Ising models with long–range dipolar and short range ferromagnetic interactions

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We study the ground state of a d–dimensional Ising model with both long range (dipole–like) and nearest neighbor ferromagnetic (FM) interactions. The long range interaction is equal to r^{-p}, $p > d$, while the FM interaction has strength J. If $p > d+1$ and J is large enough the ground state is FM, while if $d < p \leq d+1$ the FM state is not the ground state for any choice of J. In $d = 1$ we show that for any $p > 1$ the ground state has a series of transitions from an antiferromagnetic state of period 2 to $2h$–periodic states of blocks of sizes h with alternating sign, the size h growing when the FM interaction strength J is increased (a generalization of this result to the case $0 < p \leq 1$ is also discussed). In $d \geq 2$ we prove, for $d < p \leq d+1$, that the dominant asymptotic behavior of the ground state energy agrees for large J with that obtained from a periodic striped state conjectured to be the true ground state. The geometry of contours in the ground state is discussed.

I. INTRODUCTION

There has been and continues to be much interest in mesoscopic pattern formation induced by competing short and long range interactions [1–4]. Of particular interest is the competition between dipole–like interactions bewteen spins which decay like $r^{-(d+1)}$, d the dimension of the lattice, and short range exchange interactions [5, 6]. For $d = 2$, this type of competitive interaction is believed responsible for many of the observed patterns in thin magnetic films [7] and other quasi–two dimensional systems, including Langmuir monolayers [8], lipid monolayers [9], liquid crystals [10] and polymer films [11]. While the spin variables in these systems are, in most cases, Heisenberg spins, there has also been work on models in which the spin variables are of the Ising type [6, 12–16].

Surprisingly, there appears to have been very little work on proving mathematically any of the conclusions arrived at in these works via physical arguments and Monte Carlo simulations.

In the present work we begin this task. The problem is not simple. Most of the mathematically rigorous techniques developed for obtaining the low temperature phase diagram of spin systems, e.g. the Pirogov–Sinai theory [17], depend on the interaction being short range. Only methods based on reflection positivity [18] or on convexity [19–23] seem applicable to the kind of systems considered here. Our results are therefore quite meagre. They are mostly confined to the characterization of the ground states of the one–dimensional system and to partial results in higher dimensions. Hopefully, they will be a stepping stone to future work.

A. The model

Given an integer N, let Λ_N be a "simple cubic" d–dimensional torus of side $2N$ and $\underline{\sigma} \in \{\pm 1\}^{\Lambda_N}$. Any configuration $\underline{\sigma}$ is a sequence of $\sigma_i = \pm 1$ labelled by $i \in \Lambda_N$. The Hamiltonian of this Ising model with *periodic boundary conditions* is taken to be

$$H_N(\underline{\sigma}) = -J \sum_{i\in\Lambda_N} \sum_{k=1}^{d} \sigma_i \sigma_{i+e_k} + \sum_{(i,j)} \sigma_i \, J_p(j-i) \, \sigma_j \ , \qquad J_p(j-i) = \sum_{n\in\mathbb{Z}^d} \frac{1}{|i-j+2nN|^p} \tag{1.1}$$

where in the first sum e_k is the unit vector in the k–th coordinate direction (and if $i+e_k \notin \Lambda_N$ we define $\sigma_{i+e_k} \equiv \sigma_{i-2Ne_k}$), the second sum runs over generic distinct pairs of sites in Λ_N and $p > d$. The first term in (1.1) will be called the *exchange energy* term and we will take $J \geq 0$ while the second one will be called the *dipolar energy* term. In $d > 1$ we shall define $|i-j|$ to be the usual Euclidean distance between i and j. Note that the sum over n in the definition of $J_p(j-i)$ makes sense as long as $p > d$. However some of the results discussed below hold true even in the case $d-1 < p \leq d$, *provided* the definition of $J_p(j-i)$ is replaced by $J_p(j-i) = |j-i|^{-p} + (2N - |j-i|)^{-p}$,

$\forall 1 \leq |j-i| \leq 2N-1$. In the following we will restrict attention to the case $p > d$ and we will comment on possible generalizations of our results to $d-1 < p \leq d$ in the Remark after Theorem 2.

B. Main results

We wish to determine the ground state of model (1.1) for different values of p and J. A first remark is that in any dimension for $p > d+1$ and J large enough the ground state is ferromagnetic and is unique modulo a global spin flip. This is, in $d > 1$, a corollary of the contour estimates in [24], and we shall reproduce its proof as a byproduct of our analysis. For this case the low temperature Gibbs states are also known: for $d = 1$ and $p > 2$ there is a unique Gibbs state for any $\beta < +\infty$ [25] while for $d \geq 2$, $p > d+1$ and β large enough, there are two different pure states obtained as the thermodynamic limits of the equilibrium states with $+$ or $-$ boundary conditions [24]. On the contrary, for $d < p \leq d+1$, the ferromagnetic state is not the ground state in any dimension, for any value of J. Instability of the ferromagnetic state in presence of a long range antiferromagnetic interaction of the form $1/r^p$, $d < p \leq d+1$, was first noted in [26]. The reason for this instability lies in the divergence of the first moment of the long range interaction which makes the ferromagnetic state unstable towards flipping spins in a large enough domain.

Our main results give a characterization of the ground state in the whole regime $p > d$. For $d = 1$ the characterization is complete, in the sense that for any $p > 1$ we can compute the ground state energy per site and we can prove that the corresponding ground state configurations are periodic and consist of blocks of equal size and alternating signs (a block is a maximal sequence of adjacent spins of the same sign).

Theorem 1. (Ground state energy, d=1).
For $d = 1$ and $p > 1$, the ground state energy $E_0(N)$ of the Hamiltonian (1.1) satisfies

$$\lim_{N\to\infty} \frac{1}{2N} E_0(N) = \min_{h\in\mathbb{Z}^+} e(h) \tag{1.2}$$

where $e(h) \stackrel{def}{=} \lim_{N\to\infty}(2Nh)^{-1}E^{(h)}_{per}(Nh)$ and $E^{(h)}_{per}(Nh)$ is the energy of a periodic configuration on a ring of $2Nh$ sites consisting of blocks of size h with alternating signs.

The function $e(h)$ is explicitly computable, see Section III. The minimization of $e(h)$ with respect to h can be performed exactly and, as expected, for $p > 2$ and J larger than an explicit constant $J_0 \equiv \Gamma(p)^{-1} \int_0^\infty d\alpha \alpha^{p-1} e^{-\alpha}(1-e^{-\alpha})^{-2}$, the minimizer is $h = +\infty$ (i.e. the ground state is ferromagnetic). For $1 < p \leq 2$ and all $J \geq 0$ or $p > 2$ and $0 \leq J < J_0$ the minimum of $e(h)$ over the positive integers is attained at an integer $h^*(J)$ which is a piecewise constant monotone increasing function of J. Thus there is a unique integer minimizer of $e(h)$ for almost all values of J which jumps at a discrete set of values of J. At those values of J the minimum is attained at two consecutive integers $h^*(J)$ and $h^*(J)+1$ and the corresponding ground states, for suitable values of N, are the configurations with all blocks of the same size, either $h^*(J)$ or $h^*(J)+1$.

Theorem 2. (Finite volume ground states).
Let $d = 1$ and $p > 1$. At the values of J such that the minimum of $e(h)$ is attained at a single integer $h^*(J)$, then in a ring of length $2N$, such that N is divisible by $h^*(J)$, the only ground states are the periodic configurations consisting of blocks of size $h^*(J)$. At the values of J such that the minimum is attained at two consecutive integers $h^*(J)$ and $h^*(J)+1$, then in a ring of length $2N$, such that N is divisible both by $h^*(J)$ and $h^*(J)+1$, the only ground states are the periodic configurations consisting of blocks of the same size, either $h^*(J)$ or $h^*(J)+1$. In both cases there is a finite (independent of N) gap between the energies of the ground states and of any other state.

Remarks.
1) *Stability for* $0 < p \leq 1$. The proofs of Theorems 1 and 2 do not really require $p > 1$ and, provided we change the definition of $J_p(j-i)$ as described after (1.1), they can be easily adapted to cover the cases $0 < p \leq 1$. In particular for any $p > 0$ the ground state energy can be computed by minimizing energy among the periodic states of blocks with alternating signs and the computation shows that the ground state energy is *extensive* (i.e. it scales proportionally to N as $N \to \infty$), even for $0 < p \leq 1$ (a case in which thermodynamic stability is not a priori guaranteed).
2) *Infinite volume ground states.* A corollary of Theorem 2 is that the infinite volume periodic configurations $\underline{\sigma}_J \in \{\pm 1\}^{\mathbb{Z}}$ with blocks all of the same size $h^*(J)$ (or $h^*(J)+1$, if J is a value at which the minimum of $e(h)$ is not unique) and alternating sign are

infinite volume ground state configurations, in the sense that they are stable against bounded perturbations [17]; i.e. given a finite set $X \subset \mathbb{Z}$, if R_X is the operator flipping the spins in X, the (finite) energy difference between $R_X \underline{\sigma}_J$ and $\underline{\sigma}_J$ is *positive*. This can be proven by writing the energy difference between the infinite configurations $R_X \underline{\sigma}_J$ and $\underline{\sigma}_J$ as the limit of the energy differences between the finite volume configurations on rings of length $2N$ (N divisible by $h^*(J)$) obtained by restricting $R_X \underline{\sigma}_J$ and $\underline{\sigma}_J$ to the finite rings. By Theorem 2, the differences between the energies of the finite volume approximations of $R_X \underline{\sigma}_J$ and $\underline{\sigma}_J$ are positive. This implies that $\underline{\sigma}_J$ is an infinite volume ground state configuration.

A stronger result which follows from our analysis is that for any $J \geq 0$ we can choose a sequence of integers $\{N_i\}_{i\in\mathbb{N}}$, $N_i \xrightarrow[i\to\infty]{} \infty$, such that the ground states on the rings of lenghts $2N_i$ are periodic configurations of blocks of size $h^*(J)$ and the corresponding sequence of ground state energies per site $e_0(N_i)$ is the sequence of "lowest possible specific energies", i.e. any other sequence $\{N_i'\}_{i\in\mathbb{N}}$, $N_i' \xrightarrow[i\to\infty]{} \infty$, has $e_0(N_i') \geq e_0(N_i)$, eventually as $i \to \infty$.

In dimension larger then one we do not find an exact asymptotic expression for the ground state energy. Still, for large J, we find rigorous upper and lower bounds to the ground state energy. The result is the following.

Theorem 3. *Let $d \geq 2$. If $p = d+1$ and J is large enough, there exist positive, J–independent, constants C_1, C_2 such that*

$$e_{FM}(J) - C_1 e^{-J|\mathbb{S}_{d-1}|^{-1}} \leq \lim_{N\to\infty} \frac{1}{|\Lambda_N|} E_0(N) \leq e_{FM}(J) - C_2 e^{-J|\mathbb{S}_{d-1}|^{-1}} , \quad (1.3)$$

where $e_{FM}(J)$ is the energy per site of the ferromagnetic configuration $(\underline{\sigma})_i = +1$ and $|\mathbb{S}_{d-1}|$ is the volume of the $d-1$ dimensional unit sphere.
If $d < p < d+1$ and J is large enough, then there exist positive, J–independent, constants K_1, K_2 such that

$$e_{FM}(J) - K_1 J^{-\frac{p-d}{d+1-p}} \leq \lim_{N\to\infty} \frac{1}{|\Lambda_N|} E_0(N) \leq e_{FM}(J) - K_2 J^{-\frac{p-d}{d+1-p}} . \quad (1.4)$$

The upper bounds in (1.3) and (1.4) follow from a variational computation. The best known constants C_2, K_2 are obtained by minimizing over the periodic striped configurations $(\underline{\sigma}^{(h)}_{striped})_i = (-1)^{[i_1/h]}$, where i_1 is the first component of i and $[x]$ is the largest integer less than or equal to x. It is remarkable that, for $J \to \infty$, the minimum over the striped configurations provides a variational energy that is lower than the one obtained by minimizing over the periodic checkerboard configurations; this was shown in [12] for $d = 2$ and $p = 3$. A computation analogous to that in [12] allows to prove a similar result in higher dimensions for $d < p \leq d+1$. This result together with numerical studies support the conjecture that the ground state of (1.1) in $d \geq 2$ and $d < p \leq d+1$ is in fact a periodic striped configuration (at least asymptotically for large J). In this paper we prove the lower bounds in (1.3) and (1.4), which show that for large J the difference between the ground state energy and the ferromagnetic energy scales exactly as predicted by the above mentioned variational computation. The proof of the lower bound is based on an energy argument which also gives estimates on the allowed shapes and sizes of Peierls' contours in the ground state.

II. OUTLINE OF THE PROOFS

Before presenting the proofs of Theorems 1, 2 and 3 in detail, we give a short outline.

The proofs of Theorem 1 and 2 are based on *reflection positivity.* It is known [27] that for $J = 0$ the Hamiltonian (1.1) is reflection positive in any dimension. In $d = 1$ this means that, given any configuration of spins on a ring of $2N$ sites $\underline{\sigma} = (\sigma_{-N+1}, \ldots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_N)$ and a bond $b = (i, i+1)$ connecting site i with its neighbor $i+1$, the average energy of the two configurations obtained by reflecting around b is always lower than the original one; e.g. choosing $b = (0, 1)$ the average of the energies of the two reflected configurations $(-\sigma_N \ldots, -\sigma_2, -\sigma_1, \sigma_1, \sigma_2, \ldots, \sigma_N)$ and $(\sigma_{-N+1} \ldots, \sigma_{-1}, \sigma_0, -\sigma_0, -\sigma_{-1}, \ldots, -\sigma_{-N+1})$ is always lower than the energy of the original configuration $\underline{\sigma}$. Using this property and repeatedly reflecting around different bonds, one can prove that for $J = 0$ the ground state is the antiferromagnetic alternating state. Note that periodic boundary conditions are necessary for repeated reflections around different bonds.

Unfortunately, as soon as $J > 0$, the Hamiltonian (1.1) is *not* reflection positive anymore. Still, one can think of making use of the reflection symmetry around bonds $b = (i, i+1)$ separating a spin σ_i from a spin $\sigma_{i+1} = -\sigma_i$: such reflection does not change the exchange energy between σ_i and σ_{i+1} and lowers the dipole energy. By repeatedly reflecting around such sites one could expect to be able to reduce the search for the ground states just to the class of periodic configurations of blocks of the same size and alternating sign and explicitly look for periodic configuration with minimal energy (we recall that by block we mean a maximal sequence of consecutive spins all of the same sign). However there is a difficulty: because of periodic boundary conditions, in order not to increase the exchange energy in the reflection, one should reflect around bonds b not only separating a $+$ from a $-$ spin, but with the further property that the bond b' at a distance N from b also separates a $+$ from a $-$ spin: but in a generic configuration $\underline{\sigma}$ there will be no bond b with such a property!

A possible solution to this problem is to cut the ring into two uneven parts both containing the same number of blocks but not necessarily of the same length. The configurations obtained by reflecting the two uneven parts will have a lower energy but in general different lengths $2N', 2N''$. The idea is to reflect repeatedly in this fashion, keeping track of the errors due to the fact that the length of the ring is changing at each reflection. A convenient way of doing this, exploited in Section III, is to rewrite the spin Hamiltonian (1.1) as an effective Hamiltonian for new "atoms" of "charges" h_i, corresponding to the spin blocks of size h_i. The new effective Hamiltonian $E(\ldots, h_{-1}, h_0, h_1, h_2, \ldots)$ is again reflection symmetric (in a slightly different sense, though) and its explicit form allows for an easy control of the finite size errors one is left with after repeated reflections. Some technical aspects of the proofs of Theorem 1 and 2 are given in the Appendices.

The proof of Theorem 3, in particular the lower bounds in (1.3) and (1.4), is based on a Peierls' contour estimate described in Section IV. For large ferromagnetic coupling J, we shall describe the ground state configuration(s) in terms of droplets of $-$ spins surrounded by $+$ spins and of contours separating the $+$ from the $-$ phases. The requirement that the energy of the ground state configuration is minimal imposes bounds on the geometry and the energy of such droplets, implying the lower bound in (1.4). Contrary to the methods exploited in the proofs of Theorem 1 and 2, the proof of Theorem 3 is robust under modifications both of the boundary conditions and of the specific form of the interaction potential.

III. ONE DIMENSION

In this section we want to prove Theorems 1 and 2. Restricting to $d = 1$, we shall consider any configuration $\underline{\sigma} \in \{\pm 1\}^{\Lambda_N}$ on the ring of length $2N$ as a sequence of blocks of alternating sign, where by definition a block is a maximal sequence of adjacent spins of the same sign. We note that, due to the periodic boundary condition, the number M of blocks on the ring is either 1 (if the state is ferromagnetic) or an even number. We shall denote by h_i, $i = 1, \ldots, M$, the sizes of such blocks. A block will be called a + block (− block), if its spins are all of sign $+1$ (-1).

We want to prove that the sizes h_i of the blocks in the ground state are all equal, at least for N large enough. The strategy will be the following. Given any configuration $\underline{\sigma}_N$ in the ring Λ_N with $M \geq 2$ blocks of alternating signs of sizes $h_{-\frac{M}{2}+1}, \ldots, h_{\frac{M}{2}}$, we will rewrite the energy $H_N(\underline{\sigma}_N)$ as a function of M and the h_i's, i.e. $H_N(\underline{\sigma}_N) = E_N(M, (\underline{h}_L, \underline{h}_R))$, where $\underline{h}_L = (h_{-\frac{M}{2}+1}, \ldots, h_0)$ and $\underline{h}_R = (h_1, \ldots, h_{\frac{M}{2}})$. Setting $\underline{h} = (\underline{h}_L, \underline{h}_R)$, we will show that, for M and N fixed, the Hamiltonian $E_N(M, \underline{h})$ is *reflection positive* with respect to the reflection

$$\hat{\theta} h_i = h_{-i+1} \, , \qquad -\frac{M}{2} < i \leq \frac{M}{2} \, , \tag{3.5}$$

that is

$$E_N(M, \underline{h}) \geq \frac{1}{2} \left[E_N(M, (\underline{h}_L, \hat{\theta}\underline{h}_L)) + E_N(M, (\hat{\theta}\underline{h}_R, \underline{h}_R)) \right] \, , \tag{3.6}$$

where $\hat{\theta}\underline{h}_L = (h_0, \ldots, h_{-\frac{M}{2}+1})$ and $\hat{\theta}\underline{h}_R = (h_{\frac{M}{2}}, \ldots, h_1)$, with of course the signs of the spins in the reflected blocks being opposite to what they were originally. Using repeatedly this symmetry, we will get bounds from above and below for the ground state energy, in terms of the energy of periodic configurations.

A. The integral representation

In order to show reflection positivity of $E_N(M, \underline{h})$, e.g. (3.6), we need to look more closely at the structure of the Hamiltonian. A straightforward calculation in Appendix B gives the energy $E_N(M, \underline{h})$ of a configuration in Λ_N of M blocks of sizes $h_{-\frac{M}{2}+1}, \ldots, h_{\frac{M}{2}}$ (recall that M is even and p.b.c. are assumed) as:

$$E_N(M, \underline{h}) = -2J(N - M) + 2N \int_0^\infty \frac{d\alpha}{\Gamma(p)} \, \alpha^{p-1} \frac{e^{-\alpha}}{(1 - e^{-\alpha})^2 (1 - e^{-2\alpha N})} \, .$$

$$
\cdot\Big\{ \sum_{i=-\frac{M}{2}+1}^{\frac{M}{2}} [h_i(1-e^{-\alpha})(1-e^{-2\alpha N}) - (1-e^{-\alpha h_i})(1-2e^{-\alpha(2N-h_i)}+e^{-2\alpha N})] +
$$

$$
+ \sum_{-\frac{M}{2}<i<j\le\frac{M}{2}} (-1)^{j-i}(1-e^{-\alpha h_i})(1-e^{-\alpha h_j})[\prod_{i<k<j} e^{-\alpha h_k} + \prod_{\substack{-\frac{M}{2}<k<i\\ j<k\le\frac{M}{2}}} e^{-\alpha h_k}]\Big\} \,. \quad (3.7)
$$

The expression in braces in the last two lines of (3.7) can be pictorially interpreted as the energy of a system of particles on a ring of M sites with $h_i \ge 1$ particles at site i. There is an on site energy as in the second line of (3.7) and a "many body" energy as in the third line.

The energy in (3.7) has the remarkable property of being *reflection positive* with respect to the reflection (3.5). This can be proven by rewriting the expression in braces in the last two lines of (3.7) as

$$
\int_0^\infty \nu(\alpha)\Big[H_R(\alpha,\underline{h}_R) + H_R(\alpha,\hat\theta\underline{h}_L) - \sum_{i=1}^{2} F_i(\alpha,\underline{h}_R)\, F_i(\alpha,\hat\theta\underline{h}_L)\Big] d\alpha \,, \qquad (3.8)
$$

where

$$
\begin{aligned}
\nu(\alpha) &\equiv \frac{1}{\Gamma(p)}\,\alpha^{p-1}\frac{e^{-\alpha}}{(1-e^{-\alpha})^2(1-e^{-2\alpha N})}\,,\\
H_R(\alpha,\underline{h}_R) &= -\sum_{i=1}^{\frac{M}{2}}(1-e^{-\alpha h_i})(1-2e^{-\alpha(2N-h_i)}+e^{-2\alpha N}) +\\
&\quad + \sum_{1\le i<j\le\frac{M}{2}} (-1)^{j-i}(1-e^{-\alpha h_i})(1-e^{-\alpha h_j}) \prod_{i<k<j} e^{-\alpha h_k}\,,\\
F_1(\alpha,\underline{h}_R) &= \sum_{i=1}^{\frac{M}{2}}(-1)^i(1-e^{-\alpha h_i}) \prod_{1\le k<i} e^{-\alpha h_k}\,,\\
F_2(\alpha,\underline{h}_R) &= \sum_{i=1}^{\frac{M}{2}}(-1)^i(1-e^{-\alpha h_i}) \prod_{i<k\le\frac{M}{2}} e^{-\alpha h_k}\,,
\end{aligned} \qquad (3.9)
$$

Let us define $H(\alpha,\underline{h})$ as follows:

$$
H(\alpha,\underline{h}) = H_R(\alpha,\underline{h}_R) + H_R(\alpha,\hat\theta\underline{h}_L) - \sum_{i=1}^{2} F_i(\alpha,\underline{h}_R)\, F_i(\alpha,\hat\theta\underline{h}_L) \qquad (3.10)
$$

so that $E_N(M,\underline{h}) = -2J(N-M) + 2N\int_0^\infty \frac{d\alpha}{\Gamma(p)}\,\alpha^{p-1}\frac{e^{-\alpha}}{(1-e^{-\alpha})} + \int_0^\infty d\alpha\,\nu(\alpha)H(\alpha,\underline{h})$.

Using the fact that $H_R(\alpha,\underline{h}_R) = H_R(\alpha,\hat\theta\underline{h}_R)$ and that, by Schwarz inequality, $F_i(\alpha,\underline{h}_R)F_i(\alpha,\hat\theta\underline{h}_L) \le \frac{1}{2}[F_i(\alpha,\underline{h}_R)F_i(\alpha,\underline{h}_R) + F_i(\alpha,\hat\theta\underline{h}_L)\ F_i(\alpha,\hat\theta\underline{h}_L)]$, we find

$$
\int d\alpha\,\nu(\alpha)\,H(\alpha,\underline{h}) = \qquad (3.11)
$$

$$= \int d\alpha\, \nu(\alpha) \left[H_R(\alpha, \underline{h}_R) + H_R(\alpha, \hat{\theta}\underline{h}_L) - \sum_{i=1}^{2} F_i(\alpha, \underline{h}_R)\, F_i(\alpha, \hat{\theta}\underline{h}_L) \right] \geq$$
$$\geq \frac{1}{2} \int d\alpha\, \nu(\alpha) \left[H(\alpha, (\underline{h}_L, \hat{\theta}\underline{h}_L)) + H(\alpha, (\hat{\theta}\underline{h}_R, \underline{h}_R)) \right] ,$$

so that (3.6) is proved.

Reflecting repeatedly with respect to different bonds we end up with

$$\int d\alpha\, \nu(\alpha)\, H(\alpha, \underline{h}) \geq \frac{1}{M} \sum_{i=-\frac{M}{2}+1}^{\frac{M}{2}} \int d\alpha\, \nu(\alpha)\, H(\alpha, (h_i, h_i, \ldots, h_i, h_i)) , \tag{3.12}$$

and $E_N(M, \underline{h}) \geq \frac{1}{M} \sum_{i=-\frac{M}{2}+1}^{\frac{M}{2}} E_N(M, (h_i, h_i, \ldots, h_i, h_i))$, which is an example of the *chessboard inequality*, see Theorem 4.1 in [18].

B. Bounds for $1 < p \leq 2$

We now temporarily restrict to the case $1 < p \leq 2$. A key remark is that in this case there exists a p–dependent constant K_p, explicitly computable as described in Appendix A, such that the blocks in the ground state have sizes h_i all satisfying the following apriori bound:

$$\begin{cases} h_i \leq 12 e^J , & \text{if} \quad p = 2 \\ h_i \leq K_p J^{1/(2-p)} & \text{if} \quad 1 < p < 2 , \end{cases} \tag{3.13}$$

Eq. (3.13) shows in particular that, if $1 < p \leq 2$, the FM state is not the ground state for any finite J when N is sufficiently large; in particular in the ground state the number M of blocks is $M \geq 2$ (and necessarily even). Note also that since all blocks satisfy $1 \leq h_i \leq h_{max}$, with h_{max} given by (3.13), it must be $M \leq 2N \leq h_{max} M$, that is N and M are of the same order as $N \to \infty$.

To compute the ground state energy, let us introduce the auxiliary partition function

$$Q_N = \sum_{M, \underline{h}}^{*} e^{-\beta E_N(M, \underline{h})} \tag{3.14}$$

where the $*$ means that we are summing only over the choices of M and $\underline{h}$ compatible with the constraint $\sum_{i=-\frac{M}{2}+1}^{\frac{M}{2}} h_i = 2N$ and with the bounds (3.13). We can then obtain the ground state energy per site $e_0(N)$ by taking the limit $-\lim_{\beta \to \infty} \frac{1}{\beta} \log Q_N = 2N e_0(N)$.

Using (3.12) we get:

$$Q_N \leq \sum_{M,\underline{h}}^{*} \prod_{i=-\frac{M}{2}+1}^{\frac{M}{2}} e^{-\beta \frac{1}{M} E_N(M,(h_i,\ldots,h_i))} \,. \tag{3.15}$$

By (3.7), we find that $\frac{1}{M}E_N(M,(h,\ldots,h))$ can be written explicitly as

$$\frac{1}{M}E_N(M,(h,\ldots,h)) = -Jh + Ah + 2J + \int_0^\infty \frac{d\alpha}{\Gamma(p)} \frac{\alpha^{p-1}e^{-\alpha}}{(1-e^{-\alpha})^2}\Big\{-2\tanh\frac{\alpha h}{2} + \\ + \frac{1}{(1-e^{-2\alpha N})}\frac{(1-e^{-\alpha h})^2}{1+e^{-\alpha h}}(2e^{-\alpha(2N-h)} + e^{-2\alpha N} - e^{-\alpha h(M-1)})\Big\} \,. \tag{3.16}$$

where the constant A in the r.h.s. is $A = \int_0^\infty \frac{d\alpha}{\Gamma(p)}\,\alpha^{p-1}\frac{e^{-\alpha}}{(1-e^{-\alpha})}$. Note that the absolute value of the term in the last line can be bounded above by KN^{-p}, for a suitable constant K. Then, recalling the definition of $E^{(h)}_{per}(Nh)$ (see Theorem 1) and defining $e(h)$ as

$$e(h) \stackrel{def}{=} \lim_{N\to\infty}\frac{1}{2Nh}E^{(h)}_{per}(Nh) = -J + A + \frac{2J}{h} - \frac{2}{h}\int_0^\infty \frac{d\alpha}{\Gamma(p)}\,\alpha^{p-1}\frac{e^{-\alpha}}{(1-e^{-\alpha})^2}\tanh\frac{\alpha h}{2} \tag{3.17}$$

we find that the r.h.s. of (3.15) and thus Q_N can be bounded above by

$$Q_N \leq \sum_{M,\underline{h}}^{*} \prod_{i=-\frac{M}{2}+1}^{\frac{M}{2}} e^{-\beta h_i e(h_i)} e^{\beta K' N^{-p}} \tag{3.18}$$

Using the fact that the number of terms in the sum is less than 2^{2N}, we can bound the r.h.s. of (3.18) from above by $2^{2N}e^{\beta K'/N^{p-1}}e^{-\beta 2N\widetilde{e}_0}$, where $\widetilde{e}_0 \equiv \min_{h\in\mathbb{Z}^+} e(h)$, so that we find $e_0(N) \geq \widetilde{e}_0 - K'/N^p$.

It follows from the explicit expression of $e(h)$ given in (3.17) that $\min_{h\in\mathbb{Z}^+} e(h) = e(h^*(J))$, where the integer $h^*(J)$ is a piecewise constant monotone increasing function of J. Thus there is a unique integer minimizer of $e(h)$ for almost all values of J which jumps at a discrete set of values of J. At those values of J the minimum is attained at two consecutive integers $h^*(J)$ and $h^*(J)+1$. As discussed below, the corresponding ground states are, asymptotically for N large, the configurations with all blocks of the same size, either $h^*(J)$ or $h^*(J)+1$. If h is not a minimizer, then $e(h)$ is separated by a gap from $\widetilde{e}_0$. These properties follow from the fact that $\partial_h e(h) = 0$ has a unique solution for $h \in \mathbb{R}^+$ and the solution is a local minimum.

Let us now turn to the problem of finding a lower bound for Q_N. If N is divisible by h^*, then of course

$$Q_N \geq e^{-\beta 2Ne(h^*) - \beta K' N^{-p}} \tag{3.19}$$

so that we find

$$e_0(N) = \widetilde{e}_0 + O(\frac{1}{N^p}) \ . \tag{3.20}$$

If N is not divisible by h^* the error in the r.h.s. is replaced by $O(N^{-1})$. This follows from the fact that we can bound Q_N from below by restricting the sum in (3.14) to a configuration with all but one block of sizes h^*: the last block being of size $h^* < h < 2h^*$. The energy of such configuration is $2Ne(h^*) + O(1)$. This concludes the proof of Theorem 1 in the case $p \leq 2$.

C. The case $p > 2$

Let us now discuss the case $p > 2$. It is straightforward to verify that in this case, if $J \geq J_0 \equiv \int_0^\infty \frac{d\alpha}{\Gamma(p)} \alpha^{p-1} \frac{e^{-\alpha}}{(1-e^{-\alpha})^2}$, then the minimizer of $e(h)$ is $h = +\infty$. Correspondingly we can prove that the ground state of (1.1) is the ferromagnetic state. In fact let us assume by contradiction that in the ground state there is a block B_h of finite size h. Then it must be true that the energy $\Delta E(h)$ needed to reverse the sign of all the spins in B_h is nonnegative: $\Delta E(h) \geq 0$. On the other side $\Delta E(h) \leq -4J + 2E_1(h)$, where $E_1(h)$ is the dipole energy between the + block B_h and an external sea of + spins. It is straightforward to check that

$$E_1(h) = 2 \int_0^\infty \frac{d\alpha}{\Gamma(p)} \alpha^{p-1} \frac{e^{-\alpha}}{(1-e^{-\alpha})^2} (1 - e^{-\alpha h}) < 2J_0 \tag{3.21}$$

and this leads to a contradiction. This proves Theorem 1 for $p > 2$ and $J \geq J_0$.

If on the contrary $J < J_0$, a repetition of the proof in Appendix A leads to the bound $h_i \leq K_2'/(J_0 - J)^{1/(p-2)}$ on the sizes h_i of the blocks in the ground state. Then we can repeat the proof above to get the desired result (1.2) and this concludes the proof of Theorem 1. ■

D. Uniqueness of the ground state

In this subsection we prove Theorem 2. If $p > 2$ and $J \geq J_0$ the statement is a corollary of the proof above: in fact the contradiction obtained after (3.21) shows that in this case the ferromagnetic state is the unique ground state.

Let us then consider the cases $1 < p \leq 2$ or $p > 2$ and $J < J_0$, in which we have an apriori upper bound on the sizes of the blocks. Let us first consider the case in which the minimizer h^* of $e(h)$ is unique and let us choose $2N = Mh^*$. Let $\underline{h}^0$ be a configuration in Λ_N for which the set $\mathcal{I} = \{i : h_i^0 \neq h^*\}$ is non empty. Note that the number of blocks M^0 in $\underline{h}^0$ is not necessarily equal to M. Using again the chessboard estimate (see Theorem 4.1 of [18]), the energy $E_N(M^0, \underline{h}^0)$ of the configuration $\underline{h}^0$ can be bounded below as:

$$E_N(M^0, \underline{h}^0) \geq \frac{1}{M^0} \sum_{i=-\frac{M^0}{2}+1}^{\frac{M^0}{2}} E_N(M^0, (h_i^0, \ldots, h_i^0)) \geq \sum_{i=-\frac{M^0}{2}+1}^{\frac{M^0}{2}} (h_i^0 e(h_i^0) - \frac{K}{N^p}) \quad (3.22)$$

Using (3.20) we find that

$$E_N(M^0, \underline{h}^0) - 2Ne_0(N) \geq \sum_{i \in \mathcal{I}} h_i^0 (e(h_i^0) - e(h^*)) + O(\frac{1}{N^{p-1}}) , \qquad (3.23)$$

so that, since $e(h_i^0) - e(h^*) \geq \Delta e$, we have that for N big enough the unique ground state for $2N = Mh^*$ is the state with M blocks all of sizes h^*. Similarly, if $2N$ is not divisible by h^*, any configuration with a sufficiently big number of "wrong" blocks (i.e. of blocks with sizes $\neq h^*$), will be separated by a gap from the ground state.

Finally, let us consider the case in which there are two minimizers $h^*, h^* + 1$. The same proof as above goes on to show that, if N is divisible either by h^* or by $h^* + 1$, then the ground state energy is given by (3.20) and the energy of configurations with blocks of "wrong" sizes (i.e. different both from h^* and $h^* + 1$) is asymptotically larger than the energy of the ground state. Similarly, if N is not divisible either by h^* or by $h^* + 1$, the energy of configurations with a sufficiently big number of blocks with wrong sizes will be larger than the ground state energy.

We are left with considering the case of a configuration with blocks' sizes all equal either to h^* or to $h^* + 1$. Let us denote by $\underline{\sigma}(h^*)$ the $2h^*$–periodic configuration with all blocks of size h^* and by $\underline{\sigma}(h^* + 1)$ the $2(h^* + 1)$–periodic configuration with all blocks of size h^*. Can a configuration with a finite fraction of blocks of sizes h^* and $h^* + 1$ be a ground state?

Let us consider a configuration $\underline{h}^0$ with $h_i^0 \in \{h^*, h^* + 1\}$ and the total number of blocks equal to M in a volume $2N$, with N divisible by h^* or $h^* + 1$ and M divisible by 4 (this condition on M is not restrictive: if M were not divisible by 4 we could consider the configuration obtained by doubling the system). Let

$$\mathcal{I}_0 \stackrel{def}{=} \{i : h_i^0 = h_{i+1}^0 = h^*\} \;, \quad \mathcal{I}_1 \stackrel{def}{=} \{i : h_i^0 = h_{i+1}^0 = h^* + 1\} \;, \quad \mathcal{I}_2 \stackrel{def}{=} \{i : h_i^0 \neq h_{i+1}^0\} \;, \tag{3.24}$$

where of course $h^0_{\frac{M}{2}+1} = h^0_{-\frac{M}{2}+1}$. Note that $|\mathcal{I}_0|+|\mathcal{I}_2|/2 = M_0$, where M_0 is the number of blocks of size h^*, and $|\mathcal{I}_1| + |\mathcal{I}_2|/2 = M_1$, where M_1 is the number of blocks of size $h^* + 1$.

Using again the chessboard inequality, we find that

$$E_N(M, \underline{h}^0) \geq \frac{1}{M} \sum_{i=-\frac{M}{2}+1}^{\frac{M}{2}} E_N(h_i, h_{i+1}, h_{i+1}, h_i, \ldots, h_i, h_{i+1}, h_{i+1}, h_i) \tag{3.25}$$

Now, except for an error of $O(M^{-(p-1)})$, the r.h.s. of (3.25) can be rewritten as $|\mathcal{I}_0|h^*\widetilde{e}_0 + |\mathcal{I}_1|(h^*+1)\widetilde{e}_0 + |\mathcal{I}_2|(h^* + \frac{1}{2})\widetilde{e}$, where $\widetilde{e}$ is the (infinite volume) specific energy of the $(4h^*+2)$–periodic configuration obtained by repeating periodically the configuration (h^*, h^*+1, h^*+1, h^*) over the volume. The key remark is that $\widetilde{e} - \widetilde{e}_0 = \delta > 0$: this is proven in Appendix C. Then, using the fact that $|\mathcal{I}_0| + |\mathcal{I}_2|/2 = M_0$ (with M_0 the number of blocks of size h^*) and that $|\mathcal{I}_1| + |\mathcal{I}_2|/2 = M_1$ (with M_1 the number of blocks of size $h^* + 1$), we get

$$E_N(M, \underline{h}^0) - 2Ne_0(N) \geq |\mathcal{I}_2|(h^* + \frac{1}{2})(\widetilde{e} - \widetilde{e}_0) \tag{3.26}$$

and the proof of Theorem 2 is concluded. ∎

IV. HIGHER DIMENSIONS

In this section we want to prove Theorem 3. As already remarked after Theorem 3, for $d = 2$ and $p = 3$ the upper bound was obtained in [12] by minimizing over the periodic striped configurations (and checking that asymptotically for large J such an upper bound is better than the one obtained by minimizing over the periodic checkerboard configurations). The upper bound in dimension higher than 2 and for $d < p < d + 1$ can be again obtained by a variational computation, minimizing the energy over the periodic striped configurations. We do not repeat the details here; the result is provided by the upper bounds in (1.3) and (1.4).

We now focus on the lower bounds in (1.3) and (1.4). We need to introduce some definitions; in particular via the basic Peierls construction we introduce the definitions of *contours* and *droplets*. Given any possible configuration $\underline{\sigma}$ on Λ_N, and in particular the ground state configuration(s), we define Δ to be the set of sites at which $\sigma_i = -1$, i.e. $\Delta = \{i \in \Lambda : \sigma_i = -1\}$. We draw around each $i \in \Delta$ the $2d$ sides of the unit $(d-1)$ dimensional cube centered at i and suppress the faces which occur twice: we obtain in this way a *closed polyhedron* $\Gamma(\Delta)$ which can be thought as the boundary of Δ. Each face of $\Gamma(\Delta)$ separates a point $i \in \Delta$ from a point $j \notin \Delta$. Along a $d-2$ dimensional edge of $\Gamma(\Delta)$ there can be either 2 or 4 faces meeting. In the case of 4 faces, we deform slightly the polyhedron, "chopping off" the edge from the cubes containing a $-$ spin. When this is done $\Gamma(\Delta)$ splits into disconnected polyhedra $\gamma_1, \ldots, \gamma_r$ which we shall call *contours*. Note that, because of the choice of periodic boundary conditions, all contours are closed but can possibly wind around the torus Λ_N. The definition of contours naturally induces a notion of connectedness for the spins in Δ: given $i, j \in \Delta$ we shall say that i and j are connected iff there exists a sequence $(i = i_0, i_1, \ldots, i_n = j)$ such that i_m, i_{m+1}, $m = 0, \ldots, n-1$, are nearest neighbors and none of the bonds (i_m, i_{m+1}) crosses $\Gamma(\Delta)$. The maximal connected components δ_i of Δ will be called *droplets* and the set of droplets of Δ will be denoted by $\mathcal{D}(\Delta) = \{\delta_1, \ldots, \delta_s\}$. Note that the boundaries $\Gamma(\delta_i)$ of the droplets $\delta_i \in \mathcal{D}(\Delta)$ are all distinct subsets of $\Gamma(\Delta)$ with the property: $\cup_{i=1}^s \Gamma(\delta_i) = \Gamma(\Delta)$.

Given the definitions above, let us rewrite the energy in (1.1) as

$$H_N(\underline{\sigma}) = |\Lambda_N| e_{FM}(J) + 2J \sum_{\gamma \in \Gamma(\Delta)} |\gamma| - \sum_{\delta \in \mathcal{D}(\Delta)} E_{dip}(\delta) \, , \tag{4.27}$$

where $e_{FM}(J)$ is the energy per site of the ferromagnetic configuration $\sigma_i \equiv +1$ and $E_{dip}(\delta) \stackrel{def}{=} 2 \sum_{i \in \delta} \sum_{j \in \Delta^c} J_p(i-j)$. Let us arbitarily choose a number $\ell \geq 1$ (to be conveniently fixed below) and let us correspondingly rewrite $E_{dip}(\delta)$ as

$$E_{dip}(\delta) = 2 \sum_{i \in \delta} \sum_{j \in \Delta^c} \mathbb{1}(|i-j| \leq \ell) J_p(i-j) + 2 \sum_{i \in \delta} \sum_{j \in \Delta^c} \mathbb{1}(|i-j| > \ell) J_p(i-j) \tag{4.28}$$

Denoting the last term by $E^{>\ell}(\delta)$, we note that $\sum_{\delta \in \mathcal{D}(\Delta)} E^{>\ell}(\delta)$ can be bounded above by $2 \min\{|\Delta|, |\Delta^c|\} \Phi_p(\ell) \leq |\Lambda| \Phi_p(\ell)$, where $\Phi_p(\ell) = \sum_{|n|>\ell} |n|^{-p} \leq const. \ell^{-(p-d)}$. For large ℓ the constant is smaller than $2d|\mathbb{S}_d|/(p-d)$ where $|\mathbb{S}_d|$ is the volume of the d–dimensional unit sphere. The first term in the r.h.s. of (4.28) can be rewritten as

$$E^{\leq \ell}(\delta) = 2 \sum_{|n| \leq \ell} J_p(n) \sum_{i \in \delta} \sum_{j \in \Delta^c} \mathbb{1}(i-j=n) \leq 2 \sum_{|n| \leq \ell} \frac{1}{|n|^p} \sum_{i \in \delta} \sum_{j \in \mathbb{Z}^d \setminus \delta} \mathbb{1}(i-j=n) \tag{4.29}$$

where in the last inequality we neglected an error term vanishing in the thermodynamic limit.

Now, the number of ways in which $n = (n_1, \ldots, n_d)$ may occur as the difference $i-j$ or $j-i$ with $i \in \delta$ and $j \notin \delta$ is at most $\sum_{\gamma\in\Gamma(\delta)} \sum_{i=1}^d |\gamma|_i |n_i|$, where $|\gamma|_i$ is the number of faces in γ orthogonal to the i–th coordinate direction. To prove this, draw a path on the lattice of length $|n_1| + \cdots + |n_d|$ connecting i and j. Such a path must cross a face of γ and if this face is orthogonal to the i–th coordinate axis, can do so only in $|n_i|$ ways and the claim follows.

The conclusion is that

$$E^{\leq\ell}(\delta) \leq \sum_{\gamma\in\Gamma(\delta)} \sum_{|n|\leq\ell} \frac{1}{|n|^p} \sum_{i=1}^{2} |\gamma|_i |n_i| = \sum_{\gamma\in\Gamma(\delta)} |\gamma| \sum_{|n|\leq\ell} \frac{|n_1|}{|n|^p} \,. \tag{4.30}$$

The sum $\sum_{|n|\leq\ell} |n_1|/|n|^p$ can be bounded above by $\int_{1\leq|x|\leq\ell} d^d x |x_1|/|x|^p + const.$ where the constant is smaller than $2pd|\mathbb{S}_d|2^{p-d}/(p-d)$ and $|\mathbb{S}_d|$ is the volume of the d–dimensional unit sphere. Then (4.30) implies $E^{\leq\ell}(\delta) \leq 2\Psi_p(\ell) + 2pd|\mathbb{S}_d|2^{p-d}/(p-d)$, where $\Psi_p(\ell) = |\mathbb{S}_{d-1}| \log \ell$ if $p = d+1$ and $\Psi_p(\ell) = |\mathbb{S}_{d-1}|(\ell^{d+1-p} - 1)/(d+1-p)$ if $d < p < d+1$.

Putting these bounds back into (4.27), we find:

$$E_0(N) \geq |\Lambda_N| e_{FM} + 2\left(J - \frac{pd|\mathbb{S}_d|2^{p-d}}{p-d} - \Psi_p(\ell)\right) \sum_{\gamma\in\Gamma(\Delta)} |\gamma| - \frac{2d|\mathbb{S}_d|}{p-d} \frac{|\Lambda_N|}{\ell^{p-d}} \,, \tag{4.31}$$

where Δ is the set of $-$ sites in the (unknown) ground state configuration. Choosing ℓ in such a way that $\Psi_p(\ell) = J - pd|\mathbb{S}_d|2^{p-d}/(p-d)$, we find the lower bounds in (1.3) and (1.4). This concludes the proof of Theorem 3. ■

As a side remark, let us note that a discussion similar to the one above implies that if $p > d+1$ and J is large enough no droplet can appear in the ground state, i.e. the ground state is ferromagnetic. In fact, let us assume by contradiction that the ground state is not $\sigma_i \equiv -1$ and that there is at least one droplet δ in the ground state. Then the energy needed to reverse all the spins in δ must be positive: $-2J\sum_{\gamma\in\Gamma(\delta)} |\gamma| + E_{dip}(\delta) \geq 0$. Proceeding as above in the proof of (4.30) and using that $p > d+1$, we get $E_{dip}(\delta) \leq \sum_{\gamma\in\Gamma(\delta)} |\gamma| \sum_{|n|\geq 1} \frac{|n_1|}{|n|^p} \leq const. \sum_{\gamma\in\Gamma(\delta)} |\gamma|$; hence for J big enough we get a contradiction and this proves that the ground state is ferromagnetic.

The contour method implemented in the proof of Theorem 3 also allows to get some informations about the geometry of contours and droplets in the ground state for $d<p\le d+1$. In fact if $\mathcal{D}(\Delta)$ is the set of droplets in the ground state and we consider a configuration $\underline{\sigma}'$ on $\{\pm 1\}^\Lambda$ with Δ' s.t. $\mathcal{D}(\Delta')=\mathcal{D}(\Delta)\setminus\{\delta\}$, for some δ, then it must be $H_N(\underline{\sigma}')-E_0(N)\ge 0$. With the definitions introduced above we have

$$\begin{aligned} H_N(\underline{\sigma}')-E_0(N) \quad &\le -2J\sum_{\gamma\in\Gamma(\delta)}|\gamma|+E_{dip}(\delta)\le \\ &\le 2\left(-J+\frac{pd|\mathbb{S}_d|2^{p-d}}{p-d}+\Psi_p(\ell)\right)\sum_{\gamma\in\Gamma(\delta)}|\gamma|+\frac{2d|\mathbb{S}_d|}{p-d}\frac{|\delta|}{\ell^{p-d}} \end{aligned} \tag{4.32}$$

Imposing that the r.h.s. of (4.32) is positive and choosing $\Psi_p(\ell)+\frac{pd|\mathbb{S}_d|2^{p-d}}{p-d}-J=-1$, we find $\sum_{\gamma\in\Gamma(\delta)}|\gamma|\le const.|\delta|e^{-J/|\mathbb{S}_{d-1}|}$ if $p=d+1$ and $\sum_{\gamma\in\Gamma(\delta)}|\gamma|\le const.|\delta|J^{-\frac{p-d}{d+1-p}}$ if $d<p<d+1$. If δ does not wind up Λ_N, using the isoperimetric inequality $|\delta|\le(\sum_{\gamma\in\Gamma(\delta)}|\gamma|/2d)^d$ we also find that in the ground state $\sum_{\gamma\in\Gamma(\delta)}|\gamma|\ge const.e^{J/|\mathbb{S}_{d-1}|}$ if $p=d+1$ or $\sum_{\gamma\in\Gamma(\delta)}|\gamma|\ge const.J^{\frac{p-d}{d+1-p}}$ if $d<p\le d+1$.

Finally we mention that exploiting the same energy arguments above, one can prove that the ground state of Hamiltonian (1.1) with Λ_N an $N^{d-1}\times w$ cylinder with periodic boundary conditions is the ferromagnetic ground state, *provided* $w\le K_3'e^{J/|\mathbb{S}_{d-1}|}$, if $p=d+1$, or $w\le K_p'J^{\frac{p-d}{d+1-p}}$ if $d<p\le d+1$, for a suitable constant K_p'. This result strongly suggests that for $d<p\le d+1$ the droplets in the ground state are quasi–$(d-1)$ dimensional structures of width $O(e^{J/|\mathbb{S}_{d-1}|})$ or $O(J^{\frac{p-d}{d+1-p}})$, reminiscent of the conjectured stripes. If we could prove that the droplets in the ground state must necessarily be stripes, then we could exploit the methods in Section.III to prove that the ground state is indeed realized by a periodic striped configuration.

V. CONCLUDING REMARKS

A. One dimension.

Ground state. In 1D the characterization of the infinite system ground state is complete: the ground state is periodic and there is a sequence of transitions from an antiferromagnetic state of period two to $2h$–periodic states of blocks of sizes h, $h>1$, with alternating sign, the size h changing (increasing discontinuously) when the ferromagnetic interaction strength J is increased.

Our proof is based on a reflection positivity argument which relies heavily on the details of the model Hamiltonian, e.g. on the fact that the long–range repulsion is reflection positive. It is an interesting open problem to establish more general conditions the long range repulsive interaction should satisfy in order to guarantee existence of a periodic modulated ground state. Note that the problem of determining the ground state of a spin system with positive and convex potential was solved by Hubbard and by Pokrovsky and Uimin (even in presence of a magnetic field) [19, 20]: this means that for $J = 0$ the assumption of convexity of the potential is enough to determine the ground state of (1.1). The proof in [19, 20] was generalized to an "almost" convex case by Jedrzejewski and Miekisz [21, 22]. However this proof requires a small ferromagnetic interaction $1 \leq J \leq 1 + 2^{-p}$. It is an open problem to generalize our analysis (in the presence of a large ferromagnetic coupling) to the case of a long range convex interaction.

It would also be interesting to establish what would be the effect of adding a magnetic field to the model Hamiltonian. It is expected that in the presence of a large short range ferromagnetic coupling and of a positive magnetic field B of increasing strength the ground state is still periodic with the $+$ blocks larger than the $-$'s, at least if $|B|$ is not too large [16]. However for large magnetic field it could be the case that small variations of the magnetic field could induce an infinite sequence of transitions between periodic states characterized by different rational values of the magnetization (Devil's staircase): this is in fact what happens for $J = 0$ as a function of the magnetic field [28]. According to [28] the antiferromagnetic state would remain the ground state for $|B| < B_c$ for $J = 0$. For larger values of B there is a sequence of transitions from the antiferromagnetic state of period two to (complicated) periodic states with all possible rational values of the magnetization.

Positive temperature. Another interesting open problem in one dimension consists in establishing properties of the infinite volume Gibbs measures at positive temperatures. Note that for a *ferromagnetic* long range interaction with decay $1/r^p$, $1 < p \leq 2$, it is known [29, 30] that there is a phase transition for the inverse temperature β large enough. In the case of model (1.1), on the contrary, it is natural to expect a unique Gibbs measure for any finite β, even for $1 < p \leq 2$. It is in fact known that if $J = 0$ in (1.1), then for $d = 1$ and $p > 1$ there is a unique limit Gibbs state at all values of the inverse temperature β [23, 31, 32]. The proof in [23] does not extend to the case

$J > 0$. It is known however that for $J > 0$ all Gibbs states are translation invariant [31]. Similarly, using the argument in [33], one can show that the Gibbs states obtained as limits of finite volume Gibbs measures with boundary conditions $\tau^k \underline{\sigma}_{per}(h^*(J))$, where $\underline{\sigma}_{per}(h^*(J))$ is a periodic ground state configuration with blocks of size $h^*(J)$ and τ the translation operator, are all equivalent among each other, for any $k = 1, \ldots, h^*(J)$. If, on the contrary, $0 < p \leq 1$ and, say, $J = 0$, it has been conjectured [34] that model (1.1), modified as explained after (1.1), admits at least two different Gibbs states obtained as limits of finite volume Gibbs measures with boundary conditions $\cdots + - + - \cdots$ and $\cdots - + - + \cdots$ (which are presumably well defined because the Hamiltonian is thermodynamically stable, see Remark (1) after Theorem 2).

B. Higher dimensions.

Ground state. In two or more dimensions we have shown that the specific ground state energy agrees asymptotically for large J with the best variational ground state known so far, which is a periodic striped configuration [12].

As already mentioned in the introduction, the ground states are believed, on the basis of variational computations and MonteCarlo simulations, to consist of a sequence of "stripes" of size h (growing when the ferromagnetic interaction strength J is increased) and alternating signs.

If J is sufficiently large, the addition of a magnetic field is believed to lead to a thickening of the stripes with spins parallel to the external field. At very large fields a transition from a striped to a "bubble" phase is expected [16]: the bubble phase consisting of large "cilindrical" droplets of spins parallel to the external field arranged in a periodic fashion and surrounded by a "sea" of spins of opposite sign.

Positive temperature. The non trivial structure of the set of ground states is believed to have a counterpart at positive temperature: in particular in the absence of magnetic field it is believed that at low temperature there are different pure Gibbs states describing striped states with two possible orientations (horizontal and vertical). The striped states are expected to "melt" at a positive critical temperature [6] with the "stripe melting" described in terms of an effective Landau–Ginzburg free energy functional [2].

It goes without saying that it would be of great interest to substantiate these pictures by rigorous proofs. Since most standard methods (cluster expansion, correlation inequalities) seem to fail in giving any useful information for systems of spins with long range antiferromagnetic interactions, new ideas are needed to understand some of the aforementioned problems.

APPENDIX A: A PRIORI BOUNDS ON THE SIZE OF THE BLOCKS

In this appendix we prove (3.13). Given a block B_{h_i} of size h_i, let us assume that $h_i = (2K+1)h$, for two integers K and h to be chosen conveniently (it will be clear from the proof below that this assumption is not restrictive). To be definite let us assume that the block B_{h_i} of size h_i we are considering is a + block. Since $h_i = (2K+1)h$, we can think of B_{h_i} as a sequence of three contiguous blocks of sizes Kh, h and Kh respectively, to be called B^1_h, B^2_h, B^3_h. Let us now compute the energy needed to flip the block B^2_h and let us call it $\Delta E'(h)$. If we are in the ground state, then of course $\Delta E'(h) \geq 0$. Also, we have $\Delta E'(h) \leq 4J - 2E_1(h)$, where $E_1(h)$ is the dipole interaction energy between a + block B^2_h of size h and a configuration of spins in $\mathbb{Z} \setminus B^2_h$ such that the spins in B^1_h, B^3_h are all + and the spins in $\mathbb{Z} \setminus B_{h_i}$ are all $-$. $E_1(h)$ is readily computed as

$$
\begin{aligned}
E_1(h) \quad &= 2\sum_{i=1}^{h}\Big[\sum_{j=h+1}^{h+Kh} - \sum_{j\geq h+Kh+1}\Big]\frac{1}{(j-i)^p} = \\
&= \sum_{i=1}^{h}\Big[\sum_{j=h+1}^{h+Kh} - \sum_{j\geq h+Kh+1}\Big]\int_0^\infty \frac{d\alpha}{\Gamma(p)}\alpha^{p-1}e^{-\alpha(j-i)} = \\
&= 2\int_0^\infty \frac{d\alpha}{\Gamma(p)}\,\alpha^{p-1}\frac{e^{-\alpha}}{(1-e^{-\alpha})^2}(1-e^{-\alpha h})(1-2e^{-\alpha K h})
\end{aligned}
\tag{A.1}
$$

so that, using the condition $\Delta E'(h) \geq 0$, we find that in the ground state

$$
\int_0^\infty \frac{d\alpha}{\Gamma(p)}\,\alpha^{p-1}\frac{e^{-\alpha}}{(1-e^{-\alpha})^2}(1-e^{-\alpha h})(1-2e^{-\alpha K h}) \leq J \tag{A.2}
$$

Using now the fact that $\alpha e^{-\alpha/2} \leq 1 - e^{-\alpha} \leq \alpha$ for $\alpha \geq 0$, the l.h.s. of (A.2) can be bounded below by:

$$
\frac{1}{\Gamma(p)}\int_0^\infty \frac{d\alpha}{\alpha}\frac{1}{\alpha^{2-p}}e^{-\alpha}(1-e^{-\alpha h})(1-2e^{-\alpha(Kh-1)}) \tag{A.3}
$$

Now, if $p = 2$ (A.3) can be computed to give $\log(h+1)/(1+1/K)^2$. Then, choosing $K = 1$ and plugging these bounds back into (A.2), we find that, if $p = 2$, $\log h \leq J + 2\log 2$ and $h_i = 3h \leq 12e^J$.

If $1 < p < 2$, (A.3) can be further bounded below as

$$
\begin{aligned}
&\frac{1}{\Gamma(p)}\int_0^\infty \frac{d\alpha}{\alpha}\frac{1}{\alpha^{2-p}}\Big\{e^{-\alpha h}(1-e^{-\alpha h})^2 - e^{-\alpha K h}(1-e^{-\alpha h})\Big\} \geq \\
&\geq \frac{h^{2-p}}{\Gamma(p)}\int_0^\infty \frac{d\alpha}{\alpha}\frac{1}{\alpha^{2-p}}\Big\{e^{-\alpha}(1-e^{-\alpha})^2 - \alpha e^{-\alpha K}\Big\} \equiv (A' - \frac{B'}{K^{p-1}})h^{2-p}
\end{aligned}
\tag{A.4}
$$

Choosing K in such a way that $B'/K^{p-1} \leq A'/2$ and plugging the bounds into (A.2), we find that $A'h^{2-p} \leq 2J$, so that $h_i = (2K+1)h \leq (2K+1)(2J/A')^{1/(2-p)}$.

APPENDIX B: THE INTEGRAL REPRESENTATION

In this Appendix we want to give the details of the computation leading to (3.7). First of all let us note that the first two terms in the r.h.s. of (3.7) come from the short range FM interaction. In order to show that the computation of the dipole interaction energy leads to the remaining terms appearing in the r.h.s. of (3.7), let us first note that the infinite volume dipole interaction energy E_{12} between two blocks of signs ε_1 and ε_2, $\varepsilon_\alpha = \pm 1$, of sizes h_1 and h_2 and separated by a string of d spins, can be computed as follows:

$$\begin{aligned} E_{12} \quad &= \varepsilon_1\varepsilon_2 \sum_{i=1}^{h_1}\sum_{j=1}^{h_2} \frac{1}{(j-i+h_1+d)^p} = \varepsilon_1\varepsilon_2 \int_0^\infty \frac{d\alpha}{\Gamma(p)}\, \alpha^{p-1} e^{-\alpha(j-i+h_1+d)} = \\ &= \varepsilon_1\varepsilon_2 \int_0^\infty \frac{d\alpha}{\Gamma(p)}\, \alpha^{p-1} \frac{e^{-\alpha}}{(1-e^{-\alpha})^2}(1-e^{-\alpha h_1})e^{-\alpha d}(1-e^{-\alpha h_2}) \end{aligned} \tag{B.1}$$

Now, in order to compute the finite volume dipole interaction energy of two different blocks h_i and h_j, $j > i$, we can simply apply the definition and (B.1) to find:

$$\begin{aligned} &\varepsilon_i\varepsilon_j \sum_{n\in\mathbb{Z}}\sum_{l=1}^{h_i}\sum_{m=1}^{h_j} \frac{1}{|m-l+h_i+\cdots+h_{j-1}+2nN|^p} = \qquad\qquad (B.2) \\ &= \varepsilon_i\varepsilon_j \int_0^\infty \frac{d\alpha}{\Gamma(p)}\, \alpha^{p-1} \frac{e^{-\alpha}}{(1-e^{-\alpha})^2}(1-e^{-\alpha h_i})(1-e^{-\alpha h_j})\Big[\prod_{i<k<j} e^{-\alpha h_k}\Big] \sum_{n\ge 0} e^{-2\alpha n N} + \\ &+\varepsilon_i\varepsilon_j \int_0^\infty \frac{d\alpha}{\Gamma(p)}\, \alpha^{p-1} \frac{e^{-\alpha}}{(1-e^{-\alpha})^2}(1-e^{-\alpha h_i})(1-e^{-\alpha h_j})e^{-\alpha(2N-h_i-\cdots-h_j)} \sum_{n\ge 0} e^{-2\alpha n N}\ , \end{aligned}$$

which leads to the term in the third line of (3.7) (we used that $\varepsilon_i\varepsilon_j = (-1)^{j-i}$ and $\sum_{i=1}^M h_i = 2N$). Similarly, the finite volume dipole self–interaction of a block h_i can be computed as

$$\begin{aligned} &\sum_{k=1}^{h_i-1} \frac{h_i-k}{k^p} + 2\sum_{n\ge 1}\sum_{i,j=1}^{h_i} \frac{1}{(j-i+2nN)^p} = \int_0^\infty \frac{d\alpha}{\Gamma(p)}\, \alpha^{p-1} \frac{e^{-\alpha}}{(1-e^{-\alpha})^2}\cdot \qquad (B.3) \\ &\cdot\Big[h_i(1-e^{-\alpha}) - (1-e^{-\alpha h_i}) + 2e^{\alpha h_i}(1-e^{-\alpha h_i})^2 \frac{e^{-2\alpha N}}{1-e^{-2\alpha N}}\Big] \end{aligned}$$

Summing (B.3) over $i = 1, \ldots, M$, we get the last term in the first line of (3.7) and the term in the second line.

APPENDIX C: COMPUTATION OF THE 1D ENERGY GAP

In this Appendix we want to compute the specific energy $\widetilde{e}$ of the $(4h^*+2)$–periodic configuration $(h^*, h^*+1, h^*+1, h^*, \ldots)$ in the case in which h^* and h^*+1 are both minimizers of $e(h)$. In particular we will show that $\widetilde{e}-\widetilde{e}_0=\delta>0$. Using the general expression (3.7), after some algebra we find that

$$\begin{aligned}\widetilde{e} &= -J+A+\frac{4J}{2h^*+1}-\frac{2}{2h^*+1}\int_0^\infty d\alpha\,\nu(\alpha)\left\{\frac{2(1-e^{-\alpha h^*})(1-e^{-\alpha(h^*+1)})}{1-e^{-\alpha(4h^*+2)}}+\right.\\ &\left.+\frac{e^{-2\alpha h^*}(1-e^{-\alpha(h^*+1)})^2+e^{-2\alpha(h^*+1)}(1-e^{-\alpha h^*})^2}{1-e^{-\alpha(4h^*+2)}}\right\} \qquad (C.1)\end{aligned}$$

where $\nu(\alpha)\equiv(\Gamma(p))^{-1}\alpha^{p-1}e^{-\alpha}(1-e^{-\alpha})^{-2}$. We want to prove that this expression is strictly larger than $\widetilde{e}_0$. Note that, since both h^* and h^*+1 are minimizers, we have

$$\frac{J}{h^*}-\frac{1}{h^*}\int_0^\infty d\alpha\,\nu(\alpha)\tanh\frac{\alpha h^*}{2}=\frac{J}{h^*+1}-\frac{1}{h^*+1}\int_0^\infty d\alpha\,\nu(\alpha)\tanh\frac{\alpha(h^*+1)}{2} \qquad (C.2)$$

implying that

$$J=\int_0^\infty d\alpha\,\nu(\alpha)\Big[(h^*+1)\tanh\frac{\alpha h^*}{2}-h^*\tanh\frac{\alpha(h^*+1)}{2}\Big] \qquad (C.3)$$

and

$$\widetilde{e}_0=2\int_0^\infty d\alpha\,\nu(\alpha)\Big(\tanh\frac{\alpha h^*}{2}-\tanh\frac{\alpha(h^*+1)}{2}\Big) \qquad (C.4)$$

Using (C.3) and (C.4) we find

$$\begin{aligned}\widetilde{e}-\widetilde{e}_0 &= 2\int_0^\infty d\alpha\,\nu(\alpha)\left[\tanh\frac{\alpha h^*}{2}+\tanh\frac{\alpha(h^*+1)}{2}-\right. \qquad (C.5)\\ &\left.-\frac{2(1-e^{-\alpha h^*})(1-e^{-\alpha(h^*+1)})+e^{-2\alpha h^*}(1-e^{-\alpha(h^*+1)})^2+e^{-2\alpha(h^*+1)}(1-e^{-\alpha h^*})^2}{1-e^{-\alpha(4h^*+2)}}\right]\end{aligned}$$

A bit more of algebra shows that (C.5) can be rewritten as

$$\widetilde{e}-\widetilde{e}_0=2\int_0^\infty d\alpha\,\nu(\alpha)(e^{-\alpha h^*}-e^{-\alpha(h^*+1)})^2(1-e^{-\alpha h^*})(1-e^{-\alpha(h^*+1)}) \qquad (C.6)$$

and this concludes the proof.

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