Covering relations for multidimensional dynamical systems

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Abstract

We present a topological technique for analyzing dynamical systems with complex behavior, based on the general notion of covering relations. Our method can be used to study multidimensional dynamical systems with an arbitrary number of 'topologically' expanding directions.

1 Introduction

In the study of differentiable dynamical systems, Markov partitions represent a fundamental construction. A Markov partition consists of diffeomorphic copies of multidimensional rectangles that cross transversally one another under iteration. Such a construction can be exploited to obtain a coding for the orbits of the system, which is generally referred as symbolic dynamics.

It is however very difficult, in general, to rigorously establish the existence of a Markov partition when dealing with an explicitly given map or with a Poincaré map associated to an ordinary differential equation. Therefore, introducing a topological analogue of a Markov partition is of great importance. In such an attempt, one strives to subdivide a particular region of the phase space into blocks which cross one another in a topologically consistent way under iteration. What 'consistent' means really depends on context and objectives.

In the one-dimensional case, one can give a detailed account for 'consistency' by the means of covering relations, which can be described in the following straightforward fashion. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map, and let $I, J \subset \mathbb{R}$

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be closed intervals. We say that I f-covers J, and write $I \overset{f}{\to} J$, iff $J \subset f(I)$. Covering relations between intervals yield to valuable information on the existence of points with prescribed trajectories, and of periodic orbits. The following basic theorem illustrates this idea.

Theorem 1 *Let I_i, $i = 0, 1, \ldots, k$, be closed intervals, such that $I_k = I_0$. Let $f_i : I_{i-1} \to \mathbb{R}$ be continuous maps such that for every $i = 1, \ldots, k$, we have $I_{i-1} \overset{f_i}{\to} I_i$.*

Then there exists a point $x \in I_0$, such that

$$\begin{aligned} f_i \circ f_{i-1} \circ \cdots \circ f_1(x) &\in I_i, \quad i = 1, \ldots, k \\ f_k \circ f_{k-1} \circ \cdots \circ f_1(x) &= x. \end{aligned}$$

The proof of the above theorem is very simple, following directly from the Darboux property.

In this paper, we present a general notion of covering relations in multi-dimensional dynamical systems, as a practical method to analyze complicated dynamics. We develop the ideas from the papers [19, 20, 21], which allow us to generalize the notion of one-dimensional covering relations to the several dimensional case, in a way that an analogue of Theorem 1 holds true. These new tools turn out to be quite effective, as they were already used to prove several new results concerning certain phenomena reducible to low dimensional dynamics:

- stability (continuation) of Sharkovskii's ordering [20, 23] and estimates of topological entropy [14] for multidimensional perturbations of one-dimensional maps,
- computer assisted proofs of the existence of symbolic dynamics for the Hénon map [22], the Rössler equations [22], the Chua circuit [7], the Lorenz equations [8], the Hénon-Heiles hamiltonian [2], the planar restricted three body problem [1, 18].

In the above mentioned applications, the underlying dynamical systems exhibit so-called topological horseshoes - natural generalizations of Smale's horseshoe. The presence of these horseshoes can be inferred from modelling the dynamics by piecewise affine (linear) maps, which indicate hyperbolic-like expansions and contractions. In this paper we extend the notion of covering relation in order to include examples which cannot be modelled by hyperbolic maps.

We would like to make a few comments on the methods used in this paper. First, note that the definition of covering relation and the proof of Theorem 1 in the one-dimensional situation are very simple, but they cannot be directly transposed in several dimensions. Therefore, in the multidimensional situation, the condition on an interval being cover by the image of a second interval is replaced by a requirement that the image of a cube N under the map f is stretched across another cube M in a topologically nontrivial manner (see Definition 6). These cubes together with the corresponding choices of coordinate systems will be referred as h-sets (the letter h suggesting the hyperbolic-like directions). The correct crossing of the cubes under f will be referred as a covering relation. The use of the Darboux property in the proof of Theorem 1 will be replaced by the use of the local Brouwer degree.

We would like to make a few comments on connections with related work. Our approach is similar in spirit to Easton's method of windows [5], [6]. In a simple setting, a window in a manifold is a diffeomorphic copy of a multi-dimensional rectangle. A pair of windows are correctly aligned provided that each horizontal of one is transverse to each vertical of the other at a point that correspond to the interior of the rectangle. A key result in this direction says that, given a bi-infinite sequence of windows and connecting diffeomorphisms, such that the image of each window is correctly aligned with the next window under the connecting diffeomorphism, there is an orbit that runs through the windows in the prescribed order (see Corollary 12 for our version of this result). Easton's approach relies heavily on the differential structure and on transversality, so it usually requires tedious computations in applications. Our approach captures the idea of correct alignment in a much more efficient, topological way. We provide a criteria for alignment that requires only checking the behavior of certain distinguished components of the boundaries of the 'windows', and the existence of precisely one horizontal in the first 'window' whose image can be homotopically deformed onto a horizontal in the second 'window' by a map with nonzero local Brouwer degree (see Theorem 15).

There also exists a vast Conley index literature (see [16, 8, 12, 13] and the references given there) devoted to topological methods for detecting periodic orbits and chaos. Up to date, one of the strongest results in this direction has been provided by Srzednicki in [16] (see also [11]), based on the Lefschetz Fixed Point Theorem. Our results go beyond the ones in [16], since we additionally consider coverings induced by inverse maps (backcovering relations). These new type of relations turn out to be very useful in studying systems exhibiting time-reversal symmetry (see [1, 18]). Even in the case of direct coverings, which is considered in [16] (in a different, more abstract language), we have the advantage of providing a quite elegant geometric approach, which allows one to obtain deeper and stronger results (see [17]), when compared to the one based on the Lefschetz Fixed Point Theorem.

The content of paper is described as follows. In Section 2 we set up the notation that we use throughout the paper, we introduce a simple type of covering relation, and we prove an analogue of Theorem 1 for these relations. In Section 3 we extend the results form Section 2 to the 'multiple wrapped covering' case. In Section 4 we present some sufficient conditions that ensure the existence of covering relations. In Section 5 we briefly analyze a class of examples, different from topological horseshoes, to which our approach is applicable.

Finally, in the Appendix, for the convenience of the reader, we included basic properties of the local degree and the degree of mappings of the sphere, which makes our paper reasonably self-contained.

2 Covering relations, the simple case

Notation: For a given norm in $\mathbb{R}^n$, by $B_n(c,r)$ we denote the open ball of radius r centered at $c \in \mathbb{R}^n$. When the dimension n is obvious from the context, we will drop the subscript n. Let $S^n(c,r) = \partial B_{n+1}(c,r)$, by the symbol S^n we will denote $S^n(0,1)$. We set $\mathbb{R}^0 = \{0\}$, $B_0(0,r) = \{0\}$, $\partial B_0(0,r) = \emptyset$.

For a given set Z, by $\operatorname{int} Z$, $\overline{Z}$, ∂Z we denote the interior, the closure and the boundary of Z, respectively. For a map $h : [0,1] \times Z \to \mathbb{R}^n$, we set $h_t = h(t, \cdot)$. By Id we denote the identity map. For a map f, by $\operatorname{dom}(f)$ we will denote the domain of f. If $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map, we say that $X \subset \operatorname{dom}(f^{-1})$ iff the map $f^{-1} : X \to \mathbb{R}^n$ is well defined and continuous.

Definition 1 *An h-set is a quadruple consisting of*

- *a compact subset N of $\mathbb{R}^n$*
- *a pair of numbers $u(N), s(N) \in \{0,1,2,\dots\}$, with $u(N) + s(N) = n$*
- *a homeomorphism $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$, such that*

$$c_N(N) = \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1).$$

With an abuse of notation, we will denote such a quadruple by N. We denote

$$\begin{aligned} N_c &= \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1), \\ N_c^- &= \partial\overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1), \\ N_c^+ &= \overline{B_{u(N)}}(0,1) \times \partial\overline{B_{s(N)}}(0,1), \\ N^- &= c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+). \end{aligned}$$

Hence an h-set N is a product of two closed balls with respect to some coordinate system. The numbers $u(N)$ and $s(N)$ stand for the dimensions of nominally unstable and stable directions, respectively. The subscript c refers to the new coordinates given by homeomorphism c_N. Notice that if $u(N) = 0$, then $N^- = \emptyset$ and if $s(N) = 0$, then $N^+ = \emptyset$.

Definition 2 *Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : N \to \mathbb{R}^n$ be a continuous map. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$. We say that*

$$N \overset{f}{\Longrightarrow} M$$

(N f-covers M) iff the following conditions are satisfied

1. *There exists a continuous homotopy $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, such that the following conditions hold true*

$$\begin{aligned} h_0 &= f_c, \\ h([0,1], N_c^-) \cap M_c &= \emptyset, \\ h([0,1], N_c) \cap M_c^+ &= \emptyset. \end{aligned}$$

2. *There exists a linear map* $A : \mathbb{R}^u \to \mathbb{R}^u$, *such that*

$$\begin{aligned} h_1(p,q) &= (Ap,0), \quad \text{for } p \in \overline{B_u}(0,1) \text{ and } q \in \overline{B_s}(0,1), &(1)\\ A(\partial B_u(0,1)) &\subset \mathbb{R}^u \setminus \overline{B_u}(0,1). &(2)\end{aligned}$$

In the context of the above definition we will call the map h_1 *a model map* for the relation $N \stackrel{f}{\Longrightarrow} M$.

Remark 2 *When* $u > 0$, *then condition (2) is equivalent to each of the following conditions*

$$\begin{aligned} \overline{B_u}(0,1) &\subset A(B_u(0,1)), \\ \|Ap\| &> 1, \qquad \text{for } p \in \partial B_u(0,1), \\ \|Ap\| &> \|p\|, \qquad \text{for } p \neq 0. \end{aligned}$$

Remark 3 *When* $u = 0$, *then* $\mathbb{R}^u = \{0\}$ *and so* $A : \mathbb{R}^u \to \mathbb{R}^u$ *is given by* $A(0) = 0$. *Taking into account that* $\partial B_u(0,1) = \emptyset$, *we see that the second part of (2) is vacuously satisfied, and so condition (2) is equivalent to* $h_1(x) = 0$ *for all* x. *It is easy to see that, in this case,* $N \stackrel{f}{\Longrightarrow} M$ *iff* $f(N) \subset \operatorname{int} M$.

Definition 3 *Let* N *be an h-set. We define the h-set* N^T *as follows*

- *The compact subset of the quadruple* N^T *is the compact subset of the quadruple* N, *also denoted by* N,
- $u(N^T) = s(N)$, $s(N^T) = u(N)$
- *The homeomorphism* $c_{N^T} : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N^T)} \times \mathbb{R}^{s(N^T)}$ *is defined by*

  $$c_{N^T}(x) = j(c_N(x)),$$

  where $j : \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)} \to \mathbb{R}^{s(N)} \times \mathbb{R}^{u(N)}$ *is given by* $j(p,q) = (q,p)$.

Notice that $N^{T,+} = N^-$ and $N^{T,-} = N^+$. This operation is useful in the context of inverse maps, as it was first pointed out in [1].

Definition 4 *Assume* N, M *are h-sets, such that* $u(N) = u(M) = u$ *and* $s(N) = s(M) = s$. *Let* $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. *Assume that* $g^{-1} : M \to \mathbb{R}^n$ *is well defined and continuous. We say that* $N \stackrel{g}{\Longleftarrow} M$ *(*N *g-backcovers* M*) iff* $M^T \stackrel{g^{-1}}{\Longrightarrow} N^T$.

Following [1], let us point out that, although covering and backcovering occur often simultaneously, they are not equivalent, for example it can happen that the map f is not defined on N.

Theorem 4 *Let N_i, $i = 0,\dots,k$ be h-sets and $N_k = N_0$. Assume that for each $i = 1,\dots,k$ we have either*

$$N_{i-1} \overset{f_i}{\Longrightarrow} N_i, \tag{3}$$

or

$$N_i \subset dom\,(f_i^{-1}) \quad and \quad N_{i-1} \overset{f_i}{\Longleftarrow} N_i. \tag{4}$$

Then there exists a point $x \in \operatorname{int} N_0$, such that

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \in \operatorname{int} N_i, \quad i = 1,\dots,k, \tag{5}$$

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x. \tag{6}$$

Proof: Without any loss generality we can assume that

$$\begin{aligned} c_{N_i} &= \mathrm{Id}, \qquad \text{for } i = 0,\dots,k-1, \\ f_i &= f_{i,c}, \qquad \text{for } i = 1,\dots,k, \\ N_i &= N_{c,i}, \quad N_i^{\pm} = N_{i,c}^{\pm}. \end{aligned}$$

In order to simplify the exposition we set $N_{-1} = N_{k-1}$ and $N_k = N_0$ and accordingly $f_0 = f_k$. We also define $g_i = f_i^{-1}$, for those i for which we have the back-covering relation $N_{i-1} \overset{f_i}{\Longleftarrow} N_i$.

Notice that from the definition of covering relation, it follows immediately that there exist $u \geq 0$, $s \geq 0$, such that $u(N_i) = u$ and $s(N_i) = s$, for all $i = 0,\dots,k-1$.

The idea of the proof is to find a solution of the equation $x - (f_k \circ f_{k-1} \circ \cdots \circ f_1)(x) = 0$ of nonzero local Brouwer degree. Each mapping f_i corresponding to a direct covering is homotopic to some linear map, and each mapping f_i corresponding to a backcovering has its inverse g_i homotopic to some linear map. We will prove that an appropriate composition of these linear maps has a non-degenerate fixed point, and use the homotopy property of the local Brouwer degree to conclude that $f_k \circ f_{k-1} \circ \cdots \circ f_1$ has a fixed point.

As a tool for keeping track of the occurrences of coverings and backcoverings, we define the map $\delta : \{0,\dots,k\} \to \{0,1\}$ by $\delta(i) = 1$ if $N_{i-1} \overset{f_i}{\Longrightarrow} N_i$ and $\delta(i) = 0$ if $N_{i-1} \overset{f_i}{\Longleftarrow} N_i$. Let h_i be a homotopy map from the definition of covering relation for $N_{i-1} \overset{f_i}{\Longrightarrow} N_i$ or $N_{i-1} \overset{f_i}{\Longleftarrow} N_i$. In the case of a direct covering (i.e. $\delta(i) = 1$), the homotopy h_i satisfies

$$h_i(0,x) = f_i(x), \quad \text{where } x \in \mathbb{R}^{u+s}, \tag{7}$$

$$h_i(1,(p,q)) = (A_i p, 0), \quad \text{where } p \in \mathbb{R}^u \text{ and } q \in \mathbb{R}^s, \tag{8}$$

$$h_i([0,1], N_{i-1}^-) \cap N_i = \emptyset, \tag{9}$$

$$h_i([0,1], N_{i-1}) \cap N_i^+ = \emptyset. \tag{10}$$

In the case of a backcovering (i.e. $\delta(i)=0$), the homotopy h_i satisfies

$$
\begin{aligned}
h_i(0,x) &= g_i(x), \quad \text{where } x\in\mathbb{R}^{u+s}, && (11)\\
h_i(1,(p,q)) &= (0,A_i q), \quad \text{where } p\in\mathbb{R}^u \text{ and } q\in\mathbb{R}^s, && (12)\\
h_i([0,1],N_i^+)\cap N_{i-1} &= \emptyset, && (13)\\
h_i([0,1],N_i)\cap N_{i-1}^- &= \emptyset. && (14)
\end{aligned}
$$

It is enough to prove that there exists $x_i\in\operatorname{int}N_i$ for $i=0,\dots,k-1$ such that

$$
\begin{aligned}
&f_i(x_{i-1})=x_i, \quad \text{if } \delta(i)=1,\\
&g_i(x_i)=x_{i-1}, \quad \text{if } \delta(i)=0.
\end{aligned} \tag{15}
$$

We will treat (15) as a multidimensional system of equations to be solved. To this end, let us define

$$\Pi = N_0\times N_1\times\cdots\times N_{k-1}.$$

A point $x\in\Pi$ will be represented by $x=(x_0,x_1,\dots,x_{k-1})$. We set $x_{-1}=x_{k-1}$ and $x_k=x_0$.

We define a map $F=(F_0,F_1,\dots,F_{k-1}):\Pi\to\mathbb{R}^{(u+s)k}$ by

$$F_i(x_0,x_1,\dots,x_{k-1})=\begin{cases}x_i-f_i(x_{i-1}) & \text{if } \delta(i)=1,\\ x_{i-1}-g_i(x_i) & \text{if } \delta(i)=0.\end{cases}$$

With this notation, solving the system (15) is equivalent to solving the equation $F(x)=0$ in $\operatorname{int}\Pi$.

We define a homotopy $H=(H_0,\dots,H_{k-1}):[0,1]\times\Pi\to\mathbb{R}^{(u+s)k}$ by

$$H_i(\lambda,x_0,x_1,\dots,x_{k-1})=\begin{cases}x_i-h_i(\lambda,x_{i-1}) & \text{if } \delta(i)=1,\\ x_{i-1}-h_i(\lambda,x_i) & \text{if } \delta(i)=0.\end{cases}$$

Notice that $H(0,x)=F(x)$. The assertion of the theorem is a consequence of the following two lemmas, which will be proved after we complete the current proof.

Lemma 5 *For all $\lambda\in[0,1]$ the local Brouwer degree $\deg(H_\lambda,\operatorname{int}\Pi,0)$ is well defined and does not depend on λ. Namely, for all $\lambda\in[0,1]$ we have*

$$\deg(H_\lambda,\operatorname{int}\Pi,0)=\deg(H_1,\operatorname{int}\Pi,0).$$

Lemma 6

$$\deg(H_1,\operatorname{int}\Pi,0)=\pm1.$$

We continue the proof of Theorem 4. Since $F=H_0$, from the above lemmas it follows immediately that

$$\deg(F,\operatorname{int}\Pi,0)=\deg(H_0,\operatorname{int}\Pi,0)=\deg(H_1,\operatorname{int}\Pi,0)\neq 0.$$

Hence there exists $x \in \Pi$ such that $F(x) = 0$. ∎

Proof of Lemma 5: From the homotopy property (see Appendix) it is enough to prove that

$$H_\lambda(x) \neq 0, \quad \text{for all } x \in \partial\Pi \text{ and } \lambda \in [0,1]. \tag{16}$$

In order to prove (16), let us fix $x = (x_0, x_1, \ldots, x_{k-1}) \in \partial\Pi$. It is easy to see that the exists $i \in \{0, 1, \ldots k-1\}$, such that $x_i \in \partial N_i = N_i^+ \cup N_i^-$. Hence at least one of the following conditions hold true

$$x_i \in N_i^+, \tag{17}$$

$$x_i \in N_i^-. \tag{18}$$

For each of the two above cases, we have to consider the following four possibilities

$$N_{i-1} \overset{f_i}{\Longrightarrow} N_i \overset{f_{i+1}}{\Longrightarrow} N_{i+1}, \tag{19}$$

$$N_{i-1} \overset{f_i}{\Longrightarrow} N_i \overset{f_{i+1}}{\Longleftarrow} N_{i+1}, \tag{20}$$

$$N_{i-1} \overset{f_i}{\Longleftarrow} N_i \overset{f_{i+1}}{\Longrightarrow} N_{i+1}, \tag{21}$$

$$N_{i-1} \overset{f_i}{\Longleftarrow} N_i \overset{f_{i+1}}{\Longleftarrow} N_{i+1}. \tag{22}$$

Assume first that $x_i \in N_i^+$. If (19) or (20) holds true, then from (10) we obtain

$$h_i(t, x_{i-1}) \neq x_i,$$

for every $t \in [0,1]$ and every $x_{i-1} \in N_{i-1}$. If (21) or (22) is satisfied, then from (13) it results that

$$h_i(t, x_i) \neq x_{i-1},$$

for every $t \in [0,1]$ and every $x_{i-1} \in N_{i-1}$. This proves if $x_i \in N_i^+$, then $H_t(x) \neq 0$ for any $t \in [0,1]$.

Assume now that $x_i \in N_i^-$. If (19) or (21) holds true, then from (9) it follows that for every $t \in [0,1]$ and every $x_{i+1} \in N_{i+1}$ we have

$$h_{i+1}(t, x_i) \neq x_{i+1}.$$

If (20) or (22) is satisfied, then from (14) we obtain

$$h_{i+1}(t, x_{i+1}) \neq x_i,$$

for every $t \in [0,1]$ and every $x_{i+1} \in N_{i+1}$. This proves that if $x_i \in N_i^-$, then $H_t(x) \neq 0$ for any $t \in [0,1]$. ∎

Proof of Lemma 6: Let us represent x_i as a pair $x_i = (p_i, q_i)$, where $p_i \in \mathbb{R}^u$ and $q_i \in \mathbb{R}^s$. In this representation, the map $H_1(p_0, q_0, \ldots, p_{k-1}, q_{k-1}) = (\tilde{p}_0, \tilde{q}_0, \ldots, \tilde{p}_{k-1}, \tilde{q}_{k-1})$ has the following form

$$\tilde{p}_i = p_i - A_i p_{i-1}, \quad \text{if } \delta(i) = 1, \tag{23}$$

$$\tilde{q}_i = q_i, \quad \text{if } \delta(i) = 1, \tag{24}$$

$$\tilde{p}_i = p_{i-1}, \quad \text{if } \delta(i) = 0, \tag{25}$$

$$\tilde{q}_i = q_{i-1} - A_i q_i, \quad \text{if } \delta(i) = 0. \tag{26}$$

The map H_1 is linear. From (100) it follows that to prove that $\deg(H_1, \operatorname{int}\Pi, 0) = \pm 1$ it is enough to show that H_1 is an isomorphism.

Assume that $H_1(p_0, q_0, \ldots, p_{k-1}, q_{k-1}) = 0$. We have to show that for all $i = 0, \ldots, k-1$ $p_i = 0$ and $q_i = 0$. We will only show that $p_i = 0$ for all i, the proof for q_i being similar.

If $u = 0$, then there is nothing to prove, as all p_i equal to 0, by definition. Let $u > 0$. If for all $i = 0, \ldots, k-1$, we have $\delta(i) = 1$, then

$$p_0 = (A_k \circ A_{k-1} \circ \cdots \circ A_1)p_0$$

From Remark 2, it follows that $\|A_i(p)\| > \|p\|$ for all $p \neq 0$. Hence $p_0 = 0$ in this case, and then $p_i = 0$ for all i. If there exists j such that $\delta(j) = 0$, then $p_{j-1} = 0$ and an easy induction argument shows that $p_i = 0$ for $i = 0, \ldots, k-1$. ∎

It is important to remark here that the proof of Lemma 5 does not use property **2** from the definition of covering relations.

In the view of Theorem 4, it makes practical sense to make no distinction between the covering relations $N \stackrel{f}{\Longrightarrow} M$ and $N \stackrel{f}{\Longleftarrow} M$. Following [1], we introduce

Definition 5 *Let N and M be h-sets. We say that N* generically f-covers *M ($N_1 \stackrel{f}{\Longleftrightarrow} N_2$) if $N \stackrel{f}{\Longrightarrow} M$ or $N \stackrel{f}{\Longleftarrow} M$.*

We emphasize that the relation $N_1 \stackrel{f}{\Longleftrightarrow} N_2$ is not symmetric in general.

Collorary 7 *Assume that we have the following chain of covering relations*

$$N_0 \stackrel{f_1}{\Longleftrightarrow} N_1 \stackrel{f_2}{\Longleftrightarrow} N_2 \ldots \stackrel{f_k}{\Longleftrightarrow} N_k,$$

Then there exists a point $x \in \operatorname{int} N_0$, such that

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \in \operatorname{int} N_i, \quad i = 1, \ldots, k. \tag{27}$$

Moreover if $N_k = N_0$, then x can be chosen so that

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x. \tag{28}$$

Proof: The statement for a periodic loop ($N_0 = N_k$) follows directly from Theorem 4. The nonperiodic case will be reduced to the periodic one by adding a new covering relation to close the loop as follows.

Notice that from the definition of covering relation it follows immediately that $u(N_i) = u$ and $s(N_i) = s$ for some u, s and all $i = 0, \ldots, k-1$.

We can assume that for $c_{N_i} = Id$ for $i = 0, 1, \ldots, k$. It easy to find a map f_{k+1}, an affine map $A_{k+1} : \mathbb{R}^u \to \mathbb{R}^u$, and a homotopy h_{k+1} such that

$$N_k \stackrel{f_{k+1}}{\Longrightarrow} N_0, \tag{29}$$

where $f_{k+1}(p, q) = (A_{k+1}(p), 0)$ and $h_{k+1}(t, x) = f_{k+1}(x)$.

Now we have a closed loop of covering relations to which we can apply Theorem 4. This finishes the proof. ∎

3 Multiple wrapped covering relations

The goal of this section is to generalize the notion of covering relations introduced in Section 2. We will change condition **2** in the definition of covering relations in order to allow for more general maps at the end of homotopy h (this means that we allow for different *model maps*).

Definition 6 *Assume that N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : N \to \mathbb{R}^n$ be a continuous map. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$. Let w be a nonzero integer. We say that*

$$N \overset{f,w}{\Longrightarrow} M$$

(N f-covers M with degree w) iff the following conditions are satisfied

1. *there exists a continuous homotopy $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, such that the following conditions hold true*

$$\begin{aligned} h_0 &= f_c, & (30)\\ h([0,1], N_c^-) \cap M_c &= \emptyset, & (31)\\ h([0,1], N_c) \cap M_c^+ &= \emptyset. & (32) \end{aligned}$$

2. *There exists a map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that*

$$\begin{aligned} h_1(p,q) &= (A(p), 0), \text{ for } p \in \overline{B_u}(0,1) \text{ and } q \in \overline{B_s}(0,1), & (33)\\ A(\partial B_u(0,1)) &\subset \mathbb{R}^u \setminus \overline{B_u(0,1)}. & (34) \end{aligned}$$

Moreover, we require that

$$\deg(A, \overline{B_u}(0,1), 0) = w,$$

Note that in the case $u = 0$, an h-set N can cover an h-set M only with degree $w = 1$.

The previous definition of covering relation (Definition 1) is a particular case of the present one, with the degree w equal to $\operatorname{sgn}(\det(A))$ (for $u > 0$). See Figure 1 for an example of a multiple wrapped covering relation. As in Section 2, we will call the map h_1 *a model map* for the relation $N \overset{f,w}{\Longrightarrow} M$.

Remark 8 *In applications, we would like to decide whether two h-sets are correctly aligned based essentially on the information on their boundaries. Condition 1 from the above definition is stated in this spirit. In condition 2, we can express the local Brouwer degree of A as the winding number of $A(\partial B_u(0,1))$ about the origin. More precisely, in the case $u > 0$, since the map $A : \overline{B_n}(0,1) \to \mathbb{R}^n$ satisfies*

$$0 \notin A(\partial B_u(0,1)), \tag{35}$$

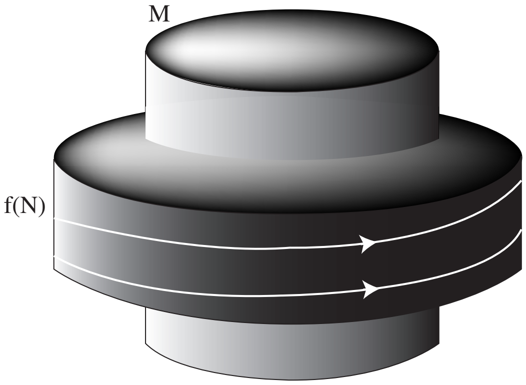

Figure 1: An illustration of a multiple wrapped covering relation in the plane. The arrows on the side of $f(N)$ merely suggest that a cross section disk in $f(N)$ wraps around twice the corresponding cross section disk in M

we can define a map $s_A : S^{u-1} \to S^{u-1}$ by

$$s_A(x) = \frac{A(x)}{\|A(x)\|}. \tag{36}$$

The degree $d(s_A)$ of a mapping of a sphere is defined in Appendix 6.2. By Lemma 23, we obtain $\deg(A, \overline{B_u}(0,1), 0) = d(s_A)$. Thus, the degree of a covering $N \overset{f,w}{\Longrightarrow} M$ can be computed as $w = d(s_A)$.

We define the corresponding notion of backcovering for this new type of covering relation.

Definition 7 *Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. Assume that $g^{-1} : |M| \to \mathbb{R}^n$ is a well defined, continuous map. We say that $N \overset{g,w}{\Longleftarrow} M$ (N g-backcovers M with degree w) iff $M^T \overset{g^{-1},w}{\Longrightarrow} N^T$.*

Theorem 9 *Let N_i, $i = 0, \dots, k$ be h-sets and $N_k = N_0$. Assume that for each $i = 1, \dots, k$ we have either*

$$N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i, \tag{37}$$

or

$$N_i \subset dom\,(f_i^{-1}) \quad and \quad N_{i-1} \overset{f_j,w_i}{\Longleftarrow} N_i. \tag{38}$$

Then there exists a point $x \in \operatorname{int} N_0$, such that

$$\begin{aligned} f_i \circ f_{i-1} \circ \cdots \circ f_1(x) &\in \operatorname{int} N_i, \quad i = 1, \dots, k, && (39)\\ f_k \circ f_{k-1} \circ \cdots \circ f_1(x) &= x. && (40)\end{aligned}$$

Proof: The proof of this theorem follows the same pattern as the proof of Theorem 4. We define Π, $\delta(i)$, F and H as it was done there. Lemma 5 is valid, with the same proof, because condition **2** in the definition of covering is not used in its proof.

Instead of Lemma 6 we will have the following

Lemma 10

$$|\deg(H_1, \operatorname{int}\Pi, 0)| = |w_1 \cdot w_2 \cdot \ldots w_k|.$$

We finish the proof by the same argument as in Theorem 4. ∎

3.1 Proof of Lemma 10.

Let us represent x_i as a pair $x_i = (p_i, q_i)$, where $p_i \in \mathbb{R}^u$ and $q_i \in \mathbb{R}^s$. As usual we set $(p_{-1}, q_{-1}) = (p_{k-1}, q_{k-1})$, $(p_k, q_k) = (p_0, q_0)$ and $A_k = A_0$. In this representation the map $H_1(p_0, q_0, \ldots, p_{k-1}, q_{k-1}) = (\tilde{p}_0, \tilde{q}_0, \ldots, \tilde{p}_{k-1}, \tilde{q}_{k-1})$ has the following form (for $\lambda = 0$)

$$\tilde{p}_i = (1-\lambda)p_i - A_i(p_{i-1}), \quad \text{if } \delta(i) = 1, \tag{41}$$

$$\tilde{q}_i = q_i, \quad \text{if } \delta(i) = 1, \tag{42}$$

$$\tilde{p}_i = p_{i-1}, \quad \text{if } \delta(i) = 0, \tag{43}$$

$$\tilde{q}_i = (1-\lambda)q_{i-1} - A_i(q_i), \quad \text{if } \delta(i) = 0. \tag{44}$$

The above equations define a homotopy $C : [0,1] \times \Pi \to \mathbb{R}^{(u+s)k}$. We will show that $\deg(C_\lambda, \operatorname{int}\Pi, 0)$ is independent of λ and then we compute the degree of C_1.

Lemma 11 *For any* $\lambda \in [0,1]$

$$\deg(C_\lambda, \operatorname{int}\Pi, 0) = \deg(C_1, \operatorname{int}\Pi, 0).$$

Proof: From the homotopy property of the local degree (see Appendix), it follows that it is enough to prove that

$$C_\lambda(x) \neq 0, \quad \text{for all } x \in \partial\Pi \text{ and } \lambda \in [0,1]. \tag{45}$$

Let us take $x = (p_0, q_0, p_1, q_1, \ldots, p_{k-1}, q_{k-1}) \in \partial\Pi$. There exists i such that one the following conditions holds true

$$p_i \in S^u, \tag{46}$$

$$q_i \in S^s. \tag{47}$$

Assume that $p_i \in S^u$. If $\delta(i+1) = 1$, then $\tilde{p}_{i+1} \neq 0$, because from condition (34) it follows that

$$\|A_{i+1}(p_i)\| > 1 \geq \|(1-\lambda)p_{i+1}\|, \tag{48}$$

for any $p_{i+1} \in \overline{B_u(0,1)}$.

If $\delta(i+1) = 0$, then obviously $\tilde{p}_{i+1} = p_i \neq 0$.
The argument for the case $q_i \in S^s$ is similar. ∎

Now we turn to the computation of the degree of C_1. Observe that C_1 has the following form

$$\tilde{p}_i = -A_i(p_{i-1}), \quad \text{if } \delta(i) = 1, \tag{49}$$
$$\tilde{q}_i = q_i, \quad \text{if } \delta(i) = 1, \tag{50}$$
$$\tilde{p}_i = p_{i-1}, \quad \text{if } \delta(i) = 0, \tag{51}$$
$$\tilde{q}_i = -A_i(q_i), \quad \text{if } \delta(i) = 0. \tag{52}$$

From the product property of the degree it follows that

$$|\deg(C_1, \Pi, 0)| =$$
$$\left| \Pi_{i\in\delta^{-1}(1)} \deg(-A_i, \overline{B_u}(0,1), 0) \cdot \Pi_{i\in\delta^{-1}(0)} \deg(-A_i, \overline{B_s}(0,1), 0) \right|.$$

In the formula above if $\delta^{-1}(i) = \emptyset$, then the corresponding product is set to be equal to 1. In the situation when $u = 0$ or $s = 0$ the corresponding product is also set equal to 1.
From (104) it follows that

$$\deg(-A, U, 0) = (-1)^u \deg(A, U, 0), \tag{53}$$

which completes the proof. ∎

The following corollary is an immediate consequence of Theorem 9.

Collorary 12 *Let N_i, $i \in \mathbb{Z}$ be h-sets. Assume that for each $i \in \mathbb{Z}$ we have either*

$$N_{i-1} \overset{f_i, w_i}{\Longrightarrow} N_i \tag{54}$$

or

$$N_i \subset dom(f_i^{-1}) \quad and \quad N_{i-1} \overset{f_i, w_i}{\Longleftarrow} N_i. \tag{55}$$

Then there exists a point $x \in \operatorname{int} N_0$, such that

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \quad \in \quad \operatorname{int} N_i, \quad i \in \mathbb{Z}. \tag{56}$$

Moreover, if $N_{i+k} = N_i$ for some $k > 0$ and all i, then the point x can be chosen so that

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x. \tag{57}$$

3.2 Stability of covering relations with respect to C^0-perturbations

We will state and prove here a very simple theorem on the stability of covering relations to C^0-perturbations. This result is very important in applications, especially in computer assisted proofs based on covering relations, as it shows that sufficiently small errors in numerical approximations of the map f do not affect the nature of a covering relation $N \overset{f}{\Longrightarrow} M$.

Theorem 13 *Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f, g : N \to \mathbb{R}^n$ be continuous. Assume that*

$$N \stackrel{f,w}{\Longrightarrow} M.$$

Then there exists $\epsilon > 0$, such that if $|f_c(x) - g_c(x)| < \epsilon$ for all $x \in N_c$, where

$$f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$$
$$g_c = c_M \circ g \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s,$$

then

$$N \stackrel{g,w}{\Longrightarrow} M.$$

Proof: The basic idea is to construct a good homotopy connecting f_c and g_c. Namely, we construct a homotopy $\tilde{h} : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, such that

$$\tilde{h}_0 = g_c, \qquad \tilde{h}_1 = f_c, \tag{58}$$
$$\tilde{h}([0,1], N_c^-) \cap M_c = \emptyset, \tag{59}$$
$$\tilde{h}([0,1], N_c) \cap M_c^+ = \emptyset, . \tag{60}$$

We set

$$\tilde{h}(t,x) = (1-t)g_c(x) + t f_c(x). \tag{61}$$

It is easy to see that for ϵ small enough, the conditions (59) and (60) are satisfied.

Let h be a homotopy from the relation $N \stackrel{f,w}{\Longrightarrow} M$. It is easy to see the homotopy $H : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$ given by

$$H(t,x) = \begin{cases} \tilde{h}(2t, x) & \text{for } t \in [0, \frac{1}{2}], \\ h(2t-1, x) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases} \tag{62}$$

satisfies all the conditions from Def. 6 for a covering relation $N \stackrel{g,w}{\Longrightarrow} M$ ∎

In above theorem the size of the perturbation i.e. ($|f_c(x) - g_c(x)|$ for $x \in N_c$) was given in c_N and c_M coordinates. Now we state the result involving the difference between f and g on N.

Theorem 14 *Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f, g : N \to \mathbb{R}^n$ be continuous. Assume that*

$$N \stackrel{f,w}{\Longrightarrow} M,$$

and that the coordinate map c_M satisfies a Lipschitz condition. Then there exists $\epsilon > 0$, such that if $|f(x) - g(x)| < \epsilon$ for all $x \in N$, then

$$N \stackrel{g,w}{\Longrightarrow} M.$$

Proof: Let L be a Lipschitz constant for c_M. Let f_c and g_c be as in Theorem 13. We have

$$\max_{x \in N_c} |f_c(x) - g_c(x)| = \max_{x \in N} |c_M(f(x)) - c_M(g(x))| \leq L \max_{x \in N} |f(x) - g(x)|.$$

The assertion follows from Theorem 13. ∎

4 How to find a homotopy for the covering relations

The goal of this section is to present sufficient conditions which ensure that $N \overset{f}{\Longrightarrow} M$, solely based on the knowledge of $f(N)$ and M.

Definition 8 *Let N be a h-set. We set*

$$S(N)_c^- = \{(p,q) \in \mathbb{R}^u \times \mathbb{R}^s \mid \|p\| > 1\}. \tag{63}$$

We define $S(N)^- = c_N^{-1}(S(N)_c^-)$.

Theorem 15 *Let N, M be two h-sets in $\mathbb{R}^n$, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : N \to \mathbb{R}^n$ be continuous. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$.*

Assume that there exists $q_0 \in \overline{B}_s(0,1)$, such that following conditions are satisfied

1.

$$\begin{aligned} f_c(\overline{B}_u(0,1) \times \{q_0\}) &\subset \operatorname{int}(S(M)_c^- \cup M_c), & (64)\\ f_c(N_c^-) \cap M_c &= \emptyset, & (65)\\ f_c(N_c) \cap M_c^+ &= \emptyset. & (66) \end{aligned}$$

2.1 *Case $u > 0$. We define a map $A_{q_0} : \mathbb{R}^u \to \mathbb{R}^u$ by*

$$A_{q_0}(p) = \pi_u(f_c(p, q_0)), \tag{67}$$

where $\pi_u : \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}^u$ is the projection onto $\mathbb{R}^u$, $\pi_u(p,q) = p$. We assume that

$$\begin{aligned} A_{q_0}(\partial B_u(0,1)) &\subset \mathbb{R}^u \setminus \overline{B_u(0,1)}, & (68)\\ \deg(A_{q_0}, \overline{B_u}(0,1), 0) = w &\neq 0. & (69) \end{aligned}$$

2.2 *Case $u = 0$. We assume that*

$$f_c(N_c) \subset \operatorname{int} M_c,$$

and set $w = 1$.

Then

$$N \overset{f,w}{\Longrightarrow} M.$$

Proof: We have to prove that there exists a homotopy $h : [0,1] \times N_c \to \mathbb{R}^n$ satisfying conditions from Definition 6.

The case $u = 0$ is trivial ($w = 1$). Thus we focus on the case $u > 0$.

For any $q_1 \in \mathbb{R}^s$ we define a deformation retraction onto $\mathbb{R}^u \times \{q_1\}$, $R_{q_1} : [0,1] \times \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}^u \times \mathbb{R}^s$, by

$$R_{q_1}(\lambda, p, q) = (p, (1-\lambda)q + \lambda q_1). \tag{70}$$

Notice that $R_{q_1}(0,p,q) = (p,q)$.

We define the homotopy $h : [0,1] \times N_c \to \mathbb{R}^n$ by

$$h(\lambda, p, q) = \begin{cases} f_c(R_{q_0}(2\lambda, p, q)) & \lambda \in [0, 0.5], \\ R_0(2\lambda - 1, f_c(p, q_0)) & \lambda \in [0.5, 1]. \end{cases} \tag{71}$$

The homotopy h is a superposition of the retraction in the domain of f_c onto $q = q_0$ with the retraction in the image onto the subspace $q = 0$.

Notice that

$$h_0(x) = f_c(x), \quad h_{1/2}(p,q) = f_c(p, q_0), \quad h_1(p,q) = (\pi_u f_c(p, q_0), 0). \tag{72}$$

To prove (31), notice that $R_{q_0}(\lambda, N_c^-) \subset N_c^-$ for $\lambda \in [0,1]$, hence from condition (65) it follows that

$$f_c(R_{q_0}(\lambda, N_c^-)) \cap M_c \subset f_c(N_c^-) \cap M_c = \emptyset. \tag{73}$$

This proves condition (31) for $\lambda \in [0, 0.5]$. For the proof for $\lambda \in [0.5, 1]$ observe that from condition (68) it follows that

$$f_c(p, q_0) \in S(M)_c^-, \qquad \text{for } (p, q_0) \in N_c^- \;. \tag{74}$$

But $R_0\left([0,1] \times S(M)_c^-\right) \subset S(M)_c^-$ and $S(M)_c^- \cap M_c = \emptyset$. Hence condition (31) is satisfied.

It remains to show that condition (32) is true for h. As above, consider two cases $\lambda \leq 1/2$ and $\lambda \geq 1/2$. For $\lambda \leq 1/2$ from (66) it follows that

$$h_\lambda(N_c) \cap M_c^+ \subset f_c(N_c) \cap M_c^+ = \emptyset. \tag{75}$$

For $\lambda \geq 1/2$ observe that from (64) it follows that

$$\begin{gathered} h_\lambda(N_c) \subset R_0(2\lambda - 1, f_c(\overline{B}_u(0,1) \times \{q\})) \subset \\ \subset R_0\left(2\lambda - 1, \operatorname{int}\left(S(M)_c^- \cup M_c\right)\right) \subset \operatorname{int}\left(S(M)_c^- \cup M_c\right). \end{gathered}$$

But $(\operatorname{int}(S(M)_c^- \cup M_c)) \cap M^+ = \emptyset$. ∎

The above theorem allows to take its assumptions as a definition of covering relation as it was done before by Zgliczyński in [2, 20] for maps with one topologically expanding direction ($u = 1$), and in [21] for maps which are close to products of one dimensional maps. Below we will discuss the case of $u = 1$.

4.1 Case of one nominally expanding direction, $u = 1$

In this section we discuss the case of $u = 1$, hence we have only one nominally expanding direction. The basic idea here is that each of the sets N^-, $S(N)^-$ consists of two disjoint components, allowing us to simplify the assumptions of Theorem 15.

Definition 9 *Let N be an h-set, such that $u(N) = 1$. We set*

$$\begin{aligned} N_c^{le} &= \{-1\} \times \overline{B}_s(0,1), \\ N_c^{re} &= \{1\} \times \overline{B}_s(0,1), \\ S(N)_c^l &= (-\infty,-1) \times \mathbb{R}^s, \\ S(N)_c^r &= (1,\infty) \times \mathbb{R}^s. \end{aligned}$$

We define

$$N^{le} = c_N^{-1}(N_c^{le}), \quad N^{re} = c_N^{-1}(N_c^{re}),$$
$$S(N)^l = c_N^{-1}(S(N)^l), \quad S(N)^r = c_N^{-1}(S(N)^r).$$

We will call N^{le}, N^{re}, $S(N)^l$ and $S(N)^r$ the left edge, the right edge, the left side and right side of N, respectively.

It is easy to see that $N^- = N^{le} \cup N^{re}$ and $S(N)^- = S(N)^l \cup S(N)^r$.

The triple $(N, \overline{S(N)^l}, \overline{S(N)^r})$ represents a t-set, as it was defined in [2].

Theorem 16 *Let N, M be two h-sets in $\mathbb{R}^n$, such that $u(N) = u(M) = 1$ and $s(N) = s(M) = s = n-1$. Let $f : N \to \mathbb{R}^n$ be continuous.*

Assume that there exists $q_0 \in \overline{B}_s(0,1)$, such that the following conditions are satisfied

$$f(c_N(\overline{B}_u(0,1) \times \{q_0\})) \subset \operatorname{int}(S(M)^l \cup M \cup S(M)^r), \tag{76}$$
$$f(N) \cap M^+ = \emptyset, \tag{77}$$

and one of the following two conditions holds true

$$f(N^{le}) \subset S(M)^l \quad \text{and} \quad f(N^{re}) \subset S(M)^r, \tag{78}$$
$$f(N^{le}) \subset S(M)^r \quad \text{and} \quad f(N^{re}) \subset S(M)^l. \tag{79}$$

Then there exists $w = \pm 1$, such that

$$N \overset{f,w}{\Longrightarrow} M.$$

Proof: We will show that assumptions of Theorem 15 are satisfied.

From the alternative conditions (78) and (79), it follows that

$$f_c(N_c^{le} \cup N_c^{re}) = f(N_c^-) \cap M_c = \emptyset.$$

We define a map A_{q_0} as in Theorem 15. It is easy to see that $\deg(A_{q_0}, \overline{B_u}(0,1), 0) = 1$ provided that condition (78) is satisfied, and $\deg(A_{q_0}, \overline{B_u}(0,1), 0) = -1$ provided that (79) holds. ∎

5 Some examples, model maps

The goal of this section is to discuss what are all possible model maps and to give nontrivial examples of chaotic behavior different from topological horseshoes.

Let us remind the reader that, in the context of Definition 6, the map h_1 is called *a model map* for the relation $N \overset{f}{\Longrightarrow} M$, and that the degree w of a covering can be computed as $w = d(s_A)$, where A is the u-component of h_1.

We have to consider three cases $u = 0$, $u = 1$ and $u \geq 2$. The case $u = 0$ is trivial, because $h_1(x) = 0$ by the definition.

For $u = 1$, from Theorem 16 it follows that we can have only $w = \pm 1$. It is easy to show (see for example [19, 22]) that in this case we can always choose $h_1 = m$ to be of the following form

$$m(p, q) = (\lambda \cdot p, 0), \tag{80}$$

where $\lambda \in \mathbb{R}$, $\lambda < -1$ if $w = -1$ and $\lambda > 1$ if $w = 1$. Topological horseshoes are present in virtually every paper devoted to detection of chaos through topological methods (see, for example [19, 20, 16, 11, 9]). A topological horseshoe is defined as a map $f : N = N_0 \cup N_1 \to \mathbb{R}^n$, satisfying the following covering relations

$$N_i \overset{f,\pm 1}{\Longrightarrow} N_j, \quad i, j = 0, 1, \tag{81}$$

where $u(N_i) = 1$ and $N_0 \cap N_1 = \emptyset$. Using Theorem 9, it is easy to show that f has symbolic dynamics on two symbols (see also [22] and Section 5.2).

5.1 Case $u > 1$.

In order to give a description of all possible model maps (up to a homotopy), we first consider maps of the sphere S^{u-1}.

For simplicity, we will represent $\mathbb{R}^u$ as $\mathbb{C} \times \mathbb{R}^{u-2}$, where $\mathbb{C}$ is a set of complex numbers. For $z \in \mathbb{C}$, by $\bar{z}$ we denote the complex conjugate of z.

For $k \in \mathbb{Z} \setminus \{0\}$ we define a map $m_k : \mathbb{R}^u \to \mathbb{R}^u$ by

$$m_k(z, x_3, \ldots, x_u) = \left(|z| \left(\frac{z}{|z|} \right)^k, x_3, \ldots, x_u \right), \qquad \text{for } k > 0 \tag{82}$$

$$m_k(z, x_3, \ldots, x_u) = \left(|z| \left(\frac{\bar{z}}{|z|} \right)^{|k|}, x_3, \ldots, x_u \right), \qquad \text{for } k < 0. \tag{83}$$

It is easy to see that $m_k(S^{u-1}) = S^{u-1}$. It is well known (see, for example, the proof of Hopf Theorem in [4]) that

$$d(m_k) = k. \tag{84}$$

From Hopf Theorem and equation (84) it follows that the maps m_k describe all possible self-maps of the sphere, up to a homotopy. Hence all possible model maps (up to a homotopy) for the covering relation $N \overset{f,w}{\Longrightarrow} M$ are given by

$$M_w(p, q) = (\lambda m_w(p), 0), \qquad \lambda \in \mathbb{R}, \quad \lambda > 1. \tag{85}$$

5.2 An example of chaotic behavior based on multiple wrapped coverings

In this section we describe a class of chaotic examples based on the model maps of higher degree introduced in previous subsection.

Let $u = 2$ and $s \geq 0$. We will represent $\mathbb{R}^{u+s}$ as $\mathbb{C} \times \mathbb{R}^s$. We define an h-set N_0 by

$$\begin{aligned} N_0 &= \overline{B}_2(0,1) \times \overline{B}_s(0,1), \\ u(N_0) &= 2, s(N_0) = s, \\ c_{N_0}(z,q) = (z,q), &\qquad z \in \mathbb{C} \text{ and } q \in \mathbb{R}^s \end{aligned}$$

and an h-set N_1

$$\begin{aligned} N_1 &= \overline{B}_2(a,1) \times \overline{B}_s(0,1), \\ u(N_1) &= 2, s(N_1) = s, \\ c_{N_1}(z,q) = (z-a,q), &\qquad z \in \mathbb{C} \text{ and } q \in \mathbb{R}^s \end{aligned}$$

where $a \in \mathbb{C}$, $|a| > 2$.

Notice that $N_0 \cap N_1 = \emptyset$. For any $w_1, w_2 \in \mathbb{Z} \setminus \{0\}$ we define a map f on $N_0 \cup N_1$ by

$$f(z,x) = \begin{cases} \lambda(m_{w_1}(z), q) & \text{for } (z,q) \in N_0, \\ \lambda(m_{w_2}(z), q) & \text{for } (z,q) \in N_1. \end{cases} \tag{86}$$

It is easy to see that for all $\lambda > |a| + 1$ we have

$$\begin{aligned} N_0 \overset{f,w_1}{\Longrightarrow} N_0, &\qquad N_0 \overset{f,w_1}{\Longrightarrow} N_1, \\ N_1 \overset{f,w_2}{\Longrightarrow} N_1, &\qquad N_1 \overset{f,w_2}{\Longrightarrow} N_0. \end{aligned}$$

Let $g : \mathbb{R}^{u+s} \to \mathbb{R}^{u+s}$ be a perturbation of f given by

$$g(z,p) = f(z,q) + (\epsilon_z(z,q), \epsilon_q(z,q)), \tag{87}$$

where $\epsilon_z : \mathbb{C} \times \mathbb{R}^s \to \mathbb{C}$ and $\epsilon_q : \mathbb{C} \times \mathbb{R}^s \to \mathbb{R}^s$ are continuous functions.

Lemma 17 *Let f and g are given as above. Assume that the following conditions are satisfied for $(z,q) \in N_0 \cup N_1$*

$$\begin{aligned} |\epsilon_z(z,q)| &< \lambda - (|a| + 1), \\ |\epsilon_q(z,q)| &< 1. \end{aligned}$$

Then

$$\begin{aligned} N_0 \overset{g,w_1}{\Longrightarrow} N_0, &\qquad N_0 \overset{g,w_1}{\Longrightarrow} N_1, \\ N_1 \overset{g,w_2}{\Longrightarrow} N_1, &\qquad N_1 \overset{g,w_2}{\Longrightarrow} N_0. \end{aligned}$$

Proof: It is easy to see that for all covering relations, the common homotopy $h(t,(z,q)) = tf(z,q) + (1-t)g(z,q)$ satisfies all assumptions of Theorem 15. ∎

Let $N = N_0 \cup N_1$. We define a forward invariant set of g by

$$\mathrm{Inv}^+(N,g) = \{x \in N \mid g^k(x) \in N, \quad \text{for } k \in \mathbb{N}\ \}.$$

To proceed further we will recall some notions from symbolic dynamics. For any $k \geq 2$, we define $\Sigma_k^+ = \{0,1,\dots,k-1\}^{\mathbb{N}}$, where $\mathbb{N}$ denotes the set of all nonnegative integers. On Σ_k^+ we define a shift map $\sigma : \Sigma_k^+ \to \Sigma_k^+$ by

$$\sigma((a_0,a_1,a_2,\dots)) = (a_1,a_2,\dots).$$

We define a map $\pi : \mathrm{Inv}^+(N,g) \to \Sigma_2$ by

$$\pi(x)_k = j, \qquad \text{iff } g^k(x) \in N_j. \tag{88}$$

From Theorem 9 we obtain the following

Theorem 18 *Let g satisfy the assumptions of Lemma 17. Then the map π defined above is onto. Moreover, for any periodic sequence $\alpha \in \Sigma_2$ there exists $x \in Inv^+(N,g)$ such that $\pi(x) = \alpha$ and x is periodic point of g with the same principal period as α.*

Proof: For any finite sequence $(\alpha_0,\alpha_1,\dots,\alpha_{l-1}) \in \{0,1\}^l$ consider a closed loop of covering relation for g (we drop here the symbol of the function and the degree)

$$N_{\alpha_0} \Longrightarrow N_{\alpha_1} \Longrightarrow \dots \Longrightarrow N_{l-1} \Longrightarrow N_0. \tag{89}$$

From Theorem 9 we obtain the existence of a point x, such that $g^i(x) \in N_{\alpha_i}$ for $i = 0,\dots,l-1$ and $g^l(x) = x$. This shows that for any periodic sequence $\alpha \in \Sigma_2$, $\pi^{-1}(\alpha)$ contains a periodic point with the same principal period as α. The surjectivity of π is obtained via passing to the limit with points of large period. ∎

Observe that if one of the numbers $|w_1|, |w_2|$ is greater than one, we can expect a richer symbolic dynamics for f than just on two symbols. Also the invariant set should look different from the Cantor set obtained from Smale's horseshoes, but we will not pursue this issue here.

6 Appendix

6.1 Local Brouwer degree

In this section we list the basic properties of *the local Brouwer degree* which are relevant for us in this paper. The proofs can be found in [15, Ch. III].

For $n = 0$ we have $\mathbb{R}^n = \{0\}$. We have only one self map for this space, namely $f(0) = 0$. We formally define *the local Brouwer degree of f at* 0 *in the set* $\{0\}$ by

$$\deg(f,\{0\},0) = 1$$

Assume $n > 0$. Let $D \subset \mathbb{R}^n$ be an open set and $f : S \to \mathbb{R}^n$ be continuous, $D \subset S$ and $c \in \mathbb{R}^n$. Suppose that

$$\text{the set} \quad f^{-1}(c) \cap D \quad \text{is compact.} \tag{90}$$

Then *the local Brouwer degree of f at c in the set D* is defined. We denote it by $\deg(f, D, c)$.

If $\overline{D} \subset \mathrm{dom}(f)$ and $\overline{D}$ is compact, then (90) follows from the condition

$$c \notin f(\partial D). \tag{91}$$

Let us summarize the properties of the local Brouwer degree

Degree is an integer.

$$\deg(f, D, c) \in \mathbb{Z}. \tag{92}$$

Solution property.

$$\text{If} \quad \deg(f, D, c) \neq 0, \quad \text{then there exists } x \in D \text{ with } f(x) = c. \tag{93}$$

Homotopy property. Let $H : [0,1] \times D \to \mathbb{R}^n$ be continuous. Suppose that

$$\bigcup_{\lambda \in [0,1]} H_\lambda^{-1}(c) \cap D \quad \text{is compact.} \tag{94}$$

Then

$$\forall \lambda \in [0,1] \quad \deg(H_\lambda, D, c) = \deg(H_0, D, c). \tag{95}$$

If $[0,1] \times \overline{D} \subset \mathrm{dom}(H)$ and $\overline{D}$ is compact, then (94) follows from the following condition

$$c \notin H([0,1], \partial D). \tag{96}$$

Local degree is a locally constant function. Assume D is bounded and open. If p and q belong to the same component of $\mathbb{R}^n \setminus f(\partial D)$, then

$$\deg(f, D, p) = \deg(f, D, q). \tag{97}$$

Excision property. Suppose that $E \subset D$, E is open and

$$f^{-1}(c) \cap D \subset E. \tag{98}$$

Then

$$\deg(f, E, c) = \deg(f, D, c). \tag{99}$$

Local degree for affine maps. Suppose that $f(x) = A(x - x_0) + c$, where A is a linear map and $x_0 \in \mathbb{R}^n$. If the equation $A(x) = 0$ has no nontrivial solutions (i.e. if $Ax = 0$, then $x = 0$) and $x_0 \in D$, then

$$\deg(f, D, c) = \mathrm{sgn}(\det A). \tag{100}$$

Product property Let $U_i \subset \mathbb{R}^{n_i}$, $c_i \in \mathbb{R}^{n_i}$, $f_i : U_i \to \mathbb{R}^{n_i}$, for $i = 1, 2$. The map $(f_1, f_2) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is given by $(f_1, f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$. We have

$$\deg((f_1, f_2), U_1 \times U_2, (c_1, c_2)) = \deg(f_1, U_1, c_1) \cdot \deg(f_2, U_2, c_2), \tag{101}$$

whenever the right hand side is defined.

Multiplication property Let $D \subset \mathbb{R}^n$ be bounded and open. Let $f : \bar{D} \subset \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ are two continuous mappings and Δ_i the bounded components of $\mathbb{R}^n \setminus f(\partial D)$. Then

$$\deg(g \circ f, D, p) = \sum_{\Delta_i} \deg(g, \Delta_i, p) \deg(f, D, \Delta_i), \tag{102}$$

where $\deg(f, D, \Delta_i) = \deg(f, D, q_i)$ for some $q_i \in \Delta_i$. From equation (97) it follows that this definition of $\deg(f, D, \Delta_i)$ does not depend on the choice of q_i.

Addition property. If $D = \bigcup_{i \in I} D_i$, where each D_i is open, the family $\{D_i\}$ is disjoint and $\partial D_i \subset \partial D$, then for every $c \notin f(\partial D)$:

$$\deg(f, D, c) = \sum_{i \in I} \deg(f, D_i, c). \tag{103}$$

From Multiplication property and formula (100) we obtain immediately

Collorary 19 *Let $D \subset \mathbb{R}^n$ be open and bounded. Let $A : D \to \mathbb{R}^n$, be continuous and $0 \notin A(\partial D)$,*

$$\deg(-A, U, 0) = (-1)^n \deg(A, U, 0). \tag{104}$$

As the consequence of Addition and Excision property we obtain the following

Collorary 20 *Suppose that D is a finite union of open sets $D = \bigcup_{i=1}^n D_i$ such that the sets $f^{-1}_{|D_i}(c)$ are mutually disjoint and $c \notin f(\partial D_i)$. Then*

$$\deg(f, D, c) = \sum_{i=1}^{n} \deg(f_{|D_i}, D_i, c). \tag{105}$$

Here is another important consequence of above properties

Collorary 21 *Assume $V \subset \mathbb{R}^n$ is bounded and open. Let $f : \overline{V} \to \mathbb{R}^n$ be a C^1-map. Assume that $c \in \mathbb{R}^n \setminus f(\partial V)$ is a regular value for f, i.e. for each $x \in f^{-1}(c)$ the Jacobian matrix of f at x denoted by $Df(x)$ is nonsingular, then*

$$\deg(f, V, c) = \sum_{x \in f^{-1}(c)} \operatorname{sgn}(\det Df(x)).$$

6.2 The degree of maps $S^n \to S^n$

In this section we recall some relevant facts on the degree of maps $S^n \to S^n$ see for example [4, Ch. 7.5].

Definition 10 *Let $n \geq 1$. The degree of a continuous map $f : S^n \to S^n$ is a unique integer $d(f)$ such that $f_*(u) = d(f) \cdot u$, for any generator $u \in H_n(S^n)$, where $H_n(S^n)$ is n-th homology group of S^n and $f_* : H_n(S^n) \to H_n(S^n)$ is the induced homomorphism.*

For $n = 0$ we define the degree, $d(f)$, as follows, $S^0 = \{-1, 1\}$. We set

$$d(f) = \begin{cases} 1, & \text{if } f(1) = 1 \text{ and } f(-1) = -1, \\ -1, & \text{if } f(1) = -1 \text{ and } f(-1) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{106}$$

Theorem 22 (H. Hopf) *Let $n \geq 1$. Then $f, g : S^n \to S^n$ are homotopic if and only if $d(f) = d(g)$.*

Lemma 23 *Let $u > 0$, Assume that $A : \overline{B_u}(0,1) \to \mathbb{R}^u$ is a continuous map, such that*

$$0 \notin A(\partial B(0,1)).$$

Let the map $s_A : S^{u-1} \to S^{u-1}$ be given by

$$s_A(x) = \frac{A(x)}{\|A(x)\|}.$$

Then

$$\deg(A, \overline{B_u}(0,1), 0) = d(s_A). \tag{107}$$

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