Algebraic Supersymmetry: A case study

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Dedicated to Daniel Kastler on the occasion of his 80th birthday

Abstract

The treatment of supersymmetry is known to cause difficulties in the C*–algebraic framework of relativistic quantum field theory; several no–go theorems indicate that super–derivations and super–KMS functionals must be quite singular objects in a C*–algebraic setting. In order to clarify the situation, a simple supersymmetric chiral field theory of a free Fermi and Bose field defined on $\mathbb{R}$ is analyzed. It is shown that a meaningful C^*–version of this model can be based on the tensor product of a CAR–algebra and a novel version of a CCR–algebra, the "resolvent algebra". The elements of this resolvent algebra serve as mollifiers for the super–derivation. Within this model, unbounded (yet locally bounded) graded KMS–functionals are constructed and proven to be supersymmetric. From these KMS–functionals, Chern characters are obtained by generalizing formulae of Kastler and of Jaffe, Lesniewski and Osterwalder. The characters are used to define cyclic cocycles in the sense of Connes' noncommutative geometry which are "locally entire".

1 Introduction

Graded (super) derivations occur in many parts of physics: supersymmetry, BRS-constraint reduction and cyclic homology, to name a few. To adequately model these in a C*-algebra setting involves notorious domain problems. Kishimoto and Nakamura [16] showed, for example, that apparently natural domain assumptions on the supersymmetry graded derivations lead to an empty theory. Similarly, supersymmetric KMS–functionals underlying the construction of cyclic cocycles as in [15, 11] cannot exist in the case of infinitely extended systems [3]. These obstructions may explain why a general C*-algebraic framework for supersymmetry has not yet emerged. It thus seems worthwhile to explore representative examples in more detail in order to identify the pertinent structures.

In the present article we aim to develop tools to define and analyze in a C*-algebra setting a simple but, with regard to the mathematical problems under investigation, generic supersymmetric quantum field theory. It is the model of a chiral Fermi– and Bose–field, defined on the light ray $\mathbb{R}$. As the construction of this model is easily accomplished in the Wightman setting of (unbounded) quantum fields, we can concentrate here on the specific problems arising in the passage to a C^*–framework.

Although the model has formally the structure of a tensor product of a CAR–algebra and a CCR–algebra, the adequate formulation of its C^*–version requires some care. It turns out that the standard Weyl algebra description of the CCR part is not suitable for the formulation of supersymmetry. We therefore introduce a more viable variant of the CCR–algebra, the resolvent algebra, which formally may be thought of as being generated by the resolvents of the underlying Bose–field. These resolvents act as mollifiers for the super–derivation and allow one to define it on a domain which is weakly dense in the underlying C*–algebra in all representations of interest. The resolvents also lead to a mollified version of the fundamental relation of supersymmetry, relating the square of the super–derivation and the generator of time translations. These rather weak variants of supersymmetry turn out to be sufficient for the further analysis.

Having clarified the C*–algebraic formulation of supersymmetry, one has the necessary tools for the analysis of the supersymmetric KMS–functionals in this model. Again, these functionals are easily constructed in the Wightman setting. Yet, as follows from general arguments [3], they cannot be extended continuously to the full underlying C*–algebra. In fact, one does not have any *a priori* information on their domains of definition.

In the present model, the restrictions of the supersymmetric KMS–functionals to any local subalgebra of the underlying C*–algebra turn out to be bounded. Thus these functionals are densely defined, but their domain of definition does not contain any non–trivial analytic elements with regard to the dynamics, as is required in the construction of cyclic cocycles given in [15, 11]. Nevertheless, by relying on techniques from the theory of analytic functions of several complex variables, it is possible to define cyclic cocycles in the present model as well. The restrictions of these cocycles to any fixed local subalgebra of the underlying C*–algebra turn out to be entire in the sense of Connes [5].

So the present field–theoretic model allows for a satisfactory C*–algebraic formulation of supersymmetry and the analysis of its consequences. There are three observations which are of interest going beyond the present model: First, a C*–algebraic formulation of supersymmetry has to rely on the concept of mollifiers or, complementary, of unbounded operators affiliated with the underlying C*–algebra [9]. Second, there is growing evidence that supersymmetric KMS–functionals, although being unbounded, are locally bounded, in accordance with the heuristic considerations in [3]. And third, although these functionals generically do not have analytic elements in their domain of definition, they can still be used to define local versions of Connes' entire cyclic cocycles by relying on techniques from complex analysis. Based on these insights, a proper C*–algebraic framework for the formulation of supersymmetry and the analysis of its consequences in quantum field theory seems within reach. We hope to return to this problem elsewhere.

The plan of our paper is as follows. We will state our results in the body of the paper, and defer almost all the proofs to the appendix. In Sect. 2 we present in the Wightman framework the basic supersymmetry model which we wish to analyze; in Sect. 3 we prepare for its analysis in a C*–setting by considering algebraic mollifying relations for the quantum fields, which leads to the study of the C*–algebras generated by the resolvents of the fields. In Sect. 4 we use these tools to present the C*–algebraic framework of the model. In Sect. 5 we define (unbounded) graded KMS–functionals on the model and prove basic properties for them, including their supersymmetry invariance and their local boundedness. In Sect. 6 we use these KMS–functionals to define a

Chern character formula (generalizing the construction in [15, 11]), from which we obtain a (locally) entire cyclic cocycle in the sense of Connes. This can then be taken as input to an index theory for supersymmetric quantum field theories, of the type proposed by Longo [18].

2 The model

We begin by presenting here our model in the Wightman framework, which we would like to model in a C*-algebra setting. It is the the simplest example for supersymmetry on noncompact spacetime, in that we have one dimension, one boson and one fermion.

We assume chiral fields, so there is only one space-time dimension, $\mathbb{R}$. The Fermi field is given by the Clifford operators $c(f) = c(f)^*$, where $f \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ and

$$\{c(f),\, c(g)\} = (f, g) := \int fg\, dx\ .$$

The boson field is $j(f) = j(f)^*$, where $f \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ and

$$[j(f),\, j(g)] = i\sigma(f, g) := i \int fg'\, dx\ .$$

The $\mathbb{Z}_2$–grading automorphism γ comes from the Fermi field by

$$\gamma(c(f)) = -c(f)\,, \qquad \gamma(j(f)) = j(f)$$

and defines even and odd parts of the polynomial field algebra by $A_\pm = (A \pm \gamma(A))/2$. The heuristic supercharge $Q := \int c(x)j(x)\, dx$ defines the supersymmetry generator δ as the graded derivation:

$$\delta(A) := [Q,\, A_+] + \{Q,\, A_-\}$$

which satisfies $\delta(AB) = \delta(A)B + \gamma(A)\delta(B)$. Note that on the generating elements of the field algebra we have:

$$\delta(c(f)) = j(f)\,, \qquad \delta(j(f)) = ic(f')\ . \tag{1}$$

Time evolution is given by translation, i.e.

$$\alpha_t(c(f)) := c(f_t) \qquad \alpha_t(j(f)) := j(f_t)$$

where $f_t(x) := f(x - t)$, $x \in \mathbb{R}$. The generator of time evolution is the derivation:

$$\delta_0(c(f)) = ic(f')\,, \qquad \delta_0(j(f)) = ij(f')\ . \tag{2}$$

The supersymmetry relation is valid on the field algebra:

$$\delta^2 = \delta_0\ . \tag{3}$$

Our problem is to realize this structure in a C*-algebra setting. Some problems already arise from the relation $\delta((c(f)) = j(f)$, in which δ takes a bounded operator to an unbounded one. We will deal with this issue in the next section. A deeper source of problems will come from the theorems of Kishimoto and Nakamura [16] which will make it hard to realize the supersymmetry relation (3) on a dense domain.

3 On Mollifiers and Resolvent Algebras

Here we develop tools to handle the unboundedness of the range elements of δ. Recall that a selfadjoint operator A on a Hilbert space $\mathcal{H}$ is *affiliated* with a C*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ if the resolvent $(i\lambda\mathbb{I} - A)^{-1} \in \mathcal{A}$ for some $\lambda \in \mathbb{R}\backslash 0$ (hence for all $\lambda \in \mathbb{R}\backslash 0$). This notion is used by Georgescu [9] e.a. (and is weaker than the one used by Woronowicz [25]) and it implies the usual one, i.e. that A commutes with all unitaries commuting with $\mathcal{A}$ (but not conversely). Observe that

$$A(i\lambda\mathbb{I} - A)^{-1} = \overline{(i\lambda\mathbb{I} - A)^{-1}A} = i\lambda(i\lambda\mathbb{I} - A)^{-1} - \mathbb{I} \in \mathcal{A}\,.$$

Thus the resolvent $(i\lambda\mathbb{I} - A)^{-1} = M$ acts as a "mollifier" for A, i.e. $\overline{MA}$ and AM are bounded and in $\mathcal{A}$, and M is invertible such that $M^{-1}\overline{MA} = A = AMM^{-1}$. This suggests that as AM and $\overline{MA}$ in $\mathcal{A}$ carries the information of A in bounded form, we can "forget" the original representation, and study the affiliated A abstractly through these elements.

We want to apply this idea to a representation of the bosonic fields $j(f) = j(f)^*$, $f \in \mathcal{S}(\mathbb{R})$ where

$$[j(f),\, j(g)] = i\sigma(f,\, g) := i\int f g'\, dx$$

on some common dense invariant core $\mathcal{D} \subset \mathcal{H}$ of the selfadjoint fields $j(f)$. It seems natural to look for mollifiers in the Weyl algebra

$$\overline{\Delta(\mathcal{S},\, \sigma)} = C^*\left\{\exp(ij(f)) \;\mid\; f \in \mathcal{S}(\mathbb{R})\right\}\,,$$

(abstractly $\overline{\Delta(\mathcal{S},\, \sigma)}$ is the C*-algebra generated by a set of unitaries $\{\,\delta_f \mid f \in \mathcal{S}(\mathbb{R})\,\}$ such that $\delta_f^* = \delta_{-f}$ and $\delta_f\delta_g = e^{-i\sigma(f,g)/2}\delta_{f+g}$). Unfortunately this is not possible because:

3.1 Proposition *The Weyl algebra $\overline{\Delta(\mathcal{S},\, \sigma)}$ contains no nonzero element M such that $j(f)M$ is bounded for some $f \in \mathcal{S}(\mathbb{R})\backslash 0$. Thus $\overline{\Delta(\mathcal{S},\, \sigma)}$ contains no mollifier for any nonzero $j(f)$, and $j(f)$ is not affiliated with $\overline{\Delta(\mathcal{S},\, \sigma)}$.*

Proof: Assume that $M \in \overline{\Delta(\mathcal{S},\, \sigma)}$ is nonzero such that $j(f)M$ is bounded for some nonzero $f \in \mathcal{S}(\mathbb{R})$. Let $U(t) := \exp(itj(f))$, and denote the spectral resolution of $j(f)$ by $j(f) = \int \lambda\, dP(\lambda)$, then

$$\begin{aligned}\|(U(t) - \mathbb{I})M\| &= \left\|\int (e^{it\lambda} - 1)dP(\lambda)M\right\| \\ &= |t|\left\|\int \frac{(e^{it\lambda} - 1)}{t\lambda}\, dP(\lambda)\int \lambda'\, dP(\lambda')M\right\| \\ &\leq C|t|\|j(f)M\| \longrightarrow 0\end{aligned}$$

as $t \to 0$, where we used the bound $|\frac{e^{ix}-1}{x}| < C$ for some constant C. Let $\mathcal{J} \subset \overline{\Delta(\mathcal{S},\, \sigma)}$ consist of all elements M such that $\|(U(t) - \mathbb{I})M\| \to 0$ as $t \to 0$. This is clearly a norm-closed linear space, and by the inequality $\|(U(t) - \mathbb{I})MA\| \leq \|(U(t) - \mathbb{I})M\|\;\|A\|$ it is also a right ideal. To see that it is a two sided ideal note that

$$\left\|(U(t) - \mathbb{I})e^{ij(g)}M\right\| = \left\|(U(t)e^{it\sigma(f,g)} - \mathbb{I})M\right\|$$

still converges to 0 as $t \to 0$, and use the fact that $\overline{\Delta(\mathcal{S},\, \sigma)}$ is the norm closure of the span of $\left\{\,e^{ij(g)} \;\mid\; g \in \mathcal{S}(\mathbb{R})\right\}$. But $\overline{\Delta(\mathcal{S},\, \sigma)}$ is simple, hence $\mathcal{J} \ni M$ must be zero. ∎

Our solution is to abandon the Weyl algebra as the appropriate C*-algebra to model the bosonic fields $j(f)$, and instead to choose the unital C*-algebra generated by the resolvents:

$$C^*\{\,\mathbb{I},\ R(\lambda,f)\ \mid\ \lambda\in\mathbb{R}\backslash 0,\ f\in\mathcal{S}(\mathbb{R})\backslash 0\,\}$$

where $R(\lambda,f):=(i\lambda\mathbb{I}-j(f))^{-1}$. Then by construction all $j(f)$ are affiliated to this C*-algebra and it contains mollifiers $R(\lambda,f)$ for all of them.

The above discussion took place in a concrete setting, i.e. represented on a Hilbert space, and we would like to abstract this. Just as the Weyl algebra can be abstractly defined by the Weyl relations, we now want to abstractly define the C*-algebra of resolvents (of the $j(f)$) by generators and relations.

3.2 Definition *Given a symplectic space $(X,\,\sigma)$, we define $\mathcal{R}_0$ to be the universal unital *-algebra generated by the set $\{\,R(\lambda,f)\ \mid\ \lambda\in\mathbb{R}\backslash 0,\ f\in X\backslash 0\,\}$ and the relations*

$$\begin{aligned}
R(\lambda,f)^* &= R(-\lambda,f) && (4)\\
R(\lambda,f) &= \frac{1}{\lambda}R(1,\,\frac{1}{\lambda}f) && (5)\\
R(\lambda,f)-R(\mu,f) &= i(\mu-\lambda)R(\lambda,f)R(\mu,f) && (6)\\
[R(\lambda,f),\,R(\mu,g)] &= i\sigma(f,g)\,R(\lambda,f)\,R(\mu,g)^2R(\lambda,f) && (7)\\
R(\lambda,f)R(\mu,g) &= R(\lambda+\mu,\,f+g)[R(\lambda,f)+R(\mu,g)+i\sigma(f,g)R(\lambda,f)^2R(\mu,g)] && (8)
\end{aligned}$$

*where $\lambda,\ \mu\in\mathbb{R}\backslash 0$ and $f,\,g\in X\backslash 0$. That is, start with the free unital *-algebra generated by $\{\,R(\lambda,f)\ \mid\ \lambda\in\mathbb{R}\backslash 0,\ f\in X\backslash 0\,\}$ and factor out by the ideal generated by the relations (4) to (8) to obtain the *-algebra $\mathcal{R}_0$.*

3.3 Remark (i) The *-algebra $\mathcal{R}_0$ is nontrivial, because it has nontrivial representations. For instance, in a Fock representation of the CCRs over $(X,\,\sigma)$ we have the CCR-fields $\varphi(f)$ from which we can define $\pi(R(\lambda,f))=(i\lambda\mathbb{I}-\varphi(f))^{-1}$ to obtain a representation of $\mathcal{R}_0$.

(ii) Obviously (4) encodes the selfadjointness of $j(f)$, (5) encodes $j(\lambda f)=\lambda j(f)$, (6) encodes that $R(\lambda,f)$ is a resolvent, (7) encodes the canonical commutation relations and (8) encodes additivity $j(f+g)=j(f)+j(g)$. Moreover, the identity was added explicitly, we do not have that $R(1,0)=-i\mathbb{I}$, in fact $R(1,0)$ is undefined.

To define our resolvent C*-algebra, we need to decide on which C*-seminorm to define on $\mathcal{R}_0$. The obvious choice is the enveloping C*-norm, however for the purpose of our model, it is more convenient to use a different norm, which we now define. We will say that a state ω on the Weyl algebra $\overline{\Delta(X,\,\sigma)}$ is **strongly regular** if the functions

$$\mathbb{R}^n\ni(\lambda_1,\ldots,\lambda_n)\to\omega(\delta_{\lambda_1 f_1}\cdots\delta_{\lambda_n f_n})$$

are smooth for all $f_1,\ldots,\,f_n\in X$ and all $n\in\mathbb{N}$. Of special importance is that the GNS-representation of a strongly regular state has a common dense invariant domain for all the generators $j(f)$ of the one parameter groups $\lambda\to\pi_\omega(\delta_{\lambda f})$ (this domain is obtained by applying the polynomial algebra of the Weyl operators $\{\,\pi_\omega(\delta_f)\ \mid\ f\in X\,\}$ to the cyclic GNS-vector). Some important classes of states, e.g. quasi-free states are strongly regular. Denote by π_S the direct sum of the GNS-representations of all strongly regular states, then as the resolvents of the fields are in $\pi_S(\overline{\Delta(X,\,\sigma)})''$, we can extend π_S to a representation of $\mathcal{R}_0$ by the Laplace transform:

$$\pi_S(R(\lambda,f)) := -i\int_0^\infty e^{-\lambda t}\pi_S(\delta_{-tf})\,dt\;, \qquad \lambda>0\,. \tag{9}$$

We define our **resolvent algebra** $\mathcal{R}(X,\,\sigma)$ as the abstract C*-algebra generated by $\pi_S(\mathcal{R}_0)$, i.e. we factor $\mathcal{R}_0$ by $\operatorname{Ker}\pi_S$ and complete w.r.t. the operator norm of π_S.

We state some elementary properties of $\mathcal{R}(X,\,\sigma)$.

3.4 Theorem *Let $(X,\,\sigma)$ be a given nondegenerate symplectic space, and define $\mathcal{R}(X,\,\sigma)$ as above. Then for all $\lambda,\,\mu\in\mathbb{R}\backslash 0$ and $f,\,g\in X\backslash 0$ we have:*

(i) $[R(\lambda,f),\,R(\mu,f)]=0$. *Substitute $\mu=-\lambda$ to see that $R(\lambda,f)$ is normal.*

(ii) $\|R(\lambda,f)\|=|\lambda|^{-1}$.

(iii) *$R(\lambda,f)$ is analytic in λ. Explicitly, the series expansion:*

$$R(\lambda,f)=\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n\,R(\lambda_0,f)^{n+1}i^n, \qquad \lambda,\,\lambda_0\neq 0 \qquad \textit{(Von Neumann series)}$$

converges in norm whenever $|\lambda_0-\lambda|<|\lambda_0|$.

(iv) *$R(\lambda,tf)$ is norm continuous in $t\in\mathbb{R}\backslash 0$.*

(v) $R(\lambda,f)R(\mu,g)^2R(\lambda,f)=R(\mu,g)R(\lambda,f)^2R(\mu,g)$.

(vi) *Let $T\in\operatorname{Sp}(X,\sigma)$ be a symplectic transformation. then $\alpha(R(\lambda,f)):=R(\lambda,Tf)$ defines a unique automorphism $\alpha\in\operatorname{Aut}\mathcal{R}(X,\,\sigma)$.*

Note that the von Neumann series for $R(\lambda,f)$ converges for any $z\in\mathbb{C}$ with $|z-\lambda_0|<|\lambda_0|$, i.e. on a disk which stays off the real line. Using different $\lambda_0's$ we can thus define $R(z,f)$ for any complex z not on the real line. and deduce the properties in the definition for these from the series. Thus we obtain also resolvents $R(z,f)$ for complex z in $\mathcal{R}(X,\,\sigma)$.

Any operator family R_λ satisfying the resolvent equation (6) is called by Hille a pseudo-resolvent (cf. p215 in [26]), and for such a family we know (cf. Theorem 1 p216 in [26]) that:

- All R_λ have a common range and a common null space.
- A pseudo resolvent R_λ is the resolvent for an operator B iff $\operatorname{Ker}R_\lambda=\{0\}$, and in this case $\operatorname{Dom}B=\operatorname{Ran}R_\lambda$ for all λ.

Thus we define:

3.5 Definition *A* **regular representation** $\pi\in\operatorname{Rep}\mathcal{R}(X,\,\sigma)$ *is a Hilbert space representation such that*

$$\operatorname{Ker}\pi(R(1,f))=\{0\} \qquad \forall\, f\in\mathcal{S}(\mathbb{R},\mathbb{R})\backslash 0\,.$$

We denote the collection of regular representations by Reg.

Obviously many regular representations are known, e.g. π_S and the Fock representation. Given a $\pi \in \operatorname{Rep}\mathcal{R}(X,\,\sigma)$ with $\operatorname{Ker}\pi(R(1,f)) = \{0\}$, we can define a field operator by

$$j_\pi(f) := i\mathbb{I} - \pi(R(1,f))^{-1}$$

with domain $\operatorname{Dom} j_\pi(f) = \operatorname{Ran}\pi(R(1,f))$. Thus for $\pi \in \mathrm{Reg}$, all the field operators $j_\pi(f)$, $f \in \mathcal{S}(\mathbb{R},\mathbb{R})$ are defined, and we have the resolvents $\pi(R(\lambda,f)) = (i\lambda\mathbb{I} - j_\pi(f))^{-1}$.

3.6 Theorem *Let $\mathcal{R}(X,\,\sigma)$ be as above, and let $\pi \in \operatorname{Rep}\mathcal{R}(X,\,\sigma)$ satisfy $\operatorname{Ker}\pi(R(1,f)) = \{0\} = \operatorname{Ker}\pi(R(1,h))$ for given $f,\ h \in X$. Then*

(i) $j_\pi(f)$ *is selfadjoint, and* $\pi(R(\lambda,f))\operatorname{Dom} j_\pi(h) \subseteq \operatorname{Dom} j_\pi(h)$.

(ii) $\lim_{\lambda\to\infty} i\lambda\pi(R(\lambda,f))\psi = \psi$ *for all* $\psi \in \mathcal{H}_\pi$,

(iii) $\lim_{s\to 0} i\pi(R(1,sf))\psi = \psi$ *for all* $\psi \in \mathcal{H}_\pi$.

(iv) The space $\mathcal{D} := \pi(R(1,f)R(1,h))\mathcal{H}_\pi$ *is a joint dense domain for* $j_\pi(f)$ *and* $j_\pi(h)$ *and we have:* $[j_\pi(f),\, j_\pi(h)] = i\sigma(f,h)$ *on* $\mathcal{D}$,

(v) $j_\pi(\lambda f + h) = \lambda j_\pi(f) + j_\pi(h)$ *for all* $\lambda \in \mathbb{R}$ *on* $\mathcal{D}$,

(vi) $j_\pi(f)\pi(R(\lambda,f)) = \pi(R(\lambda,f))j_\pi(f) = i\lambda\pi(R(\lambda,f)) - \mathbb{I}$ *on* $\operatorname{Dom} j_\pi(f)$,

(vii) $[j_\pi(f),\pi(R(\lambda,h))] = i\sigma(h,f)\pi(R(\lambda,h)^2)$ *on* $\operatorname{Dom} j_\pi(f)$,

(viii) Denote $W(f) := \exp(ij_\pi(f))$, *then*

$$\begin{aligned} W(f)W(h) &= e^{i\sigma(f,h)}W(h)W(f) \\ W(f)\pi(R(\lambda,h))W(f)^* &= \pi(R(\lambda + i\sigma(f,h),\,h)). \end{aligned}$$

Moreover $W(f)\mathcal{D} \subseteq \mathcal{D} \supseteq W(h)\mathcal{D}$, *hence* $\mathcal{D} := \pi(R(1,f)R(1,h))\mathcal{H}_\pi$ *is a common core for* $j_\pi(f)$ *and* $j_\pi(h)$.

A distinguished regular representation of $\mathcal{R}(X,\,\sigma)$ is of course the defining strongly regular representation π_S. By definition $\mathcal{R}(X,\,\sigma)$ is faithfully represented in it, and moreover, there is a common dense invariant domain $\mathcal{D}_0$ for all the field operators $j_{\pi_S}(f)$, $f \in X$. This domain can be enlarged to a dense invariant domain $\mathcal{D}_S$ for both the resolvents and the fields simply by applying all polynomials in $j_{\pi_S}(f)$ and $\pi_S(R(\lambda,f))$ to $\mathcal{D}_0$, which makes sense, because from (i) above all resolvents preserve the joint domain $\bigcap\{\operatorname{Dom} j_{\pi_S}(f) \ \mid\ f \in X\}$. Thus we can form the *-algebra of (unbounded) operators

$$\mathcal{E}_0 := *\text{–alg}\,\{\, j_{\pi_S}(f),\ \pi_S(R(\lambda,f)) \ \mid\ f \in X,\ \lambda \in \mathbb{R}\backslash 0\,\}$$

on $\mathcal{D}_S$. Then $\mathcal{E}_0$ contains of course the *-algebra $\pi_S(\mathcal{R}_0)$ generated by resolvents alone, which is dense in $\mathcal{R}(X,\,\sigma)$. We will need these *-algebras $\mathcal{E}_0 \supset \pi_S(\mathcal{R}_0)$ below, and will generally not indicate the faithful representation π_S w.r.t. which they are defined. Note that for any strongly regular state ω, its cyclic GNS-vector is in the domain of all $j_{\pi_\omega}(f)$, hence ω extends to define a functional on $\mathcal{E}_0$. Thus we give a meaning to all expressions of the form

$$\begin{aligned} &\omega(j(f_1)\cdots j(f_n)R(\lambda_1,g_1)\cdots)R(\lambda_k,g_k)) \\ &\qquad := (\Omega_\omega,\, j_{\pi_\omega}(f_1)\cdots j_{\pi_\omega}(f_n)\pi_\omega(R(\lambda_1,g_1)\cdots)R(\lambda_k,g_k))\Omega_\omega) \end{aligned}$$

as above. A very important class of states on $\overline{\Delta(X,\sigma)}$ are the quasifree states, which we will need below. They are given by

$$\omega(\delta_f) = \exp\big(-\tfrac{1}{2}\langle f|f\rangle_\omega\big), \quad f\in X,$$

where $\langle\cdot|\cdot\rangle_\omega$ is a (possibly semi–definite) scalar product on the complex linear space $X+iX$ satisfying

$$\langle f|g\rangle_\omega - \langle g|f\rangle_\omega = i\sigma(f,g), \quad f,g\in X.$$

Any quasifree state is also regular in the strong sense. By a routine computation one can represent the expectation values of products of Weyl operators in a quasifree state in the form

$$\omega(\delta_{f_1}\cdots\delta_{f_n}) = \exp\Big(-\sum_{k<l}\langle f_k|f_l\rangle_\omega - \tfrac{1}{2}\sum_l \langle f_l|f_l\rangle_\omega\Big).$$

Making use of the Laplace transform (9) for the GNS–represntation of the resolvents, we have for $\lambda_1,\ldots\lambda_n>0$

$$\begin{aligned}
&\omega(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n))\\
&= (-i)^n\int_0^\infty dt_1\ldots\int_0^\infty dt_n\, e^{-\sum_k t_k\lambda_k}\omega(\delta_{t_1f_1}\cdots\delta_{t_nf_n})\\
&= (-i)^n\int_0^\infty dt_1\ldots\int_0^\infty dt_n\, \exp\Big(-\sum_k t_k\lambda_k - \sum_{k<l} t_kt_l\langle f_k|f_l\rangle_\omega - \tfrac{1}{2}\sum_l t_l^2\langle f_l|f_l\rangle_\omega\Big). \quad (10)
\end{aligned}$$

Remark: One can replace anywhere in this equation f_k by $-f_k$, thus it does not impose any restriction of generality to assume that $\lambda_1,\ldots\lambda_n>0$. The relation (10) should be regarded as the definition of quasifree states on the resolvent algebra.

In our calculations below, we will frequently need the following differentiablility of quasifree states:

3.7 Proposition *Let ω be a quasifree state as above, and let $x_k\in\mathbb{R}\mapsto f_k(x_k)\in X$, $k=1,\ldots n$ be paths in X for which the functions $x_k,x_l\longmapsto\langle f_k(x_k)|f_l(x_l)\rangle_\omega$, $k,l=1,\ldots n$, are smooth. Then*

$$\begin{aligned}
&\frac{\partial}{\partial x_r}\,\omega\Big(R(\lambda_1,f_1(x_1))\cdots R(\lambda_n,f_1(x_n))\Big)\\
= & -\sum_{k=1}^{r-1}\frac{\partial}{\partial x_r}\Big(\langle f_k(x_k)|f_r(x_r)\rangle_\omega\Big)\,\frac{\partial^2}{\partial\lambda_r\partial\lambda_k}\,\omega\Big(R(\lambda_1,f_1(x_1))\cdots R(\lambda_n,f_1(x_n))\Big)\\
&-\tfrac{1}{2}\frac{\partial}{\partial x_r}\Big(\langle f_r(x_r)|f_r(x_r)\rangle_\omega\Big)\,\frac{\partial^2}{\partial\lambda_r^2}\,\omega\Big(R(\lambda_1,f_1(x_1))\cdots R(\lambda_n,f_1(x_n))\Big)\\
&-\sum_{k=r+1}^{n}\frac{\partial}{\partial x_r}\Big(\langle f_r(x_r)|f_k(x_k)\rangle_\omega\Big)\,\frac{\partial^2}{\partial\lambda_r\partial\lambda_k}\,\omega\Big(R(\lambda_1,f_1(x_1))\cdots R(\lambda_n,f_1(x_n))\Big)
\end{aligned}$$

and all the partial derivatives involved in this formula exist.

4 C*-algebra formulation of Supersymmetry.

Here we want to write our model of Section 2 in a C*-algebra framework. However, to motivate our choices made below, let us recall a theorem of Kishimoto and Nakamura [16]:

4.1 Theorem *Let $\mathcal{A}$ be a C^*-algebra with $\mathbb{Z}_2$–grading γ, let $\alpha : \mathbb{R} \to \mathrm{Aut}\,\mathcal{A}$ be a pointwise continuous action with generator δ_0 having a smooth domain $C^\infty(\delta_0) := \bigcap_{n=1}^{\infty} \mathrm{Dom}\,(\delta_0^n)$. Let δ be a closable graded derivation with $\mathrm{Dom}\,(\delta) \supset C^\infty(\delta_0)$, $\delta \circ \alpha_t = \alpha_t \circ \delta$ for all t, and $\delta^2 = \delta_0$ on $C^\infty(\delta_0)$. Then δ is bounded.*

Thus it will be hard to obtain the supersymmetry relation on natural dense domains.

For the fermion field, let $\mathcal{H} = L^2(\mathbb{R})$ and define $\mathrm{CAR}(\mathcal{H})$ in Araki's self-dual form (cf. [1]) as follows. On $\mathcal{K} := \mathcal{H} \oplus \mathcal{H}$ define an antiunitary involution Γ by $\Gamma(h_1 \oplus h_2) := \overline{h}_2 \oplus \overline{h}_1$. Then $\mathrm{CAR}(\mathcal{H})$ is the unique simple C*–algebra with generators $\{\, \Phi(k) \mid k \in \mathcal{K} \,\}$ such that $k \to \Phi(k)$ is antilinear, $\Phi(k)^* = \Phi(\Gamma k)$, and

$$\{\Phi(k_1),\, \Phi(k_2)^*\} = (k_1,\, k_2)\mathbb{I}\;, \qquad k_i \in \mathcal{K}\;.$$

The correspondence with the heuristic creators and annihilators of fermions is given by $\Phi(h_1 \oplus h_2) = a(h_1) + a^*(\overline{h}_2)$, where

$$a(h) = \int a(x)\,\overline{h(x)}\,dx\;, \qquad a^*(h) = \int a^*(x)\,h(x)\,dx\;.$$

To obtain the Clifford operators $c(f) = c(f)^*$, $f \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ we take $c(f) := \Phi(f \oplus f)/\sqrt{2}$, in which case we have $c(f) = c(f)^*$ and $\{c(f),\, c(g)\} = (f, g) = \int fg\,dx$. Let $\mathrm{Cliff}(\mathcal{S}(\mathbb{R})) := C^*\{\, c(f) \mid f \in \mathcal{S}(\mathbb{R}) \,\}$ and notice that $\widetilde{c}(f) := i\Phi(f \oplus -f)/\sqrt{2}$ also satisfies the Clifford relations, hence generates another copy of $\mathrm{Cliff}(\mathcal{S}(\mathbb{R}))$ in $\mathrm{CAR}(\mathcal{H})$, and together these two Clifford algebras generate all of $\mathrm{CAR}(\mathcal{H})$. In fact, since the $c(f)$ and $\widetilde{c}(g)$ anticommute, we have that $\mathrm{CAR}(\mathcal{H}) \cong \mathrm{Cliff}(\mathcal{S}(\mathbb{R}) \oplus \mathcal{S}(\mathbb{R}))$. Conversely, if we are given a real pre-Hilbert space X with complexified completion Y and a projection P and antiunitary involution Γ such that $\Gamma P \Gamma = \mathbb{I} - P$ and these preserve X, then we have an isomorphism $\mathrm{Cliff}(X) \cong \mathrm{CAR}(Y)$ given by $\Phi(x) = (c(Px) - ic(\Gamma Px) + c(P\Gamma x) + ic((\mathbb{I} - P)x))/\sqrt{2}$.

For the bosonic part we take the resolvent algebra $\mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma)$ where $\sigma(f, g) := \int fg'\,dx$, and so the full C*-algebra in which we want to define our model is

$$\mathcal{A} := \mathrm{Cliff}(\mathcal{S}(\mathbb{R})) \otimes \mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma)$$

where the tensor norm is unique because the CAR-algebra is nuclear. The grading automorphism γ is the identity on $\mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma)$, and $\gamma(\Phi(k)) = -\Phi(k)$ for all k on the CAR-part.

Next, we want to define on some suitable domain in $\mathcal{A}$ the supersymmetry graded derivation δ corresponding to the relations (1). First, considering $\delta(j(f)) = ic(f')$, since δ is a derivation on the bosonic part, it is natural to define

$$\delta(R(\lambda, f)) := ic(f')\,R(\lambda, f)^2 \quad \in \mathcal{A}\,. \tag{11}$$

However due to the unbounded rhs of $\delta(c(f)) = j(f)$ we cannot define δ directly on the $c(f)$, so we need to multiply by mollifiers. Define

$$\zeta(f) \;:=\; c(f)R(1, f)\,, \tag{12}$$

then $$\delta(\zeta(f)) \;:=\; iR(1, f) - \mathbb{I} + ic(f)c(f')R(1, f)^2 \quad \in \mathcal{A} \tag{13}$$

where we made use of the graded derivation property, the relations (1) and $j(f)R(\lambda, f) = i\lambda R(\lambda, f) - \mathbb{I}$. Next, we would like to extend δ as a graded derivation to the *-algebra generated by these basic objects:

$$\mathcal{D}_S := *\text{–alg}\,\{\,\mathbb{I},\ R(\lambda, f),\ \zeta(f)\ \mid\ \lambda \in \mathbb{R}\backslash 0,\ f \in \mathcal{S}(\mathbb{R})\backslash 0\,\} \subset \mathcal{A}\,.$$

Observe that $\mathcal{D}_S$ is not norm-dense in $\mathcal{A}$, however due to Theorem 3.6(ii) applied to $R(\lambda, f)c(f) = \zeta(f/\lambda)$, it will be strong operator dense in $\mathcal{A}$ in any regular representation. Note that δ does not preserve $\mathcal{D}_S$, it takes its image in the norm dense *-algebra

$$\mathcal{A}_0 := *\text{–alg}\,\{\,\mathbb{I},\ R(\lambda, f),\ c(f)\ \mid\ \lambda \in \mathbb{R}\backslash 0,\ f \in \mathcal{S}(\mathbb{R})\backslash 0\,\} \subset \mathcal{A}\,.$$

To see that δ extends as a graded derivation to $\mathcal{D}_S$, we proceed as follows. Let π_0 be any representation of $\mathrm{Cliff}(\mathcal{S}(\mathbb{R}))$ then $\pi_0 \otimes \pi_S$ is a faithful representation of $\mathcal{A}$ and there is a common dense invariant domain $\mathcal{D} := \mathcal{H}_{\pi_0} \otimes \mathcal{D}_S$ for all $\pi_0(c(f)) \otimes \mathbb{I}$, $\mathbb{I} \otimes j_{\pi_S}(f)$ and $\mathbb{I} \otimes \pi_S(R(\lambda, f))$ where $\mathcal{D}_S$ denotes the domain of $\mathcal{E}_0$ defined at the end of Section 3. (Henceforth we will not indicate tensoring by $\mathbb{I}$ nor the representations π_0, π_S when the context makes clear what is meant). Let

$$\mathcal{E} := *\text{–algebra}\,\{\,c(f),\ j(f),\ R(\lambda, f)\ \mid\ f \in \mathcal{S}(\mathbb{R}),\ \lambda \in \mathbb{R}\backslash 0\,\} \supset \mathcal{A}_0$$

so we have the *-algebras $\mathcal{R}_0 \subset \mathcal{E}_0 \subset \mathcal{E}$ on $\mathcal{D}$. Define on the generating elements of $\mathcal{E}$ a map $\overline{\delta}$, setting

$$\begin{aligned} \overline{\delta}(j(f)) &= ic(f'), \\ \overline{\delta}(R(\lambda, f)) &= ic(f')R(\lambda, f)^2, \\ \overline{\delta}(c(f)) &= j(f). \end{aligned}$$

We will see that this map extends to a graded derivation on $\mathcal{E}$. For the proof it suffices to show that $\overline{\delta}$ is linear and satisfies the graded Leibniz rule on any finite polynomial involving operators $j(f)$, $R(\lambda, f)$ and $c(f)$, i.e. in each instance only a finite number of test functions f and real parameters λ are involved. We will take advantage of this fact as follows.

Let $X_s \subset \mathcal{S}(\mathbb{R})$ be any *finite*–dimensional subspace and consider the subalgebra $\mathcal{E}(X_s) \subset \mathcal{E}$ generated by the elements $j(f)$, $R(\lambda, f)$ and $c(f)$ with $\lambda \in \mathbb{R}\backslash 0$, $f \in X_s$. We extend X_s to a space $X_s{}' \subset \mathcal{S}(\mathbb{R})$ by adding to the elements of X_s also their *first* derivatives. Picking in $X_s{}'$ some (finite) orthonormal basis $\{h_n\}$ with regard to the scalar product $(\cdot, \cdot)$, we have for any $f \in X_s$ the "completeness relations" $\sum_n (h_n, f)\, h_n = f$, $\sum_n (h_n, f')\, h_n = f'$.

Next, we define an operator $Q_s \in \mathcal{E}$, setting

$$Q_s = \sum\nolimits_n c(h_n) j(h_n).$$

As Q_s is of fermionic (odd) type, we can consistently define with the help of it a graded derivation $\overline{\delta}_s$ on $\mathcal{E}$, setting for even and odd elements $E_\pm \in \mathcal{E}$, respectively,

$$\overline{\delta}_s(E_+) = [Q_s, E_+] = Q_s E_+ - E_+ Q_s, \quad \overline{\delta}_s(E_-) = \{Q_s, E_-\} = Q_s E_- + E_- Q_s.$$

Computing the action of $\overline{\delta}_s$ on the even elements $j(f)$, $R(\lambda, f)$ and odd elements $c(f)$, where $\lambda \in \mathbb{R}\backslash 0$, $f \in X_s$, we obtain from the basic relations in $\mathcal{E}$ by some elementary algebraic manipulations.

$$\begin{aligned} \overline{\delta}_s(j(f)) &= i\sum\nolimits_n c(h_n)\,(h_n, f') = ic(f')\,, \\ \overline{\delta}_s(R(\lambda, f)) &= i\sum\nolimits_n c(h_n)\,(h_n, f')\,R(\lambda, f)^2 = ic(f')R(\lambda, f)^2, \\ \overline{\delta}_s(c(f)) &= \sum\nolimits_n (h_n,\, f)\, j(h_n) = j(f). \end{aligned}$$

Thus we conclude that the action of $\overline{\delta}$ on the generating elements of $\mathcal{E}(X_s)$ coincides with the action of the graded derivation $\overline{\delta}_s$. As the choice of the subspace X_s was arbitrary, it follows that $\overline{\delta}$ extends to a graded derivation on the whole polynomial algebra $\mathcal{E}$.

The final step consists in showing that the action of δ on the generating elements $R(\lambda, f)$, $\zeta(f)$ of $\mathcal{D}_S$ coincides with the action of the graded derivation $\overline{\delta}$. But this follows immediately from the relations given above. Thus δ extends to a graded derivation with domain $\mathcal{D}_S$ and range in $\mathcal{A}_0$. Uniqueness is clear from the graded derivation property, so we have proven:

4.2 Theorem *There is a unique graded derivation $\delta : \mathcal{D}_S \to \mathcal{A}$ satisfying relations (11) and (13).*

Next we need to define the time evolution derivation δ_0 in this C*-setting. From the equations (2) this suggest that we define on $\mathcal{E}$ a *-derivation $\overline{\delta}_0$ satisfying:

$$\begin{aligned} \overline{\delta}_0(j(f)) &= ij(f'), \\ \overline{\delta}_0(R(\lambda, f)) &= iR(\lambda, f)j(f')R(\lambda, f), \\ \overline{\delta}_0(c(f)) &= i\,c(f') \end{aligned}$$

and then proceed to the corresponding mollified relations in $\mathcal{A}$. For the proof that $\overline{\delta}_0$ extends to a *-derivation on $\mathcal{E}$, we proceed as in the discussion of the superderivation: We pick any finite dimensional subspace $X_s \subset \mathcal{S}(\mathbb{R})$, consider the corresponding subalgebra $\mathcal{E}(X_s) \subset \mathcal{E}$ and choose in the extended space $X_s{}' \subset \mathcal{S}(\mathbb{R})$, containing the elements of X_s and their first derivatives, some orthonormal basis $\{h_n\}$. In addition to the completeness relations mentioned above we will also make use of $\sum_n (h_n, f)\, h_n' = f'$ for $f \in X_s$.

We consider now the symmetric operator in $\mathcal{E}$

$$H_s = \frac{1}{2} \sum\nolimits_n \{ic(h_n{}')c(h_n) + j(h_n)j(h_n)\}.$$

Putting

$$\overline{\delta}_{0\,s}(\,\cdot\,) = [H_s,\, \cdot\,],$$

it induces a *–derivation on $\mathcal{E}$. Its action on the generating elements of $\mathcal{E}(X_s)$ can easily be computed:

$$\begin{aligned} \overline{\delta}_{0\,s}(j(f)) &= \sum\nolimits_n j(h_n)\, i(h_n, f') = i\, j(f'), \\ \overline{\delta}_{0\,s}(R(\lambda, f)) &= \sum\nolimits_n i(h_n, f')\, R(\lambda, f)j(h_n)R(\lambda, f) = i\, R(\lambda, f)j(f')R(\lambda, f), \\ \overline{\delta}_{0\,s}(c(f)) &= \frac{1}{2} \sum\nolimits_n i\{(h_n, f)\, c(h_n{}') + (h_n, f')\, c(h_n)\} = i\, c(f'). \end{aligned}$$

Thus we conclude as in the preceding discussion that the action of $\overline{\delta}_0$ on the generating elements of $\mathcal{E}(X_s)$ coincides with the action of the derivation $\overline{\delta}_{0\,s}$. As X_s was arbitrary, it follows that $\overline{\delta}_0$ extends to a derivation on the whole polynomial algebra $\mathcal{E}$.

The supersymmetry relation $\overline{\delta}^2 = \overline{\delta}_0$ can now be verified on the generating elements of $\mathcal{E}$ and thus holds on the whole algebra $\mathcal{E}$.

The question now is how one should define the time evolution δ_0 and the square δ^2 on the C*-algebra $\mathcal{A}$ from the unbounded versions in $\mathcal{E}$. Since $\delta : \mathcal{D}_S \to \mathcal{A}_0$, its square δ^2 does not make sense on $\mathcal{D}_S$. Note however that for every $A \in \mathcal{A}_0$ there is a monomial $M \in \mathcal{D}_S$ of resolvents $R(\lambda, f)$ such that

$$AM \in \mathcal{D}_S \ni MA\,.$$

(By Theorem 3.6(ii) we know that in regular representations we can let these mollifiers M go to $\mathbb{I}$ in the strong operator topology.)

4.3 Definition *For each $A \in \mathcal{D}_S$ let $M_A \in \mathcal{D}_S$ be a monomial of resolvents $R(\lambda, f)$ such that $M_A\delta(A) \in \mathcal{D}_S$. Define*

$$M_A\delta^2(A) := \delta(M_A\delta(A)) - \delta(M_A)\delta(A) \in \mathcal{A}_0\,.$$

Note that this definition coincides with $M_A\overline{\delta}^2(A)$ in $\mathcal{E}$, however the definition above involves only bounded quantities, so it can be defined independently in the C*-setting on $\mathcal{D}_S$. Of course we then have the mollified SUSY–relations $M_A\delta^2(A) = M_A\overline{\delta}_0(A)$ for all $A \in \mathcal{D}_S$ from the unbounded SUSY relation in $\mathcal{E}$. This is not however acceptable for a bounded SUSY–relation until we have demonstrated the connection of $M_A\overline{\delta}_0(A)$ with the time evolution. The time evolution $\alpha : \mathbb{R} \to \operatorname{Aut}\mathcal{A}$ is just translation, as this is a chiral theory

$$\alpha_t(c(f)) := c(f_t)\,, \quad \alpha_t\left(R(\lambda, f)\right) = R(\lambda, f_t)\,.$$

The desired connection

$$M_A\overline{\delta}_0(A) = -i\frac{d}{dt}M_A\,\alpha_t(A)\Big|_0 \tag{14}$$

exists only in specific regular representations on suitable domains, and for these one will then have supersymmetry. In many applications, one only needs the supersymmetry weakly, i.e.

$$\omega(BM_A\overline{\delta}_0(A)C) = -i\frac{d}{dt}\omega(BM_A\,\alpha_t(A)C)\Big|_0 = \omega(BM_A\delta^2(A)C)$$

for A, B, C in a suitable domain and ω a distinguished functional. We will verify this relation explicitly below for the functionals used in our constructions.

5 Graded KMS–functionals.

Graded KMS–functionals are used in supersymmetric theories to calculate cyclic cocycles [11, 15], and here we want to develop this theory in the current context for our simple supersymmetric model as a first application of it.

5.1 Definition *Let $\mathcal{A}$ be a unital C*-algebra with a grading automorphism $\gamma \in \operatorname{Aut}\mathcal{A}$, $\gamma^2 = \iota$, and a (pointwise continuous) action $\alpha : \mathbb{R} \to \operatorname{Aut}\mathcal{A}$ such that $\alpha_t \circ \gamma = \gamma \circ \alpha_t$ for all t. Then a* **graded KMS–functional** *is a (possibly unbounded) functional φ on $\mathcal{A}$ such that*

(i) $\operatorname{Dom}\varphi$ *is a unital dense *–subalgebra of $\mathcal{A}$ such that*

$$\gamma(\operatorname{Dom}\varphi) \subseteq \operatorname{Dom}\varphi \supseteq \alpha_t(\operatorname{Dom}\varphi) \quad \forall\, t,$$

(ii) For all A, $B \in \operatorname{Dom}\varphi$ there is a continuous complex function $F_{A,B} : S \to \mathbb{C}$ on the strip $S := \mathbb{R} + i[0, 1]$ which is analytic on the interior of S and satisfying on the boundary:

$$\begin{aligned} F_{A,B}(t) &= \varphi\left(A\,\alpha_t(B)\right) \quad \forall\, t \\ F_{A,B}(t+i) &= \varphi\left(\alpha_t(B)\gamma(A)\right) \quad \forall\, t \in \mathbb{R}\,. \end{aligned}$$

(iii) For A, $B \in \mathrm{Dom}\,\varphi$ we have

$$|F_{A,B}(t+is)| < C(1+|t|)^N \quad \forall\, t \in \mathbb{R},\ s \in (0,1)$$

and some $C \in \mathbb{R}_+$ and $N \in \mathbb{N}$ depending on A and B.

5.2 Example Below for our model, we will define on

$$\mathcal{A} = \mathrm{Cliff}(\mathcal{S}(\mathbb{R})) \otimes \mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma)$$

a functional $\varphi = \psi \otimes \omega$, with $\mathrm{Dom}\,\varphi = \mathcal{A}_0$ where ψ and ω are quasi-free with two-point functions

$$\begin{aligned} \omega(j^2(f)) &= \int \frac{p}{1-e^{-p}} \left|\widehat{f}(p)\right|^2 dp\,, \\ \psi(c(f)c(g)) &= \lim_{\varepsilon\to 0^+} \int \frac{p}{1-e^{-p}}\, \frac{p}{p^2+\varepsilon^2}\, \widehat{f}(p)\, \overline{\widehat{g}(p)}\, dp \end{aligned}$$

and we will verify that $\varphi = \psi \otimes \omega$ is a graded KMS–functional. Note that ω is a state on $\mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma)$, but ψ is unbounded and nonpositive. It does however satisfy supersymmetry, in that $\varphi \circ \delta = 0$, and equation (14) holds weakly.

The motivation for using graded KMS–functionals come from several sources:

- Physicists used graded KMS–functionals to construct supersymmetric field theories in a thermal background [8, 10].
- Jaffe e.a. [11] and Kastler [15] used graded KMS–functionals to construct cyclic cocycles in Connes' cyclic cohomology.

The reasons why one has to use nonpositive unbounded KMS–functionals for field theories on noncompact spacetime are as follows. First, there is the theorem of Buchholz and Ojima [4] that supersymmetry breaks down in spatially homogeneous KMS–states, and second there is the theorem of Buchholz and Longo [3] that if φ is a bounded graded KMS–functional of $\mathcal{A}$ with time evolution $\alpha : \mathbb{R} \to \mathrm{Aut}\,\mathcal{A}$, and if there are $\beta_n \in \mathrm{Aut}\,\mathcal{A}$ such that

$$\lim_{n\to\infty} \varphi\,(C[A,\, \beta_n(B)]) = 0 \quad \forall\, A,\, B,\, C \in \mathcal{A}$$

then $\alpha = \iota$. On noncompact spaces, the translations will produce the β_n in a local field theory. Thus in local field theories on noncompact spaces, we are inevitably led to unbounded graded KMS–functionals for supersymmetry.

First, we would like to establish a few general properties of graded KMS–functionals.

5.3 Proposition *Given a graded KMS-functional φ defined w.r.t. the data $(\mathcal{A}, \gamma, \alpha)$, then*

(i) φ is α–invariant.

(ii) φ is γ–invariant.

Moreover if a functional φ satisfies the graded KMS–property on a subset $Y \subset \mathrm{Dom}\,\varphi \subset \mathcal{A}$, then it also satisfies the graded KMS–property on $\mathrm{Span}\,Y$.

5.4 Proposition *Let $\mathcal{A} = \mathcal{C} \otimes \mathcal{B}$ where $\mathcal{C}$ and $\mathcal{B}$ are unital C^*-algebras with $\mathcal{C}$ nuclear. Let $\sigma : \mathbb{R} \to \mathrm{Aut}\,\mathcal{C}$ and $\beta : \mathbb{R} \to \mathrm{Aut}\,\mathcal{B}$ be dynamical systems, and let $\gamma \in \mathrm{Aut}\,\mathcal{C}$ be a grading automorphism, $\gamma^2 = \iota$. Let $\omega \in \mathfrak{S}(\mathcal{B})$ be a KMS–state on $\mathcal{B}$ w.r.t. β, and let ψ be a graded KMS–functional on $\mathcal{C}$ w.r.t. σ. Define a functional $\varphi := \psi \otimes \omega$ with $\mathrm{Dom}\,\varphi := \mathrm{Span}\,\{\, C \otimes B \mid C \in \mathrm{Dom}\,\psi,\ B \in \mathcal{B}\,\}$ by $\varphi(C \otimes B) := \psi(C)\omega(B)$. Then φ is a graded KMS–functional w.r.t. the grading $\gamma \otimes \iota$, and the C^*-dynamical system $\sigma \otimes \beta : \mathbb{R} \to \mathrm{Aut}\,(\mathcal{C} \otimes \mathcal{B})$.*

Thus for our model, as $\mathcal{A} = \mathrm{Cliff}(\mathcal{S}(\mathbb{R})) \otimes \mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma)$, it suffices to define a graded KMS–functional ψ on $\mathrm{Cliff}(\mathcal{S}(\mathbb{R}))$ and a KMS–state ω on $\mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma)$ from which we can then construct the graded KMS–functional $\varphi := \psi \otimes \omega$. We start by defining the KMS–state ω on $\mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma)$.

5.5 Theorem *(i) There is a quasi–free state on $\overline{\Delta(\mathcal{S},\,\sigma)}$ defined by $\omega(\delta_f) := \exp[-s(f,f)/2]$, $f \in \mathcal{S}(\mathbb{R},\mathbb{R})$ where*

$$s(f,g) := \int \frac{p}{1-e^{-p}} \widehat{f}(p)\,\overline{\widehat{g}(p)}\,dp = \langle f|g\rangle_\omega\,. \tag{15}$$

(ii) This quasi–free state ω on $\overline{\Delta(\mathcal{S},\,\sigma)}$ extends to a KMS–functional on $\pi_\omega(\overline{\Delta(\mathcal{S},\,\sigma)})''$, hence on $\mathcal{R}(\mathcal{S}(\mathbb{R}), \sigma)$, where it is defined by Equation (10). The time evolution used for the KMS–condition is translation of test functions f.

Next, we would like to define a graded KMS–functional ψ on $\mathrm{Cliff}(\mathcal{S}(\mathbb{R}))$ with $\mathrm{Dom}\,\psi =$ *-alg $\{\, c(f) \mid f \in \mathcal{S}(\mathbb{R})\,\}$. By the last part of Proposition 5.3, it suffices to define ψ and check its KMS-properties on the monomials $c(f_1)\cdots c(f_n)$. Recall that a quasi–free functional on the Clifford algebra is uniquely defined by its two point functional and the relations:

$$\psi(c(f_1)\cdots c(f_{2k+1})) = 0 \tag{16}$$

$$\psi(c(f_1)\cdots c(f_{2k})) = (-1)^{\binom{k}{2}} \sum_P (-1)^P \prod_{j=1}^{k} \psi\Big(c(f_{P(j)})\,c(f_{P(k+j)})\Big) \tag{17}$$

where $k \in \mathbb{N}$ and P is any permutation of $\{1, 2, \ldots, 2k\}$ such that $P(1) < \cdots < P(k)$ and $P(j) < P(k+j)$ for $j = 1, \ldots, k$ (cf. p89 in [21]). Using this formula, we define a quasi–free functional ψ with two point function

$$\begin{aligned} \theta(f,g) := \psi(c(f)c(g)) &= \lim_{\varepsilon\to 0^+} \int \frac{p}{1-e^{-p}}\,\frac{p}{p^2+\varepsilon^2}\,\widehat{f}(p)\,\overline{\widehat{g}(p)}\,dp \\ &= \lim_{\varepsilon\to 0^+} \Big(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\Big) \frac{1}{1-e^{-p}}\;\widehat{f}(p)\,\overline{\widehat{g}(p)}\,dp \\ &= \lim_{\varepsilon\to 0^+} \Big(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\Big) \Big(\frac{1}{p}\Big)\,\frac{p}{1-e^{-p}}\;\widehat{f}(p)\,\overline{\widehat{g}(p)}\,dp \\ &= \mathcal{P}\Big(\frac{1}{p}\Big)(G) \end{aligned} \tag{18}$$

where $\mathcal{P}\left(\frac{1}{p}\right)$ denotes the distribution consisting of the Cauchy Principal Part integral of $1/p$, and $G(p) := \frac{p}{1-e^{-p}}\,\widehat{f}(p)\,\overline{\widehat{g}(p)}$ which is differentiable everywhere and of fast decay since $\frac{p}{1-e^{-p}}$ is differentiable and of linear growth, and f, g are real-valued Schwartz functions. Thus $\theta(f,g)$ is well-defined for all $f,\, g \in \mathcal{S}(\mathbb{R})$.

We mention as an aside that the quasifreeness of a graded KMS–functional ψ on the Clifford algebra and the formula for its two–point function are a consequence of the graded KMS–condition, as can be shown by similar arguments as in [23].

This quadratic form θ is unbounded and not positive definite, because $(1-e^{-p})^{-1}$ is unbounded and not positive. θ has the following useful properties.

5.6 Theorem *Let $f,\,g\in\mathcal{S}(\mathbb{R})$, then*

(i) $\theta(f,g)=(g,\,(P+T)f)$ *where $P=2\pi\times$ projection onto positive spectrum of $D:=id/dx$, and T is an unbounded operator given explicitly by*

$$(g,Tf)=2i\int dx\int dy\,f(x)\,g(y)\,(x-y)\int_0^\infty dp\,\ln(1-e^{-p})\,\cos(p(x-y))\,.$$

Moreover P_JTP_J is trace–class and selfadjoint for all compact intervals $J\subset\mathbb{R}$ where P_J is the projection onto $L^2(J)\subset L^2(\mathbb{R})$.

(ii) For $z\in S=\mathbb{R}+i[0,1]$ define

$$G(z):=\theta(f,g_z):=\lim_{\varepsilon\to 0^+}\Big(\int_{-\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\Big)\frac{e^{ipz}}{1-e^{-p}}\,\widehat{f}(p)\,\overline{\widehat{g}(p)}\,dp\,.$$

Then G is continuous on S, analytic on its interior, and satisfies

$$|G(t+is)|\leq A+B|t|\quad \textit{for}\, t\in\mathbb{R},\;s\in[0,1]$$

and constants A, B.

Using these properties, one can now establish that:

5.7 Theorem *The quasi–free functional ψ with two point functional θ and domain $\mathrm{Dom}\,\psi=$ *-alg$\{\,c(f)\;|\;f\in\mathcal{S}(\mathbb{R})\,\}$, is a graded KMS–functional on $\mathrm{Cliff}(\mathcal{S}(\mathbb{R}))$ where time evolution is given by translation.*

Thus by Proposition 5.4 we have a KMS–functional $\varphi=\psi\otimes\omega$ on $\mathcal{A}$ with domain

$$\mathrm{Dom}\,\varphi=\mathrm{Span}\,\{\,C\otimes R\;|\;C\in\mathrm{Dom}\,\psi,\;R\in\mathcal{R}(\mathcal{S}(\mathbb{R}),\sigma)\,\}\,.$$

What makes this KMS–functional interesting, is that it satisfies supersymmetry, i.e.

5.8 Theorem *For the quasifree functional φ above, we have that*

(i) $\mathcal{A}_0\subset\mathrm{Dom}\,\varphi$,

(ii) $\varphi(\delta(A))=0$ *for all* $A\in\mathcal{D}_S$

(iii) $\varphi(BM_A\overline{\delta}_0(A)C)=-i\frac{d}{dt}\varphi(BM_A\,\alpha_t(A)C)\Big|_0=\varphi(BM_A\delta^2(A)C)$ *for all $A\in\mathcal{D}_S$ and $B,C\in\mathcal{A}_0$ where M_A is a monomial of resolvents $R(\lambda,f)$ such that $M_A\delta(A)\in\mathcal{D}_S$ (as in Definition 4.3).*

Whilst the functional φ is unbounded, it is locally bounded in the sense of the theorem below. For the local algebras, let $J\subset\mathbb{R}$ be a bounded interval and define

$$\begin{aligned}\mathcal{A}(J)\;&:=\;\mathrm{C}^*\{\,c(f),\;R(\lambda,f)\;|\;\mathrm{supp}\,f\subseteq J,\;f\in\mathcal{S}(\mathbb{R}),\;\lambda\in\mathbb{R}\backslash 0\,\}\\ \mathcal{A}_0(J)\;&:=\;\text{*-alg}\{\,c(f),\;R(\lambda,f)\;|\;\mathrm{supp}\,f\subseteq J,\;f\in\mathcal{S}(\mathbb{R}),\;\lambda\in\mathbb{R}\backslash 0\,\}\end{aligned}$$

hence $\mathcal{A}_0(J)$ is a dense *-algebra of $\mathcal{A}(J)$, and $\mathcal{A}_0(J)\subset\mathrm{Dom}\,\varphi$. Then:

5.9 Theorem *For the quasifree functional φ above, and a bounded interval $J\subset\mathbb{R}$ we have that*

$$\Big\|\varphi\restriction\mathcal{A}_0(J)\Big\|\leq\exp(K|J|^2)$$

where K is a constant (independent of J), and $|J|$ is the length of J.

Thus φ is bounded on all the local algebras $\mathcal{A}_0(J)$.

6 The JLO–cocycle.

From an assumed supersymmetry structure on a C*-algebra and a KMS-functional, Jackiw, Lesniewski and Osterwalder [11] and Kastler [15] constructed with a Chern character formula an entire cyclic cocycle in the sense of Connes [5]. Their assumed supersymmetry assumptions are too restrictive to include quantum field theories on noncompact spacetimes. Here we want to show that we can adapt the JLO cocycle formula to produce a well-defined entire cyclic cocycle for our model, using the KMS-functional in the preceding section. We first need to make sense of the Chern character formula:

$$\begin{aligned}\tau_n(a_0,\ldots,a_n) \;&:=\; i^{\epsilon_n}\int_{\sigma_n}\varphi\Big(a_0\,\alpha_{is_1}(\delta\gamma(a_1))\,\alpha_{is_2}(\delta(a_2))\,\alpha_{is_3}(\delta\gamma(a_3))\cdots\\ &\qquad\cdots\alpha_{is_n}(\delta\gamma^n(a_n))\Big)\,ds_1\cdots ds_n\,,\qquad a_i\in\mathcal{D}_S,\end{aligned}\tag{19}$$

where $\epsilon_n := n \bmod 2$, φ is the graded KMS–functional above w.r.t. the data γ, α, δ above, and

$$\sigma_n := \{\,\mathbf{s}\in\mathbb{R}^n \;\mid\; 0\le s_1\le s_2\le\cdots\le s_n\le 1\,\}\;.$$

Since only $\alpha:\mathbb{R}\to\operatorname{Aut}\mathcal{A}$ is given, the expressions $\alpha_{is},\ s\in\mathbb{R}$ in the formula are undefined, and need to be interpreted. Let $b_0,\ldots,b_n\in\mathcal{A}_0\subset\operatorname{Dom}\varphi$, then using the KMS–property of φ, the function

$$Q(t_1,\ldots,t_n) := \varphi(b_0\alpha_{t_1}(b_1\alpha_{t_2}(b_2\cdots\alpha_{t_n}(b_n)\cdots))=\varphi(b_0\alpha_{t_1}(b_1)\cdots\alpha_{t_1+\cdots+t_n}(b_n))$$

can be analytically extended in each variable t_j into the strip $\{\,z_j \mid 0\le\operatorname{Im}z_j\le 1\,\}$ keeping the other varables real. This produces n functions $Q^j : T_j\to\mathbb{C}$ where $T_j := \{\,\mathbf{z}\in\mathbb{C}^n \mid z_j\in\mathbb{R}+i[0,1],\ z_k\in\mathbb{R}\text{ for }k\neq j\,\}$ such that $Q^i\restriction\mathbb{R}^n = Q^j\restriction\mathbb{R}^n$ for all i,j. The sets T_j are *flat tubes,* i.e. of the form $T_B=\mathbb{R}^n+iB$ where the basis $B\subset\mathbb{R}^n$ is of dimension less than n. To continue, we now need the *Flat Tube Theorem* [2]:

6.1 Theorem *Let T_{B_1}, $T_{B_2}\subset\mathbb{C}^n$ be two flat tubes whose bases B_1, B_2 are convex, with closures which contain 0 and are star-shaped w.r.t. 0. Let F_1, F_2 be any two functions analytic in T_{B_1}, T_{B_2} respectively, with continuous boundary values on $\mathbb{R}^n$ and such that $F_1\restriction\mathbb{R}^n = F_2\restriction\mathbb{R}^n$. Then there is a unique function F extending F_1 and F_2 analytically into the tube $T_{\widehat{B_1\cup B_2}}$ (where $\widehat{B_1\cup B_2}$ is the convex hull of $B_1\cup B_2$) and with continuous boundary values on $\mathbb{R}^n$. We have that $T_{\widehat{B_1\cup B_2}} = \bigcup\limits_{0\le\lambda\le1} T_{B_\lambda}$ where $B_\lambda := (1-\lambda)B_1+\lambda B_2$.*

Using this inductively, we can extend Q by analytic continuation into the tube $\mathcal{T}_n := \mathbb{R}^n + i\Sigma_n$ where Σ_n is the convex hull of the unit intervals on the axes, i.e. the simplex $\Sigma_n := \{\,\mathbf{s}\in\mathbb{R}^n \mid 0\le s_i\ \forall i,\ \ s_1+\cdots+s_n\le 1\,\}$. So we will interpret

$$\varphi(b_0\alpha_{z_1}(b_1)\cdots\alpha_{z_1+\cdots+z_n}(b_n)) := Q(z_1,\ldots,z_n)$$

for $\mathbf{z}\in\mathcal{T}_n$ to be this unique analytic continuation. The change of variables $w_1 := z_1,\ w_2 := z_1+z_2,\ldots,\ w_n := z_1+\cdots+z_n$ defines an invertible complex linear map $W:\mathbb{R}^n+i\Sigma_n\to\mathbb{R}^n+i\sigma_n$, so both W and W^{-1} are analytic, and hence

$$(Q\circ W^{-1})(w_1,\ldots,w_n) =: \varphi(b_0\alpha_{w_1}(b_1)\cdots\alpha_{w_n}(b_n))$$

is analytic on $\mathbb{R}^n + i\sigma_n$. In particular

$$\int_{\Sigma_n} \varphi(b_0\alpha_{ir_1}(b_1)\cdots\alpha_{ir_1+\cdots+ir_n}(b_n))\,dr_1\cdots dr_n = \int_{\sigma_n} \varphi(b_0\alpha_{is_1}(b_1)\cdots\alpha_{is_n}(b_n))\,ds_1\cdots ds_n$$

by the change of variables $s_1 = r_1,\ s_2 = r_1 + r_2, \ldots, s_n = r_1 + \cdots + r_n$. By the substitutions $a_0 = b_0,\ b_1 = \delta\gamma(a_1), \ldots, b_n = \delta\gamma^n(a_n)$ into this formula, we arrive at a consistent interpretation of the Chern character formula (19).

Let us recall from [11, 12, 5] the definition of an entire cyclic cocycle.

6.2 Definition *Equip $\mathcal{D}_S$ with the Sobolev norm $\|a\|_* = \|a\|+\|\delta a\|$, and for any *–algebra $\mathcal{D} \subseteq \mathcal{D}_S$ let $\mathcal{C}^n(\mathcal{D})$ denote the space of $(n+1)$–linear functionals on $\mathcal{D}$ which are continuous w.r.t. the norm $\|\cdot\|_*$, and let $\|\cdot\|_*$ denote also the norm on $\mathcal{C}^n(\mathcal{D})$ w.r.t. the norm $\|\cdot\|_*$ on $\mathcal{D}$. Define the space of cochains $\mathcal{C}(\mathcal{D})$ to be the space of sequences $\rho = (\rho_0, \rho_1, \ldots)$ where $\rho_n \in \mathcal{C}^n(\mathcal{D})$ which satisfy the entire analyticity condition:*

$$\lim_{n\to\infty} n^{1/2}\|\rho_n\|_*^{1/n} = 0\ . \tag{20}$$

The entire cyclic cohomology is defined by a coboundary operator $\partial = b+B$ on $\mathcal{C}(\mathcal{D})$ for operators

$$b : \mathcal{C}^n(\mathcal{D}) \to \mathcal{C}^{n+1}(\mathcal{D})\ , \qquad B : \mathcal{C}^{n+1}(\mathcal{D}) \to \mathcal{C}^n(\mathcal{D})$$

$$\textit{i.e.} \qquad (\partial\rho)_n(a_0, \ldots, a_n) = (b\rho_{n-1})(a_0, \ldots, a_n) + (B\rho_{n+1})(a_0, \ldots, a_n) \tag{21}$$

where b and B are given by:

$$\begin{aligned}(b\rho_n)(a_0, \ldots, a_{n+1}) &= \sum_{j=0}^{n}(-1)^j \rho_n(a_0, \ldots, a_ja_{j+1}, \ldots, a_{n+1}) \\ &\qquad\qquad +(-1)^{n+1}\rho_n(\widetilde{a}_{n+1}\cdot a_0, a_1, \ldots, a_n) \qquad (22)\\ (B\rho_n)(a_0, \ldots, a_{n-1}) &= \rho_n(\mathbb{I}, a_0, \ldots, a_{n-1}) + (-1)^{n-1}\rho_n(a_0, \ldots, a_{n-1}, \mathbb{I}) \\ &\quad + \sum_{j=1}^{n-1}(-1)^{(n-1)j}\Big[\rho_n\left(\mathbb{I}, \gamma(a_{n-j}), \ldots, \gamma(a_{n-1}), a_0, \ldots, a_{n-j-1}\right) \\ &\qquad\qquad +(-1)^{n-1}\rho_n\left(\gamma(a_{n-j}), \ldots, \gamma(a_{n-1}), a_0, \ldots, a_{n-j-1}, \mathbb{I}\right)\Big] \qquad (23)\end{aligned}$$

$$\textit{where} \qquad \widetilde{a} := \begin{cases} \gamma(a) & \textit{if } \rho_n \in \mathcal{C}^n_+(\mathcal{D}) \\ a & \textit{if } \rho_n \in \mathcal{C}^n_-(\mathcal{D}) \end{cases}$$

where $\mathcal{C}^n_+(\mathcal{D})$ denotes the even part under γ, and $\mathcal{C}^n_-(\mathcal{D})$ the odd part. The **entire cyclic cocycles** *are those $\rho \in \mathcal{C}(\mathcal{D})$ for which $\partial\rho = 0$, i.e.*

$$(b\rho_{n-1})(a_0, \ldots, a_n) = -(B\rho_{n+1})(a_0, \ldots, a_n)\ , \qquad n = 1, 2, \ldots \tag{24}$$

Below we will use the even part of τ to define an entire cyclic cocycle.

6.3 Theorem *For τ_n defined in Equation (19) we have that*

$$|\tau_n(a_0, \ldots, a_n)| \leq \frac{A}{n!}\, e^{Bn}\, \|a_0\|_* \cdots \|a_n\|_*$$

*for all $a_i \in \mathcal{D}_k :=$ *–alg$\{\,\mathbb{I},\ R(1,f),\ \zeta(f)\ \mid\ \mathrm{supp}(f) \subseteq [-k,\,k]\,\} \subset \mathcal{D}_S$ and where $\|a\|_* := \|a\| + \|\delta a\|$ as above, and for some constants A and B which depend on $k > 0$ but are independent of n. Thus condition (20) holds for τ, i.e.*

$$\lim_{n\to\infty} n^{1/2}\|\tau_n\|_*^{1/n} = 0\ .$$

Using this, we can now prove that:

6.4 Theorem *The sequence* $\widetilde{\tau} := (\tau_0, 0, -\tau_2, 0, \tau_4, 0, -\tau_6, \ldots) \in \mathcal{C}(\mathcal{D}_k)$ *defines an entire cyclic cocycle for each* $k > 0$, *i.e.*

$$(b\tau_{n-1})(a_0, \ldots, a_n) = (B\tau_{n+1})(a_0, \ldots, a_n)\,, \qquad n = 1, 3, 5, \ldots$$

and the entire analyticity condition holds.

It is possible to have taken the choice $\partial = b - B$ for the cyclic coboundary operator above;- this is in fact done in [13], and would have led to the cyclic cocycle $(\tau_0, 0, \tau_2, 0, \tau_4, 0, \ldots)$ instead of $\widetilde{\tau}$ above.

Note that whilst we have obtained entire cyclic cocycles on each compact set $[-k,\, k]$ these do not define an entire cyclic cocycle on $\mathcal{D}_{\rm comp} := {}^*\text{–alg}\,\{\,\mathbb{I},\ R(1, f),\ \zeta(f)\ \mid\ \text{supp}(f) \text{ is compact}\,\} \subset \mathcal{D}_S$ because one can choose a sequence $\{a_0,\, a_1, \ldots\}$ with $a_j \in \mathcal{D}_{k_j}$ where k_j grows sufficiently fast so that through the dependencies of the constants A and B in Theorem 6.3 on k_j the entire analytic condition fails. One expects to use an inductive limit argument, to define an index on $\mathcal{D}_{\rm comp}$ from the indices on the $\mathcal{D}_k$.

From this cyclic cocycle we can calculate an index for this quantum field theory, but its physical significance is presently unclear, though one would expect it to remain stable under deformations. This type of index is discussed in more detail in Longo [18].

7 Conclusions

In this paper we have explored how supersymmetric quantum fields can be treated in a C*-algebra setting, avoiding the obstructions found by Kishimoto and Nakamura [16] and by Buchholz and Longo [3]. We did this in detail for a simple one–dimensional model.

In order to establish a reasonable domain of definition for the super–derivation, we found it necessary to analyze a notion of "mollifiers" for the quantum fields and to introduce a corresponding C*-algebra, the resolvent algebra. The full algebra $\mathcal{A}$ defining the model can then be taken as the tensor product of this resolvent algebra and the familiar CAR–algebra.

The super–derivation is defined on a subalgebra which is weakly dense in $\mathcal{A}$ in all representations of physical interest; alternatively, one can define it on a norm dense subalgebra of $\mathcal{A}$ with range in a *–algebra $\mathcal{E}$ of bounded and unbounded operators which are affiliated with $\mathcal{A}$ in the sense of [9]. Similarly, the basic supersymmetry relation can either be formulated in a mollified form on some weakly dense domain or, alternatively, as a relation between maps which have been extended to the *–algebra $\mathcal{E}$. These findings reveal some basic features of supersymmetric quantum field theories which have to be taken into account in a general C*-framework covering such theories. The tools developed here should also be useful in other areas of quantum field theory where one needs to use graded derivations, e.g. in BRS–constraint theory.

We also exhibited in the present model graded KMS–functionals for arbitrary positive temperatures which are supersymmetric. In accordance with the general results in [3], these functionals are unbounded. Yet their restrictions to any local subalgebra of the underlying C*-algebra $\mathcal{A}$ are bounded. It is an interesting question whether these functionals are also locally normal with respect to the vacuum representation of the theory, as one would heuristically expect.

The KMS–functionals were then employed to define cyclic cocycles. In view of the fact that the domain of definition of these functionals does not contain analytic elements with regard to the time evolution, the strategy outlined in [15, 11] could not be applied here. That these cocycles can be constructed, nevertheless, is due to the fact that the functionals inherit sufficiently strong analyticity properties from the KMS–condition which allow one to perform the necessary complex integrations. Moreover, the resulting cocycles are entire on all local algebras.

These functionals may thus be taken as an input for a quantum index theory as suggested by Longo [18]. Such an index should be stable under deformations, and one can easily think of possible deformations of our model, e.g. deform the supersymmetry generator Q by an appropriate function M :

$$\begin{aligned} Q_M &:= \int M(x)\, j(x)\, c(x)\, dx \\ \text{so} \qquad \delta_M(c(f)) &= j(Mf), \qquad \delta_M(j(f)) = c(Mf') \\ \text{thus:} \qquad \delta_M^2 &=: \delta_{0M} \end{aligned}$$

defines the new generator for time evolution.

8 Appendix

Proof of Theorem 3.4

(i) By (6) we have that $i(\mu-\lambda)R(\lambda,f)R(\mu,f) = R(\lambda,f) - R(\mu,f) = -(R(\mu,f)-R(\lambda,f)) = i(\mu-\lambda)R(\mu,f)R(\lambda,f)$, i.e. $[R(\lambda,f),\, R(\mu,f)] = 0$.
(ii) The Fock representation is a subrepresentation of π_S, and the resolvents of the fields $\varphi(f)$ give the Fock representation induced on $\mathcal{R}(X,\sigma)$, i.e. $\pi(R(\lambda,f)) = (i\lambda - \varphi(f))^{-1}$. Since this is nonzero, $R(\lambda,f) \neq 0$ for all nonzero f and λ. Now by $R(\lambda,f)^* = R(-\lambda,f)$ we get

$$2|\lambda|\|R(\lambda,f)\|^2 = \|2\lambda R(\lambda,f)R(\lambda,f)^*\| = \|R(\lambda,f) - R(\lambda,f)^*\| \leq 2\|R(\lambda,f)\|\,.$$

Thus by $R(\lambda,f)\neq 0$ we find $\|R(\lambda,f)\| \leq 1/|\lambda|$. Now

$$\|R(\lambda,f)\| \geq \|\pi(R(\lambda,f))\| = \|(i\lambda-\varphi(f))^{-1}\| = \sup_{t\in\sigma(\varphi(f))}\left|\frac{1}{i\lambda - t}\right| = \frac{1}{|\lambda|}$$

using the fact that the spectrum $\sigma(\varphi(f)) = \mathbb{R}$. Thus $\|R(\lambda,f)\| = 1/|\lambda|$.
(iii) Rearrange equation (6) to get:

$$R(\lambda,f)(\mathbb{I} - i(\lambda_0-\lambda)R(\lambda_0,f)) = R(\lambda_0,f)\,.$$

Now by (ii) above, if $|\lambda_0 - \lambda| < |\lambda_0|$ then $\|i(\lambda_0-\lambda)R(\lambda_0,f)\| < 1$, and hence $(\mathbb{I} - i(\lambda_0-\lambda)R(\lambda_0,f))^{-1}$ exists, and is given by a norm convergent power series in $i(\lambda_0-\lambda)R(\lambda_0,f)$. That is, we have that

$$R(\lambda,f) = R(\lambda_0,f)(\mathbb{I} - i(\lambda_0-\lambda)R(\lambda_0,f))^{-1} = \sum_{n=0}^{\infty}(\lambda_0-\lambda)^n\, R(\lambda_0,f)^{n+1} i^n$$

when $|\lambda_0-\lambda| < |\lambda_0|$, as claimed.
(iv) From equation (5) we get $R(\lambda,tf) = \frac{1}{\lambda}R(1,\frac{t}{\lambda}f) = \frac{1}{t}R(\frac{\lambda}{t},f)$ so

$$R(\lambda,sf) \quad - \quad R(\lambda,tf) = \tfrac{1}{s}R(\tfrac{\lambda}{s},f) - \tfrac{1}{t}R(\tfrac{\lambda}{t},f)$$

$$\begin{aligned} &= \tfrac{1}{s}\left(R(\tfrac{\lambda}{s},f) - R(\tfrac{\lambda}{t},f)\right) + \left(\tfrac{1}{s}-\tfrac{1}{t}\right) R(\tfrac{\lambda}{t},f) \\ &= \tfrac{i\lambda}{s}\left(\tfrac{1}{t}-\tfrac{1}{s}\right) R(\tfrac{\lambda}{s},f)R(\tfrac{\lambda}{t},f) + \left(\tfrac{1}{s}-\tfrac{1}{t}\right) R(\tfrac{\lambda}{t},f)\,. \end{aligned}$$

Thus $\|R(\lambda,sf)-R(\lambda,tf)\| \leq \left|\frac{1}{s}-\frac{1}{t}\right| 2\left|\frac{t}{\lambda}\right|$ from which continuity away from zero is clear.
(v) This follows directly from equation (7) by interchanging λ and f with μ and g resp.
(vi) Recall that we have a faithful (strongly regular) representation π_S of $\mathcal{R}(X,\sigma)$ which is an extension of a regular representation of the Weyl algebra $\overline{\Delta(X,\,\sigma)}$ such that $\pi_S\left(\overline{\Delta(X,\,\sigma)}\right)'' \supset \pi_S\left(\mathcal{R}(X,\sigma)\right)$ and such that each $\pi_S(R(\lambda,f))$ is the resolvent of the generator $j_{\pi_S}(f)$ of the one-parameter group $t\to\pi_S(\delta_{tf})$. Let $T\in\mathrm{Sp}(X,\sigma)$ then it defines an automorphism of $\overline{\Delta(X,\,\sigma)}$ by $\alpha_T(\delta_f)=\delta_{Tf}$ which preserves the set of strongly regular states, and in fact defines a bijection on the set of strongly regular states by $\omega\to\omega\circ\alpha_T$. Now π_S is the direct sum of the GNS–representations of all the strongly regular states, and hence $\pi_S\circ\alpha_T$ is just π_S where its direct summands have been permuted. Such a permutation of direct summands can be done by conjugation of a unitary, thus π_S is unitarily equivalent to $\pi_S\circ\alpha_T$, and so we can extend α_T by unitary conjugation to $\pi_S\left(\overline{\Delta(X,\,\sigma)}\right)''$. By equation (9) we get that $\alpha_T\left(\pi_S(R(\lambda,f))\right)=\pi_S(R(\lambda,Tf))$, and hence α_T preserves $\mathcal{R}(X,\sigma)$, so defines an automorphism on it.

Proof of Theorem 3.6

(i) Observe that by Theorem 1 p216 of Yosida [26], we deduce from $\mathrm{Ker}\,\pi(R(1,f))=\{0\}$ that $\pi(R(\lambda,f))$ is the resolvent of $j_\pi(f)$, i.e. we have now for all $\lambda\neq 0$ that $j_\pi(f)=i\lambda\mathbb{I}-\pi(R(\lambda,f)))^{-1}$. Then

$$\begin{aligned} j_\pi(tf) &= i\mathbb{I}-\pi(R(1,tf))^{-1} = i\mathbb{I}-t\pi(R(\tfrac{1}{t},f))^{-1} \\ &= t\left(i\tfrac{1}{t}\mathbb{I}-\pi(R(\tfrac{1}{t},f))^{-1}\right) = t\,j_\pi(f)\,. \end{aligned}$$

Thus

$$\begin{aligned} j_\pi(f)^* &= \left(i\mathbb{I}-\pi(R(1,f))^{-1}\right)^* \supseteq -i\mathbb{I}-\left(\pi(R(1,f))^{-1}\right)^* \\ &= -i\mathbb{I}-\pi(R(1,f)^*)^{-1} = -i\mathbb{I}-\pi(R(-1,f))^{-1} \\ &= -i\mathbb{I}+\pi(R(1,-f))^{-1} = -j_\pi(-f) = j_\pi(f) \end{aligned}$$

and hence $j_\pi(f)$ is symmetric. To see that it is selfadjoint note that:

$$\mathrm{Ran}\,(j_\pi(f)\pm i\mathbb{I}) = \mathrm{Ran}\,\left(-\pi(R(\pm1,f))^{-1}\right) = \mathrm{Dom}\,\left(\pi(R(\pm1,f))\right) = \mathcal{H}_\pi$$

hence the deficiency spaces $(\mathrm{Ran}\,(j_\pi(f)\pm i\mathbb{I}))^\perp=\{0\}$ and so $j_\pi(f)$ is selfadjoint.
For the domain claim, recall that $\mathrm{Dom}\,j_\pi(f)=\mathrm{Ran}\,\pi(R(1,f))$. So

$$\begin{aligned} \pi(R(\lambda,f))\mathrm{Dom}\,j_\pi(h) &= \pi(R(\lambda,f))\pi(R(1,h))\mathcal{H}_\pi \\ &= \pi\left(R(1,h)R(\lambda,f)+i\sigma(f,h)R(1,h)R(\lambda,f)^2R(1,h)\right)\mathcal{H}_\pi \\ &\subseteq \pi\left(R(1,h)\right)\mathcal{H}_\pi = \mathrm{Dom}\,j_\pi(h). \end{aligned}$$

(ii) Let $j_\pi(f)=\int\lambda dP(\lambda)$ be the spectral resolution of $j_\pi(f)$. Then $\pi(R(\mu,f))=\int\frac{1}{i\mu-\lambda}dP(\lambda)$ hence

$$i\mu\pi(R(\mu,f))\psi = \int\frac{i\mu}{i\mu-\lambda}\,dP(\lambda)\psi \quad \forall\,\psi\in\mathcal{H}_\pi\,.$$

Since $\left|\frac{i\mu}{i\mu-\lambda}\right| < 1$ (for $\mu \in \mathbb{R}\backslash 0$) the integrand is dominated by 1 which is an L^1–function with respect to $dP(\lambda)$, and as we have pointwise that $\lim\limits_{\mu\to\infty} \frac{i\mu}{i\mu-\lambda} = 1$, we can apply the dominated convergence theorem to get that

$$\lim_{\mu\to\infty} i\mu\pi(R(\mu,f))\psi = \int dP(\lambda)\psi = \psi\,.$$

(iii) $i\pi(R(1,sf))\psi = \int \frac{i}{i-s\lambda}\,dP(\lambda)\psi \to \psi$ as $s \to 0$ by the same argument as in (ii).
(iv) Let $\mathcal{D} := \pi(R(1,f)R(1,h))\mathcal{H}_\pi$, then by definition $\mathcal{D} \subseteq \operatorname{Ran}\pi(R(1,f)) = \operatorname{Dom} j_\pi(f)$. Moreover $\pi(R(1,f)R(1,h))\mathcal{H}_\pi = \pi(R(1,h)[R(1,f) + i\sigma(f,h)R(1,f)^2R(1,h)])\mathcal{H}_\pi \subseteq \operatorname{Ran}\pi(R(1,h)) = \operatorname{Dom} j_\pi(h)$, i.e. $\mathcal{D} \subseteq \operatorname{Dom} j_\pi(f) \cap \operatorname{Dom} j_\pi(h)$. That $\mathcal{D}$ is dense, follows from (iii) of this theorem, using

$$\lim_{s\to 0}\lim_{t\to 0} \pi(R(1,sf)R(1,th)\psi = -\psi$$

for all $\psi \in \mathcal{H}_\pi$, as well as $sR(1,sf) = R(1/s,\, f)$ and the fact mentioned before (cf. Theorem 1 p216 in [26]) that all $\pi(R(\lambda,f))$ have the same range for f fixed.
Let $\psi \in \mathcal{D}$, i.e. $\psi = \pi(R(1,f)R(1,h))\varphi$ for some $\varphi \in \mathcal{H}_\pi$. Then

$$\begin{aligned}
&\pi(R(1,h)R(1,f))[j_\pi(f), j_\pi(h)]\psi \\
&\qquad = \pi(R(1,h)R(1,f))[\pi(R(1,f))^{-1}, \pi(R(1,h))^{-1}]\pi(R(1,f)R(1,h))\varphi \\
&= \pi(R(1,f)R(1,h) - R(1,h)R(1,f))\varphi = i\sigma(f,h)\pi(R(1,h)R(1,f)^2R(1,h))\varphi \\
&= i\sigma(f,h)\pi(R(1,h)R(1,f))\psi\,.
\end{aligned}$$

Since $\operatorname{Ker}\pi(R(1,h)R(1,f)) = \{0\}$ it follows that $[j_\pi(f), j_\pi(h)] = i\sigma(f,h)$ on $\mathcal{D}$.
(v) From Equation (5) we have that

$$\pi(R(\lambda,f)) = (1\lambda - j_\pi(f))^{-1} = \tfrac{1}{\lambda}\pi(R(1,\tfrac{1}{\lambda}f)) = \tfrac{1}{\lambda}\cdot\left(i\mathbb{I} - j_\pi(\tfrac{1}{\lambda}f)\right)^{-1}$$

and hence that $j_\pi(f) = \lambda\, j_\pi(\frac{1}{\lambda}f)$, i.e. $j_\pi(\lambda f) = \lambda j_\pi(f)$ for all $\lambda \in \mathbb{R}\backslash 0$. In equation (8):

$$\pi\left(R(\lambda,f)R(\mu,g)\right) = \pi\left(R(\lambda+\mu,\ f+g)[R(\lambda,f) + R(\mu,g) + i\sigma(f,g)R(\lambda,f)^2R(\mu,g)]\right)$$

multiply on the left by $i(\lambda+\mu)\mathbb{I} - j_\pi(f+g)$ and apply to $(i\mu\mathbb{I} - j_\pi(g))(i\lambda\mathbb{I} - j_\pi(f))\psi$, $\psi \in \mathcal{D}$ to get

$$(i(\lambda+\mu)\mathbb{I} - j_\pi(f+g))\psi = ((i\mu\mathbb{I} - j_\pi(g)) + (i\lambda\mathbb{I} - j_\pi(f)))\,\psi$$

making use of $[(i\mu\mathbb{I} - j_\pi(g)), (i\lambda\mathbb{I} - j_\pi(f))]\,\psi = i\sigma(g,f)\psi$. Thus $j_\pi(f+g) = j_\pi(f) + j_\pi(g)$ on $\mathcal{D}$.
(vi) From the spectral resolution for $j_\pi(f)$ we have trivially that on $\operatorname{Dom} j_\pi(f)$

$$j_\pi(f)\pi(R(\mu,f)) = \pi(R(\mu,f))j_\pi(f) = \int \frac{\lambda}{i\mu\mathbb{I} - \lambda}dP(\lambda) = i\mu\pi(R(\mu,f)) - \mathbb{I}\,.$$

(vii) Let $\psi \in \operatorname{Dom} j_\pi(f) = \operatorname{Ran}\pi(R(\lambda,f))$, i.e. $\psi = \pi(R(\lambda,f))\varphi$ for some $\varphi \in \mathcal{H}_\pi$. Then

$$\begin{aligned}
\pi(R(\lambda,f))[j_\pi(f),\, \pi(R(\lambda,g))]\psi &= \pi(R(\lambda,f))[j_\pi(f),\, \pi(R(\lambda,g))]\pi(R(\lambda,f))\varphi \\
&= \pi\left([R(\lambda,g), R(\lambda,f)]\right)\varphi = i\sigma(g,f)\pi\left(R(\lambda,f)R(\lambda,g)^2R(\lambda,f)\right)\varphi \\
&= i\sigma(g,f)\pi\left(R(\lambda,f)R(\lambda,g)^2\right)\psi\,.
\end{aligned}$$

Since $\mathrm{Ker}\,\pi(R(\lambda,f))=\{0\}$, it follows that

$$[j_\pi(f),\,\pi(R(\lambda,g))]=i\sigma(g,f)\pi\left(R(\lambda,g)^2\right)$$

on $\mathrm{Dom}\,j_\pi(f)$.

(viii) We first prove the second equality. Let $\psi,\,\varphi\in\widetilde{\mathcal{D}}:=\mathrm{Span}\{\,\chi_{[-a,a]}(j_\pi(f))\mathcal{H}_\pi\;\mid\;a>0\,\}$ which is a dense subspace. Since $\left\|j_\pi(f)\restriction\chi_{[-a,a]}(j_\pi(f))\mathcal{H}_\pi\right\|\leq a$, we can use the exponential series, i.e.

$$W(f)\psi:=\exp\left(ij_\pi(f)\right)\psi=\sum_{n=0}^\infty\frac{(ij_\pi(f))^n}{n!}\psi\qquad\forall\,\psi\in\widetilde{\mathcal{D}}\,.$$

By the usual rearrangement of series we then have

$$(\varphi,\,W(f)\pi(R(\lambda,h))W(f)^*\psi)=\sum_{n=0}^\infty\frac{1}{n!}\left(\varphi,\,(\mathrm{ad}\,ij_\pi(f))^n(\pi(R(\lambda,h)))\psi\right)$$

for all $\varphi,\,\psi\in\widetilde{\mathcal{D}}$. Using part (vii) we have

$$\begin{aligned}
&& (\mathrm{ad}\,ij_\pi(f))(\pi(R(\lambda,h)^k)) &= k\,\sigma(f,h)\pi(R(\lambda,h)^{k+1})\\
\text{thus} && (\mathrm{ad}\,ij_\pi(f))^n(\pi(R(\lambda,h)) &= n!\,\sigma(f,h)^n\pi(R(\lambda,h)^{n+1})\\
\text{so} && (\varphi,\,W(f)\pi(R(\lambda,h))W(f)^*\psi) &= \sum_{n=0}^\infty\sigma(f,h)^n\left(\varphi,\,\pi(R(\lambda,h)^{n+1})\psi\right)\\
&& &= (\varphi,\,\pi(R(\lambda+i\sigma(f,h),\,h))\psi)
\end{aligned}$$

where we made use of the Von Neumann series (Theorem 3.4(iii)) in the last step. Since the operators involved are bounded and $\widetilde{\mathcal{D}}$ is dense, it follows that

$$W(f)\pi(R(\lambda,h))W(f)^*=\pi(R(\lambda+i\sigma(f,h),\,h))\,. \tag{25}$$

To prove the first equation, let us write $W(h)$ in terms of resolvents. Note that $\lim\limits_{n\to\infty}(1-it/n)^{-n}=e^{it}$, $t\in\mathbb{R}$ and so by the bound: $\sup\limits_{t\in\mathbb{R}}|(1-it/n)^{-n}|=\sup\limits_{t\in\mathbb{R}}\left(1+t^2/n^2\right)^{-n}=1$, it follows from spectral theory (cf. Theorem VIII.5(d), p262 in [22]) that

$$W(h)=e^{ij_\pi(h)}=\lim_{n\to\infty}\left(1-ij_\pi(h)/n\right)^{-n}=\lim_{n\to\infty}\pi\left(iR(1,-h/n)\right)^n$$

in strong operator topology. Apply equation (25) to this to get

$$\begin{aligned}
W(f)W(h)W(f)^* &= \operatorname*{s-lim}_{n\to\infty}\pi\left(iR(1+i\sigma(f,-\tfrac{h}{n}),-\tfrac{h}{n})\right)^n\\
&= \operatorname*{s-lim}_{n\to\infty}\left(\mathbb{I}-i(\sigma(f,h)+j_\pi(h))/n\right)^{-n}\\
&= \exp[i\sigma(f,h)+ij_\pi(h)]=e^{i\sigma(f,h)}W(h)
\end{aligned}$$

as required. Now

$$W(f)\mathcal{D}=W(f)\pi(R(\lambda,f)R(\mu,h))\mathcal{H}_\pi=\pi(R(\lambda,f)R(\mu+i\sigma(f,h),h))\mathcal{H}_\pi=\mathcal{D}$$

hence we conclude that $\mathcal{D}$ is a core for $j_\pi(f)$ (cf. Theorem VIII.11, p269 in [22]).

Proof of Proposition 3.7

Recall Equation (10)

$$\begin{aligned}
&\omega(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n))\\
&=(-i)^n\int_0^\infty dt_1\ldots\int_0^\infty dt_n\,\exp\Big(-\sum_k t_k\lambda_k-\sum_{k<l}t_kt_l\langle f_k|f_l\rangle_\omega-\tfrac12\sum_l t_l^2\langle f_l|f_l\rangle_\omega\Big)
\end{aligned}$$

then the integrand $F(\mathbf{x})$ is differentiable as a function of $\mathbf{x}\in\mathbb{R}^n$, because by assumption all $x_k,\,x_l\mapsto\langle f_k(x_k)|f_l(x_l)\rangle_\omega$ are continuously differentiable. Thus by the chain rule we obtain for its partial derivatives that

$$\begin{aligned}
\frac{\partial}{\partial x_r}F(\mathbf{x}) &= \frac{\partial}{\partial x_r}\Big[-\sum_k t_k\lambda_k-\sum_{k<l}t_kt_l\langle f_k|f_l\rangle_\omega-\tfrac12\sum_l t_l^2\langle f_l|f_l\rangle_\omega\Big]F(\mathbf{x})\\
&= \Big[-\sum_{k=1}^{r-1}t_rt_k\frac{\partial}{\partial x_r}\langle f_k(x_k)|f_r(x_r)\rangle_\omega-\tfrac12 t_r^2\frac{\partial}{\partial x_r}\langle f_r(x_r)|f_r(x_r)\rangle_\omega\\
&\qquad -\sum_{k=r+1}^{n}t_rt_k\frac{\partial}{\partial x_r}\langle f_r(x_r)|f_k(x_k)\rangle_\omega\Big]F(\mathbf{x}) \qquad (26)\\
&= -\sum_{k=1}^{r-1}\frac{\partial}{\partial x_r}\langle f_k(x_k)|f_r(x_r)\rangle_\omega\frac{\partial^2}{\partial\lambda_r\partial\lambda_k}F(\mathbf{x})-\tfrac12\frac{\partial}{\partial x_r}\langle f_r(x_r)|f_r(x_r)\rangle_\omega\frac{\partial^2}{\partial\lambda_r^2}F(\mathbf{x})\\
&\qquad -\sum_{k=r+1}^{n}\frac{\partial}{\partial x_r}\langle f_r(x_r)|f_k(x_k)\rangle_\omega\frac{\partial^2}{\partial\lambda_r\partial\lambda_k}F(\mathbf{x}) \qquad (27)
\end{aligned}$$

making use of $t\exp(-t\lambda)=-\frac{\partial}{\partial\lambda}\exp(-t\lambda)$. In the middle step (26), all the terms are integrable functions in the t_i–variables, and bounded by an integrable function (uniformly in x_r). To see this, recall that $\langle\cdot|\cdot\rangle_\omega$ is positive semidefinite and hence $|t_it_jF(\mathbf{x})|\leq t_it_j\exp\Big(-\sum_k t_k\lambda_k\Big)$ and this is integrable in the (positive) variables $t_1,\ldots,t_n$ because $\lambda_k>0$ for all k. Thus we can use the dominated convergence theorem for derivatives (cf. Theorem 2.7 in [7]) to conclude that $\omega(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n))$ is differentiable in all x_i–variables, and the partial derivatives can be taken into the integral to give via (27):

$$\begin{aligned}
&\frac{\partial}{\partial x_r}\omega\Big(R(\lambda_1,f_1(x_1))\cdots R(\lambda_n,f_1(x_n))\Big)\\
&= -\sum_{k=1}^{r-1}\frac{\partial}{\partial x_r}\Big(\langle f_k(x_k)|f_r(x_r)\rangle_\omega\Big)\int_0^\infty dt_1\ldots\int_0^\infty dt_n\,\frac{\partial^2}{\partial\lambda_r\partial\lambda_k}F(\mathbf{x})\\
&\qquad -\tfrac12\frac{\partial}{\partial x_r}\Big(\langle f_r(x_r)|f_r(x_r)\rangle_\omega\Big)\int_0^\infty dt_1\ldots\int_0^\infty dt_n\,\frac{\partial^2}{\partial\lambda_r^2}F(\mathbf{x})\\
&\qquad -\sum_{k=r+1}^{n}\frac{\partial}{\partial x_r}\Big(\langle f_r(x_r)|f_k(x_k)\rangle_\omega\Big)\int_0^\infty dt_1\ldots\int_0^\infty dt_n\,\frac{\partial^2}{\partial\lambda_r\partial\lambda_k}F(\mathbf{x})\,.
\end{aligned}$$

We need to argue that we can take the partial derivatives w.r.t. the λ_i's through the integrals above. Now if we have that $\langle f_r(x_r)|f_r(x_r)\rangle_\omega=0$, then each integrand factorises into a t_r–dependent and a t_r–independent part. Then the integrand w.r.t. the t_r–variable is of the form $t_r^k\exp(-t_r\lambda_r)$ for $k=0,1,2,$ and so we get explicitly from the Laplace transforms that we can take $\partial/\partial\lambda_r$ through the t_r–integral. This takes care of the part of the integral corresponding to those variables t_r for which $\langle f_r(x_r)|f_r(x_r)\rangle_\omega=0$. The remaining factor

$\widetilde{F}(\mathbf{x})$ of $F(\mathbf{x})$ depends only on variables t_r for which we have that $\langle f_r(x_r)|f_r(x_r)\rangle_\omega > 0$. Then $\left|\frac{\partial^2}{\partial\lambda_r\partial\lambda_k}\widetilde{F}(\mathbf{x})\right| \leq t_r t_k \exp\Big(-\frac{1}{2}\sum_l t_l^2\langle f_l|f_l\rangle_\omega\Big)$ which is integrable w.r.t. the remaining variables, and likewise we also get a dominating function for the first derivatives. Thus by dominated convergence (uniformly in the λ–variables) we can take the partial derivatives in λ_i through the integral in in the remaining variables. Thus we get

$$\begin{aligned}
&\frac{\partial}{\partial x_r}\omega\Big(R(\lambda_1, f_1(x_1))\cdots R(\lambda_n, f_1(x_n))\Big) \\
= \;& -\sum_{k=1}^{r-1}\frac{\partial}{\partial x_r}\Big(\langle f_k(x_k)|f_r(x_r)\rangle_\omega\Big)\frac{\partial^2}{\partial\lambda_r\partial\lambda_k}\omega\Big(R(\lambda_1, f_1(x_1))\cdots R(\lambda_n, f_1(x_n))\Big) \\
& -\tfrac{1}{2}\frac{\partial}{\partial x_r}\Big(\langle f_r(x_r)|f_r(x_r)\rangle_\omega\Big)\frac{\partial^2}{\partial\lambda_r^2}\omega\Big(R(\lambda_1, f_1(x_1))\cdots R(\lambda_n, f_1(x_n))\Big) \\
& -\sum_{k=r+1}^{n}\frac{\partial}{\partial x_r}\Big(\langle f_r(x_r)|f_k(x_k)\rangle_\omega\Big)\frac{\partial^2}{\partial\lambda_r\partial\lambda_k}\omega\Big(R(\lambda_1, f_1(x_1))\cdots R(\lambda_n, f_1(x_n))\Big)
\end{aligned}$$

Proof of Proposition 5.3

To prove this theorem, we first need to establish the following lemma.

8.1 Lemma *For the strip $S := \mathbb{R} + i[0,1[\subset \mathbb{C}$, let $F : S \to \mathbb{C}$ be a continuous function, analytic on the interior of S, which satisfies for some $C > 0$ and $\lambda \in \mathbb{C}$, $|\lambda| = 1$ the conditions:*

$$\begin{aligned}
|F(t+is)| &\leq C(1+|t|)^N \quad \forall\, t \in \mathbb{R},\ s \in (0,1) \\
\text{and} \qquad F(t+i) &= \lambda F(t) \quad \forall\, t \in \mathbb{R}\,.
\end{aligned}$$

Then $F = 0$ if $\lambda \neq 1$ and $F =$ constant if $\lambda = 1$.

Proof: Note that $\mathbb{C}$ is covered by the strips $S_n := S + in$. We define $G : \mathbb{C} \to \mathbb{C}$ by $G(z) := \lambda^n F(z - in)$ whenever $z \in S_n$ This is consistent, because on the joining lines $\mathbb{R}+in = S_n \cap S_{n-1}$ we have that $G(t+in) = \lambda^n F(t) = \lambda^{n-1}F(t+i)$. By the continuity of F on $S_0 = S$, G is continuous. Now G is analytic on the interior of each S_n and continuous on the boundary, i.e. continuous on the lines $\mathbb{R} + in$ and analytic on either side of them. So it follows from a well-known theorem of analytic continuation that G is analytic on the lines $\mathbb{R} + in$ (cf. [20] p183) hence entire. Moreover $G(z + i) = \lambda G(z)$ for all z. Let Γ be a closed anticlockwise circle of radius R centered at the fixed point z_0. If $z \in \Gamma \cap S_n$ then

$$\begin{aligned}
|G(z)| &= |\lambda^n F(z - in)| = |F(z - in)| \\
&\leq C(1 + |\mathrm{Re}(z - in)|)^N = C(1 + |\mathrm{Re}(z)|)^N \\
&\leq C(1 + (R + |\mathrm{Re}(z_0))|)^N \leq C(1 + R + |z_0|)^N
\end{aligned}$$

which is independent of n, i.e. $|G(z)| \leq C(1 + R + |z_0|)^N$ for all $z \in \Gamma$. Applying this to the Cauchy integral formula:

$$\begin{aligned}
G^{(k)}(z_0) &= \frac{k!}{2\pi i}\int_\gamma \frac{G(z)}{(z-z_0)^{k+1}}\,dz \qquad \text{we find:} \\
\left|G^{(k)}(z_0)\right| &\leq k! R \sup_{z\in\Gamma}\frac{|G(z)|}{R^{k+1}} \leq k!\, C\,\frac{(1 + R + |z_0|)^N}{R^k}\,.
\end{aligned}$$

When $k > N+1$ this goes to zero as $R \to \infty$, hence $G^{(k)}(z_0) = 0$ for all $k > N+1$. This is true for all $z_0 \in \mathbb{C}$, so G is a polynomial. However only a constant polynomial can satisfy $G(z+i) = \lambda G(z)$ (or else it has infinitely many zeroes), hence G is a constant, and if $\lambda \neq 1$, the only possible constant is zero. ∎

(i) The KMS-condition for $A = \mathbb{I}$ reads $F_{\mathbb{I},B}(t) = \varphi(\alpha_t(B)) = F_{\mathbb{I},B}(t+i)$. Thus by lemma 8.1 it follows that $F_{\mathbb{I},B}$ is constant, hence that $\varphi(\alpha_t(B))$ is independent of t.
(ii) Let $\gamma(A) = -A \in \mathrm{Dom}\,\varphi$, then $F_{A,\mathbb{I}}(t+i) = \varphi(\gamma(A)) = -\varphi(A) = -F_{A,\mathbb{I}}(t)$ so by lemma 8.1 we have $0 = F_{A,\mathbb{I}} = \varphi(A)$. For any $B \in \mathrm{Dom}\,\varphi$ decompose $B = B_+ + B_-$ into γ-even and odd parts then we get $\varphi(B) = \varphi(B_+) = \varphi(\gamma(B))$.
Finally, let a functional φ satisfy the graded KMS-condition on a set $Y \subset \mathrm{Dom}\,\varphi$. Let $A,\, B \in \mathrm{Span}\, Y$, i.e. $A = \sum_i \lambda_i A_i$, $B = \sum_j \mu_j B_j$ for $A_i,\, B_j \in Y$, and $\lambda_i,\, \mu_j \in \mathbb{C}$. Then

$$F_{A,B}(t) := \varphi(A\alpha_t(B)) = \sum_{i,j} \lambda_i \mu_j\, \varphi(A_i \alpha_t(B_j)) = \sum_{i,j} \lambda_i \mu_j\, F_{A_i,B_j}(t)\,.$$

Since φ is γ–KMS on Y the F_{A_i,B_j} are γ–KMS functions. Thus $F_{A,B}$ is continous on S, analytic on its interior, and

$$\begin{aligned}
F_{A,B}(t+i) &= \sum_{i,j} \lambda_i \mu_j\, F_{A_i,B_j}(t+i) = \sum_{i,j} \lambda_i \mu_j\, \varphi(\alpha_t(B_j)\gamma(A_i)) \\
&= \varphi(\alpha_t(B)\gamma(A))\,, \\
|F_{A,B}(t+is)| &\le \sum_{i,j} |\lambda_i \mu_j|\, |F_{A_i,B_j}(t+is)| \\
&\le \sum_{i,j} |\lambda_i \mu_j|\, C_{ij}(1+|t|)^{N_{ij}} \le C(1+|t|)^N
\end{aligned}$$

where $C := \sum_{i,j} |\lambda_i \mu_j|\, C_{ij}$ and $N := \max_{ij}(N_{ij})$. So φ is γ–KMS on $\mathrm{Span}\, Y$.

Proof of Proposition 5.4

Since $\mathrm{Dom}\,\psi$ and $\mathcal{B}$ are dense *–algebras, it follows that $\mathrm{Dom}\,\varphi := \mathrm{Span}\,\{\, C \otimes B \mid C \in \mathrm{Dom}\,\psi,\ B \in \mathcal{B} \,\}$ is a dense *-algebra of $\mathcal{A} = \mathcal{C} \otimes \mathcal{B}$, and that it is invariant w.r.t. $\gamma \otimes \iota$ and $\sigma \otimes \beta$. Thus by the last part of Proposition 5.3 it suffices to verify the KMS–property on $\{\, C \otimes B \mid C \in \mathrm{Dom}\,\psi,\ B \in \mathcal{B} \,\}$. Let $A_i := C_i \otimes B_i$, $i = 1,2$ for $C_i \in \mathrm{Dom}\,\psi \subset \mathcal{C}$, and $B_i \in \mathcal{B}$. Consider for $t \in \mathbb{R}$ the function

$$\begin{aligned}
F_{A_1,A_2}(t) &:= \varphi\,(A_1(\sigma \otimes \beta)_t(A_2)) \\
&= \varphi((C_1 \otimes B_1)(\sigma_t(C_2) \otimes \beta_t(B_2))) \\
&= \psi(C_1\, \sigma_t(C_2))\, \omega(B_1\, \beta_t(B_2)) = F^{\psi}_{C_1,C_2}(t)\, F^{\omega}_{B_1,B_2}(t)
\end{aligned}$$

where $F^{\psi}_{C_1,C_2}$ and $F^{\omega}_{B_1,B_2}$ are the KMS–functions of ψ and ω resp.. Thus, using their analytic properties, it follows that F_{A_1,A_2} extends to an function on the strip $S = \mathbb{R} + i[0,1]$, analytic on its interior and continuous on the boundary, given by $F_{A_1,A_2}(z) = F^{\psi}_{C_1,C_2}(z)\, F^{\omega}_{B_1,B_2}(z)$. Moreover

$$\begin{aligned}
F_{A_1,A_2}(t+i) &= F^{\psi}_{C_1,C_2}(t+i)\, F^{\omega}_{B_1,B_2}(t+i) = \psi(\sigma_t(C_2)\,\gamma(C_1))\, \omega(\beta_t(B_2)\, B_1) \\
&= \varphi\,(\sigma_t(C_2)\gamma(C_1) \otimes \beta_t(B_2)\, B_1) = \varphi\,((\sigma \otimes \beta)_t(A_2)(\gamma \otimes \iota)(A_1))
\end{aligned}$$

and thus F_{A_1,A_2} will be a $(\gamma\otimes\iota)$–KMS function if the tempered growth property also holds. We have $|F^{\psi}_{C_1,C_2}(t+is)| \leq K(1+|t|)^N$ for a constant K and $N\in\mathbb{N}$ depending on C_i. Now as ω is a state we have from the KMS–property that $|F^{\omega}_{B_1,B_2}(t+is)| \leq \|B_1\|\,\|B_2\|$ for $s=0,\,1$. So by the maximum modulus principle (apply it after first mapping S to the unit disk by the Schwartz mapping principle) it follows that $|F^{\omega}_{B_1,B_2}(z)| \leq \|B_1\|\,\|B_2\|$ for all $z\in S$. Thus for $t\in\mathbb{R}$, $s\in[0,1]$ we have

$$|F_{A_1,A_2}(t+is)| \leq \|B_1\|\,\|B_2\|\,K(1+|t|)^N$$

and so the tempered growth property holds for F_{A_1,A_2}. Thus φ is a $(\gamma\otimes\iota)$–KMS functional.

Proof of Theorem 5.5

(i) To prove there is a quasi–free state on $\overline{\Delta(\mathcal{S},\,\sigma)}$ defined by $\omega(\delta_f) := \exp[-s(f,f)/2]$, $f\in\mathcal{S}(\mathbb{R},\mathbb{R})$ where $s(f,f) := \int \frac{p}{1-e^{-p}}\left|\widehat{f}(p)\right|^2 dp$, it suffices to show that $|\sigma(f,h)|^2 \leq 4\,s(f,f)\,s(h,h)$ by [19].

$$\begin{aligned}
|\sigma(f,h)|^2 &= \left|\int ip\,\widehat{f}(p)\,\overline{\widehat{h}(p)}\,dp\right|^2 \leq \Big(\int \left|p\,\widehat{f}(p)\,\widehat{h}(p)\right| dp\Big)^2 \\
&= \Big(2\int_0^\infty p\left|\widehat{f}(p)\,\widehat{h}(p)\right| dp\Big)^2 \qquad \text{as } |p\widehat{f}(p)\,\widehat{h}(p)| \text{ is even by } \widehat{f}(-p)=\overline{\widehat{f}(p)} \\
&\leq 4\int_0^\infty p|\widehat{f}(p)|^2 dp \int_0^\infty k|\widehat{h}(k)|^2 dk \qquad \text{by Cauchy–Schwartz} \\
&< 4\int_0^\infty \frac{p}{1-e^{-p}}|\widehat{f}(p)|^2 dp \int_0^\infty \frac{k}{1-e^{-k}}|\widehat{h}(k)|^2 dk \qquad \text{as } p<\tfrac{p}{1-e^{-p}} \text{ for } p>0 \\
&< 4\int_{-\infty}^\infty \frac{p}{1-e^{-p}}|\widehat{f}(p)|^2 dp \int_{-\infty}^\infty \frac{k}{1-e^{-k}}|\widehat{h}(k)|^2 dk \qquad \text{as } 0<\tfrac{p}{1-e^{-p}} \text{ for all } p \\
&= 4\,s(f,f)\,s(h,h)
\end{aligned}$$

(ii) Next we need to show that this quasi–free state ω on $\overline{\Delta(\mathcal{S},\,\sigma)}$ extends to a KMS–functional on $\pi_\omega(\overline{\Delta(\mathcal{S},\,\sigma)})''$, and hence on $\mathcal{R}(\mathcal{S}(\mathbb{R}),\sigma)$. We first prove that ω is KMS on the *–algebra $\Delta_c := \text{Span}\left\{\delta_f \;\mid\; f\in\mathcal{S}(\mathbb{R}),\ \text{supp}\,\widehat{f}\ \text{compact}\right\}$. Let $\delta_f,\,\delta_h\in\Delta_c$ and consider

$$\begin{aligned}
F_1(t) &:= \omega(\delta_f\alpha_t(\delta_h)) = e^{-i\sigma(f,h_t)/2}\omega(\delta_{f+h_t}) \\
&= \exp\big[-\tfrac{i}{2}\,\sigma(f,h_t) - \tfrac{1}{2}\,s(f+h_t,f+h_t)\big] \\
&= \exp\Big[\tfrac{1}{2}\int\Big(-p\,\widehat{f}\,\overline{\widehat{h}_t} - \frac{p}{1-e^{-p}}|\widehat{f}+\widehat{h}_t|^2\Big)\,dp\Big] \\
&= \exp\Big[\tfrac{1}{2}\int p\Big(\frac{|\widehat{f}|^2+|\widehat{h}|^2}{e^{-p}-1} - \frac{2-e^{-p}}{1-e^{-p}}\,e^{ipt}\widehat{f}\,\overline{\widehat{h}} - \frac{e^{-ipt}}{1-e^{-p}}\,\overline{\widehat{f}}\,\widehat{h}\Big)\,dp\Big]
\end{aligned}$$

where we used $\widehat{h_t}(p) = e^{-ipt}\widehat{h}(p)$. Now put $K := \exp\Big[\frac{1}{2}\int p\frac{|\widehat{f}|^2+|\widehat{h}|^2}{e^{-p}-1}\,dp\Big]$, and substitute $p\to -p$ in the last integral, using $\widehat{f}\,\overline{\widehat{h}}(-p) = \overline{\widehat{f}}\,\widehat{h}(p)$ to get:

$$F_1(t) = K\,\exp\Big[-\int\frac{pe^{ipt}}{1-e^{-p}}\,\widehat{f}\,\overline{\widehat{h}}(p)\,dp\Big]\,.$$

By a similar calculation we find $\omega(\alpha_t(\delta_h)\delta_f) = F_1(t+i)$ and this suggests that we define the KMS–function for $z\in S = \mathbb{R}+i[0,1]$ by:

$$F(z) := K\,\exp\Big[-\int\frac{pe^{ipz}}{1-e^{-p}}\,\widehat{f}\,\overline{\widehat{h}}(p)\,dp\Big]\,. \tag{28}$$

Let $z = x + iy \in S = \mathbb{R} + i[0,1]$, then we know that the integral exists for $y = 0, 1$. For $y \in (0,1)$ the function $\left|\frac{p\, e^{ipz}}{1-e^{-p}}\right| = \frac{p\, e^{-py}}{1-e^{-p}}$ is bounded, so the integral exists for all $z \in S$, hence the definition (28) makes sense for F. It is however not clear that F is analytic. However, recall that by assumption $\operatorname{supp}\widehat{f}$ and $\operatorname{supp}\widehat{h}$ are compact, then it follows from the dominated convergence theorem that F is continuous on S and as the differential w.r.t. z of the integrand is continuous in p, it also implies that F is analytic on the interior of S. Thus F is the KMS–function for ω, hence by the last part of Proposition 5.4, ω is a KMS–state on Δ_c.

Next, we prove that $\pi_\omega(\Delta_c)$ is strong operator dense in $\pi_\omega(\overline{\Delta(\mathcal{S},\,\sigma)})''$. It suffices to show that for each $f \in \mathcal{S}(\mathbb{R})$ there is a sequence $\{f_n\} \subset \mathcal{S}(\mathbb{R})$ with Fourier transforms of compact support, such that $\omega(\delta_g\, \delta_{f_n}\, \delta_h) \to \omega(\delta_g\, \delta_f\, \delta_h)$ for all $g,\, h \in \mathcal{S}(\mathbb{R})$ as $n \to \infty$. Fix an $f \in \mathcal{S}(\mathbb{R})$ and define $f_n \in \mathcal{S}(\mathbb{R})$ by its Fourier transform $\widehat{f_n} = K_n \cdot \widehat{f}$ where each $K_n : \mathbb{R} \to [0,1]$ is a smooth bump function of compact support which is 1 on $[-n,n]$ and zero on $\mathbb{R}\backslash[-n-1,\, n+1]$. Then

$$\begin{aligned}
\omega(\delta_g\, \delta_{f_n}\, \delta_h) &= e^{-i\sigma(g+f_n,\, f_n+h)/2}\omega(\delta_{g+f_n+h}) \\
&= \exp\Big[-\tfrac{i}{2}\,\sigma(g+f_n,\, f_n+h) - \tfrac{1}{2}\,s(g+f_n+h,\, g+f_n+h)\Big] \\
&= \exp\Big[\tfrac{1}{2}\int\Big(-p(\widehat{g}+\widehat{f_n})(\overline{\widehat{f_n}}+\overline{\widehat{h}}) - \frac{p}{1-e^{-p}}|\widehat{g}+\widehat{f_n}+\widehat{h}|^2\Big)\,dp\Big] \\
&\xrightarrow[\infty]{n} \exp\Big[\tfrac{1}{2}\int\Big(-p(\widehat{g}+\widehat{f})(\overline{\widehat{f}}+\overline{\widehat{h}}) - \frac{p}{1-e^{-p}}|\widehat{g}+\widehat{f}+\widehat{h}|^2\Big)\,dp\Big] \\
&= \omega(\delta_g\, \delta_f\, \delta_h)
\end{aligned}$$

using dominated convergence as f is L^1.

Next, we want to use the strong operator denseness of $\pi_\omega(\Delta_c)$ in $\pi_\omega(\overline{\Delta(\mathcal{S},\,\sigma)})''$ to show that ω is KMS on all of $\pi_\omega(\overline{\Delta(\mathcal{S},\,\sigma)})''$. Let $A,\, B \in \pi_\omega(\overline{\Delta(\mathcal{S},\,\sigma)})''$ be selfadjoint, then by the Kaplansky density theorem (cf. Theorem 5,3.5 p329 in [14]) it follows that there are sequences $\{A_n\},\ \{B_n\} \subset \pi_\omega(\Delta_c)$ of selfadjoint elements such that $\|A_n\| \le \|A\|$, $\|B_n\| \le \|B\|$ and $A_n \to A$, $B_n \to B$, in strong operator topology. Now $\omega \circ \alpha_t = \omega$ because $s(f_t,\, f_t) = s(f,f)$, and thus there is an implementing unitary $U_t \in \mathcal{B}(\mathcal{H}_\omega)$ such that $U_t\pi_\omega(D)U_t^* = \pi_\omega(\alpha_t(D))$ for $D \in \overline{\Delta(\mathcal{S},\,\sigma)}$ and $U_t\Omega_\omega = \Omega_\omega$. Abbreviate the KMS–functions $F_n(z) := F_{A_n,B_n}(z)$ of ω, then for all $t \in \mathbb{R}$:

$$\begin{aligned}
F_n(t) - F_k(t) &= \omega(A_n\, \alpha_t(B_n)) - \omega(A_k\, \alpha_t(B_k)) \\
&= (A_n\Omega_\omega,\, U_tB_n\Omega_\omega) - (A_k\Omega_\omega,\, U_tB_k\Omega_\omega) \\
&= (U_t^*A_n\Omega_\omega,\, (B_n-B_k)\Omega_\omega) + (U_t^*(A_n-A_k)\Omega_\omega,\, B_k\Omega_\omega)
\end{aligned}$$

$$\text{Thus}\qquad \begin{aligned}
|F_n(t) - F_k(t)| &\le \|(B_n-B_k)\Omega_\omega\|\cdot\|A_n\| + \|(A_n-A_k)\Omega_\omega\|\cdot\|B_k\| \\
&\le \|(B_n-B_k)\Omega_\omega\|\cdot\|A\| + \|(A_n-A_k)\Omega_\omega\|\cdot\|B\|
\end{aligned}$$

and this is independent of t. Similarly

$$|F_n(t+i) - F_k(t+i)| \le \|(A_n-A_k)\Omega_\omega\|\cdot\|B\| + \|(B_n-B_k)\Omega_\omega\|\cdot\|A\|\,.$$

Now $F_n - F_k$ is a analytic function on the strip S, so by combining The Riemann mapping theorem with the maximum modulus principle we have that $|F_n - F_k|$ takes its maximum on the boundary of S. Thus by the inequalities above, $F_n - F_k$ converges uniformly to zero, hence the uniform limit $F := \lim\limits_n F_n$ exists and is analytic and bounded on S. Since $\omega(A_n\alpha_t(B_n)) =$

$(A_n\Omega_\omega,\, U_t B_n\Omega_\omega) \to (A\Omega_\omega,\, U_t B\Omega_\omega) = \omega(A\alpha_t(B))$ it follows that F is the KMS–function $F_{A,B}$ of ω. Thus ω is KMS on the selfadjoint part, hence on all of $\pi_\omega(\overline{\Delta(\mathcal{S},\,\sigma)})''$.

Since ω is quasifree, it is strongly regular, hence the resolvents of the generators of the one-parameter groups $t \to \pi_\omega(\delta_{tf})$ will provide a representation of $\mathcal{R}(\mathcal{S}(\mathbb{R}),\sigma)$, where it is defined by Equation (10) via spectral theory.

Proof of Theorem 5.6

For $z \in S = \mathbb{R} + i[0,1]$ consider

$$\begin{aligned}
G(z) &:= \lim_{\varepsilon\to 0^+}\Big(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\Big)\frac{e^{ipz}}{1-e^{-p}}\,\widehat{f}(p)\,\overline{\hat{g}(p)}\,dp \\
&= \lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{\infty}\Big(\frac{e^{ipz}}{1-e^{-p}}\,\widehat{f}(p)\,\overline{\hat{g}(p)} + \frac{e^{-ipz}}{1-e^{p}}\,\overline{\widehat{f}(p)}\,\hat{g}(p)\Big)\,dp \qquad \text{by } \widehat{f}(-p) = \overline{\widehat{f}(p)} \\
&= \int_0^\infty\Big[\Big(e^{ipz} + \frac{e^{-p}e^{ipz}}{1-e^{-p}}\Big)\,\widehat{f}(p)\,\overline{\hat{g}(p)} - \frac{e^{-p}e^{-ipz}}{1-e^{-p}}\,\overline{\widehat{f}(p)}\,\hat{g}(p)\Big]\,dp \\
&= (g, F(D)f) + \int_0^\infty dp\int dx\int dy\, f(x)\,g(y)\frac{e^{-p}}{1-e^{-p}}\Big(e^{ipz}e^{ip(y-x)} - e^{-ipz}e^{ip(x-y)}\Big) \\
&= (g, F(D)f) + 2i\int_0^\infty dp\int dx\int dy\, f(x)\,g(y)\,\frac{e^{-p}}{1-e^{-p}}\,\sin p(z+y-x) \qquad (29)
\end{aligned}$$

where $D = id/dx$ and $F(p) := e^{ipz}\chi_{[0,\infty)}(p)$, and we used the fact that the Fourier transform diagonalises D. To use Fubini to rearrange these integrals, we need to show that the integrand is integrable. We need to separate the low p from the high p behaviour in the last integral. For the low p behaviour, consider the integral

$$\int_0^1 dp\int dx\int dy\, f(x)\,g(y)\,\frac{e^{-p}}{1-e^{-p}}\,\sin p(z+y-x)\,.$$

Rearrange the integrand to $f(x)\,g(y)\,\frac{e^{-p}}{1-e^{-p}}\,p(z+y-x)\,\frac{\sin p(z+y-x)}{p(z+y-x)}$ and observe that $p(z+y-x) \in S = \mathbb{R}+i[0,1]$ since $p \in [0,1]$. Now $H(z) := \sin(z)/z$ is analytic in S, so $|H(z)|$ takes its maximum on the boundary. On $\mathbb{R}$, $|H(x)| \leq 1$, and for $\mathbb{R}+i$ we have

$$|H(x+i)|^2 = \left|\frac{\sin(x+i)}{x+i}\right|^2 = \frac{\cosh^2 1 - \cos^2 x}{x^2+1} \leq \cosh^2 1$$

and thus for the integrand

$$\left|f(x)\,g(y)\,\frac{e^{-p}}{1-e^{-p}}\,p(z+y-x)\,\frac{\sin p(z+y-x)}{p(z+y-x)}\right| \leq |f(x)\,g(y)|\,\frac{e^{-p}}{1-e^{-p}}\,p|z+y-x|\cosh 1 \qquad (30)$$

which is clearly integrable because f and g are Schwartz functions so take care of the linear factor in x and y. For the high p behaviour, consider the remaining part of the integral (29), i.e.

$$\int_1^\infty dp\int dx\int dy\, f(x)\,g(y)\,\frac{e^{-p}}{1-e^{-p}}\,\sin p(z+y-x)\,.$$

Now for $z = t+is \in S$, $t \in \mathbb{R}$, $s \in [0,1]$ we have

$$\begin{aligned}
\left|f(x)\,g(y)\,\frac{e^{-p}}{1-e^{-p}}\,\sin p(z+y-x)\right| &\leq \tfrac{1}{2}\,|f(x)\,g(y)|\,\frac{e^{-p}}{1-e^{-p}}\,\left|e^{ip(z+y-x)} - e^{-ip(z+y-x)}\right| \\
\leq \tfrac{1}{2}\,|f(x)\,g(y)|\,\frac{e^{-p}}{1-e^{-p}}\,\left|e^{-ps}+e^{ps}\right| &\leq |f(x)\,g(y)|\,\frac{e^{p(s-1)}}{1-e^{-p}}
\end{aligned}$$

This is integrable for $s \in [0,1)$, but not for $s=1$. However, we will see below that G is continuous on S, and this will be enough. Thus the integrand in (29) is integrable for $z \in \mathbb{R}+i[0,1)$, and we can apply the Fubini theorem to rearrange the order of integrals, and we get:

$$G(z) = (g, F(D)f) + 2i\int dx \int dy\, f(x)\, g(y) \int_0^\infty dp\, \frac{e^{-p}}{1-e^{-p}}\, \sin p(z+y-x)\,. \qquad (31)$$

To prove (i), let $z=0$, so $G(0)=\theta(f,g)$ and $F(p)=\chi_{[0,\infty)}(p)$ hence $P := F(D) = 2\pi\times$projection onto positive spectrum of D. So

$$\begin{aligned} G(0)=\theta(f,g) &= (g,Pf) + 2i\int dx \int dy\, f(x)\, g(y) \int_0^\infty dp\, \frac{e^{-p}}{1-e^{-p}}\, \sin p(y-x) \\ &= (g,Pf) - 2i\int dx \int dy\, f(x)\, g(y)\,(y-x) \int_0^\infty dp\, \ln(1-e^{-p})\, \cos p(y-x) \\ &= (g,(P+T)f) \end{aligned}$$

through an integration by parts. Consider the operator

$$(Tf)(x) := 2i\int dy\, f(y)\,(x-y)\int_0^\infty dp\, \ln(1-e^{-p})\, \cos p(x-y) =: \int dy\, f(y)\, K(x-y)$$

which is obviously a kernel operator with kernel K. Due to the factor $(x-y)$ in K, T is unbounded. Note however that

$$\left|\frac{\partial^n}{\partial t^n}\, \ln(1-e^{-p})\, \cos pt\right| \leq |p^n \ln(1-e^{-p})|$$

which is integrable in p. [To see this, note that for $0<p\leq 1$, $|p^n\ln(1-e^{-p})| \leq |\ln(1-e^{-p})| \leq |\ln p| + |\ln(\frac{1-e^{-p}}{p})|$ which is integrable, and for $e^p>2$ we have $|p^n\ln(1-e^{-p})| = p^n(e^{-p}+e^{-2p}/2+e^{-3p}/3+\cdots) \leq p^n\, e^{-p}(1+e^{-p}/2+e^{-2p}/3+\cdots) \leq p^n\, e^{-p}\sum_{k=0}^\infty (\frac{1}{2})^k \leq 2p^n\, e^{-p}$ which is integrable.] Thus by dominated convergence the function $t\to\int_0^\infty dp\, \ln(1-e^{-p})\, \cos pt$ is is smooth. Thus the kernel K of T is smooth. If J is a compact interval then $P_J T P_J$ has kernel $\chi_J(x)K(x-y)\chi_J(y)$ which is smooth and bounded on J. Thus $P_J T P_J$ is trace–class by Theorem 1 p128 in Lang [17]. We also have an explicit proof that $P_J T P_J$ is trace–class below in the proof of Theorem 5.9. Selfadjointness now follows from the fact that $\theta(f,f)$ is real by its formula. This proves (i).

To prove (ii), Note that we already proved above that $G(z)$ is well-defined for $z\in\mathbb{R}+i[0,1)$. To prove that it is well defined on all of S, it is only necessary to prove integrability for the high p part of the integral. For this

$$\begin{aligned} &\Big|2i\int_1^\infty dp \int dx \int dy\, f(x)\, g(y)\, \frac{e^{-p}}{1-e^{-p}}\, \sin p(z+y-x)\Big| \\ =\ & \Big|\int_1^\infty dp\, \frac{e^{-p}}{1-e^{-p}}\Big[e^{ipz}\widehat{f}(p)\,\overline{\widehat{g}(p)} - e^{-ipz}\overline{\widehat{f}(p)}\,\widehat{g}(p)\Big]\Big| \\ \leq\ & \int_1^\infty dp\, \frac{e^{-p}}{1-e^{-p}}(e^{-ps}+e^{ps})\,|\widehat{f}(p)\,\widehat{g}(p)| \\ \leq\ & \int_1^\infty dp\, \frac{2}{1-e^{-p}}|\widehat{f}(p)\,\widehat{g}(p)| \leq \frac{2}{1-e^{-1}}\|f\|\,\|g\| \end{aligned} \qquad (32)$$

where $z=t+is\in S$, and so $G(z)$ is well-defined for $z\in S$.

To establish the stated inequality for $z=s+it\in S$, consider Equation (31). Now $|F(p)| = e^{-sp}\chi_{[0,\infty)}(p)$ implies $\|F\|=1$, so $\|F(D\|=1$ and hence $|(g,F(D)f,)| \leq \|f\|\,\|g\|$. The high p

part of the integral in (31) has an estimate (32), so for the low p integral we have for its integrand the inequality (30) above, so that

$$\begin{aligned}
&\Big|\int_0^1 dp\int dx\int dy\, f(x)\,g(y)\,\frac{e^{-p}}{1-e^{-p}}\,\sin p(z+y-x)\Big| \\
\leq\ & \cosh 1\cdot\int dx\int dy\,|f(x)\,g(y)|\,(1+|t|+|x-y|)\int_0^1 dp\,\frac{p\,e^{-p}}{1-e^{-p}} \\
=\ & C+|t|E
\end{aligned}$$

for some finite constants C and E. Combining this with (32) and the estimate for $|(g,\,F(D)f)|$, we obtain $|G(t+is)|\leq A+B|t|$ for constants A, B as desired. It remains to prove that G is continuous on S and analytic in its interior. Now

$$\Big|\frac{d}{dz}\,f(x)\,g(y)\,\frac{e^{-p}}{1-e^{-p}}\,\sin p(z+y-x)\Big|\leq|f(x)\,g(y)|\,\frac{p\,e^{-p}}{1-e^{-p}}\,\tfrac{1}{2}(e^{-sp}+e^{sp})$$

which is L^1 for all $s\in(0,1)$. Thus the last integral in (31) is analytic on the interior of S. That $(g,F(D)f)$ is analytic in z follows from spectral theory, hence G is analytic on the interior of S. For continuity on S, we already have that $(g,F(D)f)$ is continuous in z, and by inequalities (32) and (30) we can get L^1 estimates to ensure that G is continuous on S.

Proof of Theorem 5.7

We now show that quasi–free functional ψ with two point functional θ is a graded KMS–functional on $\mathrm{Cliff}(\mathcal{S}(\mathbb{R}))$. Its domain $\mathrm{Dom}\,\psi=$ *-alg$\,\{\,c(f)\ \mid\ f\in\mathcal{S}(\mathbb{R})\,\}$ is clearly a unital dense *-algebra of $\mathrm{Cliff}(\mathcal{S}(\mathbb{R}))$ which is invariant w.r.t. both the grading γ and the time evolution α_t, so part (i) of Definition 5.1 is satisfied. For the KMS-condition (ii), it suffices to check it for the monomials $c(f_1)\cdots c(f_k)$. Let $A=c(f_1)\cdots c(f_k)$ and $B=c(g_1)\cdots c(g_m)$ where $k+m=2n$. Then from (16) and (17) we get

$$\begin{aligned}
F_{A,B}(t)\ &:=\ \psi(A\,\alpha_t(B))=\psi(c(f_1)\cdots c(f_k)c(T_tg_1)\cdots c(T_tg_m)) \\
&=\ (-1)^{\binom{n}{2}}\sum_P(-1)^P\prod_{j=1}^n\theta(h_{P(j)},\,h_{P(n+j)})
\end{aligned}\tag{33}$$

where $h_1=f_1,\ldots,h_k=f_k$, $h_{k+1}=T_tg_1,\ldots,h_{2n}=T_tg_m$ and $T_tf:=f_t$ is translation by t. Since $P(j)<P(n+j)$ always, the terms $\theta(h_{P(j)},\,h_{P(n+j)})$ can only be one of the types

$$\theta(f_i,T_tg_j)\qquad\text{or}\qquad\theta(T_tf_i,T_tg_j)\qquad\text{if } k<n,\text{ or}\qquad\theta(f_i,\,g_j)\qquad\text{if } k>n\,.$$

Since by the formula for θ we have $\theta(T_tf,T_tg)=\theta(f,\,g)$ the last two types are the same and constant in t. For the first type, we get by definition functions $G(t)=\theta(f,T_tg)$ as in Theorem 5.6(ii), which we therefore know extend analytically to the strip S. Thus since Equation (33) expresses $F_{A,B}(t)$ as a polynomial of constant functions and functions of the form G, it follows that $F_{A,B}(t)$ extends to a continuous function on S which is analytic on its interior. Now

$$\begin{aligned}
G(t+i)\ &=\ \lim_{\varepsilon\to0^+}\Big(\int_{-\infty}^{-\varepsilon}+\int_\varepsilon^\infty\Big)\frac{e^{ip(t+i)}}{1-e^{-p}}\ \widehat{f}(p)\,\overline{\widehat{g}(p)}\,dp \\
&=\ \lim_{\varepsilon\to0^+}\Big(\int_{-\infty}^{-\varepsilon}+\int_\varepsilon^\infty\Big)\frac{e^{ipt}}{e^p-1}\ \widehat{f}(p)\,\overline{\widehat{g}(p)}\,dp
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0^+}\Big(\int_{-\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\Big)\frac{e^{-ipt}}{e^{-p}-1}\,\widehat{f}(-p)\,\overline{\widehat{g}(-p)}\,dp \\
&= -\lim_{\varepsilon\to 0^+}\Big(\int_{-\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\Big)\frac{e^{-ipt}}{1-e^{-p}}\,\overline{\widehat{f}(p)}\,\hat{g}(p)\,dp \\
&= -\theta(T_t g,f)=\psi(\alpha_t(c(g))\gamma(c(f)))
\end{aligned}
\tag{34}
$$

which is the graded KMS-condition for $F_{c(f),c(g)}(t)=\psi(c(f)\alpha_t(c(g)))$. The terms $\theta(T_t g,f)$ are exactly the ones which occur in the corresponding expression (33) for $\psi(\alpha_t(B)\,\gamma(A))$ so the graded KMS-condition for $F_{A,B}$ follows from the one for G, Equation (34).

It remains to prove the growth condition (iii) of Definition 5.1. We already have

$$|G(t+is)|\leq a+b|t|\quad \text{for } t\in\mathbb{R},\ s\in[0,1]$$

by Theorem 5.6(ii). So from formula (34) we get that for $t+is\in S$:

$$|F_{A,B}(t+is)|\leq (a_1+b_1|t|)\cdots(a_n+b_n|t|)\leq C\,(1+|t|)^n$$

for suitable constants $a_i,\ b_i$ and C. Thus ψ is a graded KMS-functional.

Proof of Theorem 5.8

By construction $\mathrm{Dom}\,\varphi$ contains the *-algebra generated by all $c(f),\ f\in\mathcal{S}(\mathbb{R})$ as well as $\mathcal{R}(\mathcal{S}(\mathbb{R}),\sigma)$ and so it will certainly contain the *-algebra generated by $\mathbb{I}$ and all $c(f),\ R(\lambda,f)$, which is $\mathcal{A}_0$. So (i) is trivially true.

Next, for (ii), we need to prove the SUSY-invariance of φ, and for this, we need the following lemma.

8.2 Lemma *For all $g,\ f_i\in\mathcal{S}(\mathbb{R})\backslash 0$ and $\lambda_i\in\mathbb{R}\backslash 0$ we have*

$$
\begin{aligned}
&\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)\,j(g)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\\
&= \sum_{k=1}^{n} s(f_k,g)\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_k,f_k)^2\cdots R(\lambda_n,f_n)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\\
&\quad+\sum_{k=n+1}^{m} s(g,f_k)\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_k,f_k)^2\cdots R(\lambda_m,f_m)\Big)
\end{aligned}
$$

where φ is a strongly regular state on $\mathcal{R}(\mathcal{S}(\mathbb{R}),\sigma)$ so these expressions make sense on $\mathcal{E}_0$.

Proof: Recall by Theorem 5.5 that φ is a quasi–free state on $\overline{\Delta(\mathcal{S},\,\sigma)}$ defined by $\varphi(\delta_f):=\exp[-s(f,f)/2]$, $f\in\mathcal{S}(\mathbb{R},\mathbb{R})$ where s is given in Equation (15). Since the maps $t,r\to s(rf,tg)$ are smooth, we can apply Proposition 3.7 to φ w.r.t. the maps $t\to tf$. From the two relations $\frac{d}{dt}R(\lambda,tf)=j(f)R(\lambda,tf)^2$ and $\lim\limits_{t\to 0}\lambda i\pi_\varphi(R(\lambda,tf))\psi=\psi$, we get

$$
\begin{aligned}
&\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)\,j(g)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\\
&= -\mu^2\lim_{t\to 0}\frac{\partial}{\partial t}\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)\,R(\mu,tg)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\\
&= -\mu^2\lim_{t\to 0}\Big\{-\sum_{k=1}^{n}\frac{d}{dt}\,s(f_k,tg)\,\frac{\partial^2}{\partial\mu\partial\lambda_k}\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)\,R(\mu,tg)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)
\end{aligned}
$$

$$-\tfrac{1}{2}\,\frac{d}{dt}\,s(tg,tg)\,\frac{\partial^2}{\partial\mu^2}\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)\,R(\mu,tg)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)$$

$$-\sum_{k=n+1}^{m}\frac{d}{dt}\,s(tg,f_k)\,\frac{\partial^2}{\partial\mu\partial\lambda_k}\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)\,R(\mu,tg)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\Big\}$$

(By Proposition 3.7)

$$= \sum_{k=1}^{n} s(f_k,g)\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_k,f_k)^2\cdots R(\lambda_n,f_n)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)$$

$$+\sum_{k=n+1}^{m} s(g,f_k)\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_k,f_k)^2\cdots R(\lambda_m,f_m)\Big)$$

where we used $\frac{d}{d\lambda}R(\lambda,f) = -iR(\lambda,f)^2$. ■

Next we need to show that $\varphi\circ\delta$ is zero on $\mathcal{D}_S$, i.e. that it vanishes on all the monomials:

$$\zeta(f_1)\cdots\zeta(f_n)R(\lambda_1,g_1)\cdots R(\lambda_m,g_m) = c(f_1)\cdots c(f_n)R(1,f_1)\cdots R(1,f_n)R(\lambda_1,g_1)\cdots R(\lambda_m,g_m)\,.$$

Recall that δ is a restriction to $\mathcal{D}_S$ of a graded derivation on $\mathcal{E}$ defined by

$$\delta(j(f)) = ic(f'),\qquad \delta(R(\lambda,f)) = ic(f')R(\lambda,f)^2,\qquad \text{and}\qquad \delta(c(f)) = j(f)\,.$$

So we calculate:

$$\varphi\circ\delta\,(c(f_1)\cdots c(f_n)R(1,f_1)\cdots R(1,f_n)R(\lambda_1,g_1)\cdots R(\lambda_m,g_m))$$

$$= \sum_{k=1}^{n}(-1)^{k+1}\varphi\Big(c(f_1)\cdots\overset{k}{\vee}\cdots c(f_n)\,j(f_k)\,R(1,f_1)\cdots R(1,f_n)R(\lambda_1,g_1)\cdots R(\lambda_m,g_m)\Big)$$

$$+i(-1)^n\sum_{\ell=1}^{n}\varphi\left(c(f_1)\cdots c(f_n)c(f'_\ell)R(1,f_1)\cdots R(1,f_\ell)^2\cdots R(1,f_n)R(\lambda_1,g_1)\cdots R(\lambda_m,g_m)\right)$$

$$+i(-1)^n\sum_{p=1}^{m}\varphi\left(c(f_1)\cdots c(f_n)c(g'_p)R(1,f_1)\cdots\cdots R(1,f_n)R(\lambda_1,g_1)\cdots R(\lambda_p,g_p)^2\cdots R(\lambda_m,g_m)\right)$$

$$= \sum_{k=1}^{n}(-1)^{k+1}\varphi(c(f_1)\cdots\overset{k}{\vee}\cdots c(f_n))\Big\{$$

$$\sum_{r=1}^{n}s(f_k,f_r)\,\varphi\left(R(1,f_1)\cdots R(1,f_r)^2\cdots R(1,f_n)R(\lambda_1,g_1)\cdots R(\lambda_m,g_m)\right)$$

$$+\sum_{t=1}^{m}s(f_k,g_t)\,\varphi\left(R(1,f_1)\cdots R(1,f_n)R(\lambda_1,g_1)\cdots R(\lambda_t,g_t)^2\cdots R(\lambda_m,g_m)\right)\Big\}$$

$$+i(-1)^n\sum_{\ell=1}^{n}\varphi\left(c(f_1)\cdots c(f_n)c(f'_\ell)\right)\,\varphi\left(R(1,f_1)\cdots R(1,f_\ell)^2\cdots R(1,f_n)R(\lambda_1,g_1)\cdots R(\lambda_m,g_m)\right)$$

$$+i(-1)^n\sum_{p=1}^{m}\varphi\left(c(f_1)\cdots c(f_n)c(g'_p)\right)\,\varphi\left(R(1,f_1)\cdots R(1,f_n)R(\lambda_1,g_1)\cdots R(\lambda_p,g_p)^2\cdots R(\lambda_m,g_m)\right)$$

where we made use of the lemma 8.2. Note that as φ is quasifree, n must be odd for the last expression to be nonzero, and also:

$$\varphi(c(f_1)\cdots c(f_n)) = \sum_{k=1}^{n-1}(-1)^{k+1}\,\varphi(c(f_k)\,c(f_n))\,\varphi(c(f_1)\cdots\overset{k}{\vee}\cdots c(f_{n-1}))\,.$$

So we get:

$$
\begin{aligned}
&\varphi \circ \delta \left(c(f_1)\cdots c(f_n)R(1,f_1)\cdots R(1,f_n)R(\lambda_1,g_1)\cdots R(\lambda_m,g_m)\right) \\
= &\sum_{k=1}^{n}(-1)^{k+1}\varphi(c(f_1)\cdots \overset{k}{\vee}\cdots c(f_n))\Big\{\sum_{r=1}^{n}\Big[s(f_k,f_r)-i\varphi(c(f_k)c(f_r'))\Big] \\
&\qquad\qquad \times\varphi\Big(R(1,f_1)\cdots R(1,f_r)^2\cdots R(1,f_n)R(\lambda_1,g_1)\cdots R(\lambda_m,g_m)\Big) \\
&\qquad +\sum_{p=1}^{m}\Big[s(f_k,g_p)-i\varphi(c(f_k)\,c(g_p'))\Big] \\
&\qquad\qquad \times\varphi\Big(R(1,f_1)\cdots R(1,f_n)R(\lambda_1,g_1)\cdots R(\lambda_p,g_p)^2\cdots R(\lambda_m,g_m)\Big)\Big\}\,.
\end{aligned}
$$

However for the two-point functions we have:

$$
\begin{aligned}
\varphi(c(f)\,c(g')) &= \lim_{\varepsilon\to 0^+}\Big(\int_{-\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\Big)\frac{-ip}{1-e^{-p}}\;\widehat{f}(p)\,\overline{\widehat{g}(p)}\,dp \\
&= -i\int_{-\infty}^{\infty}\frac{p}{1-e^{-p}}\,\widehat{f}(p)\,\overline{\widehat{g}(p)}\,dp = -is(f,g)
\end{aligned}
$$

and so we get $\varphi\circ\delta=0$ as desired.

Finally, for (iii) we need to prove that

$$
\varphi(BM_A\overline{\delta}_0(A)C) = -i\frac{d}{dt}\,\varphi(BM_A\,\alpha_t(A)C)\Big|_0 \tag{35}
$$

for all $A\in\mathcal{D}_S$ and $B,C\in\mathcal{A}_0$. Since α_t and $\overline{\delta}_0$ do not mix $\mathrm{Cliff}(\mathcal{S}(\mathbb{R}))$ and $\mathcal{R}(\mathcal{S}(\mathbb{R}),\sigma)$ and φ has a product structure, it suffices to verify (35) on the CAR and CCR parts separately. First, on the Clifford algebra we have $\overline{\delta}_0(c(f))=ic(f')$, and by the derivative property we only need to check (35) for $A=c(f)$. However, φ is quasifree so it suffices to check for the two-point functions that

$$
\frac{d}{dt}\,\varphi(c(g)\,\alpha_t(c(f)))\Big|_0 = i\varphi(c(g)\overline{\delta}_0(c(f))) = -\varphi(c(g)\,c(f'))
$$

for all $f,\,g\in\mathcal{S}(\mathbb{R})$. The differentiability of $G(t)=\varphi(c(g)\,\alpha_t(c(f)))$ was proven above in Theorem 5.6. So:

$$
\begin{aligned}
\frac{d}{dt}\,\varphi(c(f)\,\alpha_t(c(g)))\Big|_0 &= \frac{d}{dt}\lim_{\varepsilon\to 0^+}\Big(\int_{-\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\Big)\frac{e^{ipt}}{1-e^{-p}}\;\widehat{f}(p)\,\overline{\widehat{g}(p)}\,dp\,\Big|_0 \\
&= \lim_{\varepsilon\to 0^+}\Big(\int_{-\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\Big)\frac{ip}{1-e^{-p}}\;\widehat{f}(p)\,\overline{\widehat{g}(p)}\,dp \\
&= -\varphi(c(f)\,c(g'))
\end{aligned}
$$

as required. Next, we need to check (35) on the resolvent algebra. By the derivative property, it suffices to do this for $A=R(\lambda,f)$, and by linearity for the remaining terms being monomials of resolvents. That is, we need to prove that

$$
\begin{aligned}
&\frac{d}{dt}\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)R(\mu,T_tg)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\Big|_0 \\
= \;& i\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)\overline{\delta}_0(R(\mu,g))\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)
\end{aligned} \tag{36}
$$

where $\overline{\delta}_0(R(\mu,g))=i\,R(\mu,g)\,j(g')\,R(\mu,g)$. This is an expression of the form of Proposition 3.7, so to apply this, we need to check that the functions $t\to s(f,T_tg)$ are smooth (note that

$s(T_tf,T_tg)=s(f,g))$, and this is an easy verification. In fact, $s(\cdot,\cdot)$ is clearly a distribution in each entry as it is an expectation value (f,Ag) where A is multiplication by a smooth function which is polynomially bounded. Applying Proposition 3.7 we get:

$$\begin{aligned}
&\frac{d}{dt}\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)R(\mu,T_tg)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\Big|_{t=0}\\
=\;& -\sum_{k=1}^n \frac{d}{dt}s(f_k,T_tg)\,\frac{\partial^2}{\partial\mu\partial\lambda_k}\,\varphi\Big(R(\lambda_1,f_1))\cdots R(\lambda_n,f_n)R(\mu,T_tg)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\Big|_{t=0}\\
&\qquad -\tfrac{1}{2}\frac{d}{dt}s(T_tg,T_tg)\,\frac{\partial^2}{\partial\mu^2}\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)R(\mu,T_tg)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\Big|_{t=0}\\
&\quad -\sum_{k=n+1}^m \frac{d}{dt}s(T_tg,f_k)\,\frac{\partial^2}{\partial\mu\partial\lambda_k}\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)R(\mu,T_tg)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\Big|_{t=0}\\
=\;& -\sum_{k=1}^n s(f_k,g')\,\varphi\Big(R(\lambda_1,f_1))\cdots R(\lambda_k,f_k)^2\cdots R(\lambda_n,f_n)R(\mu,g)^2\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\\
&\quad -\sum_{k=n+1}^m s(g',f_k)\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)\,R(\mu,g)^2\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_k,f_k)^2\cdots R(\lambda_m,f_m)\Big)
\end{aligned}$$

where we used $\frac{d}{dt}\,s(f,T_tg)|_0 = \frac{d}{dt}\int\frac{p}{1-e^{-p}}\,\widehat{f}\,e^{ip}\overline{\widehat{g}}\,dp|_0 = \int\frac{ip^2}{1-e^{-p}}\widehat{f}\,\overline{\widehat{g}}\,dp = -s(f,g')$, as well as $s(T_tf,T_tg)=s(f,g)$. On the other hand, for the right hand side of Equation (36) we have from Lemma 8.2, that:

$$\begin{aligned}
&i\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)\overline{\delta}_0(R(\mu,g))\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\\
=\;& -\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)R(\mu,g)\,j(g')\,R(\mu,g)\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\\
=\;& -\sum_{k=1}^n s(f_k,g')\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_k,f_k)^2\cdots R(\lambda_n,f_n)\,R(\mu,g)^2\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\\
&\quad -\sum_{k=n+1}^m s(g',f_k)\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)\,R(\mu,g)^2\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_k,f_k)^2\cdots R(\lambda_m,f_m)\Big)\\
&\quad -s(g,g')\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)\,R(\mu,g)^3\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\\
&\quad -s(g',g)\,\varphi\Big(R(\lambda_1,f_1)\cdots R(\lambda_n,f_n)\,R(\mu,g)^3\,R(\lambda_{n+1},f_{n+1})\cdots R(\lambda_m,f_m)\Big)\,.
\end{aligned}$$

However, $s(g,g')=-s(g',g)$ and so the last two terms cancel and hence we have proven (36).

Proof of Theorem 5.9

Recall that

$$\begin{aligned}
\mathcal{A}_0(J) &:= \text{*-alg}\,\{\,c(f),\ R(\lambda,f)\ \mid\ \operatorname{supp} f\subseteq J,\ f\in\mathcal{S}(\mathbb{R}),\ \lambda\in\mathbb{R}\backslash 0\,\}=\mathcal{C}(J)\otimes\mathcal{R}_0(J)\\
\text{where}\quad \mathcal{C}(J) &:= \text{*–alg}\,\{\,c(f)\ \mid\ \operatorname{supp} f\subseteq J,\ f\in\mathcal{S}(\mathbb{R})\,\}\\
\text{and}\quad \mathcal{R}_0(J) &:= \text{*–alg}\,\{\,R(\lambda,f)\ \mid\ \operatorname{supp} f\subseteq J,\ f\in\mathcal{S}(\mathbb{R}),\ \lambda\in\mathbb{R}\backslash 0\,\}\ .
\end{aligned}$$

Since $\varphi=\psi\otimes\omega$ is a product functional, and ω is a state, we have that $\|\varphi\restriction\mathcal{A}_0(J)\|=\|\psi\restriction\mathcal{C}(J)\|$, and so this is what we need to estimate. Without loss of generality we may assume J to be a closed interval, and also symmetrical about the origin (since φ is invariant w.r.t. translations).

Recall from Theorem 5.6 that $\psi(c(f)c(g)) = \theta(f,g) = (g,\,(P+T)f)$ where P is a projection (after a normalisation) and $P_J T P_J$ is trace-class and selfadjoint for all compact intervals J, and so this is the case for $c(f),\, c(g) \in \mathcal{C}(J)$. We will need to use the isomorphism of the Clifford algebra $\overline{\mathcal{C}(J)}$ with a self–dual CAR–algebra explicitly. First observe that $\overline{\mathcal{C}(J)} = \mathrm{Cliff}(L^2(J,\mathbb{R}))$ by continuity of $c(f)$. Since J is symmetrical about the origin, we can define $\Gamma : L^2(J,\mathbb{R}) \to L^2(J,\mathbb{R})$ by $(\widehat{\Gamma f})(p) := \widehat{f}(-p)$. Then $(\Gamma f)(x) = f(-x)$ and $\Gamma P = (\mathbb{I}-P)\Gamma$, and $\Gamma T \Gamma = -T$ by the explicit formula for T. Define for $f \in L^2(J,\mathbb{R})$

$$\Phi(f) := \tfrac{1}{\sqrt{2}}\Big(c(Pf) - ic(\Gamma P f) + c(P\Gamma f) + ic((\mathbb{I}-P)f)\Big)$$

and observe that $\Phi(\Gamma f) = \Phi(f)^*$ and $\{\Phi(f), \Phi(g)\} = (\Gamma f, g)\mathbb{I}$, which establishes the isomorphism. By requiring complex linearity for $\Phi(f)$, we get $\Phi(f) + i\Phi(g) =: \Phi(f+ig)$ hence get in fact isomorphism of $\overline{\mathcal{C}(J)}$ with $\mathrm{CAR}(L^2(J,\mathbb{C}))$. Note that by $\Phi(\Gamma f) = \Phi(f)^*$ the involution Γ has to extend to $L^2(J,\mathbb{C}) =: \mathcal{K}$ in a conjugate linear way. The image of $\mathcal{C}(J)$ under the isomorphism, is the dense *-algebra CAR_0 generated by all $\Phi(f)$, $f \in \mathcal{K}$, and this is the domain of ψ.

With respect to the decomposition $\mathcal{K} = P\mathcal{K} \oplus (\mathbb{I}-P)\mathcal{K} \ni f \oplus g$ we decompose $T = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ so by $T = T^*$ we get $A = A^*,\ D = D^*$ and $B = C^*$, and these operators are trace–class because T is. From the relation $\Gamma T \Gamma = -T$ we then find that $A = -D$ and $B^* = -B$, hence $T = \left(\begin{smallmatrix} A & B \\ -B & -A \end{smallmatrix}\right)$. Since T preserves the original real space $L^2(J,\mathbb{R})$ we have $\overline{Af} = A\overline{f}$ and $\overline{Bf} = B\,\overline{f}$. Then the two-point function of ψ on CAR_0 is for $f \oplus g,\ h \oplus k \in L^2(J,\mathbb{R})$:

$$\begin{aligned}
&\psi(\Phi(f\oplus g)\,\Phi(h\oplus k)) \\
= {}& \tfrac{1}{2}\,\psi\Big((c(f\oplus 0) - i\,c(0\oplus f) + c(g\oplus 0) + i\,c(0\oplus g))\cdot(c(h\oplus 0) - i\,c(0\oplus h) + c(k\oplus 0) + i\,c(0\oplus k))\Big) \\
= {}& \tfrac{1}{2}\,[(h,f) + (k,f) + (h,g) + (k,g)] + (h,Af) - i(k,Bf) + i(h,Bg) + (k,Ag) \\
= {}& \Big(\Gamma(h\oplus k),\, \Big[\tfrac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix} + \begin{pmatrix} -iB & A \\ A & iB \end{pmatrix}\Big](f\oplus g)\Big)\,.
\end{aligned}$$

This expression is complex linear in both entries, so we can extend it by linearity to $\mathcal{K}$ to get for all $f,\, g \in \mathcal{K}$ that

$$\begin{aligned}
\psi(\Phi(f)\,\Phi(g)) \;&=\; (\Gamma(g),\,(R+Q)f) \qquad \text{where:} \\
R \;&:=\; \tfrac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix} \qquad \text{and} \qquad Q := \begin{pmatrix} -iB & A \\ A & iB \end{pmatrix}
\end{aligned} \tag{37}$$

Define an operator S by $\frac{1}{2} + S := R + Q$, then S is a bounded selfadjoint operator which satisfies $\Gamma S \Gamma = S$. As $S_0 := \frac{1}{2}\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ has eigenvalues $\pm\frac{1}{2}$ and Q is trace class, $S = S_0 + Q$ has discrete spectrum with the only possible accumulation points $\pm\frac{1}{2}$ (cf. Theorem 9.6 [24]). We now need:

8.3 Lemma *(i)* $E([-\frac{1}{2},\frac{1}{2}]^c)\cdot(2|S|-\mathbb{I})$ *on* $L^2(J)$ *is trace class where* E *is the spectral resolution of* S. *Moreover we have for the trace–norms* $\|\cdot\|_1$ *that*

$$\Big\|E([-\tfrac{1}{2},\tfrac{1}{2}]^c)\cdot(2|S|-\mathbb{I})\Big\|_1 \leq b\,\|P_J T P_J\|_1$$

for a positive constant b *(independent of* J*).*

(ii) Let $\{e_j \,|\, j \in \mathbb{J}\}$ *be an orthonormal system of eigenvectors of* S *corresponding to the eigenvalues* $s_j \in [-\frac{1}{2},\frac{1}{2}]^c$, *and exhausting these eigenspaces. For each* $j \in \mathbb{J}$, *let*

*$\mathcal{C}_j$ be the two–dimensional abelian *-algebras generated by $\Phi(e_j)^*\Phi(e_j)$, and let $\mathcal{C}_0 :=$ *–alg$\Big\{\,\Phi(f)\;\mid\; f\in E([-\frac{1}{2},\frac{1}{2}])\mathcal{K}\Big\}$. Then*

$$\|\psi\restriction\mathcal{C}(J)\| = \prod_{j\in\{0\}\cup\mathbb{J}}\|\psi_j\| = \prod_{j\in\mathbb{J}}2|s_j| < \infty \tag{38}$$

where ψ_j denotes the restriction of ψ to $\mathcal{C}_j$.

Proof: (i) Let Let $E_0(\,\cdot\,)$ be the spectral resolution corresponding to S_0 . For any $s\in(0,\frac{1}{2})$ which is not in the spectrum $\sigma(S)$ of S , since S and S_0 are bounded, we obtain by spectral calculus that

$$\begin{aligned} E((s,\infty)) - E_0((s,\infty)) &= (2\pi i)^{-1}\int_C dz\,((z-S)^{-1} - (z-S_0)^{-1}) \\ &= (2\pi i)^{-1}\int_C dz\,(z-S_0)^{-1}Q(z-S)^{-1} \end{aligned} \tag{39}$$

where C is a suitable closed path in $\mathbb{C}$, e.g. a large anticlockwise simple contour with $\sigma(S)\cap[s,\infty)$ and $\sigma(S_0)\cap[s,\infty)$ in its interior, and crossing the real axis only at s and some $t>s$. So from (39) we conclude that $E((s,\infty)) - E_0((s,\infty))$ is a trace class operator. Taking into account that $E_0((s,\infty))(2S_0-1)=0$ we obtain

$$E((s,\infty))(2S-1) = (E((s,\infty)) - E_0((s,\infty)))(2S-1) + E_0((s,\infty))\,2Q, \tag{40}$$

showing that $E((s,\infty))(2S-1)$ and hence *a fortiori* $E((\frac{1}{2},\infty))\,(2S-1)$ is trace class. A similar argument establishes that $E((-\infty,-\frac{1}{2}))(2S+1)$ is trace class, and hence $E([-\frac{1}{2},\frac{1}{2}]^c)\cdot(2|S|-\mathbb{I})$ is trace class. From Equation (40) we get that

$$\begin{aligned} \|E((s,\infty))(2S-1)\|_1 &\le \|(E((s,\infty)) - E_0((s,\infty)))(2S-1)\|_1 + \|E_0((s,\infty))\,2Q\|_1 \\ &\le \|(E((s,\infty)) - E_0((s,\infty)))(2S-1)\|_1 + 2\|Q\|_1 \end{aligned} \tag{41}$$

since all the terms in (40) are trace class, and $\|AQ\|_1\le\|A\|\cdot\|Q\|_1$ for A bounded. Let P_0 be the projection onto the eigenspace of S_0 with eigenvalue $\frac{1}{2}$ (since $S_0=\frac{1}{2}\left(\begin{smallmatrix}0 & I\\ I & 0\end{smallmatrix}\right)$ this is just the space of even functions w.r.t. the decomposition associated with P). Then by substituting

$$(z-S_0)^{-1} = (z-\tfrac{1}{2})^{-1}P_0 + (z+\tfrac{1}{2})^{-1}(\mathbb{I}-P_0) \qquad\text{and}\qquad (z-S)^{-1}(2S-\mathbb{I}) = (2z-1)(z-S)^{-1} - 2\mathbb{I}$$

into (39)$\times(2S-\mathbb{I})$ we get that

$$\begin{aligned} &[E((s,\infty)) - E_0((s,\infty))](2S-\mathbb{I}) \\ =\;& \tfrac{1}{2\pi i}\Big[P_0Q\int_C dz\,(z-\tfrac{1}{2})^{-1}\Big(\tfrac{2z-1}{z-S}-2\Big) + (\mathbb{I}-P_0)Q\int_C dz\,(z+\tfrac{1}{2})^{-1}\Big(\tfrac{2z-1}{z-S}-2\Big)\Big] \\ =\;& \tfrac{1}{2\pi i}P_0Q\int_C dz\,2\,(z-S)^{-1} - 2P_0Q + \tfrac{1}{2\pi i}(\mathbb{I}-P_0)Q\int_C dz\,\frac{\Big(\frac{2z-1}{z+1/2}\Big)}{z-S} \\ =\;& P_0\,Q\,2E((s,\infty)) - 2P_0\,Q + (\mathbb{I}-P_0)Q\,f(S) \end{aligned}$$

where $f(z) := \frac{2z-1}{z+1/2}\,\chi_H(z)$ and $H := \{\,z\in\mathbb{C}\;\mid\;\mathrm{Re}(z)>s\,\}$. Now $\|f(S)\|\le\|f\restriction(s,\infty)\|_\infty = 2$, and so

$$\Big\|[E((s,\infty)) - E_0((s,\infty))](2S-\mathbb{I})\Big\|_1 \le 6\|Q\|_1 \le a\|P_JTP_J\|_1$$

for a constant $a > 0$, where we obtain the last inequality $\|Q\|_1 \leq \text{const.}\|P_J T P_J\|_1$ from the decomposition in (37) from which $PQP = -iB = -iP(P_J T P_J)(\mathbb{I}-P)$, $(\mathbb{I}-P)QP = P(P_J T P_J)P$ etc. Thus by (41) we get

$$\|E((\tfrac{1}{2},\infty))(2S-1)\|_1 = \|E(\tfrac{1}{2},\infty)E(s,\infty)(2S-1)\|_1 \leq \|E(s,\infty)(2S-1)\|_1 \leq a\|P_J T P_J\|_1 .$$

A similar argument establishes that $\|E(-\infty,-\frac{1}{2})(2S+1)\|_1 \leq a\|P_J T P_J\|_1$ and hence that $\left\|E([-\frac{1}{2},\frac{1}{2}]^c)\cdot(2|S|-\mathbb{I})\right\|_1 \leq b\,\|P_J T P_J\|_1$ for a positive constant b.

(ii) Recall that S has a purely discrete spectrum in $[-\frac{1}{2},\frac{1}{2}]^c$, so choose an orthonormal system $\{\, e_j \in \mathcal{K} \mid j \in \mathbb{J} \subseteq \mathbb{N}\,\}$ of eigenvectors of S, corresponding to eigenvalues $s_j \in [-\frac{1}{2},\frac{1}{2}]^c$ and exhausting these eigenspaces (some s_j will coincide for higher multiplicities). Let E_j be the one–dimensional orthogonal projection onto e_j, $j \in \mathbb{J}$, and let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots) \in \bigoplus_{j\in\mathbb{J}}\{-1,1\}$. We define on $\mathcal{K}$ the unitaries

$$V(\boldsymbol{\lambda}) := E([-\tfrac{1}{2},\tfrac{1}{2}]) + \sum_{j\in\mathbb{J}} \lambda_j E_j .$$

Since $V(\boldsymbol{\lambda})$ commutes with Γ, these unitaries induce an action $\gamma : \bigoplus_{j\in\mathbb{J}}\{-1,1\} \to \operatorname{Aut}\mathrm{CAR}(\mathcal{K})$ given by

$$\gamma_{\boldsymbol{\lambda}}(\Phi(f)) := \Phi(V(\boldsymbol{\lambda})f).$$

Since $V(\boldsymbol{\lambda})^2 = \mathbb{I}$, we can decompose $\mathrm{CAR}(\mathcal{K})$ into odd and even parts w.r.t. each $\gamma_{\boldsymbol{\lambda}}$. Moreover, since $V(\boldsymbol{\lambda})$ commutes with S we have that $\psi\circ\gamma_{\boldsymbol{\lambda}} = \psi$ and so ψ must vanish on the odd part of CAR_0 with respect to each $\gamma_{\boldsymbol{\lambda}}$. Let $\mathcal{C} \subset \mathrm{CAR}_0$ be the *–algebra generated by $\{\,\Phi(e_j) \mid j \in \mathbb{J}\,\} \cup \left\{\,\Phi(f) \mid f \in E([-\frac{1}{2},\frac{1}{2}])\mathcal{K}\,\right\}$, then $\mathcal{C}$ is mapped to itself by all $\gamma_{\boldsymbol{\lambda}}$. Since the two-point functional θ is bounded on $L^2(J)$, it suffices to calculate the norm of ψ on $\mathcal{C}$, and in fact on the intersection of all the even parts of $\mathcal{C}$ with respect to $\gamma_{\boldsymbol{\lambda}}$, and we denote this *-algebra by $\varepsilon(\mathcal{C})$. It is produced by a projection ε which we can consider as the projection defined on $\mathcal{C}$ by averaging over the action of $\gamma_{\boldsymbol{\lambda}}$ on $\mathcal{C}$. Since for each $A \in \mathcal{C}$ only a finite number of j's are involved, these averages will again be in the *-algebra $\mathcal{C}$. Now we only need to consider monomials in the $\Phi(e_j)$ and $\Phi(e_j)^*$ which are even in each index j. In a given monomial $\Phi(e_{j_1})\cdots\Phi(e_{j_n}) \in \varepsilon(\mathcal{C})$ if we collect all (even number of) terms with the same j together, we can then simplify it with the relations $2\,\Phi(e_j)^2 = \langle\Gamma e_j|e_j\rangle 1$ and $[\Phi(e_j)^*\Phi(e_j)]^2 = \Phi(e_j)^*\Phi(e_j) - \frac{1}{4}|(\Gamma e_j, e_j)|^2\mathbb{I}$. Thus $\varepsilon(\mathcal{C})$ is generated by the two-dimensional abelian *-algebras $\mathcal{C}_j := {}^*\text{–alg}\{\Phi(e_j)^*\Phi(e_j)\}$ and $\mathcal{C}_0 := {}^*\text{–alg}\left\{\,\Phi(f) \mid f \in E([-\frac{1}{2},\frac{1}{2}])\mathcal{K}\,\right\}$. Since for $i \neq j$ we have

$$\varepsilon\left([\Phi(e_i)^*\Phi(e_i),\, \Phi(e_j)^*\Phi(e_j)]\right) = \varepsilon\left(\overline{(\Gamma e_i, e_j)}\Phi(e_j)\Phi(e_i) + (\Gamma e_i, e_j)\Phi(e_j)^*\Phi(e_i)^*\right) = 0$$

it follows that all the $\mathcal{C}_i$ commute, and in fact we have the (incomplete) tensor product decomposition

$$\varepsilon(\mathcal{C}) = \bigotimes_{j\in\{0\}\cup\mathbb{J}} \mathcal{C}_j.$$

Moreover, ψ is a product functional on this tensor product. Hence its norm, if it exists, is given by

$$\|\psi\| = \prod_{j\in\{0\}\cup\mathbb{J}} \|\psi_j\|,$$

where ψ_j denotes the restriction of ψ to $\mathcal{C}_j$. Now ψ_0 is by construction a state on $\mathcal{C}_0$ because $\frac{1}{2}+S$ is positive on $E([-\frac{1}{2},\frac{1}{2}])\mathcal{K}$, hence the two-point function is positive and so $\|\psi_0\| = 1$. Since for $a,b\in\mathbb{C}$

$$\psi(a1+b\,\Phi(e_j)^*\Phi(e_j))=a+b\,(\tfrac{1}{2}+s_j)$$

and

$$\|a1+b\,\Phi(e_j)^*\Phi(e_j)\|=\max\{|a|,|a+b|\}$$

one obtains $\|\psi_j\|=2|s_j|$, $j\in\mathbb{J}$. Thus since $2|s_j|>1$, a necessary and sufficient condition for the existence of $\|\psi\|$ is $\sum_{j\in\mathbb{J}}(2|s_j|-1)<\infty$. However, this is guaranteed by part (i) ∎

Using this lemma, we can now prove:

8.4 Lemma *We have*

$$\|\varphi\restriction\mathcal{A}_0(J)\|\le\exp\left(b\|P_JTP_J\|_1\right)$$

where b is a positive constant (independent of J) and $\|\cdot\|_1$ denotes the trace–norm.

Proof: recall from Equation (38) that

$$\|\varphi\restriction\mathcal{A}_0(J)\|=\prod_{j\in\{0\}\cup\mathbb{J}}\|\psi_j\|=\prod_{j\in\mathbb{J}}2|s_j|=\prod_{j\in\mathbb{J}}(1+t_j)<\infty$$

where $t_j:=2|s_j|-1$. Now $\ln(1+x)\le x$ for $x\ge 0$, so

$$\ln\prod_{j=1}^N(1+t_j)=\sum_{j=1}^N\ln(1+t_j)\le\sum_{j=1}^N t_j\qquad\text{hence}\qquad\prod_{j\in\mathbb{J}}(1+t_j)\le\exp\Big(\sum_{j\in\mathbb{J}}t_j\Big)$$

Now $\sum\limits_{j\in\mathbb{J}}t_j=\sum\limits_{j\in\mathbb{J}}(2|s_j|-1)=\|E([-\tfrac{1}{2},\tfrac{1}{2}]^c)\cdot(2|S|-\mathbb{I})\|_1\le b\,\|P_JTP_J\|_1$ for a constant $b>0$ by Lemma 8.3(i). Combining these claims prove the lemma ∎

To conclude the proof of the theorem, we need to estimate $\|P_JTP_J\|_1$. Recall from Theorem 5.6 that

$$\begin{aligned}
(P_JTP_Jf)(x) &:= 2i\,\chi_J(x)\int dy\,f(y)\,(x-y)\int_0^\infty dp\,\ln(1-e^{-p})\,\cos p(x-y)\,\chi_J(y)\\
&= i\,\chi_J(x)\int dy\,f(y)\,(x-y)\int_{-\infty}^\infty dp\,\ln(1-e^{-|p|})\,e^{ip(x-y)}\,\chi_J(y)\\
&= \text{const.}\Big(\chi_J(X)\,\Big[X,\,\ln(1-e^{-|P|})\Big]\,\chi_J(X)\,f\Big)(x) \qquad (42)
\end{aligned}$$

where X is the multiplication operator $(Xf)(x)=xf(x)$, and P is as usual $i\frac{d}{dx}$, and the constant incorporates the 2π factors from the Fourier transforms. Now $D:=-\ln(1-e^{-|P|})$ is a positive operator, so write trivially $\chi_J(X)\,X\,D\,\chi_J(X)=(\chi_J(X)\,X\,D^{1/2})(D^{1/2}\,\chi_J(X))$, then we show that both factors are Hilbert–Schmidt. Now

$$\begin{aligned}
(\chi_J(X)\,X\,D^{1/2}f)(x) &= \int dy\,K(x,y) \qquad\text{where:}\\
K(x,y) &= \chi_J(x)\,x\int dp\,[-\ln(1-e^{-|p|})]^{1/2}\,e^{ip(x-y)}\\
\text{so}\qquad\|\chi_J(X)\,X\,D^{1/2}\|_2^2 &= \int dx\,dy\,|K(x,y)|^2=\int_J dx\,x^2\int dp\,[-\ln(1-e^{-|p|})]\\
&\le \text{const.}|J|^3,
\end{aligned}$$

using the integrability of $\ln(1-e^{-|p|})$. Likewise, we get

$$\|(D^{1/2}\,\chi_J(X)\|_2^2 = \int_J dx \int dp\,[\,-\ln(1-e^{-|p|})] \leq \text{const.}|J|\,.$$

Thus $\chi_J(X)\,X\,D\,\chi_J(X)$ is trace class, and as $P_J T P_J$ is, so is $\chi_J(X)\,D\,X\,\chi_J(X)$. For their trace norms we find

$$\|\chi_J(X)\,X\,D\,\chi_J(X)\|_1 \leq \|\chi_J(X)\,X\,D^{1/2}\|_2 \cdot \|(D^{1/2}\,\chi_J(X)\|_2 \leq \text{const.}|J|^2$$

and likewise $\|\chi_J(X)\,D\,X\,\chi_J(X)\|_1 \leq \text{const.}|J|^2$. Thus by (42) we get that

$$\|P_J T P_J\|_1 \leq \text{const.}\|\chi_J(X)\,X\,D\,\chi_J(X)\|_1 + \text{const.}\|\chi_J(X)\,D\,X\,\chi_J(X)\|_1 \leq K|J|^2$$

for a constant K (independent of J). Now from Lemma 8.4 we get the claim of the theorem, i.e. that

$$\|\varphi\restriction\mathcal{A}_0(J)\| \leq \exp\,(K\,|J|^2)\,.$$

Proof of Theorem 6.3

Fix a compact interval $J=[-k,\,k]$ and let $a_i \in \mathcal{A}_0(J)$ for all i, then as $\alpha_t(a_i) \in \mathcal{A}_0(J+t)$ we have $\alpha_{t_0}(a_0)\cdots\alpha_{t_n}(a_n) \in \mathcal{A}_0([-M,M])$ where $M := k + \sup_i |t_i|$. Thus by Theorem 5.9 we get

$$\Big|\varphi(\alpha_{t_0}(a_0)\cdots\alpha_{t_n}(a_n))\Big| \leq e^{4KM^2}\,\|a_0\|\cdots\|a_n\|\,.$$

Now
$$M^2 = k^2 + \sup_i t_i^2 + 2k\,\sup_i |t_i| \leq k^2 + \sum_i t_i^2 + 2k\,\sup_i (1+t_i^2)$$
$$\leq k^2 + \sum_i t_i^2 + 2k\Big(1+\sum_i t_i^2\Big)\Big) = k^2 + 2k + (1+2k)\,\sum_i t_i^2$$

hence:
$$\Big|\varphi(\alpha_{t_0}(a_0)\cdots\alpha_{t_n}(a_n))\Big| \leq A\,\exp\Big(B\,\sum_i t_i^2\Big)\,\|a_0\|\cdots\|a_n\|\,.$$

for suitable constants A and B depending only on k (but not on n). Now let $t_0=0,\ t_1=s_1$, $t_2=s_1+s_2,\ldots,\,t_n=s_1+\cdots+s_n$, and define for all $s_i\in\mathbb{R}$:

$$F_{a_o,\ldots a_n}(s_1,\ldots,\,s_n) := \exp\Big(-B\sum_{k=1}^n (s_1+\cdots+s_k)^2\Big)\cdot\varphi\Big(a_0\,\alpha_{s_1}(a_1)\cdots\alpha_{s_1+\cdots+s_n}(a_n)\Big)\,.$$

Then we have $|F_{a_o,\ldots a_n}(s_1,\ldots,\,s_n)| \leq A\,\|a_0\|\cdots\|a_n\|$, and by the KMS–property of φ, the function $F_{a_o,\ldots a_n}$ can be analytically continued in each variable s_j into the strip $S_j := \{z_j \in \mathbb{C}^n \,|\, \operatorname{Im} z_j \in [0,1]\}$, keeping the other variables real. This produces functions $F^{(j)}_{a_o,\ldots a_n}$ analytic in the flat tubes $T^j := \mathbb{R}^{n-1}\times S_j$, and hence by using the Flat Tube Theorem 6.1 inductively, we obtain an analytic continuation of $F_{a_o,\ldots a_n}$ into the tube $\mathcal{T}_n := \mathbb{R}^n + i\Sigma_n$ where $\Sigma_n := \{\,\mathbf{s}\in\mathbb{R}^n \ |\ 0\leq s_i\,\forall i,\ \ s_1+\cdots+s_n \leq 1\,\}$, coinciding with all $F^{(j)}_{a_o,\ldots a_n}$ on T^j. We want to obtain a bound for this analytic function F. We start by finding bounds for the $F^{(j)}$. Let $G(s_1,\ldots,\,s_n) := \varphi\Big(a_0\,\alpha_{s_1}(a_1)\cdots\alpha_{s_1+\cdots+s_n}(a_n)\Big)$ which has analytic extensions to each T^j, and by the definition of KMS–functionals we know that $|G(s_1,\ldots,\,s_j+ir_j,\ldots,\,s_n)| \leq C(1+|s_j|)^N$ where C and N are independent of s_j and $r_j\in[0,1]$. Now

$$F_{a_o,\ldots a_n}(s_1,\ldots,\,s_j+ir_j,\ldots,\,s_n) =$$
$$G(s_1,\ldots,\,s_j+ir_j,\ldots,\,s_n)\,\exp\Big[Br_j^2(n+1-j) - B\sum_{k=1}^n (s_1+\cdots+s_k)^2 + i\theta\Big] \tag{43}$$

where θ is real. Thus from the exponential damping factor in s_j we conclude that $F^{(j)}_{a_o,\ldots a_n}$ is bounded. By the maximum modulus principle (applied after first mapping S_j to a unit disk by the Schwartz mapping principle), the bound of $|F^{(j)}_{a_o,\ldots a_n}|$ is attained on the boundary of S_j (this also follows from the Phragmen Lindelöf theorem, cf. p138 in [6]). So on the real part of the boundary of S_j we have already from above that $|F^{(j)}_{a_o,\ldots a_n}(s_1, \ldots, s_n)| \leq A\,\|a_0\|\cdots\|a_n\|$ and by the KMS–condition and translation invariance of φ we have on the other part

$$|G(s_1, \ldots, s_j+i, \ldots, s_n)| = \Big|\varphi\Big(\alpha_{s_1+\cdots s_j}(a_j)\cdots\alpha_{s_1+\cdots+s_n}(a_n)a_0\,\alpha_{s_1}(a_1)\cdots\alpha_{s_1+\cdots+s_{j-1}}(a_{j-1})\Big)\Big|$$

$$\leq A\,\exp\Big(B\sum_{k=1}^n(s_1+\cdots+s_k)^2\Big)\cdot\|a_0\|\cdots\|a_n\| \qquad \text{hence by (43):}$$

$$|F^{(j)}_{a_o,\ldots a_n}(s_1, \ldots, s_j+i, \ldots, s_n)| \leq A\,e^{Bn}\,\|a_0\|\cdots\|a_n\| \qquad \text{and thus as } e^{Bn} > 1\,,$$

$$|F^{(j)}_{a_o,\ldots a_n}(s_1, \ldots, z_j, \ldots, s_n)| \leq A\,e^{Bn}\,\|a_0\|\cdots\|a_n\| =: C \qquad \text{for all } z_j \in S_j\,.$$

Now define $H_\alpha(z_1, \ldots, z_n) := [F_{a_o,\ldots a_n}(z_1, \ldots, z_n) - e^{i\alpha}C]^{-1}$ where $\alpha \in [0, 2\pi]$ for $(z_1, \ldots, z_n) \in \mathcal{T}_n$. Then by the estimates above for $|F^{(j)}_{a_o,\ldots a_n}|$, each map $z_j \to H_\alpha(s_1, \ldots, z_j, \ldots, s_n)$ is analytic on the strip S_j, and thus by the Flat Tube Theorem 6.1, H_α has a unique extension as an analytic function to $\mathcal{T}_n$, and hence cannot have any singularities in $\mathcal{T}_n$, i.e. $F_{a_o,\ldots a_n}(z_1, \ldots, z_n) \neq e^{i\alpha}C$ for all α. By continuity of F, the image set $F_{a_o,\ldots a_n}(\mathcal{T}_n)$ must be connected. By assumption, this set has some points inside the circle $|z| = C$, hence the entire image set is inside the circle $|z| = C$, i.e.

$$|F_{a_o,\ldots a_n}(z_1, \ldots, z_n)| \leq A\,e^{Bn}\,\|a_0\|\cdots\|a_n\| \qquad \forall\,(z_1, \ldots, z_n) \in \mathcal{T}_n \quad \text{and} \quad a_i \in \mathcal{A}_0(J)\,.$$

Consider now the Chern character formula (19):

$$\begin{aligned}\tau_n(a_0, \ldots, a_n) \;&:=\; i^{\epsilon_n}\int_{\sigma_n}\varphi\Big(a_0\,\alpha_{is_1}(\delta\gamma(a_1))\,\alpha_{is_2}(\delta(a_2))\,\alpha_{is_3}(\delta\gamma(a_3))\cdots \\ &\qquad\qquad \cdots\alpha_{is_n}(\delta\gamma^n(a_n))\Big)\,ds_1\cdots ds_n\,, \qquad a_i \in \mathcal{D}_c\,, \\ &=\; i^{\epsilon_n}\int_{\Sigma_n}\varphi(b_0\,\alpha_{ir_1}(b_1)\cdots\alpha_{ir_1+\cdots+ir_n}(b_n))\,dr_1\cdots dr_n\end{aligned}$$

where we made a change of variables $s_1 = r_1,\ s_2 = r_1+r_2, \ldots, s_n = r_1+\cdots+r_n$ and substitutions $a_0 = b_0,\ b_1 = \delta\gamma(a_1), \ldots, b_n = \delta\gamma^n(a_n)$ as in Section 6, making use of the Flat Tube Theorem. (Note that $b_i \in \mathcal{A}_0(J)\ \forall\, i$ for some J.) In fact, from the uniqueness part of the extensions to $\mathcal{T}_n$ we have that on $\mathcal{T}_n$

$$\varphi(b_0\,\alpha_{z_1}(b_1)\cdots\alpha_{z_1+\cdots+z_n}(b_n)) = \exp\,[B\sum_{k=1}^n(z_1+\cdots+z_n)^2]\cdot F_{b_o,\ldots b_n}(z_1, \ldots, z_n)$$

and so for $(z_1, \ldots, z_n) = i(r_1, \ldots, r_n) \in i\Sigma_n$ we have

$$\Big|\varphi(b_0\,\alpha_{ir_1}(b_1)\cdots\alpha_{ir_1+\cdots+ir_n}(b_n))\Big| \;\leq\; \exp\Big[-B\sum_{k=1}^n(r_1+\cdots+r_n)^2 + Bn\Big]\,A\,\|b_0\|\cdots\|b_n\|$$

$$\text{hence:} \qquad |\tau_n(a_0, \ldots, a_n)| \;\leq\; \frac{A}{n!}\,e^{Bn}\,\|b_0\|\cdots\|b_n\| \leq \frac{A}{n!}\,e^{Bn}\,\|a_0\|_*\cdots\|a_n\|_*$$

where we used first, that the volume of $|\Sigma_n| = 1/n!$, and second, that $\|b_j\| \leq \|a_j\|_*$ because $b_j = \delta\gamma(a_j) = -\gamma\delta(a_j)$ for $j > 0$. Thus $\|\tau_n\|_* \leq A\,e^{Bn}/n!$ and hence it is clear that $\lim\limits_{n\to\infty} n^{1/2}\|\tau\|_*^{1/n} \leq e^B \lim\limits_{n\to\infty} n^{1/2}(A/n!)^{1/n} = 0$ by Stirling's formula, which concludes the proof.

Proof of Theorem 6.4

By Theorem 6.3 we already have the entireness condition for $\widetilde{\tau}$ so it is only necessary to prove the cocycle condition for $a_i \in \mathcal{D}_c$:

$$(b\tau_{n-1})(a_0, \ldots, a_n) = (B\tau_{n+1})(a_0, \ldots, a_n)\,, \qquad n = 1, 3, 5, , \ldots$$

with b and B given by Equations (22) and (23). We will roughly follow the technique used in [12], but due to the different analytic properties of our model, we will need to go explicitly through the steps. In order to manipulate the expressions involved with Equation (24), we need the results in the following Lemma.

8.5 Lemma *Let $b_i \in \mathcal{A}_0$, then:*

(i) $\varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots\alpha_{is_n}(b_n)\Big) \;=\; \varphi\Big(\gamma(b_n)\, \alpha_{i(1-s_n)}(b_0)\, \alpha_{i(1-s_n+s_1)}(b_1)\cdots\alpha_{i(1-s_n+s_{n-1})}(b_{n-1})\Big)$ *for all* $(s_1, \ldots, s_n) \in \sigma_n$.

(ii) $\displaystyle\int_{\sigma_n} \varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots\alpha_{is_n}(b_n)\Big)\, ds_1\cdots ds_n = \int_{\sigma_n} \varphi\Big(\gamma(b_n)\, \alpha_{is_1}(b_0)\cdots\alpha_{is_n}(b_{n-1})\Big)\, ds_1\cdots ds_n$

(iii) The functions $(t_1, \ldots, t_n) \to \varphi\Big(b_0\, \alpha_{t_1}(b_1)\cdots\delta(\alpha_{t_k}(b_k))\cdots\alpha_{t_n}(b_n)\Big)$ *and* $(t_1, \ldots, t_n) \;\to\; \varphi\Big(b_0\, \alpha_{t_1}(b_1)\cdots\gamma(\alpha_{t_k}(b_k))\cdots\alpha_{s_n}(b_n)\Big)$ *both have analytic continuations to* $\mathbb{R}^n + i\sigma_n$ *, and for these we have*
$\varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots\delta(\alpha_{is_k}(b_k))\cdots\alpha_{is_n}(b_n)\Big) = \varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots\alpha_{is_k}(\delta(b_k))\cdots\alpha_{is_n}(b_n)\Big)$ *and*
$\varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots\gamma(\alpha_{is_k}(b_k))\cdots\alpha_{is_n}(b_n)\Big) = \varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots\alpha_{is_k}(\gamma(b_k))\cdots\alpha_{is_n}(b_n)\Big)$.

(iv) For $j = 2, \ldots, n$ we have:

$$\begin{aligned}&\int_{\sigma_{n+1}} \frac{\partial}{\partial s_j}\varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots\alpha_{is_{n+1}}(b_{n+1})\Big)\, ds_1\cdots ds_{n+1}\\ &= \int_{\sigma_n} \Big[\varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots\alpha_{is_j}(b_j b_{j+1})\cdots\alpha_{is_n}(b_{n+1})\Big)\\ &\qquad -\varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots\alpha_{is_{j-1}}(b_{j-1}b_j)\cdots\alpha_{is_n}(b_{n+1})\Big)\Big]\, ds_1\cdots ds_n\end{aligned} \tag{44}$$

$$\begin{aligned}&\int_{\sigma_{n+1}} \frac{\partial}{\partial s_1}\varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots\alpha_{is_{n+1}}(b_{n+1})\Big)\, ds_1\cdots ds_{n+1}\\ &= \int_{\sigma_n} \Big[\varphi\Big(b_0\, \alpha_{is_1}(b_1b_2)\, \alpha_{is_2}(b_3)\cdots\alpha_{is_n}(b_{n+1})\Big) - \varphi\Big(b_0\, b_1\, \alpha_{is_1}(b_2)\cdots\alpha_{is_n}(b_{n+1})\Big)\Big]\, ds_1\cdots ds_n\end{aligned} \tag{45}$$

$$\begin{aligned}&\int_{\sigma_{n+1}} \frac{\partial}{\partial s_{n+1}}\varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots\alpha_{is_{n+1}}(b_{n+1})\Big)\, ds_1\cdots ds_{n+1}\\ &= \int_{\sigma_n} \Big[\varphi\Big(\gamma(b_{n+1})\, b_0\, \alpha_{is_1}(b_1)\cdots\alpha_{is_n}(b_n)\Big) - \varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots\alpha_{is_n}(b_n b_{n+1})\Big)\Big]\, ds_1\cdots ds_n\end{aligned} \tag{46}$$

Proof: (i) Recall that the left hand side is defined by the analytic extension of the function $F_{t_1\cdots,t_n}(b_0, \cdots, b_n) := \varphi(b_0\, \alpha_{t_1}(b_1)\cdots\alpha_{t_n}(b_n))$ to the tube $\mathbb{R}^n + i\sigma_n$ by the KMS–condition and Flat Tube theorem, so $\varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots\alpha_{is_n}(b_n)\Big) \;:=\; F_{is_1\cdots,is_n}(b_0, \cdots, b_n)$. By the invariance $\varphi\circ\alpha_t = \varphi$ we have

$$\begin{aligned}F_{t_1\cdots,t_n}(b_0, \cdots, b_n) &= \varphi(\alpha_t(b_0)\, \alpha_{t+t_1}(b_1)\cdots\alpha_{t+t_n}(b_n))\\ &= F_{t,\, t+t_1\cdots,\, t+t_n}(\mathbb{I},\, b_0, \cdots, b_n) = F_{t+t_1\cdots,\, t+t_n}(\alpha_t(b_0),\, b_1, \cdots, b_n)\,.\end{aligned}$$

The latter function has an analytic continuation in the variables $(t+t_1,\ldots,t+t_n)$ to $\mathbb{R}^n+i\sigma_n$ and from the former function it also has an analytic extension in t to the strip $\mathbb{R}+i[0,1]$. Thus by the flat tube theorem we get a unique analytic extension to all of $\mathbb{R}^{n+1}+i\sigma_{n+1}$. Put $t_j=is_j$ where $\mathbf{s}\in\sigma_n$ and $t=i(1-s_n)$, then

$$F_{is_1\cdots,is_n}(b_0,\cdots,b_n)=F_{i(1-s_n),\,i(1-s_n+s_1),\cdots,\,i(1-s_n+s_{n-1}),\,i}(\mathbb{I},\,b_0,\cdots,\,b_n)$$

which is justified because we have that the variables

$$\mathbf{r}=(r_1,\cdots,r_n):=(1-s_n,\,1-s_n+s_1,\ldots,\,1-s_n+s_{n-1})\in\sigma_n\,.$$

Now the function $F_{ir_1,\cdots,ir_n,i}(\mathbb{I},\,b_0,\cdots,\,b_n)$ is obtained from the analytic extension of $F_{t_1,\cdots,t_n,v}(\mathbb{I},\,b_0,\cdots,\,b_n)$, $t_i,\ v\in\mathbb{R}$ to $\mathbb{R}^{n+1}+i\sigma_{n+1}$. By the KMS–condition:

$$\begin{aligned}F_{t_1,\cdots,t_n,i}(\mathbb{I},\,b_0,\cdots,\,b_n)&=\varphi\Big(b_n\,\gamma(\alpha_{t_1}(b_0)\cdots\alpha_{t_n}(b_{n-1}))\Big)\\&=\varphi\Big(\gamma(b_n)\,\alpha_{t_1}(b_0)\cdots\alpha_{t_n}(b_{n-1})\Big)=F_{t_1,\cdots,t_n}(\gamma(b_n),\,b_0,\cdots,\,b_{n-1})\end{aligned}$$

Thus by uniqueness of the analytic continuations we have

$$F_{is_1\cdots,is_n}(b_0,\cdots,b_n)=F_{ir_1,\cdots,ir_n,i}(\mathbb{I},\,b_0,\cdots,\,b_n)=F_{ir_1,\cdots,ir_n}(\gamma(b_n),\,b_0,\cdots,\,b_{n-1})$$

which is the statement (i) of the lemma.
(ii) By part (i) we have:

$$\begin{aligned}&\int_{\sigma_n}\varphi\Big(b_0\,\alpha_{is_1}(b_1)\cdots\alpha_{is_n}(b_n)\Big)\,ds_1\cdots ds_n\\&=\int_{\sigma_n}\varphi\Big(\gamma(b_n)\,\alpha_{i(1-s_n)}(b_0)\,\alpha_{i(1-s_n+s_1)}(b_1)\cdots\alpha_{i(1-s_n+s_{n-1})}(b_{n-1})\Big)\,ds_1\cdots ds_n\\&=\int_{\sigma_n}\varphi\Big(\gamma(b_n)\,\alpha_{ir_1}(b_0)\ \cdots\alpha_{ir_n)}(b_{n-1})\Big)\,dr_1\cdots dr_n\end{aligned}$$

making use of the change of variables $\mathbf{s}\to\mathbf{r}$ above (with Jacobian $=1$), and the fact that $\mathbf{r}\in\sigma_n$ iff $\mathbf{s}\in\sigma_n$.
(iii) Since $\varphi\Big(b_0\,\alpha_{t_1}(b_1)\cdots\delta(\alpha_{t_k}(b_k))\cdots\alpha_{t_n}(b_n)\Big)=\varphi\Big(b_0\,\alpha_{t_1}(b_1)\cdots\alpha_{t_k}(\delta(b_k))\cdots\alpha_{t_n}(b_n)\Big)$ and the latter obviously has an analytic extension to $\mathbb{R}^n+\sigma_n$ the claim follows. Likewise for the other one.
(iv) For $2\leq j\leq n$ we have

$$\begin{aligned}&\int_{s_{j-1}}^{s_{j+1}}\frac{\partial}{\partial s_j}\varphi\Big(b_0\,\alpha_{is_1}(b_1)\cdots\alpha_{is_{n+1}}(b_{n+1})\Big)\,ds_j\\&=\varphi\Big(b_0\,\alpha_{is_1}(b_1)\cdots\alpha_{is_{j-1}}(b_{j-1})\,\alpha_{is_{j+1}}(b_j)\,\alpha_{is_{j+1}}(b_{j+1})\cdots\alpha_{is_{n+1}}(b_{n+1})\Big)\\&\quad-\varphi\Big(b_0\,\alpha_{is_1}(b_1)\cdots\alpha_{is_{j-1}}(b_{j-1})\,\alpha_{is_{j-1}}(b_j)\,\alpha_{is_{j+1}}(b_{j+1})\cdots\alpha_{is_{n+1}}(b_{n+1})\Big)\\&=\varphi\Big(b_0\,\alpha_{is_1}(b_1)\cdots\alpha_{is_{j-1}}(b_{j-1})\,\alpha_{is_{j+1}}(b_jb_{j+1})\cdots\alpha_{is_{n+1}}(b_{n+1})\Big)\\&\quad-\varphi\Big(b_0\,\alpha_{is_1}(b_1)\cdots\alpha_{is_{j-1}}(b_{j-1}b_j)\,\alpha_{is_{j+1}}(b_{j+1})\cdots\alpha_{is_{n+1}}(b_{n+1})\Big)\end{aligned}$$

from which equation (44) follows by a change of label of the integration variables. For equation (45) we substitute $j=1,\ s_{j-1}=0$ into the last equation. For equation (46) we substitute

$j=n+1,\ s_{j+1}=1$ into the last equation to get

$$\begin{aligned}
&\int_{\sigma_{n+1}} \frac{\partial}{\partial s_{n+1}} \varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots \alpha_{is_{n+1}}(b_{n+1})\Big)\, ds_1\cdots ds_{n+1} \\
&\quad = \int_{\sigma_n} \Big[\varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots \alpha_{is_n}(b_n)\, \alpha_i(b_{n+1})\Big) - \varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots \alpha_{is_n}(b_n b_{n+1})\Big)\Big]\, ds_1\cdots ds_n \\
&\quad = \int_{\sigma_n} \Big[\varphi\Big(\gamma(b_{n+1})\, b_0\, \alpha_{is_1}(b_1)\cdots \alpha_{is_n}(b_n)\Big) - \varphi\Big(b_0\, \alpha_{is_1}(b_1)\cdots \alpha_{is_n}(b_n b_{n+1})\Big)\Big]\, ds_1\cdots ds_n
\end{aligned}$$

making use of part (i) for the KMS–condition. ■

Let us begin with the right hand side of our desired Equation (24). From the definition (23) we have for $a_i\in\mathcal{D}_c$ via $\delta(\mathbb{I})=0$ that:

$$\begin{aligned}
(B\tau_{n+1})(a_0,\ldots,a_n) &= i^{\epsilon_{n+1}} \int_{\sigma_{n+1}} ds_1\cdots ds_{n+1}\, \Big[\varphi\Big(\alpha_{is_1}(\delta\gamma a_0)\cdots \alpha_{is_{n+1}}(\delta\gamma^{n+1} a_n)\Big) \\
&+ \sum_{j=1}^{n} (-1)^{nj}\, \varphi\Big(\alpha_{is_1}(\delta\gamma^2 a_{n+1-j})\cdots \alpha_{is_j}(\delta\gamma^{j+1} a_n)\, \alpha_{is_{j+1}}(\delta\gamma^{j+1} a_0)\cdots \alpha_{is_{n+1}}(\delta\gamma^{n+1} a_{n-j})\Big)\Big]\,.
\end{aligned}$$

We can now use Lemma 8.5(ii) in all the terms on the right hand side to bring the factor with a_0 to the front:

$$\begin{aligned}
(B\tau_{n+1})(a_0,\ldots,a_n) &= i^{\epsilon_{n+1}} \int_{\sigma_{n+1}} ds_1\cdots ds_{n+1}\, \Big[\varphi\Big(\delta\gamma(a_0)\, \alpha_{is_1}(\delta\gamma^2 a_1)\cdots \\
&\cdots \alpha_{is_n}(\delta\gamma^{n+1} a_n)\, \alpha_{is_{n+1}}(\mathbb{I})\Big) + \sum_{j=1}^{n} (-1)^{nj}\, \varphi\Big(\delta\gamma^{j+1}(a_0)\, \alpha_{is_1}(\delta\gamma^{j+2} a_1)\cdots \\
&\cdots \alpha_{is_{n-j}}(\delta\gamma^{n+1} a_{n-j})\, \alpha_{is_{n-j+1}}(\mathbb{I})\, \alpha_{is_{n-j+2}}(\gamma\delta\gamma^2 a_{n+1-j})\cdots \alpha_{is_{n+1}}(\gamma\delta\gamma^{j+1} a_n)\Big)\Big]
\end{aligned}$$

now substitute $\varphi\to\varphi\circ\gamma^{j+1}$ in the last term, and use $\gamma\circ\delta=-\delta\circ\gamma$ and Lemma 8.5(iii):

$$\begin{aligned}
&= i^{\epsilon_{n+1}} \int_{\sigma_{n+1}} ds_1\cdots ds_{n+1}\, \Big[(-1)^{n+1}\varphi\Big(\delta(a_0)\, \alpha_{is_1}(\delta\gamma a_1)\cdots \alpha_{is_n}(\delta\gamma^n a_n)\, \alpha_{is_{n+1}}(\mathbb{I})\Big) \\
&+ \sum_{j=1}^{n} (-1)^{nj}\, \varphi\Big((-1)^{j+1}\delta(a_0)\, (-1)^{j+1}\alpha_{is_1}(\delta\gamma a_1)\cdots \\
&\cdots (-1)^{j+1}\alpha_{is_{n-j}}(\delta\gamma^{n-j} a_{n-j})\, \alpha_{is_{n-j+1}}(\mathbb{I})\, (-1)^j\alpha_{is_{n-j+2}}(\delta\gamma^j a_{n+1-j})\cdots (-1)^j\alpha_{is_{n+1}}(\delta\gamma a_n)\Big)\Big] \\
&= i^{\epsilon_{n+1}}(-1)^{n+1} \int_{\sigma_{n+1}} ds_1\cdots ds_{n+1}\, \Big[\varphi\Big(\delta(a_0)\, \alpha_{is_1}(\delta\gamma a_1)\cdots \alpha_{is_n}(\delta\gamma^n a_n)\, \alpha_{is_{n+1}}(\mathbb{I})\Big) \\
&+ \sum_{j=1}^{n} \varphi\Big(\delta(a_0)\, \alpha_{is_1}(\delta\gamma a_1)\cdots \\
&\cdots \alpha_{is_{n-j}}(\delta\gamma^{n-j} a_{n-j})\, \alpha_{is_{n-j+1}}(\mathbb{I})\, \alpha_{is_{n-j+2}}(\delta\gamma^j a_{n+1-j})\cdots \alpha_{is_{n+1}}(\delta\gamma a_n)\Big)\Big]\,.
\end{aligned}$$

Now recall that $\widetilde{\tau}:=(\tau_0,0,-\tau_2,0,\tau_4,\ldots)\in\mathcal{C}(\mathcal{D}_c)$, and hence we may assume that n is odd in the preceding expression (if n is even, $B\widetilde{\tau}_{n+1}=0$). Thus

$$\begin{aligned}
(B\tau_{n+1})(a_0,\ldots,a_n) &= \int_{\sigma_{n+1}} ds_1\cdots ds_{n+1}\, \Big[\sum_{j=0}^{n} \varphi\Big(\delta(a_0)\, \alpha_{is_1}(\delta\gamma a_1)\cdots \\
&\cdots \alpha_{is_{n-j}}(\delta\gamma^{n-j} a_{n-j})\, \alpha_{is_{n-j+1}}(\mathbb{I})\, \alpha_{is_{n-j+2}}(\delta\gamma^{n+1-j} a_{n+1-j})\cdots \alpha_{is_{n+1}}(\delta\gamma^n a_n)\Big)\Big]\,.
\end{aligned}$$

Since $\alpha_{is_{n-j+1}}(\mathbb{I}) = \mathbb{I}$, we can do the integrals w.r.t. s_{n-j+1}, and so using $0 \le s_1 \le s_2 \le \cdots \le s_{n+1} \le 1$ and a relabelling of variables, we get

$$\begin{aligned}(B\tau_{n+1})(a_0, \ldots, a_n) = \int_{\sigma_n} ds_1 \cdots ds_n \Big[\sum_{j=1}^{n-1} (s_{n-j+1} - s_{n-j})\, \varphi\Big(\delta(a_0)\, \alpha_{is_1}(\delta\gamma a_1) \cdots \\ \cdots \alpha_{is_{n-j}}(\delta\gamma^{n-j} a_{n-j})\, \alpha_{is_{n-j+1}}(\delta\gamma^{n+1-j} a_{n+1-j}) \cdots \alpha_{is_n}(\delta\gamma^n a_n)\Big) \\ + (s_1 + 1 - s_n)\, \varphi\Big(\delta(a_0)\, \alpha_{is_1}(\delta\gamma a_1) \cdots \alpha_{is_n}(\delta\gamma^n a_n)\Big)\Big] \\ = \int_{\sigma_n} ds_1 \cdots ds_n\, \varphi\Big(\delta(a_0)\, \alpha_{is_1}(\delta\gamma a_1) \cdots \alpha_{is_n}(\delta\gamma^n a_n)\Big)\,. \qquad (47)\end{aligned}$$

Next, we turn our attention to the left hand side of our desired Equation (24). Observe first that we have

$$\begin{aligned}\tau_n(\gamma a_0, \ldots, \gamma a_n) &= (-1)^n \tau_n(a_0, \ldots, a_n) \qquad \text{because:} \\ \varphi\left(\gamma(a_0)\, \alpha_{is_1}(\delta\gamma(\gamma a_1)) \cdots \alpha_{is_n}(\delta\gamma^n(\gamma a_1))\right) &= (-1)^n\, \varphi\left(a_0\, \alpha_{is_1}(\delta\gamma a_1) \cdots \alpha_{is_n}(\delta\gamma^n a_n)\right)\end{aligned}$$

since $\delta \circ \gamma = -\gamma \circ \delta$, $\varphi \circ \gamma = \varphi$ and by Lemma 8.5(iii). Thus $\widetilde{\tau} \circ \gamma = \widetilde{\tau}$, and so we have $\widetilde{a} = \gamma a$ in definition (22). An application of definition (22) to the left hand side of Equation (24) yields:

$$\begin{aligned}(b\,\tau_{n-1})(a_0, \ldots, a_n) = i^{\epsilon_{n-1}} \int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1} \Big[\sum_{j=0}^{n-1} (-1)^j \varphi\Big(a_0\, \alpha_{is_1}(\delta\gamma a_1) \cdots \alpha_{is_j}(\delta\gamma^j(a_j\, a_{j+1})) \cdots \\ \cdots \alpha_{is_{n-1}}(\delta\gamma^{n-1} a_n)\Big) + (-1)^n\, \varphi\Big((\gamma a_n)\, a_0\, \alpha_{is_1}(\delta\gamma a_1) \cdots \alpha_{is_{n-1}}(\delta\gamma^{n-1} a_{n-1})\Big)\Big]\,. \qquad (48)\end{aligned}$$

We examine the terms in this sum more closely:

$j = 0$: $\displaystyle\int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1}\, \varphi\Big(a_0\, a_1\, \alpha_{is_1}(\delta\gamma a_2) \cdots \alpha_{is_{n-1}}(\delta\gamma^{n-1} a_n)\Big)$

$j = 1$:

$$\begin{aligned}&-\int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1}\, \varphi\Big(a_0\, \alpha_{is_1}(\delta\gamma(a_1 a_2))\, \alpha_{is_2}(\delta a_3) \cdots \alpha_{is_{n-1}}(\delta\gamma^{n-1} a_n)\Big) \\ &= -\int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1}\, \varphi\Big(a_0\, \alpha_{is_1}((\delta\gamma(a_1)\, \gamma a_2 + a_1\, \delta\gamma a_2)\, \alpha_{is_2}(\delta a_3) \cdots \alpha_{is_{n-1}}(\delta\gamma^{n-1} a_n)\Big) \\ &= -\int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1} \Big[\varphi\Big(a_0\, \alpha_{is_1}((\delta\gamma(a_1)\, \gamma a_2)\, \alpha_{is_2}(\delta a_3) \cdots \alpha_{is_{n-1}}(\delta\gamma^{n-1} a_n)\Big) \\ &\qquad + \varphi\Big(a_0\, a_1\, \alpha_{is_1}(\delta\gamma a_2)\, \alpha_{is_2}(\delta a_3) \cdots \alpha_{is_{n-1}}(\delta\gamma^{n-1} a_n)\Big)\Big] \\ &\quad - \int_{\sigma_n} ds_1 \cdots ds_n\, \frac{\partial}{\partial s_1}\, \varphi\Big(a_0\, \alpha_{is_1}(a_1)\, \alpha_{is_2}(\delta\gamma a_2)\, \alpha_{is_3}(\delta a_3) \cdots \alpha_{is_n}(\delta\gamma^{n+1} a_n)\Big)\end{aligned}$$

where we made use of Equation (45) in the last step. Notice that we get a cancellation between the middle term and the $j = 0$ term in the sum. For $1 < j \le n-1$ we have the terms:

$$\begin{aligned}&(-1)^j \int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1}\, \varphi\Big(a_0\, \alpha_{is_1}(\delta\gamma a_1) \cdots \alpha_{is_j}(\delta\gamma^j(a_j a_{j+1})) \cdots \alpha_{is_{n-1}}(\delta\gamma^{n-1} a_n)\Big) \\ &= (-1)^j \int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1}\, \varphi\Big(a_0\, \alpha_{is_1}(\delta\gamma a_1) \cdots \alpha_{is_j}((\delta\gamma^j a_j)\gamma^j a_{j+1} + \gamma^{j+1}(a_j)\delta\gamma^j a_{j+1}) \cdots \alpha_{is_{n-1}}(\delta\gamma^{n-1} a_n)\Big) \\ &= (-1)^j \int_{\sigma_{n-1}} ds_1 \cdots ds_{n-1} \Big[\varphi\Big(a_0\, \alpha_{is_1}(\delta\gamma a_1) \cdots \alpha_{is_j}((\delta\gamma^j a_j)\gamma^j a_{j+1}) \cdots \alpha_{is_{n-1}}(\delta\gamma^{n-1} a_n)\Big) \\ &\qquad + \varphi\Big(a_0\, \alpha_{is_1}(\delta\gamma a_1) \cdots \alpha_{is_{j-1}}((\delta\gamma^{j-1} a_{j-1})\gamma^{j+1} a_j) \cdots \alpha_{is_{n-1}}(\delta\gamma^{n-1} a_n)\Big)\Big] \\ &\quad + (-1)^j \int_{\sigma_n} ds_1 \cdots ds_n\, \frac{\partial}{\partial s_j}\, \varphi\Big(a_0\, \alpha_{is_1}(\delta\gamma a_1) \cdots \alpha_{is_j}(\delta\gamma^{j+1} a_j) \cdots \alpha_{is_n}(\delta\gamma^{n-1} a_n)\Big)\end{aligned}$$

where we made use of Equation (44). Thus we get for Equation (48), taking into account cancellations between subsequent terms in the sum, that

$$\begin{aligned}(b\,\tau_{n-1})(a_0,\ldots,\,a_n) &= i^{\epsilon_{n-1}}(-1)^{n-1}\int_{\sigma_{n-1}} ds_1\cdots ds_{n-1}\,\varphi\Big(a_0\,\alpha_{is_1}(\delta\gamma a_1)\cdots\alpha_{is_{n-1}}((\delta\gamma^{n-1}a_{n-1})\,\gamma^{n-1}a_n)\Big)\\ &\quad + i^{\epsilon_{n-1}}\sum_{j=1}^{n-1}(-1)^j\int_{\sigma_n} ds_1\cdots ds_n\,\frac{\partial}{\partial s_j}\,\varphi\Big(a_0\,\alpha_{is_1}(\delta\gamma a_1)\cdots\alpha_{is_j}(\delta\gamma^{j+1}a_j)\cdots\alpha_{is_n}(\delta\gamma^{n-1}a_n)\Big)\\ &\quad + i^{\epsilon_{n-1}}(-1)^n\int_{\sigma_{n-1}} ds_1\cdots ds_{n-1}\,\varphi\Big((\gamma a_n)\,a_0\,\alpha_{is_1}(\delta\gamma a_1)\cdots\alpha_{is_{n-1}}(\delta\gamma^{n-1}a_{n-1})\Big)\\ &= \sum_{j=1}^{n}(-1)^j\int_{\sigma_n} ds_1\cdots ds_n\,\frac{\partial}{\partial s_j}\,\varphi\Big(a_0\,\alpha_{is_1}(\delta\gamma a_1)\cdots\alpha_{is_j}(\delta\gamma^{j+1}a_j)\cdots\alpha_{is_n}(\delta\gamma^{n-1}a_n)\Big)\end{aligned}$$

where we made use of Equation (46) and used the fact that since $\widetilde{\tau}=(\tau_0,\,0,\,-\tau_2,\,0,\tau_4,\ldots)$, we may take n to be odd. Then

$$\begin{aligned}&(b\,\tau_{n-1})(a_0,\ldots,\,a_n)\\ &= \sum_{j=1}^{n}(-1)^{j+n}\int_{\sigma_n} ds_1\cdots ds_n\,\frac{\partial}{\partial s_j}\,\varphi\Big((\gamma a_0)\,\alpha_{is_1}(\delta a_1)\cdots\alpha_{is_j}(\delta\gamma^j a_j)\cdots\alpha_{is_n}(\delta\gamma^n a_n)\Big)\\ &= \int_{\sigma_n} ds_1\cdots ds_n\sum_{j=1}^{n}\frac{\partial}{\partial s_j}\,\varphi\Big((\gamma a_0)\,\alpha_{is_1}(\gamma\delta\gamma a_1)\cdots\alpha_{is_{j-1}}(\gamma\delta\gamma^{j-1}a_{j-1})\,\alpha_{is_j}(\delta\gamma^j a_j)\cdots\alpha_{is_n}(\delta\gamma^n a_n)\Big)\,. \end{aligned}\tag{49}$$

To make further progress, we need the following lemma.

8.6 Lemma *Let $a_i\in\mathcal{D}_S$ and $(s_1,\ldots,\,s_n)\in\sigma_n$, then*

$$\begin{aligned}&\varphi\Big(\delta(a_0)\,\alpha_{is_1}(\delta a_1)\cdots\alpha_{is_n}(\delta a_n)\Big)\\ &= \sum_{j=1}^{n}\frac{\partial}{\partial s_j}\,\varphi\Big((\gamma a_0)\,\alpha_{is_1}(\gamma\delta a_1)\cdots\alpha_{is_{j-1}}(\gamma\delta a_{j-1})\,\alpha_{is_j}(a_j)\,\alpha_{is_{j+1}}(\delta a_{j+1})\cdots\alpha_{is_n}(\delta a_n)\Big)\,.\end{aligned}$$

Proof: A close examination of the proof of Theorem 5.8 shows that we actually proved that

$$\frac{d}{dt}\,\varphi(B\,\alpha_t(A)C)\Big|_0 = i\,\varphi(B\overline{\delta}_0(A)C) = i\,\varphi(B\overline{\delta}^2(A)C)$$

for all $A\in\mathcal{D}_S$ and $B,\ C\in\mathcal{A}_0$ where the right hand side makes sense because φ is strongly regular on $\mathcal{R}(\mathcal{S}(\mathbb{R}),\,\sigma)$ hence is well defined on $\mathcal{E}_0$. Now from the graded product rule for $\overline{\delta}$ on $\mathcal{E}_0$ we get

$$\overline{\delta}(b_0)\,B_1\cdots B_n = \overline{\delta}(b_0\,B_1\cdots B_n) - \sum_{j=1}^{n}\gamma(b_0\,B_1\cdots B_{j-1})\,\overline{\delta}(B_j)\,B_{j+1}\cdots B_n$$

for $b_0\in\mathcal{D}_S,\ B_1,\ldots,\,B_n\in\mathcal{A}_0$. Let $B_i=\overline{\delta}(b_i)$ for $b_i\in\mathcal{D}_S$, then

$$\overline{\delta}(b_0)\,\overline{\delta}(b_1)\cdots\overline{\delta}(b_n) = \overline{\delta}(b_0\,\overline{\delta}(b_1)\cdots\overline{\delta}(b_n)) - \sum_{j=1}^{n}\gamma(b_0\,\overline{\delta}(b_1)\cdots\overline{\delta}(b_{j-1}))\,\overline{\delta}^2(b_j)\,\overline{\delta}(b_{j+1})\cdots\overline{\delta}(b_n)\,.$$

Hence, using $\varphi\circ\overline{\delta}=0$ we get:

$$\begin{aligned}\varphi\Big(\overline{\delta}(b_0)\,\overline{\delta}(b_1)\cdots\overline{\delta}(b_n)\Big) &= -\sum_{j=1}^{n}\varphi\Big(\gamma(b_0\,\overline{\delta}(b_1)\cdots\overline{\delta}(b_{j-1}))\,\overline{\delta}^2(b_j)\,\overline{\delta}(b_{j+1})\cdots\overline{\delta}(b_n)\Big)\\ &= i\frac{d}{dt}\sum_{j=1}^{n}\varphi\Big(\gamma(b_0\,\overline{\delta}(b_1)\cdots\overline{\delta}(b_{j-1}))\,\alpha_t(b_j)\,\overline{\delta}(b_{j+1})\cdots\overline{\delta}(b_n)\Big)\Big|_0\,.\end{aligned}$$

Now make the replacements $b_0 \to a_0$, $b_i \to \alpha_{t_i}(a_i)$, $i = 1, \ldots, n$ for $a_i \in \mathcal{D}_S$ and use the fact that $\alpha_t \circ \overline{\delta} = \overline{\delta} \circ \alpha_t$ to find that:

$$\varphi\Big(\delta(a_0)\,\alpha_{t_1}(\delta\,a_1)\cdots\alpha_{t_n}(\delta\,a_n)\Big)$$
$$= i\sum_{j=1}^{n}\frac{\partial}{\partial t_j}\,\varphi\Big(\gamma(a_0)\,\alpha_{t_1}(\gamma\delta\,a_1)\cdots\alpha_{t_{j-1}}(\gamma\delta\,a_{j-1})\,\alpha_{t_j}(a_j)\,\alpha_{t_{j+1}}(\delta\,a_{j+1})\cdots\alpha_{t_n}(\delta\,a_n)\Big)$$

where we replaced $\overline{\delta}$ by δ because it is now evaluated on $\mathcal{D}_S$ only. Now by the KMS–condition, analyticity, flat tube theorem and a complex linear change of variables, we find as in Section 6 that the functions

$$(t_1, \ldots, t_n) \to \varphi\Big(\delta(a_0)\,\alpha_{t_1}(\delta\,a_1)\cdots\alpha_{t_n}(\delta\,a_n)\Big)$$
$$(t_1, \ldots, t_n) \to \varphi\Big(\gamma(a_0)\,\alpha_{t_1}(\gamma\delta\,a_1)\cdots\alpha_{t_{j-1}}(\gamma\delta\,a_{j-1})\,\alpha_{t_j}(a_j)\,\alpha_{t_{j+1}}(\delta\,a_{j+1})\cdots\alpha_{t_n}(\delta\,a_n)\Big)$$

extend analytically to the flat tube $\mathcal{T}_n := \mathbb{R}^n + i\sigma^n$ such that

$$\varphi\Big(\delta(a_0)\,\alpha_{z_1}(\delta\,a_1)\cdots\alpha_{z_n}(\delta\,a_n)\Big)$$
$$= i\sum_{j=1}^{n}\frac{\partial}{\partial z_j}\,\varphi\Big(\gamma(a_0)\,\alpha_{z_1}(\gamma\delta\,a_1)\cdots\alpha_{z_{j-1}}(\gamma\delta\,a_{j-1})\,\alpha_{z_j}(a_j)\,\alpha_{z_{j+1}}(\delta\,a_{j+1})\cdots\alpha_{z_n}(\delta\,a_n)\Big)\,.$$

In the case that $z_k = is_k$ where $(s_1, \ldots, s_n) \in \sigma^n$ we can use $\partial/\partial z_k = -i\,\partial/\partial s_k$ to obtain from the last equation the statement of the Lemma. ■

Application of the Lemma to Equation (49) then produces

$$(b\,\tau_{n-1})(a_0, \ldots, a_n) = \int_{\sigma_n} ds_1\cdots ds_n\,\varphi\Big(\delta(a_0)\,\alpha_{is_1}(\delta\gamma a_1)\cdots\alpha_{is_n}(\delta\gamma^n a_n)\Big)$$
$$= (B\tau_{n+1})(a_0, \ldots, a_n)$$

by Equation (47) and hence $\widetilde{\tau}$ is a cyclic cocycle.

Acknowledgements.

We gratefully acknowledge discussions with Arthur Jaffe, Roberto Longo and Hajime Moriya on various aspects of supersymmetry. DB wishes to thank the Department of Mathematics of the University of New South Wales and HG wishes to thank the Institute for Theoretical Physics of the University of Göttingen for hospitality and financial support which facilitated this research. The work was also supported in part by the FRG grant PS01583.

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