathcal{F}(\Gamma(e),P)(n)=O(r^{-\frac{\kappa}{2}}) \quad\text{and}$$

$$\text{p.v.}\int_{-\infty}^{\infty} \frac{1}{\eta-e}\mathcal{F}(\Gamma(\eta),\chi_{e_0}P)(n)\,\mathrm{d}\eta = O(r^{-\frac{\kappa}{2}})$$

uniformly on $[e_0-\delta,e_0+\delta]\times\overline{B}$. Moreover, by the scholium of Theorem 1

$$\int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}n\cdot x}\frac{1-\chi_{e_0}(x)}{\Phi(x)-e}\,\mathrm{d}x = O(r^{-\infty})$$

uniformly on $[e_0-\delta,e_0+\delta]\times\overline{B}$. Since (e_0,ω_0) is arbitrary in $[a,b]\times\mathrm{S}^{d-1}$, which is compact, the result follows. □

6 Application to Generalized Laplacians

In this section we apply the previous results to Green's functions of concrete Laplacians on $\mathbb{Z}^d$. We focus on two specific examples: the *standard Laplacian* and the *Molchanov–Vainberg Laplacian.*

At a higher level of generality let us consider a simple graph without loop, whose set of vertices is denoted by X (where X is allowed to be infinite). For $m, n \in X$, $\mathrm{d}(m, n)$ denotes the *graph distance* between m and n, that is, the length of the minimal chain joining m and n in the graph (∞ if m and n lie in two different connected components). Of course, d is a metric distance with values in $\mathbb{N} \cup \{\infty\}$. Notice also that (X, d) determines the graph completely, since $\{m, n\}$ is an edge if and only if $\mathrm{d}(m, n) = 1$.

We are interested in the Hilbert space, $l^2(X)$, consisting of square summable sequences indexed by X. Its usual basis is $\{\delta_n\}_{n\in X}$, where δ denotes the Kronecker delta:

$$\delta_n(m) = \mathbf{1}_{\{n\}}(m) = \begin{cases} 1 & \text{if} \quad m = n, \\ 0 & \text{if} \quad m \neq n. \end{cases}$$

The *adjacency operator* on $l^2(X)$ with respect to (X, d), sometimes called *Laplacian*, is defined as

$$(\Delta\psi)(n) = \sum_{\mathrm{d}(m,n)=1} \psi(m),$$

where $\psi \in l^2(X)$. In particular, $(\Delta\delta_n)(m) = \mathbf{1}_{\mathcal{V}_n}(m)$, where

$$\mathcal{V}_n = \{m \in X \ ; \ \mathrm{d}(m, n) = 1\}.$$

In the sequel we suppose that the degrees of the vertices of the considered graph are bounded, in other words, $\sup_n \#\mathcal{V}_n \leqslant B$ for a certain $B < \infty$.[12] Then, Δ *is a bounded, selfadjoint operator on* $l^2(X)$, as the reader may easily verify.

We are now interested in the case where $X = \mathbb{Z}^d$ and the graph distance, $\mathrm{d}(m, n)$, is translational invariant. In these circumstances we call Δ a *generalized Laplacian.* Then, $\mathrm{d}(m, n)$ is a function of $m - n$ only. Hence, letting $\mathcal{V} = \mathcal{V}_0$ (where $0 \in \mathbb{Z}^d$ denotes the origin), the considered graph is clearly determined by $(\mathbb{Z}^d, \mathcal{V})$. In particular,

$$\Delta\psi(n) = \sum_{v\in\mathcal{V}} \psi(n + v),$$

where $\#\mathcal{V} \leqslant B$ is still assumed to be finite.

Recall that the Fourier transform of $\psi \in l^2(\mathbb{Z}^d)$ is the following function, defined for $x \in \mathbb{T}^d$:

$$\widehat{\psi}(x) = (\mathcal{F}\psi)(x) = (2\pi)^{-\frac{d}{2}} \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}n\cdot x}\psi(n).$$

[12] The cardinality of a given set, A, is denoted by $\#A$.

The *symbol* of Δ, denoted by $\widehat{\Delta}$, is the lifting of Δ via the Fourier transform:

$$\widehat{\Delta} = \mathcal{F}\Delta\mathcal{F}^{-1}.$$

Theorem 22 *Let Δ be the generalized Laplacian associated with a given $\mathcal{V} \subset \mathbb{Z}^d$. Then, its symbol is the multiplication by*

$$\Phi(x) = \sum_{v\in\mathcal{V}} \mathrm{e}^{\mathrm{i}v\cdot x} = \sum_{v\in\mathcal{V}} \cos(v\cdot x),$$

where $x \in \mathbb{T}^d$.

Proof: $\widehat{\Delta}$ maps $\widehat{\psi}(x)$ to the following function:

$$\begin{aligned}(2\pi)^{-\frac{d}{2}} \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}n\cdot x} \sum_{v\in\mathcal{V}} \psi(n+v) &= (2\pi)^{-\frac{d}{2}} \sum_{v\in\mathcal{V}} \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}(n-v)\cdot x}\psi(n) \\ &= \sum_{v\in\mathcal{V}} \mathrm{e}^{-\mathrm{i}v\cdot x}\widehat{\psi}(x).\end{aligned}$$

Notice that

$$v \in \mathcal{V} \iff \mathrm{d}(v,0) = 1 \iff \mathrm{d}(0,-v) = 1 \iff -v \in \mathcal{V}.$$

Hence, denoting by $\mathcal{V}^+$ the set of $v \in \mathcal{V}$ whose first non zero coordinate is positive,

$$\sum_{v\in\mathcal{V}} \mathrm{e}^{-\mathrm{i}v\cdot x} = \sum_{v\in\mathcal{V}} \mathrm{e}^{\mathrm{i}v\cdot x} = \sum_{v\in\mathcal{V}^+} 2\cos(v\cdot x) = \sum_{v\in\mathcal{V}} \cos v\cdot x,$$

which completes the proof. $\square$

Since the range of $\Phi(x)$ is equal to the spectrum of Δ, one obtains:

Corollary $\mathrm{spec}(\Delta) = [\min\Phi, \#\mathcal{V}]$.

Notice that $\Phi(x)$ is analytic at each $x \in \mathbb{T}^d$. Hence,

$$E = \{x \in \mathbb{T}^d\ ;\ \nabla_x\Phi(x) = 0\}$$

has Lebesgue measure zero. In particular, $\mathrm{spec}(\Delta)$ is purely absolutely continuous.[13]

The *Green's function* of Δ is defined as

$$G(m,n;z) = \langle \delta_m \,|\, (\Delta - z)^{-1}\delta_n\rangle$$

[13] This may be seen in the following way: $L^2(\mathbb{T}^d) = L^2(\mathbb{T}^d\setminus E)$. Moreover, by the Inverse Function Theorem $\Phi(x)$ is invertible on $\mathbb{T}^d\setminus E$. Let $\Psi(y)$ be its inverse, so the change of variables $y = \Phi(x)$ gives a unitary equivalence between $L^2(\mathbb{T}^d)$ with the Lebesgue measure and $L^2(\Phi(\mathbb{T}^d\setminus E))$ with the measure $\|\nabla_y\Psi(y)\|\,\mathrm{d}y$. Via this unitary equivalence the operator of multiplication by $\Phi(x)$ is lifted to the operator of multiplication by y, so by definition $\|\nabla_y\Psi(y)\|\,\mathrm{d}y\restriction\Phi(\mathbb{T}^d\setminus E)$ is a spectral measure for Δ—and is clearly absolutely continuous.

for $m, n \in \mathbb{Z}^d$ and $z \in \mathbb{C}_+$. Since Δ is translational invariant, this last function depends on $m - n$ and z only. In fact, $G(m, n; z) = G(0, n - m; z)$ so we denote this latter simply by $G(n - m, z)$. Since the Fourier transform is unitary, for any $n \in \mathbb{Z}^d$ and $z \in \mathbb{C}_+$

$$\begin{aligned} G(n, z) &= \langle \widehat{\delta_0}(x) \,|\, (\widehat{\Delta} - z)^{-1} \widehat{\delta_n}(x) \rangle_2 \\ &= (2\pi)^{-d} \int_{\mathbb{T}^d} \frac{\mathrm{e}^{\mathrm{i} n \cdot x}}{\Phi(x) - z} \, \mathrm{d}x. \end{aligned}$$

By Theorem 20 the limit

$$G(n, e) = \lim_{\substack{z \to e \\ z \in \mathbb{C}_+}} G(n, z)$$

exists for any $e \notin E$. We will compute its decay when $|n| \to \infty$ in two particular examples: the *standard discrete Laplacian* and the *Molchanov–Vainberg Laplacian*, which are described in the rest of the section. Both are important in mathematical physics, since both are discretizations of the continuous Laplacian on $\mathbb{R}^d$ (up to an additive constant).

Standard Laplacian The *standard Laplacian* is the adjacency operator of the usual grid on $\mathbb{Z}^d$; it is specified by the graph distance

$$\mathrm{d}(m, n) = |m - n|_1 = \sum_{j=1}^{d} |m^{(j)} - n^{(j)}|,$$

so the set of immediate neighbors of the origin is

$$\mathcal{V} = \{(\pm 1, 0, \dots, 0), (0, \pm 1, \dots, 0), \dots, (0, 0, \dots, \pm 1)\}.$$

Hence, by Theorem 22 the symbol of the standard Laplacian is the operator of multiplication by

$$\Phi(x) = 2 \sum_{j=1}^{d} \cos x^{(j)}.$$

Thus, the spectrum of Δ is purely absolutely continuous and equal to $[-2d, 2d]$. Notice that

$$\nabla_x \Phi(x) = (-2 \sin x^{(1)}, \dots, -2 \sin x^{(d)}),$$

which vanishes only if $\cos x^{(j)} = \pm 1$ for all $j = 1, \dots, d$, in which case $\Phi(x) \in \{-2d, -2d + 4, \dots, 2d - 4, 2d\}$. In particular, the level surfaces of $\Phi(x)$,

$$\Gamma(e) = \{x \in \mathbb{T}^d \;;\; \Phi(x) = e\},$$

are regular for all $e \notin \{-2d, -2d + 4, \dots, 2d - 4, 2d\}$.

Let us show that for such e's the level surfaces are exempt of planarity, except for $e = 0$:

Theorem 23 *Let $E = \{-2d, -2d+4, \ldots, 2d-4, 2d\} \cup \{0\}$ and suppose $e \in [-2d, 2d] \setminus E$. Then, $\Gamma(e)$ admits at least one non vanishing principal curvature at any point.*

Proof: Let $x_0 \in \Gamma(e)$ be fixed. By choice of e, $\Gamma(e)$ is regular, so there exists a j such that $\sin x_0^{(j)} \neq 0$. After renaming the variables, the equation defining $\Gamma(e)$ in a neighborhood of x_0 thus becomes

$$2\cos u^{(1)} + \cdots + 2\cos u^{(d-1)} + 2\cos h = e, \tag{13}$$

where $\sin h \neq 0$ on a certain interval. In particular, the derivative with respect to h of the left-hand side in (13) is not zero. Therefore, writing $u = (u^{(1)}, \ldots, u^{(d-1)})$, there exists an implicit function, $\mathrm{h}(u)$, such that

$$2\cos u^{(1)} + \cdots + 2\cos u^{(d-1)} + 2\cos \mathrm{h}(u) = e$$

for all u in a neighborhood of u_0, where $(u_0, \mathrm{h}(u_0))$ is the permuted x_0. The number of non vanishing principal curvatures at x_0 is then given by the rank of $\mathrm{D}_u^2\mathrm{h}(u_0)$. Indeed, for any $j = 1, \ldots, d-1$, differentiating the previous equation with respect to $u^{(j)}$ gives

$$-\sin u^{(j)} - \sin h\, \partial_{u^{(j)}} h = 0, \tag{14}$$

where $h = \mathrm{h}(u)$. Consequently,

$$-\sin h\, \mathrm{D}_u^2\mathrm{h}(u) = \mathrm{diag}(\cos u^{(1)}, \ldots, \cos u^{(d-1)}) + \cos h\, [\partial_{u^{(j)}}\mathrm{h}(u)\partial_{u^{(k)}}\mathrm{h}(u)]_{j,k=1}^{d-1}.$$

Since $\sin h \neq 0$, it suffices to show that the right side in the previous equation does not vanish in a neighborhood of u_0. By the equation (14) this matrix is equal to

$$\mathrm{diag}(\cos u^{(1)}, \ldots, \cos u^{(d-1)}) + \frac{\cos h}{\sin^2 h}[\sin u^{(j)} \sin u^{(k)}]_{j,k=1}^{d-1}. \tag{15}$$

Suppose by contradiction there exists an u in the considered neighborhood such that the above vanishes. Then, for all $j = 1, \ldots, d-1$

$$\cos u^{(j)} \sin^2 h + \cos h \sin^2 u^{(j)} = 0,$$

which is equivalent to

$$(1 - \cos h \cos u^{(j)})(\cos u^{(j)} + \cos h) = 0.$$

Moreover, since $\sin h \neq 0$, it follows that $1 - \cos h \cos u^{(j)} \neq 0$, so

$$\cos u^{(j)} = -\cos h \quad \text{for } j = 1, \ldots, d-1$$

at such an u. Hence, by the equation (13), $2(2-d)\cos h = e$. If $d = 2$, then $e = 0$, contrary to our assumption. Thus, the previous situation would occur only when $d > 2$, in which case $\cos h = -\frac{e}{2d-4}$ and $\cos u^{(j)} = \frac{e}{2d-4}$ for $j = 1, \ldots, d-1$.

Then, $\cos^2 h = \cos^2 u^{(j)}$ for any j, so $\sin u^{(j)} \sin u^{(k)} = \pm \sin^2 h$. Consequently, the (j,k)-th element of the considered matrix, (15), is 0 when $j = k$, but $\pm\frac{e}{2d-4}$ otherwise. This last quantity differs from zero (since $e \notin E$), which provides a contradiction.

In conclusion, at any point in a neighborhood of x_0, $\Gamma(e)$ admits at least one non vanishing principal curvature. Since x_0 is arbitrary, this completes the proof. □

Theorem 21 thus gives a polynomial decay for the Green's function, $G(n,e)$, associated with the standard Laplacian:

Corollary *Let $E = \{-2d, -2d+4, \ldots, 2d-4, 2d\} \cup \{0\}$ and suppose $e \in [-2d, 2d] \setminus E$. Then,*

$$G(n,e) = \lim_{\substack{z \to e \\ z \in \mathbb{C}_+}} G(n,z) = O(|n|^{-\frac{1}{2}})$$

when $|n| \to \infty$, uniformly in e on each compact and uniformly in $\omega \in \mathrm{S}^{d-1}$, where $n = |n|\omega$ is the polar form of $n \neq 0$.

Molchanov–Vainberg Laplacian In order to avoid convexity problems, Molchanov and Vainberg have suggested to change the discretization of the Laplacian. They have based their construction on the 2^d full-diagonal neighbors of elements in $\mathbb{Z}^d$, instead of their $2d$ immediate neighbors. The constant energy surfaces of the resulting operator are strictly convex in any dimension, as shown below.

Explicitly, the *Molchanov–Vainberg Laplacian* (or *diagonal Laplacian*) is the adjacency operator of the translational invariant graph specified by the following set of points adjacent to the origin:

$$\mathcal{V} = \{(v^{(1)}, \ldots, v^{(d)})\ ;\ v^{(j)} \in \{1,-1\} \ \text{ for } j = 1, \ldots, d\}.$$

By an elementary combinatorial argument, $n \in \mathbb{Z}^d$ is in the component of the origin if and only if the $n^{(j)}$'s are all even or all odd. Indeed, the considered graph consists of 2^{d-1} connected components with set of representatives

$$\{(0, n^{(2)}, \ldots, n^{(d)})\ ;\ n^{(j)} \in \{0,1\} \ \text{ for } j = 2, \ldots, d\}.$$

The graph is also specified by the following metric:

$$\mathrm{d}(m,n) = \begin{cases} |m-n|_\infty & \text{if the components of } m-n \text{ have the same parity,} \\ \infty & \text{otherwise,} \end{cases}$$

where $|n|_\infty = \max_{j=1}^{d} |n^{(j)}|$.

Remarkably, the symbol of the Molchanov–Vainberg Laplacian factorizes:

Theorem 24 *$\widehat{\Delta}$ is the operator of multiplication by $\Phi(x) = 2^d \cos x^{(1)} \ldots \cos x^{(d)}$, where $x = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{T}^d$.*

Proof: Let us denote by $\{e_1, \ldots, e_d\}$ the standard basis of $\mathbb{Z}^d$. By Theorem 22 the symbol of Δ is the multiplication by

$$\begin{aligned}\Phi(x) &= \sum_{v \in \mathcal{V}} \mathrm{e}^{\mathrm{i}x \cdot v} = \sum_{v \in \mathcal{V}} \mathrm{e}^{\mathrm{i}x \cdot \sum_{j=1}^d v^{(j)} e_j} = \sum_{v \in \mathcal{V}} \prod_{j=1}^d \mathrm{e}^{\mathrm{i}x \cdot v^{(j)} e_j} \\ &= \prod_{j=1}^d (\mathrm{e}^{\mathrm{i}x \cdot e_j} + \mathrm{e}^{-\mathrm{i}x \cdot e_j}) = 2^d \prod_{j=1}^d \cos x^{(j)},\end{aligned}$$

as claimed. □

Consequently, the spectrum of Δ is purely absolutely continuous and equal to $\left[-2^d, 2^d\right]$. Moreover,

Lemma *Suppose $0 < |e| < 2^d$. Then, for all $x \in \Gamma(e)$, $\nabla_x \Phi(x) \neq 0$. In particular, $\Gamma(e)$ defines a regular surface for such an e.*

Proof: If $x \in \Gamma(e)$, then $\cos x^{(j)} \neq 0$ for $j = 1, \ldots, d$, so $\partial_{x^{(j)}} \Phi(x) = -2^d e \tan x^{(j)}$. Thus, $\|\nabla_x \Phi(x)\|^2 = 4^d e^2 \sum_{j=1}^d \tan^2 x^{(j)}$, which differs from 0, since $e \neq \pm 2^d$. □

Let us investigate the constant energy surfaces associated with $\Phi(x)$. Firstly, let us consider the covering $\tilde{\Gamma}(e) = \{x \in \mathbb{R}^d\ ;\ \Phi(x) = e\}$.

If $e = 0$, $\tilde{\Gamma}(e)$ consists of the hyperplanes of equation $x^{(j)} = (2k+1)\frac{\pi}{2}$ for $k \in \mathbb{Z}$ and $j = 1, \ldots, d$. These hyperplanes divide $\mathbb{R}^d$ into open hypercubes, which we call *cells*. The cells admit a good bicoloration in the following sense: starting from a set of two colors, say, red and blue, it is possible to paint each cell in such a way that the $2d$ neighbors of any red cell are blue and *vice versa*. Let us accomplish this, the cell containing the origin being painted in red.

If $e = 2^d$, then $\tilde{\Gamma}(e)$ is a discrete set consisting of the centers of the red cells. On the other hand, if $e = -2^d$, then $\tilde{\Gamma}(e)$ consists of the centers of the blue cells.

When x varies continuously, $\Phi(x)$ changes sign each time one of the previous hyperplanes is crossed. It follows that the connected components of $\tilde{\Gamma}(e)$ are enclosed in the red cells when $e > 0$, each red cell containing one component. Moreover, these components are all congruent. The situation is the same when $e < 0$, but replacing the red cells with the blue ones.

Finally, $\Gamma(e)$ is obtained from the previous surface by restricting $\tilde{\Gamma}(e)$ to the torus, where $e \in \left]-2^d, 2^d\right[\setminus \{0\}$ is fixed. It follows that $\Gamma(e)$ consists of 2^{d-1} identical connected components.

As Molchanov and Vainberg conjectured,

Theorem 25 *For $0 < |e| < 2^d$ any component of $\Gamma(e)$ is strictly convex.*

Proof: Suppose $0 < e < 2^d$, the other case being similar. Then, without loss of generality the considered component is

$$\{x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[^d \; ; \; 2^d \prod_{j=1}^{d} \cos x^{(j)} = e\}.$$

Let $m = d - 1$ and $h = x^{(d)}$. The equation defining the previous component becomes

$$2^d \cos x^{(1)} \ldots \cos x^{(m)} \cos h = e.$$

Since each factor in the above is positive,

$$d \ln 2 + \ln \cos x^{(1)} + \cdots + \ln \cos x^{(m)} + \ln \cos h - \ln e = 0. \tag{16}$$

Since the considered component is symmetric with respect to the hyperplanes $x^{(j)} = 0$ and $x^{(j)} = x^{(l)}$, where $j, l \in \{1, \ldots, d\}$ are distinct, it suffices to show the result on the fundamental domain $h \leqslant x^{(1)} \leqslant \cdots \leqslant x^{(m)} \leqslant 0$. There, $h \neq 0$ since $e \neq 2^d$.

The derivative with respect to h of the left side in (16) is $-\tan h$, which does not vanish. Consequently, an implicit function, $h = \mathrm{h}(x, e)$, satisfying (16) in a neighborhood of an arbitrarily fixed point in the fundamental domain exists, is analytic, and induces a local parameterization of the previous component.

It thus suffices to show that $\mathrm{D}_x^2 \mathrm{h}(x, e)$ is positive definite. Differentiating (16) with respect to $x^{(j)}$ gives

$$-\tan x^{(j)} - \tan h \, \partial_{x^{(j)}} h = 0,$$

where $h = \mathrm{h}(x, e)$. Differentiating this last equation with respect to $x^{(l)}$ gives

$$\begin{array}{rcll} -\sec^2 x^{(j)} - \sec^2 h \, (\partial_{x^{(j)}} h)^2 - \tan h \, \partial^2_{x^{(j)}} h & = & 0 & \text{if } l = j, \\ -\sec^2 h \, \partial_{x^{(j)}} h \partial_{x^{(l)}} h - \tan h \, \partial_{x^{(j)}} \partial_{x^{(l)}} h & = & 0 & \text{otherwise.} \end{array}$$

Let $a_{jl} = -\tan h \, \partial_{x^{(j)}} \partial_{x^{(l)}} h$. Since $-\tan h > 0$, it suffices to show $[a_{jl}] > 0$. By the above,

$$[a_{jl}] = \mathrm{diag}(\sec^2 x^{(1)}, \ldots, \sec^2 x^{(d-1)}) + \sec^2 h \, [\partial_{x^{(j)}} h \partial_{x^{(l)}} h].$$

Clearly, the first term of the right-hand side in the previous equation is strictly positive. Moreover, the second term is non negative, since all its principal minors are zero except the first, which is a square. Hence, $[a_{jl}] > 0$, which completes the proof. □

Theorem 21 then gives an optimal decay for the Green's function, $G(n, e)$, associated with the Molchanov–Vainberg Laplacian. Explicitly:

Corollary *Let $E = \{-2^d, 0, 2^d\}$ and suppose $e \in \left[-2^d, 2^d\right] \setminus E$. Then,*

$$G(n, e) = \lim_{\substack{z \to e \\ z \in \mathbb{C}_+}} G(n, z) = O(|n|^{-\frac{d-1}{2}})$$

when $|n| \to \infty$, uniformly in e on each compact and uniformly in $\omega \in \mathrm{S}^{d-1}$, where $n = |n|\omega$ is the polar form of $n \neq 0$.

* * *

Appendix: Estimate Using Bessel Function

Let Δ be the *standard* discrete Laplacian in dimension $d \geqslant 4$. Its Green's function is denoted by

$$G(n-m, z) = \langle \delta_m \,|\, (\Delta - z)^{-1} \delta_n \rangle,$$

where $m, n \in \mathbb{Z}^d$, $z \in \mathbb{C}_+$, and δ is the Kronecker delta. By Kato's formula

$$G(n, z) = \mathrm{i} \int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}tz} \langle \delta_0 \,|\, \mathrm{e}^{\mathrm{i}t\Delta} \delta_n \rangle \, \mathrm{d}t. \tag{17}$$

Recall that the symbol of Δ is the multiplication by $\Phi(x) = 2 \sum_{j=1}^{d} \cos x^{(j)}$, where $x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{T}^d$. Hence,

$$\begin{aligned} \langle \delta_0 \,|\, \mathrm{e}^{\mathrm{i}t\Delta} \delta_n \rangle &= (2\pi)^{-d} \prod_{j=1}^{d} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n^{(j)}k} \mathrm{e}^{2\mathrm{i}t \cos k} \, \mathrm{d}k \\ &= \prod_{j=1}^{d} \mathrm{i}^{(n^{(j)})} J_{n^{(j)}}(2t) \end{aligned}$$

at any $n = (n^{(1)}, \dots, n^{(d)})$ and $z \in \mathbb{C}_+$, where $J_m(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}mk} \mathrm{e}^{-\mathrm{i}t \sin k} \, \mathrm{d}k$ is the Bessel function.

It is well known that there exist a universal constant, C, such that

$$|J_m(t)| \leqslant \frac{C}{|m|^{\frac{1}{3}}} \quad \text{and} \quad |J_m(t)| \leqslant \frac{C}{|t|^{\frac{1}{3}}}$$

for any $m \in \mathbb{Z}$ and $t \in \mathbb{R}$ (see [L]). Consequently, since $d \geqslant 4$, for an arbitrarily fixed $\varepsilon > 0$

$$|\langle \delta_0 \,|\, \mathrm{e}^{\mathrm{i}t\Delta} \delta_n \rangle| \leqslant C \, |t|^{-\frac{1}{3}} |t|^{-\frac{1}{3}} |t|^{-\frac{1}{3}} (|t|^{-\frac{\varepsilon}{3}} |n^{(j_4)}|^{-\frac{1-\varepsilon}{3}}) |n^{(j_5)}|^{-\frac{1}{3}} \dots |n^{(j_d)}|^{-\frac{1}{3}},$$

where $(n^{(j_1)}, \dots, n^{(j_d)})$ is a permutation of $(n^{(1)}, \dots, n^{(d)})$. This last estimate, the equation (17), and the dominated convergence theorem then give

$$|G(n, e + \mathrm{i}0)| \leqslant \mathit{Const} \, |n^{(j_4)}|^{-\frac{1-\varepsilon}{3}} |n^{(j_5)}|^{-\frac{1}{3}} \dots |n^{(j_d)}|^{-\frac{1}{3}}.$$

Since ε and the permutation of $(n^{(1)}, \dots, n^{(d)})$ are arbitrary, it follows *a fortiori* that

$$G(n, e + \mathrm{i}0) = O(|n|^{-\frac{1}{3}})$$

in dimension at least 4.

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