THE ALGEBRO-GEOMETRIC TODA HIERARCHY INITIAL VALUE PROBLEM FOR COMPLEX-VALUED INITIAL DATA

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Abstract. We discuss the algebro-geometric initial value problem for the Toda hierarchy with complex-valued initial data and prove unique solvability globally in time for a set of initial (Dirichlet divisor) data of full measure. To this effect we develop a new algorithm for constructing stationary complex-valued algebro-geometric solutions of the Toda hierarchy, which is of independent interest as it solves the inverse algebro-geometric spectral problem for generally non-self-adjoint Jacobi operators, starting from a suitably chosen set of initial divisors of full measure. Combined with an appropriate first-order system of differential equations with respect to time (a substitute for the well-known Dubrovin equations), this yields the construction of global algebro-geometric solutions of the time-dependent Toda hierarchy.

The inherent non-self-adjointness of the underlying Lax (i.e., Jacobi) operator associated with complex-valued coefficients for the Toda hierarchy poses a variety of difficulties that, to the best of our knowledge, are successfully overcome here for the first time. Our approach is not confined to the Toda hierarchy but applies generally to 1+1-dimensional completely integrable (discrete and continuous) soliton equations.

1. Introduction

The principal aim of this paper is an explicit construction of unique global solutions of the algebro-geometric initial value problem for the Toda hierarchy with complex-valued initial data. More precisely, we intend to describe a solution of the following problem: Given $p \in \mathbb{N}_0$, assume $a^{(0)}, b^{(0)}$ to be complex-valued solutions of the pth stationary Toda equation $\text{s-Tl}_p(a,b) = 0$ associated with a prescribed nonsingular hyperelliptic curve $\mathcal{K}_p$ of genus p and let $r \in \mathbb{N}_0$; we want to construct unique global solutions $a = a(t_r), b = b(t_r)$ of the rth Tl flow $\text{Tl}_r(a,b) = 0$ with $a(t_{0,r}) = a^{(0)}, b(t_{0,r}) = b^{(0)}$ for some $t_{0,r} \in \mathbb{R}$. Thus, we seek a unique global solution of the initial value problem

$$\begin{aligned} &\text{Tl}_r(a,b) = 0, \\ &(a,b)\big|_{t_r = t_{0,r}} = \big(a^{(0)}, b^{(0)}\big), \end{aligned} \tag{1.1}$$

$$\text{s-Tl}_p\left(a^{(0)}, b^{(0)}\right) = 0 \tag{1.2}$$

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for some $t_{0,r} \in \mathbb{R}$, $p, r \in \mathbb{N}_0$, where $a = a(n, t_r), b = b(n, t_r)$ satisfy

$$\begin{aligned} &a\colon \mathbb{Z}\times\mathbb{R} \to \mathbb{C}\setminus\{0\}, \quad b\colon \mathbb{Z}\times\mathbb{R}\to\mathbb{C},\\ &a(\,\cdot\,,t), b(\,\cdot\,,t) \in \mathbb{C}^{\mathbb{Z}},\ t\in\mathbb{R}, \quad a(n,\,\cdot\,), b(n,\,\cdot\,)\in C^1(\mathbb{R}),\ n\in\mathbb{Z}. \end{aligned} \tag{1.3}$$

In the special case of a self-adjoint Lax (i.e., Jacobi) operator L, where a and b are real-valued and bounded, the actual solution of this algebro-geometric initial value problem consists of the following two-step procedure discussed in detail in [6] (see also [14, Sect. 1.3], [32, Sect. 8.3]):[1]

(i) An algorithm that constructs finite nonspecial divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \operatorname{Sym}^p(\mathcal{K}_p)$ in real position for all $n \in \mathbb{Z}$ starting from an initial Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \operatorname{Sym}^p(\mathcal{K}_p)$ in an appropriate real position (i.e., with Dirichlet eigenvalues in appropriate spectral gaps of L). "Trace formulas" of the type (3.25) and (3.26) then construct the stationary real-valued solutions $a^{(0)}, b^{(0)}$ of $\text{s-Tl}_p(a,b) = 0$.

(ii) The first-order Dubrovin-type system of differential equations (5.41), augmented by the initial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})} = \mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ together with the analogous "trace formulas" (5.39), (5.40) then yield unique global real-valued solutions $a = a(t_r), b = b(t_r)$ of the rth Tl flow $\mathrm{Tl}_r(a,b) = 0$ satisfying $a(t_{0,r}) = a^{(0)}, b(t_{0,r}) = b^{(0)}$.

This approach works perfectly in the special self-adjoint case where the Dirichlet divisors $\underline{\hat{\mu}}(n,t_r) = (\hat{\mu}_1(n,t_r),\dots,\hat{\mu}_p(n,t_r)) \in \operatorname{Sym}^p(\mathcal{K}_p)$, $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$, yield Dirichlet eigenvalues $\mu_1(n,t_r),\dots,\mu_p(n,t_r)$ of the Lax operator L situated in p different spectral gaps of L on the real axis. In particular, for fixed $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$, the Dirichlet eigenvalues $\mu_j(n,t_r)$, $j=1,\dots,p$, are pairwise distinct and formulas (3.26) for a and (5.41) for $(d/dt_r)\mu_j(n,t_r)$, $j=1,\dots,p$, are well-defined.

This situation drastically changes if complex-valued initial data $a^{(0)}, b^{(0)}$ or $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}$ are permitted. In this case the Dirichlet eigenvalues $\mu_j(n,t_r)$, $j = 1,\dots,p$, are no longer confined to well separated spectral gaps of L on the real axis and, in particular, they are in general no longer pairwise distinct and "collisions" between them can occur at certain values of $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$. Thus, the stationary algorithm in step (i) as well as the Dubrovin-type first-order system of differential equations (5.41) in step (ii) above, breaks down at such values of (n,t_r).A priori, one has no control over such collisions, especially, it is not possible to identify initial conditions $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}$ at some $(n_0,t_{0,r})\in\mathbb{Z}\times\mathbb{R}$ which avoid collisions for all $(n,t)\in\mathbb{Z}\times\mathbb{R}$. We solve this problem head on by explicitly permitting collisions in the stationary as well as time-dependent context from the outset. In the stationary context, we properly modify the algorithm described above in step (i) in the self-adjoint case by alluding to a more general interpolation formalism (cf. Appendix B) for polynomials, going beyond the usual Lagrange interpolation formulas. In the time-dependent context we replace the first-order system of Dubrovin-type equations (5.41), augmented with the initial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}$, by a different first-order system of differential equations (6.27) with initial conditions (6.28) which focuses on symmetric functions of $\mu_1(n,t_r),\dots,\mu_p(n,t_r)$ rather than individual Dirichlet eigenvalues $\mu_j(n,t_r)$, $j=1,\dots,p$. In this manner it will be shown that collisions of Dirichlet eigenvalues no longer pose a problem.

In addition, there is a second nontrivial complication in the non-self-adjoint case: Since the Dirichlet eigenvalues $\mu_j(n,t_r)$, $j=1,\dots,p$, are no longer confined to spectral gaps of L on the real axis as (n,t_r) vary in $\mathbb{Z}\times\mathbb{R}$, it can no longer

[1] We freely use the notation of divisors of degree p as introduced in Appendix A.

be guaranteed that $\mu_j(n,t_r)$, $j=1,\dots,p$, stay finite for all $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$. As discussed in Section 4 in the stationary case, this phenomenon is related to certain deformations of the algebraic curve $\mathcal{K}_p$ under which for some $n_0\in\mathbb{Z}$, $a(n_0)\to 0$ and $\mu_j(n_0+1)\to\infty$ for some $j\in\{1,\dots,p\}$. We solve this particular problem in the stationary as well as time-dependent case by properly restricting the initial Dirichlet divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}\in\operatorname{Sym}^p(\mathcal{K}_p)$ to a dense set of full measure.

Summing up, we offer a new algorithm to solve the inverse algebro-geometric spectral problem for generally non-self-adjoint Jacobi operators, starting from a properly chosen dense set of initial divisors of full measure. Combined with an appropriate first-order system of differential equations with respect to time (a substitute for the well-known Dubrovin equations), this yields the construction of global algebro-geometric solutions of the time-dependent Toda hierarchy.

We emphasize that the approach described in this paper is not limited to the Today hierarchy but applies universally to constructing algebro-geometric solutions $1+1$-dimensional integrable soliton equations. In particular, it applies to differential-difference (i.e., lattice) systems and we are now in the process of applying it to the Ablowitz–Ladik hierarchy. Moreover, the principal idea of replacing Dubrovin-type equations by a first-order system of the type (6.27) is also relevant in the context of general non-self-adjoint Lax operators for the continuous models in $1+1$-dimensions. (In particular, the models studied in detail in [13] can be revisited from this point of view.) We also note that while the periodic case with complex-valued a,b is of course included in our analysis, we throughout consider the more general algebro-geometric case (in which a,b need not even be quasi-periodic).

Finally we briefly describe the content of each section. Section 2 presents a quick summary of the basics of the Toda hierarchy, its recursive construction, Lax pairs, and zero-curvature equations. The stationary algebro-geometric Toda hierarchy solutions, the underlying hyperelliptic curve, trace formulas, etc., are the subject of Section 3. A new algorithm solving the algebro-geometric inverse spectral problem for generally non-self-adjoint Jacobi operators is presented in Section 4. In Section 5 we briefly summarize the properties of algebro-geometric time-dependent solutions of the Toda hierarchy and formulate the algebro-geometric initial value problem. Uniqueness and existence of global solutions of the algebro-geometric initial value problem as well as their explicit construction are then presented in our final and principal Section 6. Appendix A reviews the basics of hyperelliptic Riemann surfaces of the Toda-type and sets the stage of much of the notation used in this paper. Various interpolation formulas of fundamental importance to our stationary inverse spectral algorithm developed in Section 4 are summarized in Appendix B. Finally, Appendix C summarizes asymptotic spectral parameter expansions of various quantities fundamental to the polynomial recursion formalism presented in Section 2. These appendices support our intention to make this paper reasonably self-contained.

2. The Toda Hierarchy in a Nutshell

In this section we briefly review the recursive construction of the Toda hierarchy and associated Lax pairs and zero-curvature equations following [6], [14, Sect. 1.2], and [32, Ch. 12].

Throughout this section we make the following assumption:

Hypothesis 2.1. *Suppose*

$$a, b \in \mathbb{C}^{\mathbb{Z}} \ \textit{ and } \ a(n) \neq 0 \ \textit{ for all } \ n \in \mathbb{Z}. \tag{2.1}$$

Here $\mathbb{C}^{\mathbb{Z}}$ denotes the set of complex-valued sequences indexed by $\mathbb{Z}$.

We consider the second-order Jacobi difference expression

$$L = aS^+ + a^- S^- + b, \tag{2.2}$$

where $S^\pm$ denote the shift operators

$$(S^\pm f)(n) = f^\pm(n) = f(n\pm 1), \quad n \in \mathbb{Z}, \ f \in \mathbb{C}^{\mathbb{Z}}. \tag{2.3}$$

To construct the stationary Toda hierarchy we need a second difference expression of order $2p+2$, $p \in \mathbb{N}_0$, defined recursively in the following. We take the quickest route to the construction of P_{2p+2}, and hence to the Toda hierarchy, by starting from the recursion relations (2.4)–(2.6) below.

Define $\{f_\ell\}_{\ell\in\mathbb{N}_0}$ and $\{g_\ell\}_{\ell\in\mathbb{N}_0}$ recursively by

$$f_0 = 1, \quad g_0 = -c_1, \tag{2.4}$$

$$2f_{j+1} + g_j + g_j^- - 2bf_j = 0, \quad j \in \mathbb{N}_0, \tag{2.5}$$

$$g_{j+1} - g_{j+1}^- + 2\big(a^2 f_j^+ - (a^-)^2 f_j^-\big) - b(g_j - g_j^-) = 0, \quad j \in \mathbb{N}_0. \tag{2.6}$$

Explicitly, one finds

$$\begin{aligned}
&f_0 = 1,\\
&f_1 = b + c_1,\\
&f_2 = a^2 + (a^-)^2 + b^2 + c_1 b + c_2, \ \text{ etc.},\\
&g_0 = -c_1,\\
&g_1 = -2a^2 - c_2,\\
&g_2 = -2a^2(b^+ + b) + c_1(-2a^2) - c_3, \ \text{ etc.}
\end{aligned} \tag{2.7}$$

Here $\{c_\ell\}_{\ell\in\mathbb{N}}$ denote undetermined summation constants which naturally arise when solving (2.4)–(2.6).

Subsequently, it will be convenient to also introduce the corresponding homogeneous coefficients $\hat f_j$ and $\hat g_j$, defined by vanishing of the constants c_k, $k \in \mathbb{N}$,

$$\begin{aligned}
&\hat f_0 = 1, \quad \hat f_\ell = f_\ell\big|_{c_k=0,\, k=1,\dots,\ell}, \quad \ell \in \mathbb{N},\\
&\hat g_0 = 0, \quad \hat g_\ell = g_\ell\big|_{c_k=0,\, k=1,\dots,\ell+1}, \quad \ell \in \mathbb{N}_0.
\end{aligned} \tag{2.8}$$

Hence,

$$f_\ell = \sum_{k=0}^{\ell} c_{\ell-k} \hat f_k, \quad g_\ell = \sum_{k=1}^{\ell} c_{\ell-k} \hat g_k - c_{\ell+1}, \quad \ell \in \mathbb{N}_0, \tag{2.9}$$

introducing

$$c_0 = 1. \tag{2.10}$$

Next we define difference expressions P_{2p+2} of order $2p+2$ by

$$P_{2p+2} = -L^{p+1} + \sum_{\ell=0}^{p} \Big(g_\ell + 2af_\ell S^+\Big) L^{p-\ell} + f_{p+1}, \quad p \in \mathbb{N}_0. \tag{2.11}$$

Introducing the corresponding homogeneous difference expressions $\widehat{P}_{2p+2}$ defined by

$$\widehat{P}_{2\ell+2} = P_{2\ell+2}\big|_{c_k=0,\, k=1,\dots,\ell}, \quad \ell \in \mathbb{N}_0, \tag{2.12}$$

one finds

$$P_{2p+2} = \sum_{\ell=0}^{p} c_{p-\ell} \widehat{P}_{2\ell+2}. \tag{2.13}$$

Using the recursion relations (2.4)–(2.6), the commutator of P_{2p+2} and L can be explicitly computed and one obtains

$$\begin{aligned} [P_{2p+2}, L] = & -a\big(g_p^+ + g_p + f_{p+1}^+ + f_{p+1} - 2b^+ f_p^+\big) S^+ \\ & + 2\big(-b(g_p + f_{p+1}) + a^2 f_p^+ - (a^-)^2 f_p^- + b^2 f_p\big) \\ & - a^-\big(g_p + g_p^- + f_{p+1} + f_{p+1}^- - 2b f_p\big) S^-, \quad p \in \mathbb{N}_0. \end{aligned} \tag{2.14}$$

In particular, (L, P_{2p+2}) represents the celebrated *Lax pair* of the Toda hierarchy. Varying $p \in \mathbb{N}_0$, the stationary Toda hierarchy is then defined in terms of the vanishing of the commutator of P_{2p+2} and L in (2.14), that is,

$$[P_{2p+2}, L] = \text{s-Tl}_p(a, b) = 0, \quad p \in \mathbb{N}_0. \tag{2.15}$$

Thus one finds

$$g_p + g_p^- + f_{p+1} + f_{p+1}^- - 2b f_p = 0, \tag{2.16}$$

$$-b(g_p + f_{p+1}) + a^2 f_p^+ - (a^-)^2 f_g^- + b^2 f_p = 0. \tag{2.17}$$

Using (2.5) with $j = p$ one concludes that (2.16) reduces to

$$f_{p+1} = f_{p+1}^-, \tag{2.18}$$

that is, f_{p+1} is a lattice constant. Similarly, one infers by subtracting b times (2.16) from twice (2.17) and using (2.6) with $j = p$, that g_{p+1} is a lattice constant as well, that is,

$$g_{p+1} = g_{p+1}^-. \tag{2.19}$$

Equations (2.18) and (2.19) give rise to the pth stationary equation in the Toda hierarchy, which is introduced as follows

$$\text{s-Tl}_p(a, b) = \begin{pmatrix} f_{p+1}^+ - f_{p+1} \\ g_{p+1} - g_{p+1}^- \end{pmatrix} = 0, \quad p \in \mathbb{N}_0. \tag{2.20}$$

Explicitly,

$$\begin{aligned} \text{s-Tl}_0(a, b) &= \begin{pmatrix} b^+ - b \\ 2\big((a^-)^2 - a^2\big) \end{pmatrix} = 0, \\ \text{s-Tl}_1(a, b) &= \begin{pmatrix} (a^+)^2 - (a^-)^2 + (b^+)^2 - b^2 \\ 2(a^-)^2(b + b^-) - 2a^2(b^+ + b) \end{pmatrix} \\ & \quad + c_1 \begin{pmatrix} b^+ - b \\ 2\big((a^-)^2 - a^2\big) \end{pmatrix} = 0, \text{ etc.}, \end{aligned} \tag{2.21}$$

represent the first few equations of the stationary Toda hierarchy. By definition, the set of solutions of (2.20), with p ranging in $\mathbb{N}_0$ and $c_\ell \in \mathbb{C}$, $\ell \in \mathbb{N}$, represents the class of algebro-geometric Toda solutions.

In the following we will frequently assume that a, b satisfy the pth stationary Toda equations. By this we mean it satisfies one of the pth stationary Toda equations after a particular choice of summation constants $c_\ell \in \mathbb{C}$, $\ell = 1, \dots, p$, $p \in \mathbb{N}$, has been made.

In accordance with our notation introduced in (2.8) and (2.12), the corresponding homogeneous stationary Toda equations are defined by

$$\text{s-}\widehat{\text{Tl}}_p(a,b) = \text{s-Tl}_p(a,b)\big|_{c_\ell=0,\, \ell=1,\dots,p} = 0, \quad p \in \mathbb{N}_0. \tag{2.22}$$

Next, we introduce polynomials F_p and G_{p+1} of degree p and $p+1$, with respect to the spectral parameter $z \in \mathbb{C}$ by

$$F_p(z) = \sum_{\ell=0}^{p} f_{p-\ell} z^\ell = \sum_{\ell=0}^{p} c_{p-\ell} \widehat{F}_\ell(z), \tag{2.23}$$

$$G_{p+1}(z) = -z^{p+1} + \sum_{\ell=0}^{p} g_{p-\ell} z^\ell + f_{p+1} = \sum_{\ell=1}^{p+1} c_{p+1-\ell} \widehat{G}_\ell(z) \tag{2.24}$$

with $\widehat{F}_\ell$ and $\widehat{G}_\ell$ denoting the corresponding homogeneous polynomials defined by

$$\begin{aligned} &\widehat{F}_0(z) = F_0(z) = 1, \\ &\widehat{F}_\ell(z) = F_\ell(z)\big|_{c_k=0,\, k=1,\dots,\ell} = \sum_{k=0}^{\ell} \hat{f}_{\ell-k} z^k, \quad \ell \in \mathbb{N}_0, \end{aligned} \tag{2.25}$$

$$\begin{aligned} &\widehat{G}_0(z) = G_0(z)\big|_{c_1=0} = 0, \quad \widehat{G}_1(z) = G_1(z) = -z - b, \\ &\widehat{G}_{\ell+1}(z) = G_{\ell+1}(z)\big|_{c_k=0,\, k=1,\dots,\ell} = -z^{\ell+1} + \sum_{k=0}^{\ell} \hat{g}_{\ell-k} z^k + \hat{f}_{\ell+1}, \quad \ell \in \mathbb{N}. \end{aligned} \tag{2.26}$$

Explicitly, one obtains

$$\begin{aligned} &F_0 = 1, \\ &F_1 = z + b + c_1, \\ &F_2 = z^2 + bz + a^2 + (a^-)^2 + b^2 + c_1(z+b) + c_2, \text{ etc.}, \\ &G_1 = -z + b, \\ &G_2 = -z^2 + (a^-)^2 - a^2 + b^2 + c_1(-z+b), \text{ etc.} \end{aligned} \tag{2.27}$$

Next, we study the restriction of the difference expression P_{2p+2} to the two-dimensional kernel (i.e., the formal null space in an algebraic sense as opposed to the functional analytic one) of $(L-z)$. More precisely, let

$$\ker(L-z) = \{\psi\colon \mathbb{Z} \to \mathbb{C} \cup \{\infty\} \mid (L-z)\psi = 0\}. \tag{2.28}$$

Then (2.11) implies

$$P_{2p+2}\mid_{\ker(L-z)} = \big(2aF_p(z)S^+ + G_{p+1}(z)\big)\big|_{\ker(L-z)}. \tag{2.29}$$

Therefore, the Lax relation (2.14) becomes

$$2(z-b^+)F_p^+ - 2(z-b)F_p + G_{p+1}^+ - G_{p+1}^- = 0, \tag{2.30}$$

$$2a^2F_p^+ - 2(a^-)^2F_p^- + (z-b)(G_{p+1} - G_{p+1}^-) = 0. \tag{2.31}$$

Additional manipulations yield

$$2(z-b)F_p + G_{p+1} + G_{p+1}^- = 0, \tag{2.32}$$

$$(z-b)^2F_p + (z-b)G_{p+1} + a^2F_p^+ - (a^-)^2F_p^- = 0. \tag{2.33}$$

Indeed, adding $G_{p+1} - G_{p+1}$ to the left-hand side of (2.30) (neglecting a trivial summation constant) yields (2.32) and inserting (2.30) into (2.31) then implies (2.33). Varying $p \in \mathbb{N}_0$, equations (2.32), (2.33) provide an alternative description of the stationary Toda hierarchy.

Combining equations (2.31) and (2.32) one concludes that the quantity

$$R_{2p+2}(z) = G_{p+1}(z,n)^2 - 4a(n)^2 F_p(z,n) F_p^+(z,n) \tag{2.34}$$

is a lattice constant, and hence depends on z only. Thus, one can write

$$R_{2p+2}(z) = \prod_{m=0}^{2p+1} (z - E_m), \quad \{E_m\}_{m=0}^{2p+1} \subset \mathbb{C}. \tag{2.35}$$

One can decouple (2.32) and (2.33) to obtain separate equations for F_p and G_{p+1}. For instance, computing G_{p+1} from (2.33) and inserting the result into (2.32) yields the following linear difference equation for F_p

$$\begin{aligned}(z-b)^2(z-b^-)F_p - (z-b^-)^2(z-b)F_p^- + \big((a^-)^2F_p^- - a^2F_p^+\big)(z-b^-)\\ + \big((a^{--})^2F_p^{--} - (a^-)^2F_p\big)(z-b) = 0.\end{aligned} \tag{2.36}$$

Similarly, insertion of (2.33) into (2.34) permits one to eliminate G_{p+1} and results in the following nonlinear difference equation for F_p,

$$\begin{aligned}(z-b)^4F_p^2 - 2a^2(z-b)^2F_pF_p^+ - 2(a^-)^2(z-b)^2F_pF_p^- + a^4(F_p^+)^2\\ + (a^-)^4(F_p^-)^2 - 2a^2(a^-)^2F_p^+F_p^- = (z-b)^2R_{2p+2}(z).\end{aligned} \tag{2.37}$$

On the other hand, computing F_p in terms of G_{p+1} and G_{p+1}^+ using (2.32) and inserting the result into (2.33) yields the following linear difference equation for G_{p+1}

$$\begin{aligned}a^2(z-b^-)(G_{p+1}^+ + G_{p+1}) - (a^-)^2(z-b^+)(G_{p+1}^- + G_{p+1}^{--})\\ + (z-b^-)(z-b)(z-b^+)(G_{p+1}^- - G_{p+1}) = 0.\end{aligned} \tag{2.38}$$

Finally, inserting the result for F_p into (2.34) yields the following nonlinear difference equation for G_{p+1}

$$\begin{aligned}(z-b)(z-b^+)G_{p+1}^2 - a^2(G_{p+1}^- + G_{p+1})(G_{p+1} + G_{p+1}^+)\\ = (z-b)(z-b^+)R_{2p+2}.\end{aligned} \tag{2.39}$$

Equations (2.37) and (2.39) can be used to derive nonlinear recursion relations for the homogeneous coefficients $\hat{f}_\ell$ and $\hat{g}_\ell$ (i.e., the ones satisfying (2.8) in the case of vanishing summation constants) as proved in Theorem C.1 in Appendix C. This has interesting applications to the asymptotic expansion of the Green's function of L with respect to the spectral parameter. In addition, as proven in Theorem C.1, (2.37) leads to an explicit determination of the summation constants $c_1, \dots, c_p$ in

$$\text{s-Tl}_p(a,b) = 0, \quad p \in \mathbb{N}_0, \tag{2.40}$$

in terms of the zeros $E_0, \dots, E_{2p+1}$ of the associated polynomial R_{2p+2} in (2.35). In fact, one can prove (cf. Theorem C.1) that

$$c_k = c_k(\underline{E}), \quad k = 1, \dots, p, \tag{2.41}$$

where

$$c_k(\underline{E}) = -\sum_{\substack{j_0,\dots,j_{2p+1}=0\\ j_0+\cdots+j_{2p+1}=k}}^{k} \frac{(2j_0)!\cdots(2j_{2p+1})!}{2^{2k}(j_0!)^2\cdots(j_{2p+1}!)^2(2j_0-1)\cdots(2j_{2p+1}-1)} E_0^{j_0}\cdots E_{2p+1}^{j_{2p+1}}, \quad k=1,\dots,p \tag{2.42}$$

are symmetric functions of $\underline{E} = (E_0, \dots, E_{2p+1})$.

We emphasize that the result (2.29) is valid independently of whether or not P_{2p+2} and L commute. However, the fact that the two difference expressions P_{2p+2} and L commute implies the existence of an algebraic relationship between them. This gives rise to the Burchnall–Chaundy polynomial for the Toda hierarchy first discussed in the discrete context by Naĭman [28], [29].

Theorem 2.2. *Assume Hypothesis 2.1, fix $p \in \mathbb{N}_0$ and suppose that P_{2p+2} and L commute, $[P_{2p+2}, L] = 0$, or equivalently, assume that* s-$\mathrm{Tl}_p(a,b) = 0$. *Then L and P_{2p+2} satisfy an algebraic relationship of the type* (*cf.* (2.35))

$$\begin{aligned} &\mathcal{F}_p(L, P_{2p+2}) = P_{2p+2}^2 - R_{2p+2}(L) = 0, \\ &R_{2p+2}(z) = \prod_{m=0}^{2p+1}(z - E_m), \quad z \in \mathbb{C}. \end{aligned} \tag{2.43}$$

The expression $\mathcal{F}_p(L, P_{2p+2})$ is called the Burchnall–Chaundy polynomial of the Lax pair (L, P_{2p+2}) and it will be used in Section 3 to introduce the underlying hyperelliptic curve associated with the stationary Toda equation s-$\mathrm{Tl}_p(a,b) = 0$ (cf. (3.1)).

Next we turn to the time-dependent Toda hierarchy. For that purpose the functions a and b are now considered as functions of both the lattice point and time. For each equation in the hierarchy, that is, for each p, we introduce a deformation (time) parameter $t_p \in \mathbb{R}$ in a, b, replacing $a(n), b(n)$ by $a(n, t_p), b(n, t_p)$. The second-order difference expression L (cf. (2.2)) now reads

$$L(t_p) = a(\,\cdot\,, t_p)S^+ + a^-(\,\cdot\,, t_p)S^- + b(\,\cdot\,, t_p). \tag{2.44}$$

The quantities $\{f_\ell\}_{\ell\in\mathbb{N}_0}$, $\{g_\ell\}_{\ell\in\mathbb{N}_0}$, and P_{2p+2}, $p \in \mathbb{N}_0$ are still defined by (2.4)–(2.6) and (2.11), respectively. The time-dependent Toda hierarchy is then obtained by imposing the Lax commutator equations

$$L_{t_p}(t_p) - [P_{2p+2}(t_p), L(t_p)] = 0, \quad t_p \in \mathbb{R}, \tag{2.45}$$

varying $p \in \mathbb{N}_0$. Relation (2.45) implies

$$\begin{aligned} &\big(a_{t_p} + a(g_p^+ + g_p + f_{p+1}^+ + f_{p+1} - 2b^+ f_p^+)\big)\, S^+ \\ &\quad - \big(-b_{t_p} + 2\big(-b(g_p + f_{p+1}) + a^2 f_p^+ - (a^-)^2 f_p^- + b^2 f_p\big)\big) \\ &\quad + \big(a_{t_p} + a(g_p^+ + g_p + f_{p+1}^+ + f_{p+1} - 2b^+ f_p^+)\big)^-\, S^- = 0. \end{aligned} \tag{2.46}$$

Applying the same method we used to derive (2.18) and (2.19) one concludes

$$\begin{aligned} 0 &= L_{t_p} - [P_{2p+2}, L] \\ &= \big(a_{t_p} - a(f_{p+1}^+ - f_{p+1})\big)\, S^+ - \big(-b_{t_p} - g_{p+1} + g_{p+1}^-\big) \\ &\quad + \big(a_{t_p} - a(f_{p+1}^+ - f_{p+1})\big)^-\, S^-. \end{aligned} \tag{2.47}$$

Varying $p \in \mathbb{N}_0$, the collection of evolution equations

$$\mathrm{Tl}_p(a,b) = \begin{pmatrix} a_{t_p} - a(f_{p+1}^+ - f_{p+1}) \\ b_{t_p} + g_{p+1} - g_{p+1}^- \end{pmatrix} = 0, \quad (n,t_p) \in \mathbb{Z} \times \mathbb{R},\ p \in \mathbb{N}_0 \tag{2.48}$$

then defines the time-dependent Toda hierarchy. Explicitly,

$$\begin{aligned} \mathrm{Tl}_0(a,b) &= \begin{pmatrix} a_{t_0} - a(b^+ - b) \\ b_{t_0} - 2\big(a^2 - (a^-)^2\big) \end{pmatrix} = 0, \\ \mathrm{Tl}_1(a,b) &= \begin{pmatrix} a_{t_1} - a\big((a^+)^2 - (a^-)^2 + (b^+)^2 - b^2\big) \\ b_{t_1} - 2a^2(b^+ + b) + 2(a^-)^2(b + b^-) \end{pmatrix} \\ &\quad + c_1 \begin{pmatrix} -a(b^+ - b) \\ -2\big(a^2 - (a^-)^2\big) \end{pmatrix} = 0, \text{ etc.,} \end{aligned} \tag{2.49}$$

represent the first few equations of the time-dependent Toda hierarchy. The system of equations, $\mathrm{Tl}_0(a,b) = 0$, is of course *the* Toda system.

The corresponding homogeneous Toda equations obtained by taking all summation constants equal to zero, $c_\ell = 0$, $\ell = 1, \dots, p$, are then denoted by

$$\widehat{\mathrm{Tl}}_p(a,b) = \mathrm{Tl}_p(a,b)\big|_{c_\ell = 0,\, \ell = 1,\dots,p}. \tag{2.50}$$

Restricting the Lax relation (2.45) to the kernel $\ker(L - z)$ one finds that

$$0 = \big(L_{t_p} - [P_{2p+2}, L]\big)\big|_{\ker(L-z)} = \big(L_{t_p} + (L - z)P_{2p+2}\big)\big|_{\ker(L-z)} \tag{2.51}$$

$$\begin{aligned} &= \Big(a\Big(\frac{a_{t_p}}{a} - \frac{a_{t_p}^-}{a^-} + 2(z - b^+)F_p^+ - 2(z - b)F_p + G_{p+1}^+ - G_{p+1}^-\Big) S^+ \\ &\quad + \Big(b_{t_p} + (z - b)\frac{a_{t_p}^-}{a^-} + 2(a^-)^2 F_p^- - 2a^2 F_p^+ \\ &\qquad + (z - b)(G_{p+1}^- - G_{p+1})\Big)\Big)\Big|_{\ker(L-z)}. \end{aligned} \tag{2.52}$$

Hence one obtains

$$\frac{a_{t_p}}{a} - \frac{a_{t_p}^-}{a^-} = -2(z - b^+)F_p^+ + 2(z - b)F_p + G_{p+1}^- - G_{p+1}^+, \tag{2.53}$$

$$b_{t_p} = -(z - b)\frac{a_{t_p}^-}{a^-} - 2(a^-)^2 F_p^- + 2a^2 F_p^+ - (z - b)(G_{p+1}^- - G_{p+1}). \tag{2.54}$$

Further manipulations then yield,

$$a_{t_p} = -a\big(2(z - b^+)F_p^+ + G_{p+1}^+ + G_{p+1}\big), \tag{2.55}$$

$$b_{t_p} = 2\big((z - b)^2 F_p + (z - b)G_{p+1} + a^2 F_p^+ - (a^-)^2 F_p^-\big). \tag{2.56}$$

Indeed, (2.55) follows by adding $G_{p+1} - G_{p+1}$ to (2.53) (neglecting a trivial summation constant), and an insertion of (2.55) into (2.54) implies (2.56). Varying $p \in \mathbb{N}_0$, equations (2.55) and (2.56) provide an alternative description of the time-dependent Toda hierarchy.

Remark 2.3. *From* (2.4)–(2.6) *and* (2.24) *one concludes that the coefficient a enters quadratically in F_p and G_{p+1}, and hence the Toda hierarchy* (2.48) (*respectively* (2.20)) *is invariant under the substitution*

$$a \to a_\varepsilon = \{\varepsilon(n)a(n)\}_{n \in \mathbb{Z}}, \quad \varepsilon(n) \in \{1, -1\},\ n \in \mathbb{Z}. \tag{2.57}$$

We conclude this section by pointing out an alternative construction of the Toda hierarchy using a zero-curvature approach instead of Lax pairs (L, P_{2p+2}). To this end one defines the 2×2 matrices

$$U(z) = \begin{pmatrix} 0 & 1 \\ -a^-/a & (z-b)/a \end{pmatrix}, \tag{2.58}$$

$$V_{p+1}(z) = \begin{pmatrix} G^-_{p+1}(z) & 2a^- F^-_p(z) \\ -2a^- F_p(z) & 2(z-b)F_p(z) + G_{p+1}(z) \end{pmatrix}, \quad p \in \mathbb{N}_0. \tag{2.59}$$

Then the stationary part of this section can equivalently be based on the zero-curvature equation

$$\begin{aligned} 0 &= UV_{p+1} - V^+_{p+1}U \qquad (2.60) \\ &= \frac{2}{a}\begin{pmatrix} 0 & 0 \\ \begin{matrix} a^-\big((z-b^+)F^+_p - (z-b)F_p \\ +2^{-1}(G^+_{p+1} - G^-_{p+1})\big) \end{matrix} & \begin{matrix} a^2F^+_p - (a^-)^2F^-_p \\ +2^{-1}(z-b)(G_{p+1} - G^+_{p+1}) \\ +(z-b)^2F_p - (z-b^+)(z-b)F^+_p \end{matrix} \end{pmatrix}. \end{aligned}$$

Thus, one obtains (2.30) from the $(2,1)$-entry in (2.60). Insertion of (2.30) into the $(2,2)$-entry of (2.60) then yields (2.31). Thus, one also obtains (2.32) and hence the $(2,2)$-entry of V_{p+1} in (2.59) simplifies to

$$V_{p+1,2,2}(z) = -G^-_{p+1}(z) \tag{2.61}$$

in the stationary case. Since $\det(U(z,n)) = a^-(n)/a(n) \neq 0$, $n \in \mathbb{Z}$, the zero-curvature equation (2.60) yields that $\det(V_{p+1}(z,n))$ is a lattice constant (i.e., independent of $n \in \mathbb{Z}$) the Burchnall–Chaundy polynomial $\mathcal{F}_p(y,z)$ (cf. (2.43) and especially, the hyperelliptic curve (3.1)) is then obtained from the characteristic equation of $V_{p+1}(z)$ by

$$\begin{aligned} &\det(yI_2 - V_{p+1}(z,n)) \\ &\quad = y^2 + \det(V_{p+1}(z,n)) \\ &\quad = y^2 - G^-_{p-1}(z,n)^2 + 4a^-(n)^2F^-_p(z,n)F_p(z,n) \\ &\quad = y^2 - R_{2p+2}(z) = 0, \end{aligned} \tag{2.62}$$

using (2.61). (Here I_2 denotes the identity matrix in $\mathbb{C}^2$.) Similarly, the time-dependent part (2.44)–(2.56) can equivalently be developed from the zero-curvature equation

$$\begin{aligned} 0 &= U_{t_p} + UV_{p+1} - V^+_{p+1}U \qquad (2.63) \\ &= \frac{1}{a}\begin{pmatrix} 0 & 0 \\ \begin{matrix} a^-((a_{t_p}/a) - (a^-_{t_p}/a^-)) \\ +a^-\big(2(z-b^+)F^+_p - 2(z-b)F_p \\ +(G^+_{p+1} - G^-_{p+1})\big) \end{matrix} & \begin{matrix} -b_{t_p} - (z-b)(a_{t_p}/a) \\ +2a^2F^+_p - 2(a^-)^2F^-_p \\ +(z-b)(G_{p+1} - G^+_{p+1}) \\ +2(z-b)^2F_p - 2(z-b^+)(z-b)F^+_p \end{matrix} \end{pmatrix}. \end{aligned}$$

The $(2,1)$-entry in (2.63) yields (2.53), and inserting (2.53) into the $(2,2)$-entry of (2.63) yields (2.54) and hence also the basic equations defining the time-dependent Toda hierarchy in (2.55), (2.56).

3. Properties of Stationary Algebro-Geometric Solutions of the Toda Hierarchy

In this section we present a quick review of properties of algebro-geometric solutions of the stationary Toda hierarchy. Since this material is standard we omit all proofs and just refer to [6] (cf. also [14, Sect. 1.3], [32, Chs. 8, 9]) for detailed presentations and an extensive list of references to the literature.

For the notation employed in connection with elementary concepts in algebraic geometry (more precisely, the theory of compact Riemann surfaces), we refer to Appendix A.

Returning to Theorem 2.2, we note that (2.43) naturally leads to the hyperelliptic curve $\mathcal{K}_p$ of genus $p \in \mathbb{N}_0$, where

$$\begin{aligned} &\mathcal{K}_p \colon \mathcal{F}_p(z,y) = y^2 - R_{2p+2}(z) = 0, \\ &R_{2p+2}(z) = \prod_{m=0}^{2p+1} (z - E_m), \quad \{E_m\}_{m=0}^{2p+1} \subset \mathbb{C}. \end{aligned} \tag{3.1}$$

Throughout this section we make the following assumption:

Hypothesis 3.1. *Suppose that the hyperelliptic curve $\mathcal{K}_p$ in* (3.1) *is nonsingular, that is, assume that*

$$E_m \neq E_{m'} \text{ for } m \neq m', \; m, m' = 0, \dots, 2p+1. \tag{3.2}$$

In addition, we suppose that

$$a, b \in \mathbb{C}^{\mathbb{Z}} \text{ and } a(n) \neq 0 \text{ for all } n \in \mathbb{Z}. \tag{3.3}$$

The curve $\mathcal{K}_p$ is compactified by joining two points $P_{\infty_\pm}$, $P_{\infty_+} \neq P_{\infty_-}$, at infinity. For notational simplicity, the resulting curve is still denoted by $\mathcal{K}_p$. Points P on $\mathcal{K}_p \setminus P_{\infty_\pm}$ are represented as pairs $P = (z, y)$, where $y(\,\cdot\,)$ is the meromorphic function on $\mathcal{K}_p$ satisfying $\mathcal{F}_p(z, y) = 0$. The complex structure on $\mathcal{K}_p$ is then defined in the usual way, see Appendix A. Hence, $\mathcal{K}_p$ becomes a two-sheeted hyperelliptic Riemann surface of genus $p \in \mathbb{N}_0$ in a standard manner.

We also emphasize that by fixing the curve $\mathcal{K}_p$ (i.e., by fixing $E_0, \dots, E_{2p+1}$), the summation constants $c_1, \dots, c_p$ in the corresponding stationary s-Tl_p equation are uniquely determined as is clear from (2.41) and (2.42), which establish the summation constants c_k as symmetric functions of $E_0, \dots, E_{2p+1}$.

For notational simplicity we will usually tacitly assume that $p \in \mathbb{N}$. The trivial case $p = 0$, which leads to $a(n)^2 = (E_1 - E_0)^2/16$, $b(n) = (E_0 + E_1)/2$, $n \in \mathbb{Z}$, is of no interest to us in this paper.

In the following, the zeros[2] of the polynomial $F_p(\,\cdot\,, n)$ (cf. (2.23)) will play a special role. We denote them by $\{\mu_j(n)\}_{j=1}^p$ and write

$$F_p(z, n) = \prod_{j=1}^{p} (z - \mu_j(n)). \tag{3.4}$$

The next step is crucial; it permits us to "lift" the zeros μ_j of F_p from $\mathbb{C}$ to the curve $\mathcal{K}_p$. From (2.34) and (3.4) one infers

$$R_{2p+2}(z) - G_{p+1}(z)^2 = 0, \quad z \in \{\mu_j, \mu_k^+\}_{j,k=1,\dots,p}. \tag{3.5}$$

[2] If $a, b \in \ell^\infty(\mathbb{Z})$, these zeros are the Dirichlet eigenvalues of a bounded operator on $\ell^2(\mathbb{Z})$ associated with the difference expression L and a Dirichlet boundary condition at $n \in \mathbb{Z}$.

We now introduce $\{\hat{\mu}_j(n)\}_{j=1,\dots,p} \subset \mathcal{K}_p$ by

$$\hat{\mu}_j(n) = (\mu_j(n), -G_{p+1}(\mu_j(n), n)) \in \mathcal{K}_p, \quad j = 1, ..., p, \ n \in \mathbb{Z}. \tag{3.6}$$

Next, we recall equation (2.34) and define the fundamental meromorphic function $\phi(\,\cdot\,, n)$ on $\mathcal{K}_p$ by

$$\phi(P, n) = \frac{y - G_{p+1}(z, n)}{2a(n)F_p(z, n)} \tag{3.7}$$

$$= \frac{-2a(n)F_p(z, n+1)}{y + G_{p+1}(z, n)}, \tag{3.8}$$

$$P = (z, y) \in \mathcal{K}_p, \ n \in \mathbb{Z}$$

with divisor $(\phi(\,\cdot\,, n))$ of $\phi(\,\cdot\,, n)$ given by

$$\big(\phi(\,\cdot\,, n)\big) = \mathcal{D}_{P_{\infty_+}\underline{\hat{\mu}}(n+1)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n)}, \tag{3.9}$$

using (3.4) and (3.6). Here we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_p\} \in \operatorname{Sym}^p(\mathcal{K}_p) \tag{3.10}$$

(cf. the notation introduced in Appendix A). We note that several $\mu_j(n)$ may be equal for a given lattice point $n \in \mathbb{Z}$. Moreover, since $-G_{p+1}(\mu_j(n), n)$ takes on the same value for all coinciding zeros $\mu_j(n)$, no finite special divisors $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ can ever arise in ϕ (cf. also Lemma 3.4).

The stationary Baker–Akhiezer function $\psi(\,\cdot\,, n, n_0)$ on $\mathcal{K}_p \setminus \{P_{\infty_\pm}\}$ is then defined in terms of $\phi(\,\cdot\,, n)$ by

$$\psi(P, n, n_0) = \begin{cases} \prod_{m=n_0}^{n-1} \phi(P, m) & \text{for} \quad n \geq n_0 + 1, \\ 1 & \text{for} \quad n = n_0, \\ \prod_{m=n}^{n_0-1} \phi(P, m)^{-1} & \text{for} \quad n \leq n_0 - 1, \end{cases} \tag{3.11}$$

$$P \in \mathcal{K}_p \setminus \{P_{\infty_\pm}\}, \ (n, n_0) \in \mathbb{Z}^2$$

with divisor $\big(\psi(\,\cdot\,, n, n_0)\big)$ of $\psi(P, n, n_0)$ given by

$$\big(\psi(\,\cdot\,, n, n_0)\big) = \mathcal{D}_{\underline{\hat{\mu}}(n)} - \mathcal{D}_{\underline{\hat{\mu}}(n_0)} + (n - n_0)(\mathcal{D}_{P_{\infty_+}} - \mathcal{D}_{P_{\infty_-}}). \tag{3.12}$$

For future purposes we also introduce the following Baker–Akhiezer vector,

$$\Psi(P, n, n_0) = \begin{pmatrix} \psi^-(P, n, n_0) \\ \psi(P, n, n_0) \end{pmatrix}, \quad P \in \mathcal{K}_p \setminus \{P_{\infty_\pm}\}, \ (n, n_0) \in \mathbb{Z}^2. \tag{3.13}$$

Basic properties of ϕ, ψ, and Ψ are summarized in the following result. We abbreviate by

$$W(f, g) = a(fg^+ - f^+g) \tag{3.14}$$

the Wronskian of two complex-valued sequences f and g, and denote $P^* = (z, -y)$ for $P = (z, y) \in \mathcal{K}_p$.

Lemma 3.2. *Assume Hypothesis 3.1 and suppose that a, b satisfy the pth stationary Toda equation* (2.20). *Moreover, let $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty_\pm}\}$ and $(n, n_0) \in \mathbb{Z}^2$. Then ϕ satisfies the Riccati-type equation*

$$a\phi(P) + a^-\phi^-(P)^{-1} = z - b, \tag{3.15}$$

as well as

$$\phi(P)\phi(P^*) = \frac{F_p^+(z)}{F_p(z)}, \tag{3.16}$$

$$\phi(P) + \phi(P^*) = -\frac{G_{p+1}(z)}{aF_p(z)}, \tag{3.17}$$

$$\phi(P) - \phi(P^*) = \frac{y(P)}{aF_p(z)}. \tag{3.18}$$

Moreover, ψ *and* Ψ *satisfy*

$$\big(L - z(P)\big)\psi(P) = 0, \quad \big(P_{2p+2} - y(P)\big)\psi(P) = 0, \tag{3.19}$$

$$\Psi^+(P) = U(z)\Psi(P), \quad y\Psi(P) = V_{p+1}\Psi(P), \tag{3.20}$$

$$\psi(P,n,n_0)\psi(P^*,n,n_0) = \frac{F_p(z,n)}{F_p(z,n_0)}, \tag{3.21}$$

$$\begin{aligned} &a(n)\big(\psi(P,n,n_0)\psi(P^*,n+1,n_0) + \psi(P^*,n,n_0)\psi(P,n+1,n_0)\big) \\ &\quad = -\frac{G_{p+1}(z,n)}{F_p(z,n_0)}, \end{aligned} \tag{3.22}$$

$$W(\psi(P,\,\cdot\,,n_0),\psi(P^*,\,\cdot\,,n_0)) = -\frac{y(P)}{F_p(z,n_0)}. \tag{3.23}$$

Combining the polynomial recursion approach with (3.4) readily yields trace formulas for the Toda invariants, which are expressions of a and b in terms of the zeros μ_j of F_p. We introduce the abbreviation,

$$b^{(k)}(n) = \frac{1}{2}\sum_{m=0}^{2p+1} E_m^k - \sum_{j=1}^{p} \mu_j^k(n), \quad k \in \mathbb{N}. \tag{3.24}$$

Lemma 3.3. *Assume Hypothesis 3.1 and suppose that* a, b *satisfy the pth stationary Toda equation* (2.20)*. Then,*

$$b(n) = \frac{1}{2}\sum_{m=0}^{2p+1} E_m - \sum_{j=1}^{p} \mu_j(n), \quad n \in \mathbb{Z}. \tag{3.25}$$

In addition, if for all $n \in \mathbb{Z}$*,* $\mu_j(n) \neq \mu_k(n)$ *for* $j \neq k$*,* $j,k = 1,\dots,p$*, then,*

$$a(n)^2 = \frac{1}{2}\sum_{j=1}^{p} y(\hat\mu_j(n)) \prod_{\substack{k=1\\k\neq j}}^{p} (\mu_j(n) - \mu_k(n))^{-1} + \frac{1}{4}\big(b^{(2)}(n) - b(n)^2\big), \quad n \in \mathbb{Z}. \tag{3.26}$$

The case where some of the μ_j coincide in (3.26) requires a more elaborate argument that will be presented in Section 4.

Since nonspecial Dirichlet divisors $\mathcal{D}_{\underline{\hat\mu}}$ and the linearization property of the Abel map when applied to $\mathcal{D}_{\underline{\hat\mu}}$ will play a fundamental role later on, we also recall the following facts.

Lemma 3.4. *Assume Hypothesis 3.1 and suppose that* a, b *satisfy the pth stationary Toda equation* (2.20)*. Let* $\mathcal{D}_{\underline{\hat\mu}}$*,* $\underline{\hat\mu} = \{\hat\mu_1,\dots,\hat\mu_p\} \in \mathrm{Sym}^p(\mathcal{K}_p)$*, be the Dirichlet divisor of degree p associated with* a*,* b *defined according to* (3.6)*, that is,*

$$\hat\mu_j(n) = \big(\mu_j(n), -G_{p+1}(\mu_j(n),n)\big) \in \mathcal{K}_p, \quad j = 1,\dots,p,\ n \in \mathbb{Z}. \tag{3.27}$$

Then $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is nonspecial for all $n \in \mathbb{Z}$. Moreover, the Abel map linearizes the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) - (n - n_0)\underline{A}_{P_{\infty_-}}(P_{\infty_+}), \tag{3.28}$$

where $Q_0 \in \mathcal{K}_p$ is a given base point.
If in addition, $a, b \in \ell^\infty(\mathbb{Z})$, then there exists a constant $C_\mu > 0$ such that

$$|\mu_j(n)| \leq C_\mu, \quad j = 1, \dots, p, \ n \in \mathbb{Z}. \tag{3.29}$$

Remark 3.5. *We note that by construction, the divisors $\mathcal{D}_{\underline{\hat{\mu}}(n)}$, $n \in \mathbb{Z}$, as introduced in* (3.6) *are all finite and hence nonspecial by Lemma 3.4. On the other hand, as we will see in the next Section 4, given a nonspecial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$, the solution $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ of equation* (3.28) *may cease to be a finite divisor at some $n \in \mathbb{Z}$.*

4. An Algorithm for Solving the Inverse Algebro-Geometric Spectral Problem for (Non-self-adjoint) Jacobi Operators

The aim of this section is to derive an algorithm that enables one to construct algebro-geometric solutions for the stationary Toda hierarchy for complex-valued initial data. Equivalently, we offer a solution of the inverse algebro-geometric spectral problem starting with initial divisors in general complex position.

Up to the end of Section 3 the material was standard (see [6] and [14, Sect. 1.3], [32, Chs. 8, 9] for details) and based on the assumption that $a, b \in \mathbb{C}^{\mathbb{Z}}$ satisfy the pth stationary Toda equation (2.20). Now we embark on the corresponding inverse problem consisting of constructing a solution of (2.20) given certain initial data. More precisely, we seek to construct solutions $a, b \in \mathbb{C}^{\mathbb{Z}}$ satisfying the pth stationary Toda equation (2.20) starting from a properly restricted set $\mathcal{M}_0$ of finite nonspecial Dirichlet divisor initial data $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ at some fixed $n_0 \in \mathbb{Z}$,

$$\begin{aligned} &\underline{\hat{\mu}}(n_0) = \{\hat{\mu}_1(n_0), \dots, \hat{\mu}_p(n_0)\} \in \mathcal{M}_0, \quad \mathcal{M}_0 \subset \mathrm{Sym}^p(\mathcal{K}_p), \\ &\hat{\mu}_j(n_0) = \big(\mu_j(n_0), -G_{p+1}(\mu_j(n_0), n_0)\big), \quad j = 1, \dots, p. \end{aligned} \tag{4.1}$$

Of course we would like to ensure that the sequences obtained via our algorithm do not blow up. To investigate when this happens, we study the image of our divisors under the Abel map. The key ingredient in our analysis will be (3.28) which yields a linear discrete dynamical system on the Jacobi variety $J(\mathcal{K}_p)$. In particular, we will be led to investigate solutions $\mathcal{D}_{\underline{\hat{\mu}}}$ of the discrete initial value problem

$$\begin{aligned} &\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) - (n - n_0)\underline{A}_{P_{\infty_-}}(P_{\infty_+}), \\ &\underline{\hat{\mu}}(n_0) = \{\hat{\mu}_1(n_0), \dots, \hat{\mu}_p(n_0)\} \in \mathrm{Sym}^p(\mathcal{K}_p), \end{aligned} \tag{4.2}$$

where $Q_0 \in \mathcal{K}_p$ is a given base point. Eventually, we will be interested in solutions $\mathcal{D}_{\underline{\hat{\mu}}}$ of (4.2) with initial data $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ satisfying (4.1) and $\mathcal{M}_0$ to be specified as in (the proof of) Lemma 4.2.

Before proceeding to develop the stationary Toda algorithm, we briefly analyze the dynamics of (4.2).

Lemma 4.1. *Let $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ be defined via* (4.2) *for some divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \mathrm{Sym}^p(\mathcal{K}_p)$.*
(i) If $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is finite and nonspecial and $\mathcal{D}_{\underline{\hat{\mu}}(n+1)}$ is infinite, then $\mathcal{D}_{\underline{\hat{\mu}}(n+1)}$ contains P_{∞_+} but not P_{∞_-}.
(ii) If $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is nonspecial and $\mathcal{D}_{\underline{\hat{\mu}}(n+1)}$ is special, then $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ contains P_{∞_+} at least

twice.
Items (i) and (ii) hold if $n+1$ is replaced by $n-1$ and P_{∞_+} by P_{∞_-}.

Proof. (i) Suppose one point in $\mathcal{D}_{\underline{\hat{\mu}}(n+1)}$ equals P_{∞_-} and denote the remaining ones by $\mathcal{D}_{\underline{\tilde{\mu}}(n+1)}$. Then (4.2) tells us $\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\tilde{\mu}}(n+1)}) + \underline{A}_{P_0}(P_{\infty_+}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)})$. Since we assumed $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ to be nonspecial, we have $\mathcal{D}_{\underline{\hat{\mu}}(n)} = \mathcal{D}_{\underline{\tilde{\mu}}(n+1)} + \mathcal{D}_{P_{\infty_+}}$ contradicting finiteness of $\mathcal{D}_{\underline{\hat{\mu}}(n)}$.
(ii) We choose P_0 to be a branch point such that $\underline{A}_{P_0}(P^*) = -\underline{A}_{P_0}(P)$. In particular, if $\mathcal{D}_{\underline{\hat{\mu}}(n+1)}$ is special, then it contains a pair of points (Q, Q^*) whose contribution will cancel under the Abel map, that is, $\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(n+1)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(n+1)})$ for some $\mathcal{D}_{\underline{\hat{\nu}}(n+1)} \in \mathrm{Sym}^{p-2}(\mathcal{K}_p)$. But invoking (4.2) shows that $\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(n+1)}) + 2\underline{A}_{P_0}(P_{\infty_+})$. As $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ was assumed nonspecial, this shows that $\mathcal{D}_{\underline{\hat{\mu}}(n+1)} = \mathcal{D}_{\underline{\hat{\nu}}(n+1)} + 2\mathcal{D}_{P_{\infty_+}}$ as claimed. □

This yields the following behavior of $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ if we start with some nonspecial finite initial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$: As n increases, $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ stays nonspecial as long as it remains finite. If it becomes infinite, then it is still nonspecial and contains P_{∞_+} at least once (but not P_{∞_-}). Further increasing n, all instances of P_{∞_+} will be rendered into P_{∞_-} step by step, until we have again a nonspecial divisor that has the same number of P_{∞_-} as the first infinite one had P_{∞_+}. Generically, we expect the subsequent divisor to be finite and nonspecial again.

Next we show that most initial divisors are nice in the sense that their iterates stay away from infinity. Since we want to show that this set is of full measure, it will be convenient for us to identify $\mathrm{Sym}^p(\mathcal{K}_p)$ with the Jacobi variety $J(\mathcal{K}_p)$ via the Abel map and take the Haar measure on $J(\mathcal{K}_p)$. Of course, the Abel map is only injective when restricted to the set of nonspecial divisors, but these are the only ones we are interested in.

Lemma 4.2. *The set $\mathcal{M}_0 \subset \mathrm{Sym}^p(\mathcal{K}_p)$ of initial divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ for which $\mathcal{D}_{\underline{\hat{\mu}}(n)}$, defined via (4.2), is finite and hence nonspecial for all $n \in \mathbb{Z}$, forms a dense set of full measure in the set $\mathrm{Sym}^p(\mathcal{K}_p)$ of nonnegative divisors of degree p.*

Proof. Let $\mathcal{M}_\infty$ be the set of divisors in $\mathrm{Sym}^p(\mathcal{K}_p)$ for which (at least) one point is equal to P_{∞_+}. The image $\underline{\alpha}_{P_0}(\mathcal{M}_\infty)$ of $\mathcal{M}_\infty$ is given by

$$\underline{\alpha}_{P_0}(\mathcal{M}_\infty) = \underline{A}_{P_0}(P_{\infty_+}) + \underline{\alpha}_{P_0}(\mathrm{Sym}^{p-1}(\mathcal{K}_p)) \subset J(\mathcal{K}_p). \tag{4.3}$$

Since the (complex) dimension of $\mathrm{Sym}^{p-1}(\mathcal{K}_p)$ is $p-1$, its image must be of measure zero by Sard's theorem (see, e.g., [1, Sect. 3.6]). Similarly, let $\mathcal{M}_{\mathrm{sp}}$ be the set of special divisors, then its image is given by

$$\underline{\alpha}_{P_0}(\mathcal{M}_{\mathrm{sp}}) = \underline{\alpha}_{P_0}(\mathrm{Sym}^{p-2}(\mathcal{K}_p)), \tag{4.4}$$

assuming P_0 to be a branch point. In particular, we conclude that $\underline{\alpha}_{P_0}(\mathcal{M}_{\mathrm{sp}}) \subset \underline{\alpha}_{P_0}(\mathcal{M}_\infty)$ and thus $\underline{\alpha}_{P_0}(\mathcal{M}_{\mathrm{sing}}) = \underline{\alpha}_{P_0}(\mathcal{M}_\infty)$ has measure zero, where

$$\mathcal{M}_{\mathrm{sing}} = \mathcal{M}_\infty \cup \mathcal{M}_{\mathrm{sp}}. \tag{4.5}$$

Hence,

$$\bigcup_{n\in\mathbb{Z}} \big(\underline{\alpha}_{P_0}(\mathcal{M}_{\mathrm{sing}}) + n\underline{A}_{P_{\infty_-}}(P_{\infty_+})\big) \tag{4.6}$$

is of measure zero as well. But this last set contains all initial divisors which will hit P_{∞_+} or become special at some n. We denote by $\mathcal{M}_0$ the inverse image of the complement of the set (4.6) under the Abel map,

$$\mathcal{M}_0 = \underline{\alpha}_{P_0}^{-1}\bigg(\mathrm{Sym}^p(\mathcal{K}_p)\Big\backslash \bigcup_{n\in\mathbb{Z}} \big(\underline{\alpha}_{P_0}(\mathcal{M}_{\mathrm{sing}}) + n\underline{A}_{P_{\infty_-}}(P_{\infty_+})\big)\bigg). \tag{4.7}$$

Since $\mathcal{M}_0$ is of full measure, it is automatically dense in $\mathrm{Sym}^p(\mathcal{K}_p)$. $\square$

We briefly illustrate some aspects of this analysis in the special case $p = 1$ (i.e., the case case where (3.1) represents an elliptic Riemann surface) in more detail.

Example 4.3. The case $p = 1$.
In this case we have

$$\begin{aligned} F_1(z,n) &= z - \mu_1(n), \\ G_2(z,n) &= R_4(\hat\mu_1(n))^{1/2} + (z - b(n))F_1(z,n), \\ R_4(z) &= \prod_{m=0}^{3}(z - E_m), \end{aligned} \tag{4.8}$$

and hence a quick calculations shows that

$$\begin{aligned} G_2(z,n)^2 - R_4(z) &= 4a(n)^2(z - \mu_1(n))(z - \mu_1(n+1)) \\ &= (z - \mu_1(n))(4a(n)^2 z - 4a(n)^2 b(n) + \tilde{E}), \end{aligned} \tag{4.9}$$

where

$$\tilde{E} = \frac{1}{8}(E_0 + E_1 - E_2 - E_3)(E_0 - E_1 + E_2 - E_3)(E_0 - E_1 - E_2 + E_3). \tag{4.10}$$

Solving for $\mu_1(n+1)$, one obtains

$$\mu_1(n+1) = b(n) - \frac{\tilde{E}}{4a(n)^2}. \tag{4.11}$$

This shows that $\mu_1(n_0+1) \to \infty$, in fact, $\mu_1(n_0+1) = O(a(n_0)^{-2})$ as $a(n_0) \to 0$ during an appropriate deformation of the parameters E_m, $m = 0,\dots,3$. In particular, as $a(n_0) \to 0$, one thus infers $b(n_0+1) \to \infty$ during such a deformation since

$$b(n) = \frac{1}{2}\sum_{m=0}^{3} E_m - \mu_1(n), \quad n \in \mathbb{Z}, \tag{4.12}$$

specializing to $p = 1$ in the trace formula (3.25). Next, we illustrate the set $\mathcal{M}_\infty$ in the case $p = 1$. (We recall that $\mathcal{M}_{\mathrm{sp}} = \emptyset$ and hence $\mathcal{M}_{\mathrm{sing}} = \mathcal{M}_\infty$ if $p = 1$.) By (4.2) one infers

$$A_{P_{\infty_+}}(\hat\mu_1(n)) = A_{P_{\infty_+}}(\hat\mu_1(n_0)) + (n - n_0)A_{P_{\infty_+}}(P_{\infty_-}), \quad n, n_0 \in \mathbb{Z}. \tag{4.13}$$

We note that $\hat\mu_1 \in \mathcal{M}_\infty$ is equivalent to

$$\text{there is an } n \in \mathbb{Z} \text{ such that } \hat\mu_1(n) = P_{\infty_+} \text{ (or } P_{\infty_-}). \tag{4.14}$$

By (4.13), relation (4.14) is equivalent to

$$A_{P_{\infty_+}}(\hat\mu_1(n_0)) + A_{P_{\infty_+}}(P_{\infty_-})\,\mathbb{Z} = 0 \pmod{L_1}. \tag{4.15}$$

Thus, $\mathcal{D}_{\hat\mu_1(n_0)} \in \mathcal{M}_0 \subset \mathcal{K}_1$ if and only if

$$A_{P_{\infty_+}}(\hat\mu_1(n_0)) + A_{P_{\infty_+}}(P_{\infty_-})\,\mathbb{Z} \neq 0 \pmod{L_1} \tag{4.16}$$

or equivalently, if and only if

$$A_{P_{\infty_-}}(\hat{\mu}_1(n_0)) + A_{P_{\infty_-}}(P_{\infty_+})\,\mathbb{Z} \neq 0 \pmod{L_1}. \tag{4.17}$$

Next, we describe the stationary Toda algorithm. Since this is a somewhat lengthy affair, we will break it up into several steps.

The Stationary (Complex) Today Algorithm:

We prescribe the following data:

(i) The set

$$\{E_m\}_{m=0}^{2p+1} \subset \mathbb{C}, \quad E_m \neq E_{m'} \text{ for } m \neq m',\ m, m' = 0, \dots, 2p+1 \tag{4.18}$$

for some fixed $p \in \mathbb{N}$. Given $\{E_m\}_{m=0}^{2p+1}$, we introduce the function R_{2p+2} and the (nonsingular) hyperelliptic curve $\mathcal{K}_p$ as in (3.1).

(ii) The nonspecial divisor

$$\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \operatorname{Sym}^p(\mathcal{K}_p), \tag{4.19}$$

where $\underline{\hat{\mu}}(n_0)$ is of the form

$$\begin{aligned} \underline{\hat{\mu}}(n_0) &= \{\hat{\mu}_1(n_0), \dots, \hat{\mu}_p(n_0)\} \\ &= \{\underbrace{\hat{\mu}_1(n_0), \dots, \hat{\mu}_1(n_0)}_{p_1(n_0) \text{ times}}, \dots, \underbrace{\hat{\mu}_{q(n_0)}, \dots, \hat{\mu}_{q(n_0)}}_{p_{q(n_0)}(n_0) \text{ times}}\} \end{aligned} \tag{4.20}$$

with

$$\hat{\mu}_k(n_0) = (\mu_k(n_0), y(\hat{\mu}_k(n_0))), \quad \mu_k(n_0) \neq \mu_{k'}(n_0) \text{ for } k \neq k',\ k, k' = 1, \dots, q(n_0) \tag{4.21}$$

and

$$p_k(n_0) \in \mathbb{N},\ k = 1, \dots, q(n_0), \quad \sum_{k=1}^{q(n_0)} p_k(n_0) = p. \tag{4.22}$$

With $\{E_m\}_{m=0}^{2p+1}$ and $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ prescribed, we next introduce the following quantities (for $z \in \mathbb{C}$):

$$F_p(z, n_0) = \prod_{k=1}^{q(n_0)} (z - \mu_k(n_0))^{p_k(n_0)}, \tag{4.23}$$

$$\begin{aligned} T_{p-1}(z, n_0) = -F_p(z, n_0) \sum_{k=1}^{q(n_0)} \sum_{\ell=0}^{p_k(n_0)-1} \frac{\big(d^\ell \big(R_{2p+2}(\zeta)^{1/2}\big)/d\zeta^\ell\big)\big|_{\zeta=\mu_k(n_0)}}{\ell!(p_k(n_0) - \ell - 1)!} \\ \times \left(\frac{d^{p_k(n_0)-\ell-1}}{d\zeta^{p_k(n_0)-\ell-1}} \left((z-\zeta)^{-1} \prod_{k'=1,\, k' \neq k}^{q(n_0)} (\zeta - \mu_{k'}(n_0))^{-p_{k'}(n_0)} \right) \right) \Bigg|_{\zeta=\mu_k(n_0)}, \end{aligned} \tag{4.24}$$

$$b(n_0) = \frac{1}{2} \sum_{m=0}^{2p+1} E_m - \sum_{k=1}^{q(n_0)} p_k(n_0)\mu_k(n_0), \tag{4.25}$$

$$G_{p+1}(z, n_0) = -(z - b(n_0))F_p(z, n_0) + T_{p-1}(z, n_0). \tag{4.26}$$

Here the sign of the square root in (4.24) is chosen according to (4.21),

$$\hat{\mu}_k(n_0) = (\mu_k(n_0), y(\hat{\mu}_k(n_0))) = \big(\mu_k(n_0), R_{2p+2}(\mu_k(n_0))^{1/2}\big), \quad k = 1, \dots, q(n_0). \tag{4.27}$$

Next we record a series of facts:
(I) By construction (cf. Lemma B.1),

$$T_{p-1}^{(k)}(\mu_k(n_0), n_0) = -\frac{d^\ell \big(R_{2p+2}(z)^{1/2}\big)}{dz^\ell}\bigg|_{z=\mu_k(n_0)} = G_{p+1}^{(k)}(\mu_k(n_0), n_0), \quad \ell = 0, \dots, p_k(n_0)-1, \ k = 1, \dots, q(n_0), \tag{4.28}$$

(here the superscript (k) denotes k derivatives w.r.t. z) and hence

$$\hat{\mu}_k(n_0) = (\mu_k(n_0), -G_{p+1}(\mu_k(n_0), n_0)), \quad k = 1, \dots, q(n_0). \tag{4.29}$$

(II) Since $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ is nonspecial by hypothesis, one concludes that

$$p_k(n_0) \geq 2 \ \text{implies} \ R_{2p+2}(\mu_k(n_0)) \neq 0, \quad k = 1, \dots, q(n_0). \tag{4.30}$$

(III) By **(I)** and **(II)** one computes

$$\frac{d^\ell \big(G_{p+1}(z, n_0)^2\big)}{dz^\ell}\bigg|_{z=\mu_k(n_0)} = \frac{d^\ell R_{2p+2}(z)}{dz^\ell}\bigg|_{z=\mu_k(n_0)}, \quad z \in \mathbb{C}, \quad \ell = 0, \dots, p_k(n_0)-1, \ k = 1, \dots, q(n_0). \tag{4.31}$$

(IV) By (4.26) and (4.31) one infers that F_p divides $R_{2p+2} - G_{p+1}^2$.
(V) By (4.25) and (4.26) one verifies that

$$R_{2p+2}(z) - G_{p+1}(z, n_0)^2 \underset{z\to\infty}{=} O(z^{2p}). \tag{4.32}$$

By **(IV)** and (4.32) we may write

$$R_{2p+2}(z) - G_{p+1}(z, n_0)^2 = F_p(z, n_0)\check{F}_{p-r}(z, n_0+1), \quad z \in \mathbb{C}, \tag{4.33}$$

for some $r \in \{0, \dots, p\}$, where the polynomial $\check{F}_{p-r}$ has degree $p-r$. If in fact $\check{F}_0 = 0$, then $R_{2p+2}(z) = G_{p+1}(z, n_0)^2$ would yield double zeros of R_{2p+2}, contradicting our basic hypothesis (4.18). Thus we conclude that in the case $r = p$, $\check{F}_0$ cannot vanish identically and hence we may break up (4.33) in the following manner

$$\check{\phi}(P, n_0) = \frac{y - G_{p+1}(z, n_0)}{F_p(z, n_0)} = \frac{\check{F}_{p-r}(z, n_0+1)}{y + G_{p+1}(z, n_0)}, \quad P = (z, y) \in \mathcal{K}_p. \tag{4.34}$$

Next we decompose

$$\check{F}_{p-r}(z, n_0+1) = \check{C} \prod_{j=1}^{p-r} (z - \mu_j(n_0+1)), \quad z \in \mathbb{C}, \tag{4.35}$$

where $\check{C} \in \mathbb{C}\backslash\{0\}$ and $\{\mu_j(n_0+1)\}_{j=1}^{p-r} \subset \mathbb{C}$ (if $r = p$ we follow the usual convention and replace the product in (4.35) by 1). By inspection of the local zeros and poles as well as the behavior near $P_{\infty_\pm}$ of the function $\check{\phi}(\,\cdot\,, n_0)$, its divisor, $\big(\check{\phi}(\,\cdot\,, n_0)\big)$, is given by

$$\big(\check{\phi}(\,\cdot\,, n_0)\big) = \mathcal{D}_{P_{\infty_+}\underline{\hat{\mu}}(n_0+1)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n_0)}, \tag{4.36}$$

where

$$\underline{\hat{\mu}}(n_0+1) = \{\hat{\mu}_1(n_0+1), \dots, \hat{\mu}_{p-r}(n_0+1), \underbrace{P_{\infty_+}, \dots, P_{\infty_+}}_{r \text{ times}}\}. \tag{4.37}$$

In particular,

$$\mathcal{D}_{\underline{\hat{\mu}}(n_0+1)} \ \text{is a finite divisor if and only if} \ r = 0. \tag{4.38}$$

We note that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0+1)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) - \underline{A}_{P_{\infty_-}}(P_{\infty_+}), \tag{4.39}$$

in accordance with (4.2).

(VI) *Assuming* that (4.32) is precisely of order z^{2p} as $z \to \infty$, that is, assuming $r = 0$ in (4.33), we rewrite (4.33) in the more appropriate manner

$$R_{2p+2}(z) - G_{p+1}(z, n_0)^2 = -4a(n_0)^2 F_p(z, n_0) F_p(z, n_0 + 1), \quad z \in \mathbb{C}, \tag{4.40}$$

where we introduced the coefficient $a(n_0)^2$ to make $F_p(\,\cdot\,, n_0+1)$ a monic polynomial of degree p. (We will later discuss conditions which indeed guarantee that $r = 0$, cf. (4.38) and the discussion in step **(XI)** below.) By construction, $F_p(\,\cdot\,, n_0 + 1)$ is then of the type

$$\begin{gathered} F_p(z, n_0 + 1) = \prod_{k=1}^{q(n_0+1)} (z - \mu_k(n_0 + 1))^{p_k(n_0+1)}, \quad \sum_{k=1}^{q(n_0+1)} p_k(n_0 + 1) = p, \\ \mu_k(n_0 + 1) \neq \mu_{k'}(n_0 + 1) \text{ for } k \neq k', \; k, k' = 1, \dots, q(n_0 + 1), \;\; z \in \mathbb{C}, \end{gathered} \tag{4.41}$$

and we *define*

$$\hat{\mu}_k(n_0 + 1) = (\mu_k(n_0 + 1), G_{p+1}(\mu_k(n_0 + 1), n_0)), \quad k = 1, \dots, q(n_0 + 1). \tag{4.42}$$

Moreover, we introduce the divisor

$$\mathcal{D}_{\underline{\hat{\mu}}(n_0+1)} \in \operatorname{Sym}^p(\mathcal{K}_p) \tag{4.43}$$

by

$$\begin{aligned} \underline{\hat{\mu}}(n_0 + 1) &= \{\hat{\mu}_1(n_0 + 1), \dots, \hat{\mu}_p(n_0 + 1)\} \\ &= \{\underbrace{\hat{\mu}_1(n_0 + 1), \dots, \hat{\mu}_1(n_0 + 1)}_{p_1(n_0+1) \text{ times}}, \dots, \underbrace{\hat{\mu}_{q(n_0+1)}, \dots, \hat{\mu}_{q(n_0+1)}}_{p_{q(n_0+1)}(n_0+1) \text{ times}}\}. \end{aligned} \tag{4.44}$$

In particular, because of the definition (4.42), $\mathcal{D}_{\underline{\hat{\mu}}(n_0+1)}$ is nonspecial and hence

$$p_k(n_0 + 1) \geq 2 \text{ implies } R_{2p+2}(\mu_k(n_0 + 1)) \neq 0, \quad k = 1, \dots, q(n_0 + 1). \tag{4.45}$$

Again we note that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0+1)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) - \underline{A}_{P_{\infty_-}}(P_{\infty_+}), \tag{4.46}$$

in accordance with (4.2).

(VII) Introducing

$$b(n_0 + 1) = \frac{1}{2} \sum_{m=0}^{2p+1} E_m - \sum_{k=1}^{q(n_0+1)} p_k(n_0 + 1)\mu_k(n_0 + 1), \tag{4.47}$$

and interpolating $G_{p+1}(\,\cdot\,, n_0)$ with $F_p(\,\cdot\,, n_0 + 1)$ rather than $F_p(\,\cdot\,, n_0)$ yields

$$G_{p+1}(z, n_0) = -(z - b(n_0 + 1))F_p(z, n_0 + 1) - T_{p-1}(z, n_0 + 1), \quad z \in \mathbb{C}, \tag{4.48}$$

where

$$\begin{aligned} &T_{p-1}(z, n_0 + 1) = F_p(z, n_0 + 1) \\ &\times \sum_{k=1}^{q(n_0+1)} \sum_{\ell=0}^{p_k(n_0+1)-1} \frac{\big(d^\ell \big(R_{2p+2}(\zeta)^{1/2}\big)/d\zeta^\ell\big)\big|_{\zeta=\mu_k(n_0+1)}}{\ell!(p_k(n_0 + 1) - \ell - 1)!} \end{aligned} \tag{4.49}$$

$$\times \left(\frac{d^{p_k(n_0+1)-\ell-1}}{d\zeta^{p_k(n_0+1)-\ell-1}} \left((z-\zeta)^{-1} \prod_{k'=1,\, k'\neq k}^{q(n_0+1)} (\zeta - \mu_{k'}(n_0+1))^{-p_{k'}(n_0+1)} \right) \right) \Big|_{\zeta=\mu_k(n_0+1)}.$$

Here the sign of the square root in (4.49) is chosen in accordance with (4.42), that is,

$$\begin{aligned} \hat{\mu}_k(n_0+1) &= (\mu_k(n_0+1), y(\hat{\mu}_k(n_0+1))) \\ &= (\mu_k(n_0+1), G_{p+1}(\mu_k(n_0+1), n_0)) \\ &= \big(\mu_k(n_0+1), R_{2p+2}(\mu_k(n_0+1))^{1/2}\big), \quad k = 1, \dots, q(n_0+1). \end{aligned} \tag{4.50}$$

(VIII) An explicit computation of $a(n_0)^2$ then yields

$$\begin{aligned} a(n_0)^2 = \frac{1}{2} \sum_{k=1}^{q(n_0)} & \frac{\big(d^\ell\big(R_{2p+2}(z)^{1/2}\big)/dz^\ell\big)\big|_{z=\mu_k(n_0)}}{(p_k(n_0)-1)!} \\ & \times \prod_{k'=1,\, k'\neq k}^{q(n_0)} (\mu_k(n_0) - \mu_{k'}(n_0))^{-p_k(n_0)} + \frac{1}{4}\big(b^{(2)}(n_0) - b(n_0)^2\big). \end{aligned} \tag{4.51}$$

Here and in the following we abbreviate

$$b^{(2)}(n) = \frac{1}{2} \sum_{m=0}^{2p+1} E_m^2 - \sum_{k=1}^{q(n)} p_k(n) \mu_k(n)^2 \tag{4.52}$$

for an appropriate range of $n \in \mathbb{N}$.

The result (4.51) is obtained as follows: One starts from the identity (4.40), inserts the expressions (4.23) and (4.26) for $F_p(\,\cdot\,, n_0)$ and $G_{p+1}(\,\cdot\,, n_0)$, respectively, then inserts the explicit form (4.24) of $T_{p-1}(\,\cdot\,, n_0)$, and finally collects all terms of order z^{2p} as $z \to \infty$. An entirely elementary but fairly tedious calculation then produces (4.51).

In the special case $q(n_0) = p$, $p_k(n_0) = 1$, $k = 1, \dots, p$, (4.51) and (4.52) reduce to (3.26) and (3.24) (for $k = 2$).

(IX) Introducing

$$G_{p+1}(z, n_0+1) = -(z - b(n_0+1))F_p(z, n_0+1) + T_{p-1}(z, n_0+1) \tag{4.53}$$

one then obtains

$$G_{p+1}(z, n_0+1) = -G_{p+1}(z, n_0) - 2(z - b(n_0+1))F_p(z, n_0+1). \tag{4.54}$$

(X) At this point one can iterate the procedure step by step to construct $F_p(\,\cdot\,, n)$, $G_{p+1}(\,\cdot\,, n)$, $T_{p-1}(\,\cdot\,, n)$, $a(n)$, $b(n)$, $\mu_k(n)$, $k = 1, \dots, q(n)$, etc., for $n \in [n_0, \infty) \cap \mathbb{Z}$, subject to the following assumption (cf. (4.38)) at each step:

$$\mathcal{D}_{\underline{\hat{\mu}}(n+1)} \text{ is a finite divisor (and hence } a(n) \neq 0) \text{ for all } n \in [n_0, \infty) \cap \mathbb{Z}. \tag{4.55}$$

The formalism is symmetric with respect to n_0 and can equally well be developed for $n \in (-\infty, n_0] \cap \mathbb{Z}$ subject to the analogous assumption

$$\mathcal{D}_{\underline{\hat{\mu}}(n-1)} \text{ is a finite divisor (and hence } a(n) \neq 0) \text{ for all } n \in (-\infty, n_0] \cap \mathbb{Z}. \tag{4.56}$$

Indeed, one first interpolates $G_{p+1}(\,\cdot\,, n_0-1)$ with the help of $F_p(\,\cdot\,, n_0)$, then with $F_p(\,\cdot\,, n_0-1)$, etc.

Moreover, we once again remark for consistency reasons that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) - (n - n_0)\underline{A}_{P_{\infty_-}}(P_{\infty_+}), \quad n \in \mathbb{Z}, \tag{4.57}$$

in agreement with our starting point (4.2).
(XI) Choosing the initial data $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ such that

$$\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \mathcal{M}_0, \tag{4.58}$$

where $\mathcal{M}_0 \subset \operatorname{Sym}^p(\mathcal{K}_p)$ is the set of finite divisors introduced in Lemma 4.2, then guarantees that assumptions (4.55) and (4.56) are satisfied for all $n \in \mathbb{Z}$.
(XII) Performing these iterations for all $n \in \mathbb{Z}$, one then arrives at the following set of equations for F_p and G_{p+1} after the following elementary manipulations: Utilizing

$$G_{p+1}^2 - 4a^2 F_p F_p^+ = R_{2p+2} = (G_{p+1}^-)^2 - 4(a^-)^2 F_p^- F_p, \tag{4.59}$$

and inserting

$$G_{p+1}^+ = -G_{p+1} - 2(z-b^+)F_p^+ \tag{4.60}$$

into

$$G_{p+1}^2 - (G_{p+1}^-)^2 - 4a^2 F_p F_p^+ + 4(a^-)^2 F_p^- F_p = 0 \tag{4.61}$$

then yields

$$2a^2 F_p^+ - 2(a^-)^2 F_p^- + (z-b)(G_{p+1} - G_{p+1}^-) = 0. \tag{4.62}$$

Subtracting (4.60) from its shifted version $G_{p+1} = -G_{p+1}^- - 2(z-b)F_p$ then also yields

$$2(z-b^+)F_p^+ - 2(z-b)F_p + G_{p+1}^+ - G_{p+1}^- = 0. \tag{4.63}$$

As discussed in Section 2, (4.62) and (4.63) are equivalent to the stationary Lax and zero-curvature equations (2.15) and (2.60). At this stage we have verified the basic hypotheses of Section 3 (i.e., Hypothesis 3.1 and the assumption that a, b satisfy the pth stationary Toda equation (2.20)) and hence all results of Section 3 apply.

Finally, we briefly summarize these considerations:

Theorem 4.4. *Suppose the set $\{E_m\}_{m=0}^{2p+1} \subset \mathbb{C}$ satisfies $E_m \neq E_{m'}$ for $m \neq m'$, $m, m' = 0, \dots, 2p+1$, and introduce the function R_{2p+2} and the hyperelliptic curve $\mathcal{K}_p$ as in* (3.1). *Choose a nonspecial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \mathcal{M}_0$, where $\mathcal{M}_0 \subset \operatorname{Sym}^p(\mathcal{K}_p)$ is the set of finite divisors introduced in Lemma 4.2. Then the stationary* (*complex*) *Toda algorithm as outlined in steps* (***I***)–(***XII***) *produces solutions a, b of the pth stationary Toda equation,*

$$\text{s-Tl}_p(a,b) = \begin{pmatrix} f_{p+1}^+ - f_{p+1} \\ g_{p+1} - g_{p+1}^- \end{pmatrix} = 0, \quad p \in \mathbb{N}_0, \tag{4.64}$$

satisfying (3.3) *and*

$$\begin{aligned} a(n)^2 = \frac{1}{2} \sum_{k=1}^{q(n)} & \frac{\big(d^\ell \big(R_{2p+2}(z)^{1/2}\big)/dz^\ell\big)\big|_{z=\mu_k(n)}}{(p_k(n)-1)!} \\ & \times \prod_{k'=1,\, k'\neq k}^{q(n)} (\mu_k(n) - \mu_{k'}(n))^{-p_k(n)} + \frac{1}{4}\big(b^{(2)}(n) - b(n)^2\big), \end{aligned} \tag{4.65}$$

$$b(n) = \frac{1}{2} \sum_{m=0}^{2p+1} E_m - \sum_{k=1}^{q(n)} p_k(n)\mu_k(n), \quad n \in \mathbb{Z}. \tag{4.66}$$

Moreover, Lemmas 3.2–3.4 apply.

Theta function representations of a and b can now be derived in complete analogy to the self-adjoint case. Since the final results are formally the same as in the self-adjoint case we just refer, for instance, to [6], [7], [9], [10], [14, Sect. 1.3], [18], [19], [20], [21] (cf. also the appendix written in [8]), [25], [30, Appendix, Sect. 9], [32, Sect. 9.2], [33, Sect. 4.5].

The stationary (complex) Toda algorithm as outlined in steps (**I**)–(**XII**), starting from a nonspecial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \mathcal{M}_0$, represents a solution of the inverse algebro-geometric spectral problem for generally non-self-adjoint Jacobi operators. While we do not assume periodicity (or even quasi-periodicity), let alone real-valuedness of the coefficients of the underlying Jacobi operator, once can view this algorithm a continuation of the inverse periodic spectral problem started in 1975 (in the self-adjoint context) by Kac and van Moerbeke [15], [16] and Flaschka [12], continued in the seminal papers by van Moerbeke [24], Date and Tanaka [7], and Dubrovin, Matveev, and Novikov [10], and further developed by Krichever [18], McKean [23], van Moerbeke and Mumford [25], Mumford [26], and others, in part in the more general quasiperiodic algebro-geometric case.

We note that in general (i.e., unless one is, e.g., in the special periodic or self-adjoint case), $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ will get arbitrarily close to P_{∞_+} since straight motions on the torus are generically dense (see e.g. [2, Sect. 51] or [17, Sects. 1.4, 1.5]). Thus, no uniform bound on the sequences $a(n), b(n)$ exists as n varies in $\mathbb{Z}$. In particular, these complex-valued algebro-geometric solutions of some of the equations of the stationary Toda hierarchy, generally, will not be quasiperiodic (cf. the usual definition of quasiperiodic functions, e.g., in [31, p. 31]) with respect to n. For the special case of complex-valued and quasi-periodic Jacobi matrices where all quasiperiods are real-valued, we refer to [4] (cf. also [3]).

5. Properties of Algebro-Geometric Solutions of the Time-Dependent Toda Hierarchy

In this section we present a quick review of properties of algebro-geometric solutions of the time-dependent Toda hierarchy. Since this material is standard we omit all proofs and just refer to [6] (cf. also [14, Sect. 1.4], [32, Chs. 12, 13]) for detailed presentations and an extensive list references to the literature.

Throughout this section we will make the following assumption:

Hypothesis 5.1. *Assume that* $a\colon \mathbb{Z}\times\mathbb{R} \to \mathbb{C}\setminus\{0\}$ *and* $b\colon \mathbb{Z}\times\mathbb{R}\to\mathbb{C}$ *satisfy*

$$a(\,\cdot\,,t), b(\,\cdot\,,t) \in \mathbb{C}^{\mathbb{Z}},\ t\in\mathbb{R}, \quad a(n,\,\cdot\,), b(n,\,\cdot\,)\in C^1(\mathbb{R}),\ n\in\mathbb{Z}, \tag{5.1}$$

and assume that the hyperelliptic curve $\mathcal{K}_p$, $p\in\mathbb{N}_0$, *is nonsingular.*

In order to briefly analyze algebro-geometric solutions of the time-dependent Toda hierarchy we proceed as follows. Given $p\in\mathbb{N}_0$, consider a complex-valued solution $a^{(0)}, b^{(0)}$ of the pth stationary Toda equation $\text{s-Tl}_p(a,b)=0$, associated with $\mathcal{K}_p$ and a given set of summation constants $\{c_\ell\}_{\ell=1,\dots,p}\subset\mathbb{C}$. Next, let $r\in\mathbb{N}_0$; we intend to consider solutions $a=a(t_r), b=b(t_r)$ of the rth Tl flow $\text{Tl}_r(a,b)=0$ with $a(t_{0,r})=a^{(0)}, b(t_{0,r})=b^{(0)}$ for some $t_{0,r}\in\mathbb{R}$. To emphasize that the summation constants in the definitions of the stationary and the time-dependent Tl equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence we shall employ the notation $\widetilde{P}_{2r+2}$, $\widetilde{V}_{r+1}$, $\widetilde{F}_r$, $\widetilde{G}_{r+1}$, $\tilde{f}_s$, $\tilde{g}_s$, $\tilde{c}_s$, in order to distinguish them from P_{2p+2}, V_{p+1}, F_p, G_{p+1},

f_ℓ, g_ℓ, c_ℓ, in the following. In addition, we will follow a more elaborate notation inspired by Hirota's τ-function approach and indicate the individual rth Tl flow by a separate time variable $t_r \in \mathbb{R}$. More precisely, we will review properties of solutions a, b of the time-dependent algebro-geometric initial value problem

$$\begin{aligned} &\widetilde{\mathrm{Tl}}_r(a,b) = \begin{pmatrix} a_{t_r} - a\big(\tilde f_{p+1}^+(a,b) - \tilde f_{p+1}(a,b)\big) \\ b_{t_r} + \tilde g_{p+1}(a,b) - \tilde g_{p+1}^-(a,b) \end{pmatrix} = 0, \\ &(a,b)\big|_{t_r = t_{0,r}} = \big(a^{(0)}, b^{(0)}\big), \end{aligned} \tag{5.2}$$

$$\text{s-Tl}_p\left(a^{(0)}, b^{(0)}\right) = \begin{pmatrix} -a\big({f^+}_{p+1}\big(p^{(0)}, q^{(0)}\big) - f_{p+1}\big(p^{(0)}, q^{(0)}\big)\big) \\ g_{p+1}\big(a^{(0)}, b^{(0)}\big) - g_{p+1}^-\big(a^{(0)}, b^{(0)}\big) \end{pmatrix} = 0 \tag{5.3}$$

for some $t_{0,r} \in \mathbb{R}$, $p, r \in \mathbb{N}_0$, where $a = a(n,t_r), b = b(n,t_r)$ satisfy (5.1) and a fixed curve $\mathcal{K}_p$ is associated with the stationary solutions $a^{(0)}, b^{(0)}$ in (5.3). In terms of Lax pairs this amounts to solving

$$\frac{d}{dt_r} L(t_r) - [\widetilde P_{2r+2}(t_r), L(t_r)] = 0, \quad t_r \in \mathbb{R}, \tag{5.4}$$

$$[P_{2p+2}(t_{0,r}), L(t_{0,r})] = 0. \tag{5.5}$$

Anticipating that the Tl flows are isospectral deformations of $L(t_{0,r})$, we are going a step further replacing (5.5) by

$$[P_{2p+2}(t_r), L(t_r)] = 0, \quad t_r \in \mathbb{R}. \tag{5.6}$$

This then implies

$$P_{2p+2}(t_r)^2 = R_{2p+2}(L(t_r)) = \prod_{m=0}^{2p+1} (L(t_r) - E_m), \quad t_r \in \mathbb{R}. \tag{5.7}$$

Actually, instead of working with (5.4), (5.5), and (5.6), one can equivalently take the zero-curvature equations (2.63) as one's point of departure, that is, one can also start from

$$U_{t_r} + U\widetilde V_{r+1} - \widetilde V_{r+1}^+ U = 0, \tag{5.8}$$

$$UV_{p+1} - V_{p+1}^+ U = 0, \tag{5.9}$$

where (cf. (2.23), (2.24), (2.58), (2.59))

$$\begin{aligned} U(z) &= \begin{pmatrix} 0 & 1 \\ -a^-/a & (z-b)/a \end{pmatrix}, \\ V_{p+1}(z) &= \begin{pmatrix} G_{p+1}^-(z) & 2a^- F_p^-(z) \\ -2a^- F_p(z) & 2(z-b)F_p + G_{p+1}(z) \end{pmatrix}, \\ \widetilde V_{r+1}(z) &= \begin{pmatrix} \widetilde G_{r+1}^-(z) & 2a^- \widetilde F_r^-(z) \\ -2a^- \widetilde F_r(z) & 2(z-b)\widetilde F_r(z) + \widetilde G_{r+1}(z) \end{pmatrix}, \end{aligned} \tag{5.10}$$

and

$$F_p(z) = \sum_{\ell=0}^{p} f_{p-\ell} z^\ell = \prod_{j=1}^{p} (z - \mu_j), \quad f_0 = 1, \tag{5.11}$$

$$G_{p+1}(z) = -z^{p+1} + \sum_{\ell=0}^{p} g_{p-\ell} z^\ell + f_{p+1}, \quad g_0 = -c_1, \tag{5.12}$$

$$\widetilde{F}_r(z) = \sum_{s=0}^{r} \tilde{f}_{r-s} z^s, \quad \tilde{f}_0 = 1, \tag{5.13}$$

$$\widetilde{G}_{r+1}(z) = -z^{r+1} + \sum_{s=0}^{r} \tilde{g}_{r-s} z^s + \tilde{f}_{r+1}, \ \ \tilde{g}_0 = -\tilde{c}_1, \tag{5.14}$$

for fixed $p, r \in \mathbb{N}_0$. Here f_ℓ, $\tilde{f}_s$, g_ℓ, and $\tilde{g}_s$, $\ell = 0, \dots, p$, $s = 0, \dots, r$, are defined as in (2.4)–(2.6) with appropriate sets of summation constants. Explicitly, (5.8) and (5.9) are equivalent to (cf. (2.55), (2.56), (2.32), (2.33))

$$\begin{aligned}
a_{t_r} &= -a\big(2(z-b^+)\widetilde{F}_r^+ + \widetilde{G}_{r+1}^+ + \widetilde{G}_{r+1}\big), && (5.15)\\
b_{t_r} &= 2\big((z-b)^2\widetilde{F}_r + (z-b)\widetilde{G}_{r+1} + a^2\widetilde{F}_r^+ - (a^-)^2\widetilde{F}_r^-\big), && (5.16)\\
0 &= 2(z-b^+)F_p^+ + G_{p+1}^+ + G_{p+1}, && (5.17)\\
0 &= (z-b)^2 F_p + (z-b)G_{p+1} + a^2F_p^+ - (a^-)^2F_p^-, && (5.18)
\end{aligned}$$

respectively. In particular, (2.34) holds in the present t_r-dependent setting, that is,

$$G_{p+1}^2 - 4a^2 F_p F_p^+ = R_{2p+2}. \tag{5.19}$$

As in (3.6) one introduces

$$\hat{\mu}_j(n,t_r) = (\mu_j(n,t_r), -G_{p+1}(\mu_j(n,t_r),n,t_r)) \in \mathcal{K}_p, \quad j = 1,\dots,p, \ (n,t_r) \in \mathbb{Z}\times\mathbb{R} \tag{5.20}$$

and notes that the regularity assumptions (5.1) on a, b imply continuity of μ_j with respect to $t_r \in \mathbb{R}$ (away from collisions of zeros, μ_j are of course C^∞).

In analogy to (3.7), (3.8), one defines the meromorphic function $\phi(\,\cdot\,,n,t_r)$ on $\mathcal{K}_p$,

$$\begin{aligned}
\phi(P,n,t_r) &= \frac{y - G_{p+1}(z,n,t_r)}{2a(n,t_r)F_p(z,n,t_r)} && (5.21)\\
&= \frac{-2a(n,t_r)F_p(z,n+1,t_r)}{y + G_{p+1}(z,n,t_r)}, && (5.22)\\
& \qquad P(z,y) \in \mathcal{K}_p, \ (n,t_r) \in \mathbb{Z}\times\mathbb{R}
\end{aligned}$$

with divisor $(\phi(\,\cdot\,,n,t_r))$ of $\phi(\,\cdot\,,n,t_r)$ given by

$$(\phi(\,\cdot\,,n,t_r)) = \mathcal{D}_{P_{\infty_+}\underline{\hat{\mu}}(n+1,t_r)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n,t_r)}, \tag{5.23}$$

using (5.11) and (5.20).

The time-dependent Baker–Akhiezer function $\psi(P,n,n_0,t_r,t_{0,r})$, meromorphic on $\mathcal{K}_p$, is then defined in terms of ϕ by

$$\begin{aligned}
&\psi(P,n,n_0,t_r,t_{0,r})\\
&\quad = \exp\bigg(\int_{t_{0,r}}^{t_r} ds\big(2a(n_0,s)\widetilde{F}_r(z,n_0,s)\phi(P,n_0,s) + \widetilde{G}_{r+1}(z,n_0,s)\big)\bigg)\\
&\qquad \times \begin{cases} \prod_{m=n_0}^{n-1} \phi(P,m,t_r) & \text{for } n \geq n_0+1,\\ 1 & \text{for } n = n_0,\\ \prod_{m=n}^{n_0-1} \phi(P,m,t_r)^{-1} & \text{for } n \leq n_0-1,\end{cases} && (5.24)\\
&\quad P \in \mathcal{K}_p\setminus\{P_{\infty_\pm}\}, \ (n,n_0,t_r,t_{0,r}) \in \mathbb{Z}^2\times\mathbb{R}^2.
\end{aligned}$$

For subsequent purposes we also introduce the following Baker–Akhiezer vector,

$$\begin{aligned}\Psi(P,n,n_0,t_r,t_{0,r}) &= \begin{pmatrix} \psi^-(P,n,n_0 t_r,t_{0,r}) \\ \psi(P,n,n_0,t_r,t_{0,r}) \end{pmatrix}, \\ P\in\mathcal{K}_p\setminus\{P_{\infty_\pm}\},\ &(n,n_0,t_r,t_{0,r})\in\mathbb{Z}^2\times\mathbb{R}^2.\end{aligned} \tag{5.25}$$

Basic properties of ϕ, ψ, and Ψ are summarized in the following lemma.

Lemma 5.2. *Assume Hypothesis 5.1 and suppose that a,b satisfy* (5.15)–(5.18). *In addition, let $P=(z,y)\in\mathcal{K}_p\setminus\{P_{\infty_\pm}\}$, $(n,n_0,t_r,t_{0,r})\in\mathbb{Z}^2\times\mathbb{R}^2$, and $r\in\mathbb{N}_0$. Then ϕ satisfies*

$$a\phi(P)+a^-(\phi^-(P))^{-1}=z-b, \tag{5.26}$$

$$\begin{aligned}\phi_{t_r}(P) &= -2a\big(\widetilde{F}_r(z)\phi(P)^2+\widetilde{F}_r^+(z)\big)+2(z-b^+)\widetilde{F}_r^+(z)\phi(P)\\ &\quad +\big(\widetilde{G}_{r+1}^+(z)-\widetilde{G}_{r+1}(z)\big)\phi(P),\end{aligned} \tag{5.27}$$

$$\phi(P)\phi(P^*)=\frac{F_p^+(z)}{F_p(z)}, \tag{5.28}$$

$$\phi(P)-\phi(P^*)=\frac{y(P)}{aF_p(z)}, \tag{5.29}$$

$$\phi(P)+\phi(P^*)=-\frac{G_{p+1}(z)}{aF_p(z)}. \tag{5.30}$$

Moreover, ψ and Ψ satisfy

$$(L-z(P))\psi(P)=0,\quad (P_{2p+2}-y(P))\psi(P)=0, \tag{5.31}$$

$$\psi_{t_r}(P)=\widetilde{P}_{2r+2}\psi(P) \tag{5.32}$$

$$\qquad =2a\widetilde{F}_r(z)\psi^+(P)+\widetilde{G}_{r+1}(z)\psi(P), \tag{5.33}$$

$$\Psi^+(P)=U(z)\Psi(P),\quad y\Psi(P)=V_{p+1}\Psi(P), \tag{5.34}$$

$$\Psi_{t_r}(P)=\widetilde{V}_{r+1}(z)\Psi(P), \tag{5.35}$$

$$\psi(P,n,n_0,t_r,t_{0,r})\psi(P^*,n,n_0,t_r,t_{0,r})=\frac{F_p(z,n,t_r)}{F_p(z,n_0,t_{0,r})}, \tag{5.36}$$

$$\begin{aligned}&a(n,t_r)\big(\psi(P,n,n_0,t_r,t_{0,r})\psi(P^*,n+1,n_0,t_r,t_{0,r})\\ &\quad+\psi(P^*,n,n_0,t_r,t_{0,r})\psi(P,n+1,n_0,t_r,t_{0,r})\big)=-\frac{G_{p+1}(z,n,t_r)}{F_p(z,n_0,t_{0,r})},\end{aligned} \tag{5.37}$$

$$W(\psi(P,\cdot\,,n_0,t_r,t_{0,r}),\psi(P^*,\cdot\,,n_0,t_r,t_{0,r}))=-\frac{y(P)}{F_p(z,n_0,t_{0,r})}. \tag{5.38}$$

In complete analogy to the case of stationary trace formulas one obtains trace formulas in the time-dependent setting (cf. the abbreviation (3.24)).

Lemma 5.3. *Assume Hypothesis 5.1 and suppose that a,b satisfy* (5.15)–(5.16). *Then,*

$$b=\frac{1}{2}\sum_{m=0}^{2p+1}E_m-\sum_{j=1}^{p}\mu_j. \tag{5.39}$$

In addition, if for all $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$, $\mu_j(n,t_r)\neq\mu_k(n,t_r)$ *for* $j\neq k$, $j,k=1,\dots,p$, *then,*

$$a^2=\frac{1}{2}\sum_{j=1}^{p}y(\hat\mu_j)\prod_{\substack{k=1\\k\neq j}}^{p}(\mu_j-\mu_k)^{-1}+\frac{1}{4}\big(b^{(2)}-b^2\big). \tag{5.40}$$

For completeness we next mention the Dubrovin equations for the time variation of the Dirichlet eigenvalues of the Toda lattice.

Lemma 5.4. *Assume Hypothesis 5.1 and suppose that* a,b *satisfy* (5.15)–(5.16). *In addition, assume that the zeros* $\mu_j(n,t_r)$, $j=1,\dots,p$, *of* $F_p(\,\cdot\,,n,t_r)$ *remain distinct for all* $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$. *Then,*

$$\begin{aligned}\frac{d}{dt_r}\mu_j(n,t_r)=-2\tilde F_r(\mu_j(n,t_r),n,t_r)&\frac{y(\hat\mu_j(n,t_r))}{\prod_{\substack{\ell=1\\ \ell\neq j}}^{p}(\mu_j(n,t_r)-\mu_\ell(n,t_r))},\\ &j=1,\dots,p,\ (n,t_r)\in\mathbb{Z}\times\mathbb{R}.\end{aligned} \tag{5.41}$$

When attempting to solve the Dubrovin system (5.41), it must be augmented with appropriate divisors $\mathcal{D}_{\underline{\hat\mu}(n_0,t_{0,r})}\in\operatorname{Sym}^p\mathcal{K}_p$ as initial conditions.

For the t_r-dependence of F_p and G_{p+1} one obtains the following result.

Lemma 5.5. *Assume Hypothesis 5.1 and suppose that* a,b *satisfy* (5.15)–(5.16). *In addition, let* $(z,n,t_r)\in\mathbb{C}\times\mathbb{Z}\times\mathbb{R}$. *Then,*

$$F_{p,t_r}=2\big(F_p\widetilde G_{r+1}-G_{p+1}\widetilde F_r\big), \tag{5.42}$$

$$G_{p+1,t_r}=4a^2\big(F_p\widetilde F_r^{+}-F_p^{+}\widetilde F_r\big). \tag{5.43}$$

In particular, (5.42) *and* (5.43) *are equivalent to*

$$V_{p+1,t_r}=[\widetilde V_{r+1},V_{p+1}]. \tag{5.44}$$

It will be shown in Section 6 that Lemma 5.5 in conjunction with the fundamental identity (5.19) yields a first-order system of differential equations for f_ℓ, g_ℓ, $\ell=1,\dots,p$, that serves as a pertinent substitute for the Dubrovin equations (5.41) even (in fact, especially) when some of the μ_j coincide.

As in the case of trace formulas, also Lemma 3.4 on nonspecial Dirichlet divisors $\mathcal{D}_{\underline{\hat\mu}}$ and the linearization property of the Abel map when applied to $\mathcal{D}_{\underline{\hat\mu}}$ extends to the present time-dependent setting. For the latter fact we need to introduce a particular differential of the second kind, $\widetilde\Omega_r^{(2)}$, defined as follows. Let $\omega^{(2)}_{P_{\infty_\pm},q}$ be the normalized Abelian differential of the second kind (i.e., with vanishing a-periods) with a single pole at $P_{\infty_\pm}$ of the form

$$\omega^{(2)}_{P_{\infty_\pm},q}=\big(\zeta^{-2-q}+O(1)\big)\,d\zeta \text{ near } P_{\infty_\pm},\quad q\in\mathbb{N}_0. \tag{5.45}$$

Given the summation constants $\tilde c_1,\dots,\tilde c_r$ in $\widetilde F_r$ (cf. (5.13)), we then define

$$\widetilde\Omega_r^{(2)}=\sum_{q=0}^{r}(q+1)\tilde c_{r-q}\big(\omega^{(2)}_{P_{\infty_+},q}-\omega^{(2)}_{P_{\infty_-},q}\big),\quad \tilde c_0=1. \tag{5.46}$$

Since the differentials $\omega^{(2)}_{P_{\infty_\pm},q}$ were supposed to be normalized we have

$$\int_{a_j}\widetilde\Omega_r^{(2)}=0,\quad j=1,\dots,p. \tag{5.47}$$

Moreover, writing

$$\omega_j = \bigg(\sum_{m=0}^{\infty} d_{j,m}(P_{\infty_\pm})\zeta^m\bigg)\,d\zeta = \pm\bigg(\sum_{m=0}^{\infty} d_{j,m}(P_{\infty_+})\zeta^m\bigg)\,d\zeta \text{ near } P_{\infty_\pm}, \tag{5.48}$$

relation (A.20) yields for the vector of b-periods $\underline{\widetilde{U}}_r^{(2)}$ of $\widetilde{\Omega}_r^{(2)}$,

$$\underline{\widetilde{U}}_r^{(2)} = \big(\widetilde{U}_{r,1}^{(2)}, \dots, \widetilde{U}_{r,p}^{(2)}\big), \tag{5.49}$$

$$\widetilde{U}_{r,j}^{(2)} = \frac{1}{2\pi i}\int_{b_j} \widetilde{\Omega}_r^{(2)} = 2\sum_{q=0}^{r} \tilde{c}_{r-q} d_{j,q}(P_{\infty_+}), \quad j = 1, \dots, p. \tag{5.50}$$

The time-dependent analog of Lemma 3.4 then reads as follows.

Lemma 5.6. *Assume Hypothesis 5.1 and suppose that a, b satisfy* (5.15)–(5.16)*. Let $\mathcal{D}_{\underline{\hat{\mu}}}$, $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_p\} \in \mathrm{Sym}^p(\mathcal{K}_p)$, be the Dirichlet divisor of degree p associated with a, b defined according to* (5.20)*, that is,*

$$\hat{\mu}_j(n,t_r) = (\mu_j(n,t_r), -G_{p+1}(\mu_j(n,t_r),n,t_r)) \in \mathcal{K}_p, \quad j = 1, \dots, p,\ (n,t_r) \in \mathbb{Z}\times\mathbb{R}. \tag{5.51}$$

Then $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ is nonspecial for all $(n,t_r) \in \mathbb{Z}\times\mathbb{R}$. Moreover, the Abel map linearizes the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}) - (n-n_0)\underline{A}_{P_{\infty_-}}(P_{\infty_+}) - (t_r - t_{0,r})\underline{\widetilde{U}}_r^{(2)}, \tag{5.52}$$

where $Q_0 \in \mathcal{K}_p$ is a given base point and $\underline{\widetilde{U}}_r^{(2)}$ is the vector of b-periods of the differential of the second kind $\widetilde{\Omega}_r^{(2)}$ introduced in (5.50)*.*
In addition, if $a, b \in L^\infty(\mathbb{Z}\times\mathbb{R})$, then there exists a constant $C_\mu > 0$ such that

$$|\mu_j(n,t_r)| \le C_\mu, \quad j = 1, \dots, p,\ (n,t_r) \in \mathbb{Z}\times\mathbb{R}. \tag{5.53}$$

Proof. We will prove that

$$\begin{aligned}\psi(P,n,n_0,t_r,t_{0,r}) =& C(n,t_r)\frac{\theta(\underline{z}(P,n,t_r))}{\theta(\underline{z}(P,n_0,t_{0,r}))}\\ &\times \exp\bigg((n-n_0)\int_{Q_0}^{P} \omega_{P_{\infty_+},P_{\infty_-}}^{(3)} + (t_r - t_{0,r})\int_{Q_0}^{P} \widetilde{\Omega}_r^{(2)}\bigg),\end{aligned} \tag{5.54}$$

where

$$\underline{z}(P,n,t_r) = \underline{A}_{Q_0}(P) - \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}) + \underline{\Xi}_{Q_0}. \tag{5.55}$$

By Lemma 13.4 of [32] it suffices to show that the essential singularities at $P_{\infty_\pm}$ are equal. That is, by (5.24) we need to show that

$$\begin{aligned}&\psi(P,n_0,n_0,t_r,t_{0,r})\\ &\quad= \exp\bigg(\int_{t_{0,r}}^{t_r} ds\big(2a(n_0,s)\widetilde{F}_r(z,n_0,t_r)\phi(P,n_0,t_{0,r}) + \widetilde{G}_{r+1}(z,n_0,s)\big)\bigg)\\ &\quad= \exp\bigg(\pm(t_r - t_{0,r})\sum_{k=0}^{r} c_{r-k}\zeta^{-k-1} + O(1)\bigg) \quad \text{for } P \text{ near } P_{\infty_\pm}.\end{aligned} \tag{5.56}$$

Using (5.21) and (5.42) one obtains

$$\psi(P, n_0, n_0, t_r, t_{0,r}) = \bigg(\frac{F_p(z, n_0, t_r)}{F_p(z, n_0, t_{0,r})}\bigg)^{1/2} \exp\bigg(y \int_{t_{0,r}}^{t_r} ds \, \frac{\widetilde{F}_r(z, n_0, s)}{F_p(z, n_0, s)}\bigg) \tag{5.57}$$

and the desired asymptotics follow from Theorem C.1, which tells us that

$$\frac{y}{F_p(z, n_0, s)} \widehat{F}_k(z, n_0, s) = \pm\zeta^{-k-1} + O(1) \text{ for } P \text{ near } P_{\infty_\pm}, \tag{5.58}$$

together with (2.23). $\square$

Again the analog of Remark 3.5 applies in the present time-dependent context.

6. The Algebro-Geometric Toda Hierarchy Initial Value Problem

In this section we consider the algebro-geometric Toda hierarchy initial value problem (5.2), (5.3) with complex-valued initial data. For a generic set of initial data we will prove unique solvability of the initial value problem globally in time.

While it is naturally in the special self-adjoint case to base the solution of the algebro-geometric initial value problem on the Dubrovin equations (5.41) (and the trace formula (5.39) for b and formula (5.40) for a^2), this strategy meets with difficulties in the non-self-adjoint case as Dirichlet eigenvalues $\hat{\mu}_j$ may now collide on $\mathcal{K}_p$ and hence the denominator of (5.41) can blow up. Hence, we will develop an alternative strategy based on the use of elementary symmetric functions of the variables $\{\mu_j\}_{j=1,\dots,p}$ in this section, which can accommodate collisions of $\hat{\mu}_j$. In short, our strategy will be as follows:

(*i*) Replace the first-order autonomous Dubrovin system (5.41) of differential equations in t_r for the Dirichlet eigenvalues $\mu_j(n, t_r)$, $j = 1, \dots, p$, augmented by appropriate initial conditions, by the first-order autonomous system (6.27), (6.28) for the coefficients f_j, $j = 1, \dots, p$, g_j, $j = 1, \dots, p-1$, and $g_p + f_{p+1}$ with respect to t_r. (We note that f_j, $j = 1, \dots, p$, are symmetric functions of $\mu_1, \dots, \mu_p$.) Solve this first-order autonomous system in some time interval $(t_{0,r} - T_0, t_{0,r} + T_0)$ under appropriate initial conditions at $(n_0, t_{0,r})$ derived from an initial (nonspecial) Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0, t_{0,r})}$.

(*ii*) Use the stationary algorithm derived in Section 4 to extend the solution of step (*i*) from $\{n_0\} \times (t_{0,r} - T_0, t_{0,r} + T_0)$ to $\mathbb{Z} \times (t_{0,r} - T_0, t_{0,r} + T_0)$ (cf. Lemma 6.2).

(*iii*) Prove consistency of this approach, that is, show that the discrete algorithm of Section 4 is compatible with the time-dependent Lax and zero-curvature equations in the sense that first solving the autonomous system (6.27), (6.28) and then applying the discrete algorithm, or first applying the discrete algorithm and then solving the autonomous system (6.27), (6.28) yields the same result whenever the same endpoint (n, t_r) is reached (cf. the discussion following Lemma 6.2 and the subsequent Lemma 6.3 and Theorem 6.4).

(*iv*) Prove that there is a dense set of initial conditions of full measure for which this strategy yields global solutions of the algebro-geometric Toda hierarchy initial value problem (cf. Lemma 6.5 and Theorem 6.6).

To set up this formalism we need some preparations. From the outset we make the following assumption.

Hypothesis 6.1. *Suppose the hyperelliptic curve $\mathcal{K}_p$ in* (3.1) *is nonsingular, that*

$$a, b \in \mathbb{C}^{\mathbb{Z}} \text{ and } a(n) \neq 0 \text{ for all } n \in \mathbb{Z}, \tag{6.1}$$

and assume that a, b satisfy the pth stationary Toda equation (2.20).

Assuming Hypothesis 6.1, we consider the polynomials F_p, G_{p+1}, $\widetilde{F}_r$, and $\widetilde{G}_{r+1}$ given by (5.11)–(5.14) for fixed $p, r \in \mathbb{N}_0$. Here f_ℓ, $\tilde{f}_s$, g_ℓ, and $\tilde{g}_s$, $\ell = 0, \dots, p$, $s = 0, \dots, r$, are defined as in (2.4)–(2.6) with appropriate sets of summation constants.

Our aim will be to find an autonomous first-order system of ordinary differential equations with respect to t_r of f_ℓ and g_ℓ rather than μ_j. Indeed, we will take the coupled system of differential equations (5.42), (5.43), properly rewritten next, as our point of departure. In order to turn (5.42), (5.43) into a system of first-order ordinary differential equations for f_ℓ and g_ℓ, we first need to eliminate f_ℓ^+, $\tilde{f}_s$, $\tilde{g}_s$, and $\tilde{f}_s^+$ in terms of f_ℓ and g_ℓ as follows.

Using (2.9), (2.23), (2.25), and Theorem C.1 one infers

$$\widetilde{F}_r(z) = \sum_{s=0}^{r} \tilde{f}_{r-s} z^s = \sum_{s=0}^{r} \tilde{c}_{r-s} \widehat{F}_s(z), \tag{6.2}$$

$$\widehat{F}_\ell(z) = \sum_{k=0}^{\ell} \hat{f}_{\ell-k} z^k, \quad \hat{f}_0 = 1, \quad \hat{f}_\ell = \sum_{k=0}^{\ell\wedge p} \hat{c}_{\ell-k}(\underline{E}) f_k, \quad \ell \in \mathbb{N}_0, \tag{6.3}$$

where $m \wedge n = \min\{m, n\}$ and $\hat{c}_\ell(\underline{E})$ has been introduced in (C.4). Hence one obtains

$$\tilde{f}_0 = 1, \quad \tilde{f}_s = \mathcal{F}_{1,s}(f_1, \dots, f_p), \quad s = 1, \dots, r, \tag{6.4}$$

where $\mathcal{F}_{1,s}$, $s = 1, \dots, r$, are polynomials in p variables.

Next, using (2.9), (2.24), (2.26), and Theorem C.1 one concludes

$$\begin{aligned}
\widetilde{G}_{r+1}(z) &= -z^{r+1} + \sum_{s=0}^{r} \tilde{g}_{r-s} z^s + \tilde{f}_{r+1} = \sum_{s=1}^{r+1} \tilde{c}_{r+1-s} \widehat{G}_s(z), \\
\widehat{G}_0(z) &= G_0(z)\big|_{c_1=0} = 0, \quad \widehat{G}_1(z) = G_1(z) = -z - b, \\
\widehat{G}_{\ell+1}(z) &= G_{\ell+1}(z)\big|_{c_k=0,\, k=1,\dots,\ell} = -z^{\ell+1} + \sum_{k=0}^{\ell} \hat{g}_{\ell-k} z^k + \hat{f}_{\ell+1}, \quad \ell \in \mathbb{N},
\end{aligned} \tag{6.5}$$

$$\hat{g}_0 = 0, \quad \hat{g}_\ell = \sum_{k=0}^{\ell\wedge p} \hat{c}_{\ell-k}(\underline{E})(g_k + f_{p+1}\delta_{p,k}) - \hat{c}_{\ell+1}(\underline{E}), \quad \ell \in \mathbb{N}. \tag{6.6}$$

Hence one concludes

$$\tilde{g}_0 = -\tilde{c}_1, \quad \tilde{g}_s = \mathcal{F}_{2,s}(f_1, \dots, f_p, g_1, \dots, g_{p-1}, (g_p + f_{p+1})), \quad s = 1, \dots, r, \tag{6.7}$$

where $\mathcal{F}_{2,s}$, $s = 1, \dots, r$, are polynomials in $2p$ variables. We also recall (cf. (2.18)) that f_{p+1} is a lattice constant, that is,

$$f_{p+1} = f_{p+1}^-. \tag{6.8}$$

Next we invoke the fundamental identity (2.34) in the form

$$-4a^2 F_p^+ = \frac{R_{2p+2} - G_{p+1}^2}{F_p}. \tag{6.9}$$

While (6.9) at this point only holds in the stationary context, we will use it later on also in the t_r-dependent context and verify after the time-dependent solutions of (5.2), (5.3) have been obtained that (6.9) indeed is valid for all $(n, t_r) \in \mathbb{Z} \times \mathbb{R}$. A comparison of powers of z in (6.9) then yields

$$\begin{aligned} &4a^2 f_0^+ = -2g_1 - 2c_2, \\ &4a^2 f_\ell^+ = \mathcal{F}_{3,\ell}(f_1, \dots, f_p, g_1, \dots, g_{p-1}, (g_p + f_{p+1})), \quad \ell = 1, \dots, p, \end{aligned} \tag{6.10}$$

where $\mathcal{F}_{3,\ell}$, $\ell = 1, \dots, p$, are polynomials in $2p$ variables.

Finally, combining (6.2), (6.3), (6.9), and (6.10), one obtains

$$\begin{aligned} &4a^2 \tilde{f}_0^+ = -2g_1 - 2c_2, \\ &4a^2 \tilde{f}_s^+ = \mathcal{F}_{4,s}(f_1, \dots, f_p, g_1, \dots, g_{p-1}, (g_p + f_{p+1})), \quad s = 1, \dots, r, \end{aligned} \tag{6.11}$$

where $\mathcal{F}_{4,s}$, $s = 1, \dots, 3$, are polynomials in $2p$ variables.

We emphasize that also the Dubrovin equations (5.41) require an analogous rewriting of $\widetilde{F}_r$ in terms of (symmetric functions of) μ_j in order to represent a first-order system of differential equations for μ_j, $j = 1, \dots, p$.

Next, we make the transition to the algebro-geometric initial value problem (5.2), (5.3). We introduce a deformation (time) parameter $t_r \in \mathbb{R}$ in $a = a(t_r)$ and $b = b(t_r)$, and hence obtain t_r-dependent quantities $f_\ell = f_\ell(t_r)$, $g_\ell = g_\ell(t_r)$, $F_p(z) = F_p(z, t_r)$, $G_{p+1}(z) = G_{p+1}(z, t_r)$, etc. At a fixed initial time $t_{0,r} \in \mathbb{R}$ we require that

$$(a, b)|_{t_r = t_{0,r}} = \big(a^{(0)}, b^{(0)}\big), \tag{6.12}$$

where $a^{(0)} = a(\,\cdot\,, t_{0,r}), b^{(0)} = b(\,\cdot\,, t_{0,r})$ satisfy the pth stationary Toda equation (2.20) as in (6.1)–(6.11). As discussed in Section 4, in order to guarantee that the stationary solutions (6.12) can be constructed for all $n \in \mathbb{Z}$ one starts from a particular divisor

$$\mathcal{D}_{\underline{\hat{\mu}}(n_0, t_{0,r})} \in \mathcal{M}_0 \tag{6.13}$$

where $\underline{\hat{\mu}}(n_0, t_{0,r})$ is of the form

$$\begin{aligned} \underline{\hat{\mu}}(n_0, t_{0,r}) &= \{\hat{\mu}_1(n_0, t_{0,r}, \dots, \hat{\mu}_p(n_0, t_{0,r})\} \\ &= \{\underbrace{\hat{\mu}_1(n_0, t_{0,r}), \dots, \hat{\mu}_1(n_0, t_{0,r})}_{p_1(n_0, t_{0,r}) \text{ times}}, \dots, \underbrace{\hat{\mu}_{q(n_0, t_{0,r})}(n_0, t_{0,r}), \dots, \hat{\mu}_{q(n_0, t_{0,r})}(n_0, t_r)}_{p_{q(n_0, t_{0,r})}(n_0, t_{0,r}) \text{ times}}\} \end{aligned} \tag{6.14}$$

with

$$\begin{aligned} &\hat{\mu}_k(n_0, t_{0,r}) = (\mu_k(n_0, t_{0,r}), y(\hat{\mu}_k(n_0, t_{0,r}))), \\ &\mu_k(n_0, t_{0,r}) \neq \mu_{k'}(n_0, t_{0,r}) \text{ for } k \neq k', \ k, k' = 1, \dots, q(n_0, t_{0,r}) \end{aligned} \tag{6.15}$$

and

$$p_k(n_0, t_{0,r}) \in \mathbb{N}, \ k = 1, \dots, q(n_0, t_{0,r}), \qquad \sum_{k=1}^{q(n_0, t_{0,r})} p_k(n_0, t_{0,r}) = p. \tag{6.16}$$

Next we recall

$$\begin{aligned} &F_p(z, n_0, t_{0,r}) = \sum_{\ell=0}^{p} f_{p-\ell}(n_0, t_{0,r}) z^\ell = \prod_{k=1}^{q(n_0, t_{0,r})} (z - \mu_k(n_0, t_{0,r}))^{p_k(n_0, t_{0,r})}, \\ &T_{p-1}(z, n_0, t_{0,r}) = -F_p(z, n_0, t_{0,r}) \end{aligned} \tag{6.17}$$

$$\begin{aligned}
&\times \sum_{k=1}^{q(n_0,t_{0,r})} \sum_{\ell=0}^{p_k(n_0,t_{0,r})-1} \frac{\big(d^\ell \big(R_{2p+2}(\zeta)^{1/2}\big)/d\zeta^\ell\big)\big|_{\zeta=\mu_k(n_0,t_{0,r})}}{\ell!(p_k(n_0,t_{0,r})-\ell-1)!} \\
&\times \Bigg(\frac{d^{p_k(n_0,t_{0,r})-\ell-1}}{d\zeta^{p_k(n_0,t_{0,r})-\ell-1}}\Bigg((z-\zeta)^{-1} \\
&\quad \times \prod_{k'=1,\,k'\neq k}^{q(n_0,t_{0,r})} (\zeta-\mu_{k'}(n_0,t_{0,r}))^{-p_{k'}(n_0,t_{0,r})}\Bigg)\Bigg)\Bigg|_{\zeta=\mu_k(n_0,t_{0,r})},
\end{aligned} \tag{6.18}$$

$$b(n_0,t_{0,r}) = \frac{1}{2}\sum_{m=0}^{2p+1} E_m - \sum_{k=1}^{q(n_0,t_{0,r})} p_k(n_0,t_{0,r})\mu_k(n_0,t_{0,r}), \tag{6.19}$$

$$\begin{aligned}
G_{p+1}(z,n_0,t_{0,r}) &= -z^{p+1} + \sum_{\ell=0}^{p} g_{p-\ell}(n_0,t_{0,r})z^\ell + f_{p+1}(t_{0,r}), \\
&= -(z-b(n_0,t_{0,r}))F_p(z,n_0,t_{0,r}) + T_{p-1}(z,n_0,t_{0,r}).
\end{aligned} \tag{6.20}$$

Here the sign of the square root in (6.18) is chosen as usual by

$$\begin{aligned}
\hat\mu_k(n_0,t_{0,r}) &= (\mu_k(n_0,t_{0,r}), y(\hat\mu_k(n_0,t_{0,r}))) \\
&= \big(\mu_k(n_0,t_{0,r}), R_{2p+2}(\mu_k(n_0,t_{0,r}))^{1/2}\big) \\
&= (\mu_k(n_0,t_{0,r}), -G_{p+1}(\mu_k(n_0,t_{0,r}),n_0,t_{0,r})), \quad k=1,\dots,q(n_0,t_{0,r}).
\end{aligned} \tag{6.21}$$

By (6.17) one concludes that (6.14) uniquely determines $F_p(z,n_0,t_{0,r})$ and hence

$$f_1(n_0,t_{0,r}),\dots,f_p(n_0,t_{0,r}). \tag{6.22}$$

By (6.18)–(6.22) one concludes that also $G_{p+1}(z,n_0,t_{0,r})$ and hence

$$g_1(n_0,t_{0,r}),\dots,g_{p-1}(n_0,t_{0,r}), g_p(n_0,t_{0,r})+f_{p+1}(t_{0,r}) \tag{6.23}$$

are uniquely determined by the initial divisor $\mathcal{D}_{\underline{\hat\mu}(n_0,t_{0,r})}$ in (6.13).

Summing up the discussion in (6.2)–(6.23), we can transform the differential equations

$$\begin{aligned}
F_{p,t_r}(z,n_0,t_r) &= 2\big(F_p(z,n_0,t_r)\widetilde G_{r+1}(z,n_0,t_r) \\
&\qquad - G_{p+1}(z,n_0,t_r)\widetilde F_r(z,n_0,t_r)\big),
\end{aligned} \tag{6.24}$$

$$\begin{aligned}
G_{p+1,t_r}(z,n_0,t_r) &= 4a(n_0,t_r)^2\big(F_p(z,n_0,t_r)\widetilde F_r^+(z,n_0,t_r) \\
&\qquad - F_p^+(z,n_0,t_r)\widetilde F_r(z,n_0,t_r)\big)
\end{aligned} \tag{6.25}$$

subject to the constraint

$$-4a^2F_p^+(z,n_0,t_r) = \frac{R_{2p+2}(z) - G_{p+1}(z,n_0,t_r)^2}{F_p(z,n_0,t_r)}, \tag{6.26}$$

and associated with an initial divisor $\mathcal{D}_{\underline{\hat\mu}(n_0,t_{0,r})}$ in (6.13) into the following autonomous first-order system of ordinary differential equations (for fixed $n=n_0$),

$$\begin{aligned}
f_{j,t_r} &= \mathcal{F}_j(f_1,\dots,f_p,g_1,\dots,g_{p-1},g_p+f_{p+1}), \quad j=1,\dots,p, \\
g_{j,t_r} &= \mathcal{G}_j(f_1,\dots,f_p,g_1,\dots,g_{p-1},g_p+f_{p+1}), \quad j=1,\dots,p-1, \\
(g_p+f_{p+1})_{t_r} &= \mathcal{G}_p(f_1,\dots,f_p,g_1,\dots,g_{p-1},g_p+f_{p+1})
\end{aligned} \tag{6.27}$$

with initial condition

$$\begin{aligned} &f_j(n_0, t_{0,r}), \quad j = 1, \dots, p, \\ &g_j(n_0, t_{0,r}), \quad j = 1, \dots, p-1, \\ &g_p(n_0, t_{0,r}) + f_{p+1}(t_{0,r}), \end{aligned} \tag{6.28}$$

where $\mathcal{F}_j$, $\mathcal{G}_j$, $j = 1, \dots, p$, are polynomials in $2p$ variables. As just discussed, the initial conditions (6.28) are uniquely determined by the initial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0, t_{0,r})}$ in (6.13).

Being autonomous with polynomial right-hand sides, there exists a $T_0 > 0$, such that the first-order initial value problem (6.27), (6.28) has a unique solution

$$\begin{aligned} &f_j = f_j(n_0, t_r), \quad j = 1, \dots, p, \\ &g_j = g_j(n_0, t_r), \quad j = 1, \dots, p-1, \\ &g_p + f_{p+1} = g_p(n_0, t_r) + f_{p+1}(t_r) \\ &\text{for all } \ t_r \in (t_{0,r} - T_0, t_{0,r} + T_0) \end{aligned} \tag{6.29}$$

(cf., e.g., [35, Sect. III.10]). Given the solution (6.29), we next introduce the following quantities (where $t_r \in (t_{0,r} - T_0, t_{0,r} + T_0)$):

$$F_p(z, n_0, t_r) = \sum_{\ell=0}^{p} f_{p-\ell}(n_0, t_r) z^\ell = \prod_{k=1}^{q(n_0,t_r)} (z - \mu_k(n_0, t_r))^{p_k(n_0,t_r)}, \tag{6.30}$$

$$\begin{aligned} T_{p-1}(z, n_0, t_r) &= -F_p(z, n_0, t_r) \\ &\quad \times \sum_{k=1}^{q(n_0,t_r)} \sum_{\ell=0}^{p_k(n_0,t_r)-1} \frac{\big(d^\ell \big(R_{2p+2}(\zeta)^{1/2}\big)/d\zeta^\ell\big)\big|_{\zeta=\mu_k(n_0,t_r)}}{\ell!(p_k(n_0,t_r) - \ell - 1)!} \\ &\quad \times \Bigg(\frac{d^{p_k(n_0,t_r)-\ell-1}}{d\zeta^{p_k(n_0,t_r)-\ell-1}} \Bigg((z-\zeta)^{-1} \\ &\qquad \times \prod_{k'=1,\, k'\neq k}^{q(n_0,t_r)} (\zeta - \mu_{k'}(n_0, t_r))^{-p_{k'}(n_0,t_r)} \Bigg)\Bigg)\Bigg|_{\zeta=\mu_k(n_0,t_r)}, \end{aligned} \tag{6.31}$$

$$b(n_0, t_r) = \frac{1}{2} \sum_{m=0}^{2p+1} E_m - \sum_{k=1}^{q(n_0,t_r)} p_k(n_0, t_r) \mu_k(n_0, t_r), \tag{6.32}$$

$$\begin{aligned} G_{p+1}(z, n_0, t_r) &= -z^{p+1} + \sum_{\ell=0}^{p} g_{p-\ell}(n_0, t_r) z^\ell + f_{p+1}(t_r) \\ &= -(z - b(n_0, t_r)) F_p(z, n_0, t_r) + T_{p-1}(z, n_0, t_r). \end{aligned} \tag{6.33}$$

In particular, this leads to the divisor

$$\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_r)} \in \mathrm{Sym}^p(\mathcal{K}_p) \tag{6.34}$$

and the sign of the square root in (6.31) is chosen as usual by

$$\begin{aligned} \hat{\mu}_k(n_0, t_r) &= (\mu_k(n_0, t_r), -G_{p+1}(\mu_k(n_0, t_r), n_0, t_r)) \\ &= \big(\mu_k(n_0, t_r), R_{2p+2}(\mu_k(n_0, t_r))^{1/2}\big), \quad k = 1, \dots, q(n_0, t_r) \end{aligned} \tag{6.35}$$

and

$$\begin{aligned}\underline{\hat{\mu}}(n_0,t_r) &= \{\mu_1(n_0,t_r,\dots,\mu_p(n_0,t_r))\} \\ &= \{\underbrace{\mu_1(n_0,t_r),\dots,\mu_1(n_0,t_r)}_{p_1(n_0,t_r)\text{ times}},\dots,\underbrace{\mu_{q(n_0,t_r)}(n_0,t_r),\dots,\mu_{q(n_0,t_r)}(n_0,t_r)}_{p_{q(n_0,t_r)}(n_0,t_r)\text{ times}}\}\end{aligned} \tag{6.36}$$

with

$$\mu_k(n_0,t_r) \neq \mu_{k'}(n_0,t_r) \text{ for } k\neq k',\ k,k'=1,\dots,q(n_0,t_r) \tag{6.37}$$

and

$$p_k(n_0,t_r)\in\mathbb{N},\ k=1,\dots,q(n_0,t_r),\quad \sum_{k=1}^{q(n_0,t_r)} p_k(n_0,t_r)=p. \tag{6.38}$$

By construction (cf. (6.35)), the divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_r)}$ is nonspecial for all $t_r\in(t_{0,r}-T_0,t_{0,r}+T_0)$.

In exactly the same manner as in (4.28)–(4.31) one then infers that $F_p(\,\cdot\,,n_0,t_r)$ divides $R_{2p+2}-G_{p+1}^2$ (since t_r is just a fixed additional parameter). Moreover, arguing as in (4.32)–(4.38) we now assume that the polynomial

$$R_{2p+2}(z)-G_{p+1}(z,n_0,t_r)^2 \underset{z\to\infty}{=} O(z^{2p}) \tag{6.39}$$

is precisely of maximal order $2p$ for all $t_r\in(t_{0,r}-T_0,t_{0,r}+T_0)$. One then obtains

$$\begin{aligned}R_{2p+2}(z)-G_{p+1}(z,n_0,t_r)^2 = -4a(n_0,t_r)^2F_p(z,n_0,t_r)F_p(z,n_0+1,t_r),& \\ (z,t_r)\in\mathbb{C}\times(t_{0,r}-T_0,t_{0,r}+T_0),&\end{aligned} \tag{6.40}$$

where we introduced the coefficient $a(n_0,t_r)^2$ to make $F_p(\,\cdot\,,n_0+1,t_r)$ a monic polynomial of degree p. As in Section 4, the assumption that the polynomial $F_p(\,\cdot\,,n_0+1,t_r)$ is precisely of order p is implied by the hypothesis that

$$\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_r)}\in\mathcal{M}_0 \text{ for all } t_r\in(t_{0,r}-T_0,t_{0,r}+T_0), \tag{6.41}$$

a point we will revisit later (cf. Lemma 6.5). Given (6.40), we obtain consistency with (6.9) for $n=n_0$ and $t_r\in(t_{0,r}-T_0,t_{0,r}+T_0)$.

The explicit formula for $a(n_0,t_r)^2$ then reads (for $t_r\in(t_{0,r}-T_0,t_{0,r}+T_0)$)

$$\begin{aligned}a(n_0,t_r)^2 = \frac{1}{2}\sum_{k=1}^{q(n_0,t_r)} &\frac{\big(d^\ell\big(R_{2p+2}(z)^{1/2}\big)/dz^\ell\big)\big|_{z=\mu_k(n_0,t_r)}}{(p_k(n_0,t_r)-1)!} \\ &\times\prod_{k'=1,\,k'\neq k}^{q(n_0,t_r)}(\mu_k(n_0,t_r)-\mu_{k'}(n_0,t_r))^{-p_k(n_0,t_r)} \\ &+\frac{1}{4}\big(b^{(2)}(n_0,t_r)-b(n_0,t_r)^2\big).\end{aligned} \tag{6.42}$$

Here and in the following we use the abbreviation

$$b^{(2)}(n,t_r)=\frac{1}{2}\sum_{m=0}^{2p+1}E_m^2-\sum_{k=1}^{q(n,t_r)}p_k(n,t_r)\mu_k(n,t_r)^2 \tag{6.43}$$

for appropriate ranges of $(n,t_r)\in\mathbb{N}\times\mathbb{R}$.

With (6.30)–(6.43) in place, we can now apply the stationary formalism as summarized in Theorem 4.4, subject to the additional hypothesis (6.41), for each fixed $t_r\in(t_{0,r}-T_0,t_{0,r}+T_0)$. This yields, in particular, the quantities

$$F_p,\ G_{p+1},\ a,\ b,\ \text{and } \underline{\hat{\mu}} \text{ for } (n,t_r)\in\mathbb{Z}\times(t_{0,r}-T_0,t_{0,r}+T_0), \tag{6.44}$$

which are of the form (6.30)–(6.43), replacing the fixed $n_0 \in \mathbb{Z}$ by an arbitrary $n \in \mathbb{Z}$. In addition, one has the following fundamental identities (cf. (4.54), (4.59), (4.62), and (4.63)), which we summarize in the following result.

Lemma 6.2. *Assume Hypothesis 6.1 and condition* (6.41). *Then the following relations are valid:*

$$R_{2p+2} - G_{p+1}^2 + 4a^2 F_p F_p^+ = 0, \tag{6.45}$$

$$2(z-b^+)F_p^+ + G_{p+1}^+ + G_{p+1} = 0, \tag{6.46}$$

$$2a^2 F_p^+ - 2(a^-)^2 F_p^- + (z-b)(G_{p+1} - G_{p+1}^-) = 0, \tag{6.47}$$

$$2(z-b^+)F_p^+ - 2(z-b)F_p + G_{p+1}^+ - G_{p+1}^- = 0, \tag{6.48}$$

$$\text{on } \mathbb{C}\times\mathbb{Z}\times(t_{0,r}-T_0, t_{0,r}+T_0)$$

and hence the stationary part, (5.9)*, of the algebro-geometric initial value problem holds*

$$UV_{p+1} - V_{p+1}^+ U = 0 \text{ on } \mathbb{C}\times\mathbb{Z}\times(t_{0,r}-T_0, t_{0,r}+T_0). \tag{6.49}$$

In particular, Lemmas 3.2–3.4 apply.

Lemma 6.2 now raises the following important consistency issue: On one hand, on can solve the initial value problem (6.27), (6.28) at $n = n_0$ in some interval $t_r \in (t_{0,r}-T_0, t_{0,r}+T_0)$, and then extended the quantities F_p, G_{p+1} to all $\mathbb{C}\times\mathbb{Z}\times(t_{0,r}-T_0, t_{0,r}+T_0)$ using the stationary algorithm summarized in Theorem 4.4 as just recorded in Lemma 6.2. On the other hand, one can solve the initial value problem (6.27), (6.28) at $n = n_1$, $n_1 \neq n_0$, in some interval $t_r \in (t_{0,r}-T_1, t_{0,r}+T_1)$ with the initial condition obtained by applying the discrete algorithm to the quantities F_p, G_{p+1} starting at $(n_0, t_{0,r})$ and ending at $(n_1, t_{0,r})$. Consistency then requires that the two approaches yield the same result at $n = n_1$ for t_r in some open neighborhood of $t_{0,r}$.

Equivalently, and pictorially speaking, envisage a vertical t_r-axis and a horizontal n-axis. Then, consistency demands that first solving the initial value problem (6.27), (6.28) at $n = n_0$ in some t_r-interval around $t_{0,r}$ and using the stationary algorithm to extend F_p, G_{p+1} horizontally to $n = n_1$ and the same t_r-interval around $t_{0,r}$, or first applying the stationary algorithm starting at $(n_0, t_{0,r})$ to extend F_p, G_{p+1} horizontally to $(n_1, t_{0,r})$ and then solving the initial value problem (6.27), (6.28) at $n = n_1$ in some t_r-interval around $t_{0,r}$ should produce the same result at $n = n_1$ in a sufficiently small open t_r interval around $t_{0,r}$.

To settle this consistency issue, we will prove the following result. To this end we find it convenient to replace the initial value problem (6.27), (6.28) by the original t_r-dependent zero-curvature equation (5.8), $U_{t_r} + U\widetilde{V}_{r+1} - \widetilde{V}_{r+1}^+ U = 0$ on $\mathbb{C}\times\mathbb{Z}\times(t_{0,r}-T_0, t_{0,r}+T_0)$.

Lemma 6.3. *Assume Hypothesis 6.1 and condition* (6.41)*. Moreover, suppose that* (6.24)*–*(6.26) *hold on* $\mathbb{C}\times\{n_0\}\times(t_{0,r}-T_0, t_{0,r}+T_0)$*. Then* (6.24)*–*(6.26) *hold on* $\mathbb{C}\times\mathbb{Z}\times(t_{0,r}-T_0, t_{0,r}+T_0)$*, that is,*

$$\begin{aligned} F_{p,t_r}(z,n,t_r) = 2\big(&F_p(z,n,t_r)\widetilde{G}_{r+1}(z,n,t_r) \\ &- G_{p+1}(z,n,t_r)\widetilde{F}_r(z,n,t_r)\big), \end{aligned} \tag{6.50}$$

$$\begin{aligned} G_{p+1,t_r}(z,n,t_r) = 4a(n,t_r)^2\big(&F_p(z,n,t_r)\widetilde{F}_r^+(z,n,t_r) \\ &- F_p^+(z,n,t_r)\widetilde{F}_r(z,n,t_r)\big), \end{aligned} \tag{6.51}$$

$$R_{2p+2}(z) = G_{p+1}(z,n,t_r)^2 - 4a(n,t_r)^2 F_p(z,n,t_r) F_p^+(z,n,t_r), \quad (z,n,t_r) \in \mathbb{C}\times\mathbb{Z}\times(t_{0,r}-T_0, t_{0,r}+T_0). \tag{6.52}$$

Moreover,

$$\begin{aligned}\phi_{t_r}(P,n,t_r) &= -2a(n,t_r)\big(\widetilde F_r(z,n,t_r)\phi(P,n,t_r)^2 + \widetilde F_r^+(z,n,t_r)\big)\\ &\quad + 2(z-b^+(n,t_r))\widetilde F_r^+(z,n,t_r)\phi(P,n,t_r)\\ &\quad + \big(\widetilde G_{r+1}^+(z,n,t_r) - \widetilde G_{r+1}(z,n,t_r)\big)\phi(P,n,t_r),\end{aligned} \tag{6.53}$$

$$\begin{aligned}a_{t_r}(n,t_r) &= -a(n,t_r)\big(2(z-b^+(n,t_r))\widetilde F_r^+(z,n,t_r)\\ &\quad + \widetilde G_{r+1}^+(z,n,t_r) + \widetilde G_{r+1}(z,n,t_r)\big),\end{aligned} \tag{6.54}$$

$$\begin{aligned}b_{t_r}(n,t_r) &= 2\big((z-b(n,t_r))^2 \widetilde F_r(z,n,t_r) + (z-b(n,t_r))\widetilde G_{r+1}(z,n,t_r)\\ &\quad + a^2(n,t_r)\widetilde F_r^+(z,n,t_r) - (a^-(n,t_r))^2 \widetilde F_r^-(z,n,t_r)\big),\\ &\quad (z,n,t_r)\in\mathbb{C}\times\mathbb{Z}\times(t_{0,r}-T_0,t_{0,r}+T_0).\end{aligned} \tag{6.55}$$

Proof. By Lemma 6.2 we have (5.21)–(5.22), (5.26), (5.28)–(5.30), and (6.45)–(6.48) for $(n,t_r)\in\mathbb{Z}\times(t_{0,r}-T_0,t_{0,r}+T_0)$ at our disposal. Differentiating (6.52) at $n=n_0$ with respect to t_r, inserting (6.50) and (6.51) at $n=n_0$, yields

$$\begin{aligned}2F_p^+ a_{t_r} + aF^+_{p,t_r} &= 2a\big(G_{p+1}\widetilde F_r^+ - F_p^+\widetilde G_{r+1}\big)\\ &= 2F_p^+ a\big(-2(z-b^+)\widetilde F_r^+ - \widetilde G_{r+1}^+ - \widetilde G_{r+1}\big) + a\big(F_p^+\widetilde G_{r+1}^+ - G_{p+1}^+\widetilde F_r^+\big)\end{aligned} \tag{6.56}$$

at $n=n_0$. By inspection,

$$F_p^+(z)\widetilde G_{r+1}^+(z) - G_{p+1}^+(z)\widetilde F_r^+(z) \underset{|z|\to\infty}{=} O(z^{p-1}). \tag{6.57}$$

This can be shown directly using formulas such as (2.23)–(2.26), (6.2), (6.3), (6.5), and (6.6). It also follows from (5.42) and the fact that F_p is a monic polynomial of degree p. Thus one concludes that

$$2F_p^+ a_{t_r} = 2F_p^+ a\big(-2(z-b^+)\widetilde F_r^+ - \widetilde G_{r+1}^+ - \widetilde G_{r+1}\big) \tag{6.58}$$

at $n=n_0$, and upon cancelling $2F_p^+$ that (6.54) holds at $n=n_0$. This and (6.56) then also proves that (6.50) holds at $n=n_0+1$.

Next, differentiating $2aF_p\phi = y - G_{p+1}$ at $n=n_0$ with respect to t_r inserting (6.50), (6.51), and (6.54) at $n=n_0$, and using (5.22) to replace $2aF_p^+$ by $-(y+G_{p+1})\phi$ and (5.21) to replace $(G_{p+1}-y)$ by $-2aF_p\phi$, yields (6.53) at $n=n_0$ upon cancelling the factor $2aF_p$.

Differentiating (6.46) with respect to t_r (fixing $n=n_0$), inserting (6.46) (to replace G_{p+1}^+), (6.51) at $n=n_0$, and (6.50) at $n=n_0+1$ yields

$$\begin{aligned}0 &= -2F_p^+\big(b_{t_r}^+ - 2(z-b^+)^2\widetilde F_r^+ + 2a^2\widetilde F_r - 2(z-b^+)\widetilde G_{r+1}^+\big)\\ &\quad + 4(z-b^+)^2F_p^+\widetilde F_r^+ + 4(z-b^+)G_{p+1}\widetilde F_r^+ + 4(a)^2F_p\widetilde F_r^+ + G^+_{p+1,t_r}\\ &= -2F_p^+\big(b_{t_r}^+ - 2(z-b^+)^2\widetilde F_r^+ - 2(z-b^+)\widetilde G_{r+1}^+ + 2a^2\widetilde F_r - 2(a^+)^2\widetilde F_r^{++}\big)\\ &\quad - 4(a^+)^2F_p^+\widetilde F_r^{++} + 4(z-b^+)^2F_p^+\widetilde F_r^+ + 4(z-b^+)G_{p+1}\widetilde F_r^+\\ &\quad + 4a^2F_p\widetilde F_r^+ + G^+_{p+1,t_r}\\ &= -2F_p^+\big(b_{t_r}^+ - 2(z-b^+)^2\widetilde F_r^+ - 2(z-b^+)\widetilde G_{r+1}^+ + 2a^2\widetilde F_r - 2(a^+)^2\widetilde F_r^{++}\big)\\ &\quad + G^+_{p+1,t_r} - 4(a^+)^2F_p^+\widetilde F_r^{++} + \big(4a^2F_p + 4(z-b^+)^2F_p^+ + 4(z-b^+)G_{p+1}\big)\widetilde F_r^+\end{aligned} \tag{6.59}$$

at $n = n_0$. Combining (6.46) and (6.47) at $n = n_0$ one computes

$$4(a^+)^2 F_p^{++} = 4a^2 F_p + 4(z-b^+)^2 F_p^+ + 4(z-b^+) G_{p+1} \tag{6.60}$$

at $n = n_0$. Insertion of (6.60) into (6.59) then yields

$$\begin{aligned} 0 = &-2F_p^+\big(b^+_{t_r} - 2(z-b^+)^2 \widetilde F_r^+ - 2(z-b^+)\widetilde G^+_{r+1} + 2a^2 \widetilde F_r - 2(a^+)^2 \widetilde F_r^{++}\big) \\ &+ G^+_{p+1,t_r} - 4(a^+)^2 F_p^+ \widetilde F_r^{++} + 4(a^+)^2 F_p^{++} \widetilde F_r^+ \end{aligned} \tag{6.61}$$

at $n = n_0$. In close analogy to (6.57) one observes that

$$F_p^+(z)\widetilde F_r^{++}(z) - F_p^{++}(z)\widetilde F_r^+(z) \underset{|z|\to\infty}{=} O(z^{p-1}) \text{ for } p \in \mathbb{N}. \tag{6.62}$$

Thus, since F_p^+ is a monic polynomial of degree p, (6.61) proves that

$$b^+_{t_r} - 2(z-b^+)^2 \widetilde F_r^+ - 2(z-b^+)\widetilde G^+_{r+1} + 2a^2 \widetilde F_r - 2(a^+)^2 \widetilde F_r^{++} = 0 \tag{6.63}$$

at $n = n_0$, upon cancelling F_p^+. Thus, (6.55) holds at $n = n_0 + 1$. Simultaneously, this proves (6.51) at $n = n_0 + 1$.

Iterating the arguments just presented (and performing the analogous considerations for $n < n_0$) then extends these results to all lattice points $n \in \mathbb{Z}$ and hence proves (6.50)–(6.55) for $(z, n, t_r) \in \mathbb{C} \times \mathbb{Z} \times (t_{0,r} - T_0, t_{0,r} + T_0)$. □

We summarize Lemmas 6.2 and 6.3 next.

Theorem 6.4. *Assume Hypothesis 6.1 and condition* (6.41)*. Moreover, suppose that*

$$\begin{aligned} &f_j = f_j(n_0, t_r), \quad j = 1, \dots, p, \\ &g_j = g_j(n_0, t_r), \quad j = 1, \dots, p-1, \\ &g_p + f_{p+1} = g_p(n_0, t_r) + f_{p+1}(t_r), \\ &\text{for all } t_r \in (t_{0,r} - T_0, t_{0,r} + T_0) \end{aligned} \tag{6.64}$$

satisfies the autonomous first-order system of ordinary differential equations (6.27) *(for fixed $n = n_0$),*

$$\begin{aligned} &f_{j,t_r} = \mathcal{F}_j(f_1, \dots, f_p, g_1, \dots, g_{p-1}, g_p + f_{p+1}), \quad j = 1, \dots, p, \\ &g_{j,t_r} = \mathcal{G}_j(f_1, \dots, f_p, g_1, \dots, g_{p-1}, g_p + f_{p+1}), \quad j = 1, \dots, p-1, \\ &(g_p + f_{p+1})_{t_r} = \mathcal{G}_p(f_1, \dots, f_p, g_1, \dots, g_{p-1}, g_p + f_{p+1}) \end{aligned} \tag{6.65}$$

with initial condition

$$\begin{aligned} &f_j(n_0, t_{0,r}), \quad j = 1, \dots, p, \\ &g_j(n_0, t_{0,r}), \quad j = 1, \dots, p-1, \\ &g_p(n_0, t_{0,r}) + f_{p+1}(t_{0,r}). \end{aligned} \tag{6.66}$$

Then F_p and G_{p+1} as constructed in (6.2)*–*(6.44) *on $\mathbb{C} \times \mathbb{Z} \times (t_{0,r} - T_0, t_{0,r} + T_0)$ satisfy the zero-curvature equations* (5.8)*,* (5.9)*, and* (5.44)*,*

$$U_{t_r} + U\widetilde V_{r+1} - \widetilde V^+_{r+1} U = 0, \tag{6.67}$$

$$U V_{p+1} - V^+_{p+1} U = 0, \tag{6.68}$$

$$V_{p+1,t_r} - [\widetilde V_{r+1}, V_{p+1}] = 0, \tag{6.69}$$

on $\mathbb{C} \times \mathbb{Z} \times (t_{0,r} - T_0, t_{0,r} + T_0)$

with U, V_{p+1}, and $\widetilde{V}_{r+1}$ given by (5.10). *In particular, a, b satisfy the algebro-geometric initial value problem* (5.2), (5.3)

$$\begin{aligned}
&\widetilde{\mathrm{Tl}}_r(a,b) = \begin{pmatrix} a_{t_r} - a\big(\tilde{f}^+_{p+1}(a,b) - \tilde{f}_{p+1}(a,b)\big) \\ b_{t_r} + \tilde{g}_{p+1}(a,b) - \tilde{g}^-_{p+1}(a,b) \end{pmatrix} = 0, \\
&(a,b)\big|_{t_r=t_{0,r}} = \big(a^{(0)}, b^{(0)}\big),
\end{aligned} \tag{6.70}$$

$$\text{s-Tl}_p\left(a^{(0)}, b^{(0)}\right) = \begin{pmatrix} -a\big({f^+}_{p+1}\big(p^{(0)}, q^{(0)}\big) - f_{p+1}\big(p^{(0)}, q^{(0)}\big)\big) \\ g_{p+1}\big(a^{(0)}, b^{(0)}\big) - g^-_{p+1}\big(a^{(0)}, b^{(0)}\big) \end{pmatrix} = 0, \tag{6.71}$$

$$\text{on } \mathbb{Z} \times (t_{0,r} - T_0, t_{0,r} + T_0),$$

and are given by

$$\begin{aligned}
a(n,t_r)^2 = \frac{1}{2} \sum_{k=1}^{q(n,t_r)} &\frac{\big(d^\ell\big(R_{2p+2}(z)^{1/2}\big)/dz^\ell\big)\big|_{z=\mu_k(n,t_r)}}{(p_k(n,t_r)-1)!} \\
&\times \prod_{k'=1,\, k'\neq k}^{q(n,t_r)} s(\mu_k(n,t_r) - \mu_{k'}(n,t_r))^{-p_k(n,t_r)} \\
+ &\frac{1}{4}\big(b^{(2)}(n,t_r) - b(n,t_r)^2\big),
\end{aligned} \tag{6.72}$$

$$b(n,t_r) = \frac{1}{2}\sum_{m=0}^{2p+1} E_m - \sum_{k=1}^{q(n,t_r)} p_k(n,t_r)\mu_k(n,t_r), \tag{6.73}$$

$$(z,n,t_r) \in \mathbb{Z} \times (t_{0,r} - T_0, t_{0,r} + T_0).$$

Moreover, Lemmas 3.2–3.4 and 5.2–5.5 apply.

As in the stationary case, the theta function representations of a and b in the time-dependent context can be derived in complete analogy to the self-adjoint case. Since the final results are formally the same as in the self-adjoint case we again just refer, for instance, to [6], [7], [9], [10], [14, Sect. 1.4], [18] (cf. also the appendix written in [8]), [25], [30, Appendix, Sect. 9], [32, Sect. 13.2], [33, Sect. 4.6], [34, Ch. 28].

As in Lemma 4.2 we now show that also in the time-dependent case, most initial divisors are nice in the sense that the corresponding divisor trajectory stays away from infinity for all $(n, t_r) \in \mathbb{Z} \times \mathbb{R}$.

Lemma 6.5. *The set $\mathcal{M}_1$ of initial divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}$ for which $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$, defined via* (5.52)*, is nonspecial and finite for all $(n,t_r) \in \mathbb{Z} \times \mathbb{R}$, forms a dense set of full measure in the set $\mathrm{Sym}^p(\mathcal{K}_p)$ of nonnegative divisors of degree p.*

Proof. Let $\mathcal{M}_{\mathrm{sing}}$ be as introduced in the proof of Lemma 4.2. Then

$$\begin{aligned}
&\bigcup_{t_r\in\mathbb{R}} \Big(\underline{\alpha}_{P_0}(\mathcal{M}_{\mathrm{sing}}) + t_r \underline{\widetilde{U}}_r^{(2)}\Big) \\
&\quad = \bigcup_{t_r\in\mathbb{R}} \Big(\underline{A}_{P_0}(P_{\infty_+}) + \underline{\alpha}_{P_0}(\mathrm{Sym}^{p-1}(\mathcal{K}_p)) + t_r \underline{\widetilde{U}}_r^{(2)}\Big)
\end{aligned} \tag{6.74}$$

is of measure zero as well, since it is the image of $\mathbb{R} \times \mathrm{Sym}^{p-1}(\mathcal{K}_p)$ which misses one real dimension in comparison to the $2p$ real dimensions of $J(\mathcal{K}_p)$. But then

$$\bigcup_{(n,t_r)\in\mathbb{Z}\times\mathbb{R}} \Big(\underline{\alpha}_{P_0}(\mathcal{M}) + n\underline{A}_{P_{\infty_-}}(P_{\infty_+}) + t_r \underline{\widetilde{U}}_r^{(2)}\Big) \tag{6.75}$$

is also of measure zero. Applying $\underline{\alpha}_{P_0}^{-1}$ to the complement of the set in (6.75) then yields a set $\mathcal{M}_1$ of full measure in $\mathrm{Sym}^p(\mathcal{K}_p)$. In particular, $\mathcal{M}_1$ is necessarily dense in $\mathrm{Sym}^p(\mathcal{K}_p)$. □

Theorem 6.6. *Let $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})} \in \mathcal{M}_1$ be an initial divisor as in Lemma 6.5. Then the sequences a, b constructed from $\underline{\hat{\mu}}(n_0, t_{0,r})$ as described in Theorem 6.4 satisfy Hypothesis 5.1. In particular, the solution a, b of the algebro-geometric initial value problem* (6.70)*,* (6.71) *is global in $(n, t_r) \in \mathbb{Z} \times \mathbb{R}$.*

Proof. Starting with $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})} \in \mathcal{M}_1$, the procedure outlined in this section and summarized in Theorem 6.4 leads to $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ for all $(n, t_r) \in \mathbb{Z} \times (t_{0,r} - T_0, t_{0,r} + T_0)$ such that (5.52)holds. But if a, b should blow up, then $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ must hit P_{∞_+} which is impossible by our choice of initial condition. □

Note, however, that in general (i.e., unless one is, e.g., in the special periodic or self-adjoint case), $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ will get arbitrarily close to P_{∞_+} since straight motions on the torus are generically dense (see e.g. [2, Sect. 51] or [17, Sects. 1.4, 1.5]) and hence no uniform bound on the sequences $a(n, t_r), b(n, t_r)$ exists as (n, t_r) vary in $\mathbb{Z} \times \mathbb{R}$. In particular, these complex-valued algebro-geometric solutions of the Toda hierarchy initial value problem, in general, will not be quasiperiodic (cf. the usual definition of quasiperiodic functions, e.g., in [31, p. 31]) with respect to n or t_r.

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Appendix A. Hyperelliptic Curves of the Toda-Type

We provide a brief summary of some of the fundamental notations needed from the theory of hyperelliptic Riemann surfaces. More details can be found in some of the standard textbooks [11] and [27] as well as in monographs and surveys dedicated to integrable systems such as [5, Ch. 2], [8], [13, App. A, B], [32, App. A].

Fix $p \in \mathbb{N}$. We intend to describe the hyperelliptic Riemann surface $\mathcal{K}_p$ of genus p of the Toda-type curve (2.43), associated with the polynomial

$$\begin{aligned} &\mathcal{F}_p(z,y) = y^2 - R_{2p+2}(z) = 0, \\ &R_{2p+2}(z) = \prod_{m=0}^{2p+1} (z - E_m), \quad \{E_m\}_{m=0}^{2p+1} \subset \mathbb{C}. \end{aligned} \tag{A.1}$$

To simplify the discussion we will assume that the affine part of $\mathcal{K}_p$ is nonsingular, that is, we assume that

$$E_m \neq E_{m'} \text{ for } m \neq m', \ m, m' = 0, \dots, 2p+1 \tag{A.2}$$

throughout this appendix. Next we introduce an appropriate set of (nonintersecting) cuts $\mathcal{C}_j$ joining $E_{m(j)}$ and $E_{m'(j)}$, $j = 1, \dots, p+1$, and denote

$$\mathcal{C} = \bigcup_{j=1}^{p+1} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k. \tag{A.3}$$

Define the cut plane

$$\Pi = \mathbb{C} \setminus \mathcal{C}, \tag{A.4}$$

and introduce the holomorphic function

$$R_{2p+2}(\,\cdot\,)^{1/2} \colon \Pi \to \mathbb{C}, \quad z \mapsto \bigg(\prod_{m=0}^{2p+1} (z - E_m) \bigg)^{1/2} \tag{A.5}$$

on Π with an appropriate choice of the square root branch in (A.5). Next we define the set

$$\mathcal{M}_p = \{(z, \sigma R_{2p+2}(z)^{1/2}) \mid z \in \mathbb{C}, \ \sigma \in \{1, -1\}\} \cup \{P_{\infty_+}, P_{\infty_-}\} \tag{A.6}$$

by extending $R_{2p+2}(\,\cdot\,)^{1/2}$ to $\mathcal{C}$. The hyperelliptic curve $\mathcal{K}_p$ is then the set $\mathcal{M}_p$ with its natural complex structure obtained upon gluing the two sheets of $\mathcal{M}_p$ crosswise along the cuts. Moreover, we introduce the set of branch points

$$\mathcal{B}(\mathcal{K}_p) = \{(E_m, 0)\}_{m=0}^{2p+1}. \tag{A.7}$$

Points $P \in \mathcal{K}_p \setminus \{P_{\infty_\pm}\}$ are denoted by

$$P = (z, \sigma R_{2p+2}(z)^{1/2}) = (z, y), \tag{A.8}$$

where $y(P)$ denotes the meromorphic function on $\mathcal{K}_p$ satisfying $\mathcal{F}_p(z,y) = y^2 - R_{2p+2}(z) = 0$ and

$$y(P) \underset{\zeta \to 0}{=} \mp \bigg(1 - \frac{1}{2}\bigg(\sum_{m=0}^{2p+1} E_m\bigg)\zeta + O(\zeta^2)\bigg)\zeta^{-p-1} \text{ as } P \to P_{\infty_\pm}, \, \zeta = 1/z. \tag{A.9}$$

In addition, we introduce the holomorphic sheet exchange map (involution)

$$* \colon \mathcal{K}_p \to \mathcal{K}_p, \quad P = (z,y) \mapsto P^* = (z, -y), \ P_{\infty_\pm} \mapsto P_{\infty_\pm}^* = P_{\infty_\mp} \tag{A.10}$$

and the two meromorphic projection maps

$$\tilde{\pi} \colon \mathcal{K}_p \to \mathbb{C} \cup \{\infty\}, \quad P = (z,y) \mapsto z, \ P_{\infty_\pm} \mapsto \infty \tag{A.11}$$

and

$$y \colon \mathcal{K}_p \to \mathbb{C} \cup \{\infty\}, \quad P = (z,y) \mapsto y, \ P_{\infty_\pm} \mapsto \infty. \tag{A.12}$$

Thus the map $\tilde{\pi}$ has a pole of order 1 at $P_{\infty_\pm}$ and y has a pole of order $p+1$ at $P_{\infty_\pm}$. Moreover,

$$\tilde{\pi}(P^*) = \tilde{\pi}(P), \quad y(P^*) = -y(P), \quad P \in \mathcal{K}_p. \tag{A.13}$$

As a result, $\mathcal{K}_p$ is a two-sheeted branched covering of the Riemann sphere $\mathbb{CP}^1$ ($\cong \mathbb{C} \cup \{\infty\}$) branched at the $2p+4$ points $\{(E_m, 0)\}_{m=0}^{2p+1}, P_{\infty_\pm}$. $\mathcal{K}_p$ is compact since $\tilde{\pi}$ is open and $\mathbb{CP}^1$ is compact. Therefore, the compact hyperelliptic Riemann surface resulting in this manner has topological genus p.

Next we introduce the upper and lower sheets $\Pi_\pm$ by

$$\Pi_\pm = \{(z, \pm R_{2p+2}(z)^{1/2}) \in \mathcal{M}_p \mid z \in \Pi\} \tag{A.14}$$

and the associated charts

$$\zeta_\pm \colon \Pi_\pm \to \Pi, \quad P \mapsto z. \tag{A.15}$$

Let $\{a_j, b_j\}_{j=1}^p$ be a homology basis for $\mathcal{K}_p$ with intersection matrix of the cycles satisfying

$$a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j,k = 1, \dots, p. \tag{A.16}$$

Associated with the homology basis $\{a_j, b_j\}_{j=1}^p$ we also recall the canonical dissection of $\mathcal{K}_p$ along its cycles yielding the simply connected interior $\widehat{\mathcal{K}}_p$ of the fundamental polygon $\partial\widehat{\mathcal{K}}_p$ given by

$$\partial\widehat{\mathcal{K}}_p = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_p^{-1} b_p^{-1}. \tag{A.17}$$

Let $\mathcal{M}(\mathcal{K}_n)$ and $\mathcal{M}^1(\mathcal{K}_n)$ denote the set of meromorphic functions (0-forms) and meromorphic differentials (1-forms) on $\mathcal{K}_n$. The residue of a meromorphic differential $\nu \in \mathcal{M}^1(\mathcal{K}_n)$ at a point $Q \in \mathcal{K}_n$ is defined by

$$\mathrm{res}_Q(\nu) = \frac{1}{2\pi i} \int_{\gamma_Q} \nu, \tag{A.18}$$

where γ_Q is a counterclockwise oriented smooth simple closed contour encircling Q but no other pole of ν. Holomorphic differentials are also called Abelian differentials of the first kind. Abelian differentials of the second kind $\omega^{(2)} \in \mathcal{M}^1(\mathcal{K}_n)$ are characterized by the property that all their residues vanish. They will usually be normalized by demanding that all their a-periods vanish, that is,

$$\int_{a_j} \omega^{(2)} = 0, \quad j = 1, \dots, n. \tag{A.19}$$

If $\omega^{(2)}_{P_1,n}$ is a differential of the second kind on $\mathcal{K}_n$ whose only pole is $P_1 \in \widehat{\mathcal{K}}_n$ with principal part $\zeta^{-n-2}\, d\zeta$, $n \in \mathbb{N}_0$ near P_1 and $\omega_j = \left(\sum_{m=0}^\infty d_{j,m}(P_1)\zeta^m\right) d\zeta$ near P_1, then

$$\frac{1}{2\pi i} \int_{b_j} \omega^{(2)}_{P_1,m} = \frac{d_{j,m}(P_1)}{m+1}, \quad m = 0, 1, \dots \tag{A.20}$$

Using local charts one infers that dz/y is a holomorphic differential on $\mathcal{K}_p$ with zeros of order $p-1$ at $P_{\infty_\pm}$ and hence

$$\eta_j = \frac{z^{j-1} dz}{y}, \quad j = 1, \dots, p, \tag{A.21}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_p$. Introducing the invertible matrix C in $\mathbb{C}^p$

$$C = \big(C_{j,k}\big)_{j,k=1,\dots,p}, \quad C_{j,k} = \int_{a_k} \eta_j, \tag{A.22}$$

$$\underline{c}(k) = (c_1(k), \dots, c_p(k)), \quad c_j(k) = \big(C^{-1}\big)_{j,k}, \quad j,k = 1,\dots,p, \tag{A.23}$$

the normalized differentials ω_j for $j = 1,\dots,p$,

$$\omega_j = \sum_{\ell=1}^{p} c_j(\ell)\eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j,k = 1,\dots,p, \tag{A.24}$$

form a canonical basis for the space of holomorphic differentials on $\mathcal{K}_p$.

In the chart $(U_{P_{\infty_\pm}}, \zeta_{P_{\infty_\pm}})$ induced by $1/\tilde{\pi}$ near $P_{\infty_\pm}$ one infers,

$$\begin{aligned} \underline{\omega} = (\omega_1,\dots,\omega_p) &= \mp \sum_{j=1}^{p} \frac{\underline{c}(j)\zeta^{p-j}d\zeta}{\big(\prod_{m=0}^{2p+1}(1-\zeta E_m)\big)^{1/2}} \\ &= \pm\bigg(\underline{c}(p) + \zeta\bigg(\frac{1}{2}\underline{c}(p)\sum_{m=0}^{2p+1} E_m + \underline{c}(p-1)\bigg) + O(\zeta^2)\bigg)d\zeta \text{ as } P \to P_{\infty_\pm}, \\ &\qquad \zeta = 1/z. \end{aligned} \tag{A.25}$$

The matrix $\tau = \big(\tau_{j,\ell}\big)_{j,\ell=1}^{p}$ in $\mathbb{C}^{p\times p}$ of b-periods defined by

$$\tau_{j,\ell} = \int_{b_j} \omega_\ell, \quad j,\ell = 1,\dots,p \tag{A.26}$$

satisfies

$$\operatorname{Im}(\tau) > 0 \text{ and } \tau_{j,\ell} = \tau_{\ell,j}, \ j,\ell = 1,\dots,p. \tag{A.27}$$

Associated with the matrix τ one introduces the period lattice

$$L_p = \{\underline{z} \in \mathbb{C}^p \mid \underline{z} = \underline{m} + \underline{n}\tau, \ \underline{m},\underline{n} \in \mathbb{Z}^p\} \tag{A.28}$$

and the Riemann theta function associated with $\mathcal{K}_n$ and the given homology basis $\{a_j, b_j\}_{j=1,\dots,n}$,

$$\theta(\underline{z}) = \sum_{\underline{n}\in\mathbb{Z}^n} \exp\big(2\pi i(\underline{n},\underline{z}) + \pi i(\underline{n},\underline{n}\tau)\big), \quad \underline{z} \in \mathbb{C}^n, \tag{A.29}$$

where $(\underline{u},\underline{v}) = \overline{\underline{u}}\,\underline{v}^\top = \sum_{j=1}^{n} \overline{u_j}\, v_j$ denotes the scalar product in $\mathbb{C}^n$. It has the fundamental properties

$$\theta(z_1,\dots,z_{j-1},-z_j,z_{j+1},\dots,z_n) = \theta(\underline{z}), \tag{A.30}$$

$$\theta(\underline{z}+\underline{m}+\underline{n}\tau) = \exp\big(-2\pi i(\underline{n},\underline{z}) - \pi i(\underline{n},\underline{n}\tau)\big)\theta(\underline{z}), \quad \underline{m},\underline{n} \in \mathbb{Z}^n. \tag{A.31}$$

Next, fixing a base point $Q_0 \in \mathcal{K}_p \setminus \{P_{\infty_\pm}\}$, one denotes by $J(\mathcal{K}_p) = \mathbb{C}^p/L_p$ the Jacobi variety of $\mathcal{K}_p$, and defines the Abel map $\underline{A}_{Q_0}$ by

$$\underline{A}_{Q_0}\colon \mathcal{K}_n \to J(\mathcal{K}_p), \quad \underline{A}_{Q_0}(P) = \bigg(\int_{Q_0}^{P}\omega_1,\dots,\int_{Q_0}^{P}\omega_p\bigg) \pmod{L_p}, \quad P \in \mathcal{K}_p. \tag{A.32}$$

Similarly, one introduces

$$\underline{\alpha}_{Q_0}\colon \operatorname{Div}(\mathcal{K}_p) \to J(\mathcal{K}_p), \quad \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P\in\mathcal{K}_p} \mathcal{D}(P)\underline{A}_{Q_0}(P), \tag{A.33}$$

where $\operatorname{Div}(\mathcal{K}_p)$ denotes the set of divisors on $\mathcal{K}_p$. Here a map $\mathcal{D}\colon \mathcal{K}_p \to \mathbb{Z}$ is called a divisor on $\mathcal{K}_p$ if $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_p$. (In the main body of this paper we will choose Q_0 to be one of the branch points, i.e., $Q_0 \in \mathcal{B}(\mathcal{K}_p)$, and for simplicity we will always choose the same path of integration from Q_0 to P in all Abelian integrals.)

In connection with divisors on $\mathcal{K}_p$ we will employ the following (additive) notation,

$$\begin{aligned} &\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_m}, \\ &\underline{Q} = \{Q_1, \dots, Q_m\} \in \operatorname{Sym}^m \mathcal{K}_p, \quad Q_0 \in \mathcal{K}_p, \ m \in \mathbb{N}, \end{aligned} \tag{A.34}$$

where for any $Q \in \mathcal{K}_p$,

$$\mathcal{D}_Q\colon \mathcal{K}_p \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_p \setminus \{Q\}, \end{cases} \tag{A.35}$$

and $\operatorname{Sym}^m \mathcal{K}_p$ denotes the mth symmetric product of $\mathcal{K}_p$. In particular, $\operatorname{Sym}^m \mathcal{K}_p$ can be identified with the set of nonnegative divisors $0 \leq \mathcal{D} \in \operatorname{Div}(\mathcal{K}_p)$ of degree $m \in \mathbb{N}$. A divisor $\mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_m}$ will be called finite if $Q_k \in \mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}\}$, $k = 1, \dots, m$.

For $f \in \mathcal{M}(\mathcal{K}_p) \setminus \{0\}$, $\omega \in \mathcal{M}^1(\mathcal{K}_p) \setminus \{0\}$ the divisors of f and ω are denoted by (f) and (ω), respectively. Two divisors $\mathcal{D}$, $\mathcal{E} \in \operatorname{Div}(\mathcal{K}_p)$ are called equivalent, denoted by $\mathcal{D} \sim \mathcal{E}$, if and only if $\mathcal{D} - \mathcal{E} = (f)$ for some $f \in \mathcal{M}(\mathcal{K}_p) \setminus \{0\}$. The divisor class $[\mathcal{D}]$ of $\mathcal{D}$ is then given by $[\mathcal{D}] = \{\mathcal{E} \in \operatorname{Div}(\mathcal{K}_p) \mid \mathcal{E} \sim \mathcal{D}\}$. We recall that

$$\deg((f)) = 0, \quad \deg((\omega)) = 2(p-1), \quad f \in \mathcal{M}(\mathcal{K}_p)\setminus\{0\}, \ \omega \in \mathcal{M}^1(\mathcal{K}_p)\setminus\{0\}, \tag{A.36}$$

where the degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is given by $\deg(\mathcal{D}) = \sum_{P \in \mathcal{K}_p} \mathcal{D}(P)$. It is customary to call (f) (respectively, (ω)) a principal (respectively, canonical) divisor.

Introducing the complex linear spaces

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(\mathcal{K}_p) \mid f = 0 \text{ or } (f) \geq \mathcal{D}\}, \quad r(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}(\mathcal{D}), \tag{A.37}$$

$$\mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(\mathcal{K}_p) \mid \omega = 0 \text{ or } (\omega) \geq \mathcal{D}\}, \quad i(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}^1(\mathcal{D}) \tag{A.38}$$

(with $i(\mathcal{D})$ the index of specialty of $\mathcal{D}$), one infers that $\deg(\mathcal{D})$, $r(\mathcal{D})$, and $i(\mathcal{D})$ only depend on the divisor class $[\mathcal{D}]$ of $\mathcal{D}$. Moreover, we recall the following fundamental facts.

Theorem A.1. *Let $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_p)$, $\omega \in \mathcal{M}^1(\mathcal{K}_p) \setminus \{0\}$. Then,*

$$i(\mathcal{D}) = r(\mathcal{D} - (\omega)), \quad p \in \mathbb{N}_0. \tag{A.39}$$

The Riemann-Roch theorem reads

$$r(-\mathcal{D}) = \deg(\mathcal{D}) + i(\mathcal{D}) - p + 1, \quad n \in \mathbb{N}_0. \tag{A.40}$$

By Abel's theorem, $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_p)$, $p \in \mathbb{N}$, is principal if and only if

$$\deg(\mathcal{D}) = 0 \text{ and } \underline{\alpha}_{Q_0}(\mathcal{D}) = \underline{0}. \tag{A.41}$$

Finally, assume $p \in \mathbb{N}$. Then $\underline{\alpha}_{Q_0} : \operatorname{Div}(\mathcal{K}_p) \to J(\mathcal{K}_p)$ is surjective (Jacobi's inversion theorem).

Theorem A.2. *Let* $\mathcal{D}_{\underline{Q}} \in \operatorname{Sym}^p \mathcal{K}_p$, $\underline{Q} = \{Q_1, \dots, Q_p\}$. *Then,*

$$1 \le i(\mathcal{D}_{\underline{Q}}) = s \tag{A.42}$$

if and only if there are s *pairs of the type* $\{P, P^*\} \subseteq \{Q_1, \dots, Q_p\}$ (*this includes, of course, branch points for which* $P = P^*$). *Obviously, one has* $s \le p/2$.

Next, we denote by $\underline{\Xi}_{Q_0} = (\Xi_{Q_{0,1}}, \dots, \Xi_{Q_{0,p}})$ the vector of Riemann constants,

$$\Xi_{Q_{0,j}} = \frac{1}{2}(1 + \tau_{j,j}) - \sum_{\substack{\ell=1 \\ \ell \ne j}}^{p} \int_{a_\ell} \omega_\ell(P) \int_{Q_0}^{P} \omega_j, \quad j = 1, \dots, p. \tag{A.43}$$

Theorem A.3. *Let* $\underline{Q} = \{Q_1, \dots, Q_p\} \in \operatorname{Sym}^p \mathcal{K}_p$ *and assume* $\mathcal{D}_{\underline{Q}}$ *to be nonspecial, that is,* $i(\mathcal{D}_{\underline{Q}}) = 0$. *Then,*

$$\theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \alpha_{Q_0}(\mathcal{D}_{\underline{Q}})\big) = 0 \text{ if and only if } P \in \{Q_1, \dots, Q_p\}. \tag{A.44}$$

Appendix B. Some Interpolation Formulas

In this appendix we recall a useful interpolation formula which goes beyond the standard Lagrange interpolation formula for polynomials in the sense that the zeros of the interpolating polynomial need not be distinct.

Lemma B.1. *Let* $p \in \mathbb{N}$ *and* S_{p-1} *be a polynomial of degree* $p-1$. *In addition, let* F_p *be a monic polynomial of degree* p *of the form*

$$F_p(z) = \prod_{k=1}^{q} (z - \mu_k)^{p_k}, \quad p_j \in \mathbb{N},\ \mu_j \in \mathbb{C},\ j = 1, \dots, q, \quad \sum_{k=1}^{q} p_k = p. \tag{B.1}$$

Then,

$$\begin{aligned} S_{p-1}(z) &= F_p(z) \sum_{k=1}^{q} \sum_{\ell=0}^{p_k-1} \frac{S_{p-1}^{(\ell)}(\mu_k)}{\ell!(p_k - \ell - 1)!} \\ &\quad \times \bigg(\frac{d^{p_k-\ell-1}}{d\zeta^{p_k-\ell-1}} \bigg((z-\zeta)^{-1} \prod_{k'=1,\, k'\ne k}^{q} (\zeta - \mu_{k'})^{-p_{k'}}\bigg)\bigg)\bigg|_{\zeta=\mu_k}, \quad z \in \mathbb{C}. \end{aligned} \tag{B.2}$$

In particular, S_{p-1} *is uniquely determined by prescribing the* p *values*

$$S_{p-1}(\mu_k), S'_{p-1}(\mu_k), \dots, S_{p-1}^{(p_k-1)}(\mu_k), \quad k = 1, \dots, q, \tag{B.3}$$

at the given points $\mu_1 \dots, \mu_q$.
Conversely, prescribing the p *complex numbers*

$$\alpha_k^{(0)}, \alpha_k^{(1)}, \dots, \alpha_k^{(p_k-1)}, \quad k = 1, \dots, q, \tag{B.4}$$

there exists a unique polynomial T_{p-1} *of degree* $p-1$,

$$T_{p-1}(z) = F_p(z) \sum_{k=1}^{q} \sum_{\ell=0}^{p_k-1} \frac{\alpha_k^{(\ell)}}{\ell!(p_k - \ell - 1)!} \tag{B.5}$$

$$\times \bigg(\frac{d^{p_k-\ell-1}}{d\zeta^{p_k-\ell-1}} \bigg((z-\zeta)^{-1} \prod_{k'=1,\, k'\ne k}^{q} (\zeta - \mu_{k'})^{-p_{k'}}\bigg)\bigg)\bigg|_{\zeta=\mu_k}, \quad z \in \mathbb{C}, \tag{B.6}$$

such that

$$T_{p-1}(\mu_k) = \alpha_k^{(0)}, T'_{p-1}(\mu_k) = \alpha_k^{(1)}, \dots, T_{p-1}^{(p_k-1)}(\mu_k) = \alpha_k^{(p_k-1)}, \quad k = 1, \dots, q. \tag{B.7}$$

Proof. Our starting point for proving (B.2) is the following formula derived, for instance, by Markushevich [22, Part 2, Sect. 2.11, p. 68],

$$S_{p-1}(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{d\zeta\, S_{p-1}(\zeta)}{F_p(\zeta)} \frac{F_p(\zeta) - F_p(z)}{\zeta - z}, \quad z \in \mathbb{C}, \tag{B.8}$$

where Γ is a simple, smooth, counterclockwise oriented curve encircling $\mu_1, \dots, \mu_q$ strictly in its interior. Since the integrand in (B.8) is analytic at the point $\zeta = z$, we may, without loss of generality, assume that Γ does not encircle z. With this assumption one obtains

$$\frac{1}{2\pi i} \oint_\Gamma \frac{d\zeta\, S_{p-1}(\zeta)}{\zeta - z} = 0 \tag{B.9}$$

and hence deforming Γ into sufficiently small counterclockwise oriented circles Γ_k with center at μ_k, $k = 1, \dots, q$, such that no $\mu_{k'}$, $k' \neq k$, is encircled by Γ_k, one obtains

$$\begin{aligned}
S_{p-1}(z) &= -\frac{F_p(z)}{2\pi i} \oint_\Gamma \frac{d\zeta\, S_{p-1}(\zeta)}{F_p(\zeta)(\zeta - z)} \\
&= -\frac{F_p(z)}{2\pi i} \sum_{k=1}^q \oint_{\Gamma_k} \frac{d\zeta\, S_{p-1}(\zeta)}{F_p(\zeta)(\zeta - z)} \\
&= -\frac{F_p(z)}{2\pi i} \sum_{k=1}^q \sum_{\ell=0}^{p-1} \frac{S_{p-1}^{(\ell)}(\mu_k)}{\ell!} \oint_{\Gamma_k} \frac{d\zeta\, (\zeta - \mu_k)^\ell}{F_p(\zeta)(\zeta - z)} \\
&= -\frac{F_p(z)}{2\pi i} \sum_{k=1}^q \sum_{\ell=0}^{p-1} \frac{S_{p-1}^{(\ell)}(\mu_k)}{\ell!} \oint_{\Gamma_k} \frac{d\zeta\, (\zeta - \mu_k)^\ell}{(\zeta - z) \prod_{k'=1}^q (\zeta - \mu_{k'})^{p_{k'}}} \\
&= -\frac{F_p(z)}{2\pi i} \sum_{k=1}^q \sum_{\ell=0}^{p-1} \frac{S_{p-1}^{(\ell)}(\mu_k)}{\ell!} \oint_{\Gamma_k} \frac{d\zeta\, (\zeta - \mu_k)^{\ell - p_k}}{(\zeta - z) \prod_{\substack{k'=1 \\ k' \neq k}}^q (\zeta - \mu_{k'})^{p_{k'}}} \\
&= -\frac{F_p(z)}{2\pi i} \sum_{k=1}^q \sum_{\ell=0}^{p_k-1} \frac{S_{p-1}^{(\ell)}(\mu_k)}{\ell!} \oint_{\Gamma_k} \frac{d\zeta\, (\zeta - \mu_k)^{\ell - p_k}}{(\zeta - z) \prod_{\substack{k'=1 \\ k' \neq k}}^q (\zeta - \mu_{k'})^{p_{k'}}},
\end{aligned} \tag{B.10}$$

where we used

$$\oint_{\Gamma_k} d\zeta\, (\zeta - \mu_k)^{\ell - p_k} f(\zeta) = 0 \text{ if } \ell \geq p_k,\ \ell \in \mathbb{N}, \tag{B.11}$$

for any function f analytic in a neighborhood of the disk D_k with boundary Γ_k, $k = 1, \dots, q$, to arrive at the last line of (B.10). An application of Cauchy's formula for derivatives of analytic functions to (B.10) then yields

$$\begin{aligned}
S_{p-1}(z) = -F_p(z) &\sum_{k=1}^q \sum_{\ell=0}^{p_k-1} \frac{S_{p-1}^{(\ell)}(\mu_k)}{\ell!} \\
&\times \frac{1}{2\pi i} \oint_{\Gamma_k} d\zeta \frac{1}{(\zeta - \mu_k)^{(p_k - \ell - 1) + 1}} \frac{1}{(\zeta - z) \prod_{k'=1,\, k' \neq k}^q (\zeta - \mu_{k'})^{p_{k'}}}
\end{aligned}$$

$$= F_p(z) \sum_{k=1}^{q} \sum_{\ell=0}^{p_k-1} \frac{S_{p-1}^{(\ell)}(\mu_k)}{\ell!(p_k-\ell-1)!} \\ \times \left(\frac{d^{p_k-\ell-1}}{d\zeta^{p_k-\ell-1}} \left(\frac{1}{(z-\zeta)\prod_{k'=1,\,k'\neq k}^{q}(\zeta-\mu_{k'})^{p_{k'}}} \right) \right) \bigg|_{\zeta=\mu_k}, \quad z \in \mathbb{C}, \tag{B.12}$$

and hence (B.2). Conversely, a linear algebraic argument shows that any polynomial T_{p-1} of degree $p-1$ is uniquely determined by data of the type

$$T_{p-1}(\mu_k), T'_{p-1}(\mu_k), \dots, T_{p-1}^{(p_k-1)}(\mu_k), \quad k = 1, \dots, q. \tag{B.13}$$

Uniqueness of the representation (B.2) then proves (B.5). □

We briefly mention two special cases of (B.2). First, assume the generic case where all zeros of F_p are distinct, that is,

$$q = p, \quad p_k = 1, \quad \mu_k \neq \mu_{k'} \text{ for } k \neq k',\ k, k' = 1, \dots, p. \tag{B.14}$$

In this case (B.2) reduces to the classical Lagrange interpolation formula

$$S_{p-1}(z) = F_p(z) \sum_{k=1}^{p} \frac{S_{p-1}(\mu_k)}{((dF_p(\zeta)/d\zeta)|_{\zeta=\mu_k})(z-\mu_k)}, \quad z \in \mathbb{C}. \tag{B.15}$$

Second, we consider the other extreme case where all zeros of F_p coincide, that is,

$$q = 1, \quad p_1 = p, \quad F_p(z) = (z-\mu_1)^p, \quad z \in \mathbb{C}. \tag{B.16}$$

In this case (B.2) reduces of course to the Taylor expansion of S_{p-1} around $z = \mu_1$

$$S_{p-1}(z) = \sum_{\ell=0}^{p-1} \frac{S_{p-1}^{(\ell)}(\mu_1)}{\ell!}(z-\mu_1)^{\ell}, \quad z \in \mathbb{C}. \tag{B.17}$$

Appendix C. Asymptotic Spectral Parameter Expansions and Nonlinear Recursion Relations

In this appendix we discuss asymptotic spectral parameter expansions for F_p/y and G_{p+1}/y as well as nonlinear recursion relations for the corresponding homogeneous coefficients $\hat{f}_\ell$ and $\hat{g}_\ell$ and analogous quantities fundamental to the polynomial recursion formalism for the Toda hierarchy.

We start by recalling the following elementary results (which are consequences of the binomial expansion). Let

$$\{E_m\}_{m=0,\dots,2p+1} \subset \mathbb{C} \text{ for some } p \in \mathbb{N}_0 \\ \text{and } \eta \in \mathbb{C} \text{ such that } |\eta| < \min\{|E_0|^{-1}, \dots, |E_{2p+1}|^{-1}\}. \tag{C.1}$$

Then

$$\left(\prod_{m=0}^{2p+1} \big(1 - E_m \eta\big) \right)^{-1/2} = \sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) \eta^k, \tag{C.2}$$

where

$$\hat{c}_0(\underline{E}) = 1,$$

$$\hat{c}_k(\underline{E}) = \sum_{\substack{j_0,\dots,j_{2p+1}=0\\ j_0+\cdots+j_{2p+1}=k}}^{k} \frac{(2j_0)!\cdots(2j_{2p+1})!}{2^{2k}(j_0!)^2\cdots(j_{2p+1}!)^2} E_0^{j_0}\cdots E_{2p+1}^{j_{2p+1}}, \quad k\in\mathbb{N}. \tag{C.3}$$

The first few coefficients explicitly read

$$\begin{aligned} &\hat{c}_0(\underline{E}) = 1, \quad \hat{c}_1(\underline{E}) = \frac{1}{2}\sum_{m=0}^{2p+1} E_m, \\ &\hat{c}_2(\underline{E}) = \frac{1}{4}\sum_{\substack{m_1,m_2=0\\ m_1<m_2}}^{2p+1} E_{m_1}E_{m_2} + \frac{3}{8}\sum_{m=0}^{2p+1} E_m^2, \quad \text{etc.} \end{aligned} \tag{C.4}$$

Similarly,

$$\bigg(\prod_{m=0}^{2p+1}\big(1-E_m\eta\big)\bigg)^{1/2} = \sum_{k=0}^{\infty} c_k(\underline{E})\eta^k, \tag{C.5}$$

where

$$\begin{aligned} &c_0(\underline{E}) = 1, \\ &c_k(\underline{E}) = \sum_{\substack{j_0,\dots,j_{2p+1}=0\\ j_0+\cdots+j_{2p+1}=k}}^{k} \frac{(2j_0)!\cdots(2j_{2p+1})!\, E_0^{j_0}\cdots E_{2p+1}^{j_{2p+1}}}{2^{2k}(j_0!)^2\cdots(j_{2p+1}!)^2(2j_0-1)\cdots(2j_{2p+1}-1)}, \quad k\in\mathbb{N}. \end{aligned} \tag{C.6}$$

The first few coefficients are given explicitly by

$$\begin{aligned} &c_0(\underline{E}) = 1, \quad c_1(\underline{E}) = -\frac{1}{2}\sum_{m=0}^{2p+1} E_m, \\ &c_2(\underline{E}) = \frac{1}{4}\sum_{\substack{m_1,m_2=0\\ m_1<m_2}}^{2p+1} E_{m_1}E_{m_2} - \frac{1}{8}\sum_{m=0}^{2p+1} E_m^2, \quad \text{etc.} \end{aligned} \tag{C.7}$$

Theorem C.1. *Assume* (2.1), $\text{s-Tl}_p(a,b)=0$, *and suppose* $P=(z,y)\in\mathcal{K}_p\setminus\{P_{\infty_+},P_{\infty_-}\}$. *Then* F_p/y *and* G_{p+1}/y *have the following convergent expansions as* $P\to P_{\infty_\pm}$,

$$\frac{F_p(z)}{y} = \mp\sum_{\ell=0}^{\infty}\hat{f}_\ell\zeta^{\ell+1}, \quad \frac{G_{p+1}(z)}{y} = \mp\sum_{\ell=-1}^{\infty}\hat{g}_\ell\zeta^{\ell+1}, \tag{C.8}$$

where $\zeta=1/z$ *is the local coordinate near* $P_{\infty_\pm}$ *and* $\hat{f}_\ell$ *and* $\hat{g}_\ell$ *are the homogeneous coefficients* f_ℓ *and* g_ℓ *introduced in* (2.8). *In particular,* $\hat{f}_\ell$ *and* $\hat{g}_\ell$ *can be computed from the nonlinear recursion relations*

$$\begin{aligned} &\hat{f}_0 = 1, \quad \hat{f}_1 = -b, \quad \hat{f}_2 = a^2+(a^-)^2+b^2, \\ &\hat{f}_{\ell+2} = -\frac{1}{2}\sum_{k=1}^{\ell+1}\hat{f}_{\ell+2-k}\hat{f}_k - 2b\sum_{k=0}^{\ell+1}\hat{f}_{\ell+1-k}\hat{f}_k \\ &\qquad + \sum_{k=0}^{\ell}\big(-3b^2\hat{f}_{\ell-k}\hat{f}_k + a^2\hat{f}^+_{\ell-k}\hat{f}_k + (a^-)^2\hat{f}_{\ell-k}\hat{f}^-_k\big) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{\ell-1}\big(-2b^3\hat f_{\ell-1-k}\hat f_k+2a^2b\hat f^+_{\ell-1-k}\hat f_k+2(a^-)^2b\hat f_{\ell-1-k}\hat f^-_k\big)\\
&+\sum_{k=0}^{\ell-2}\big(a^2b^2\hat f^+_{\ell-2-k}\hat f_k+(a^-)^2b^2\hat f_{\ell-2-k}\hat f^-_k+a^2(a^-)^2\hat f^+_{\ell-2-k}\hat f^-_k\\
&\qquad-\frac12 a^4\hat f^+_{\ell-2-k}\hat f^+_k-\frac12(a^-)^4\hat f^-_{\ell-2-k}\hat f^-_k\big),\quad \ell\in\mathbb{N},
\end{aligned}\tag{C.9}
$$

and

$$
\begin{aligned}
&\hat g_{-1}=-1,\quad \hat g_0=0,\quad \hat g_1=-2a^2,\\
&\hat g_{\ell+1}=\frac12\sum_{k=-1}^{\ell}(b+b^+)\hat g_{\ell-1-k}\hat g_k+\frac12\sum_{k=0}^{\ell}\hat g_{\ell-k}\hat g_k\\
&\quad+\frac12\sum_{k=-1}^{\ell-1}\big(bb^+\hat g_{\ell-2-k}\hat g_k-a^2(\hat g^-_{\ell-2-k}+\hat g_{\ell-2-k})(\hat g_k+\hat g^+_k)\big),\quad \ell\in\mathbb{N}.
\end{aligned}\tag{C.10}
$$

Moreover, one infers for the E_m-dependent summation constants c_ℓ, $\ell=0,\dots,p+1$, in F_p and G_{p+1} that

$$c_\ell=c_\ell(\underline{E}),\quad \ell=0,\dots,p+1 \tag{C.11}$$

and[3]

$$f_\ell=\sum_{k=0}^{\ell}c_{\ell-k}(\underline{E})\hat f_k,\quad \ell=0,\dots,p, \tag{C.12}$$

$$g_\ell+f_{p+1}\delta_{p,\ell}=\sum_{k=0}^{\ell}c_{\ell-k}(\underline{E})\hat g_k-c_{\ell+1}(\underline{E}),\quad \ell=0,\dots,p, \tag{C.13}$$

$$\hat f_\ell=\sum_{k=0}^{\ell\wedge p}\hat c_{\ell-k}(\underline{E})f_k,\quad \ell\in\mathbb{N}_0, \tag{C.14}$$

$$\hat g_\ell=\sum_{k=0}^{\ell\wedge p}\hat c_{\ell-k}(\underline{E})(g_k+f_{p+1}\delta_{p,k})-\hat c_{\ell+1}(\underline{E}),\quad \ell\in\mathbb{N}_0. \tag{C.15}$$

Proof. Dividing F_p and G_{p+1} by $R_{2p+2}^{1/2}$ (temporarily fixing the branch of $R_{2p+2}^{1/2}$ as z^{p+1} near infinity) one obtains

$$\frac{F_p(z)}{R_{2p+2}(z)^{1/2}}\underset{|z|\to\infty}{=}\bigg(\sum_{k=0}^{\infty}\hat c_k(\underline{E})z^{-k}\bigg)\bigg(\sum_{\ell=0}^{p}f_\ell z^{-\ell-1}\bigg)=\sum_{\ell=0}^{\infty}\check f_\ell z^{-\ell-1}, \tag{C.16}$$

$$\frac{G_{p+1}(z)}{R_{2p+2}(z)^{1/2}}\underset{|z|\to\infty}{=}\bigg(\sum_{k=0}^{\infty}\hat c_k(\underline{E})z^{-k}\bigg)\bigg(\sum_{\ell=0}^{p+1}\tilde g_\ell z^{-\ell}\bigg)=z^{-1}\sum_{\ell=-1}^{\infty}\check g_\ell z^{-\ell} \tag{C.17}$$

for some coefficients $\check f_\ell$ and $\check g_\ell$ to be determined next. Here we have temporarily introduced the notation

$$G_{p+1}(z)=-z^{p+1}+\sum_{\ell=0}^{p}g_{p-\ell}z^\ell+f_{p+1}=\sum_{\ell=0}^{p+1}\tilde g_{p-\ell}z^\ell. \tag{C.18}$$

[3] $m\wedge n=\min\{m,n\}$.

Dividing (2.37) and (2.39) by R_{2p+2} and inserting expansions (C.16) and (C.17) into the resulting equations then yield the nonlinear recursion relations (C.9) and (C.10) (with $\hat{f}_\ell$ and $\hat{g}_\ell$ replaced by $\check{f}_\ell$ and $\check{g}_\ell$, respectively). More precisely, one first obtains $|\check{f}_0| = |\check{g}_{-1}| = 1$ and upon choosing the signs of $\check{f}_0$ and $\check{g}_{-1}$ such that $\check{f}_0 = \hat{f}_0 = 1$ and $\check{g}_{-1} = -1$ one obtains (C.9) and (C.10). Next, dividing (2.31) and (2.32) by $R_{2p+2}^{1/2}$, inserting the expansions (C.16) and (C.17), and comparing powers of $z^{-\ell}$ as $z \to \infty$, one infers that $\check{f}_\ell$ and $\check{g}_\ell$ satisfy the linear recursion relations (2.4)–(2.6). Hence one concludes that

$$\check{f}_\ell = f_\ell, \quad \check{g}_\ell = g_\ell, \quad \ell \in \mathbb{N}_0 \tag{C.19}$$

for certain values of the summation constants c_ℓ. To show that $\check{f}_\ell = \hat{f}_\ell$, $\check{g}_\ell = \hat{g}_\ell$, and hence all c_ℓ, $\ell \in \mathbb{N}$, vanish, we recall the notion of degree as used in the proof of Lemma 5.4, which serves as an efficient tool to distinguish between homogeneous and nonhomogeneous quantities. To this end we employ the notation

$$f^{(r)} = S^{(r)} f, \quad f = \{f(n)\}_{n\in\mathbb{Z}} \subset \mathbb{C}, \quad S^{(r)} = \begin{cases} (S^+)^r, & r \geq 0, \\ (S^-)^{-r}, & r < 0, \end{cases} \quad r \in \mathbb{Z} \tag{C.20}$$

and introduce

$$\deg(a^{(r)}) = \deg(b^{(r)}) = 1, \quad r \in \mathbb{Z}. \tag{C.21}$$

This results in

$$\deg(\hat{f}_\ell) = \ell, \quad \deg(\hat{g}_\ell) = \ell + 1, \quad \ell \in \mathbb{N}. \tag{C.22}$$

using induction in the linear recursion relations (2.4)–(2.6). Similarly, the nonlinear recursion relations (C.9) and (C.10) yield inductively that

$$\deg(\check{f}_\ell) = \ell, \quad \deg(\check{g}_\ell) = \ell + 1, \quad \ell \in \mathbb{N}. \tag{C.23}$$

Hence one concludes that

$$\check{f}_\ell = \hat{f}_\ell, \quad \check{g}_\ell = \hat{g}_\ell, \quad \ell \in \mathbb{N}_0. \tag{C.24}$$

A comparison of coefficients in (C.16) proves (C.14). Similarly, we use (C.17) to establish (C.15). Next, multiplying (C.2) and (C.5), a comparison of coefficients of η^k yields

$$\sum_{\ell=0}^{k} \hat{c}_{k-\ell}(\underline{E}) c_\ell(\underline{E}) = \delta_{k,0}, \quad k \in \mathbb{N}_0. \tag{C.25}$$

Thus, one computes

$$\begin{aligned}\sum_{m=0}^{\ell} c_{\ell-m}(\underline{E}) \hat{f}_m &= \sum_{m=0}^{\ell} \sum_{k=0}^{m} c_{\ell-m}(\underline{E}) \hat{c}_{m-k}(\underline{E}) f_k = \sum_{k=0}^{\ell} \sum_{p=k}^{\ell} c_{\ell-p}(\underline{E}) \hat{c}_{p-k}(\underline{E}) f_k \\ &= \sum_{k=0}^{\ell} \bigg(\sum_{m=0}^{\ell-k} c_{\ell-k-m}(\underline{E}) \hat{c}_m(\underline{E}) \bigg) f_k = f_\ell, \quad \ell = 0, \dots, p, \end{aligned} \tag{C.26}$$

applying (C.25). Hence one obtains (C.12) and thus (C.11) (cf. (2.9)). The corresponding proof of (C.13) is similar to that of f_ℓ. $\square$

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