SCHRÖDINGER OPERATORS WITH MANY BOUND STATES

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ABSTRACT. Consider the Schrödinger operators $H_\pm = -d^2/dx^2 \pm V(x)$. We present a method for estimating the potential in terms of the negative eigenvalues of these operators. Among the applications are inverse Lieb-Thirring inequalities and several sharp results concerning the spectral properties of $H_\pm$.

1. INTRODUCTION

We are interested in Schrödinger equations

$$-y''(x) + V(x)y(x) = Ey(x) \tag{1.1}$$

and the associated self-adjoint operators $H_+ = -d^2/dx^2 + V(x)$ on $L_2(0,\infty)$. The potential V is assumed to be locally integrable on $[0,\infty)$. One also needs a boundary condition of the following form:

$$y(0)\cos\alpha - y'(0)\sin\alpha = 0. \tag{1.2}$$

The basic question we would like to address is the following: *What is the influence of the structure of the discrete spectrum on the potential V and on the spectral properties of H_+?*

It is then necessary to consider H_+ and $H_- = -d^2/dx^2 - V(x)$ simultaneously because otherwise sign definite potentials provide counterexamples to any possible positive result one might imagine. So our basic assumption is the following:

$$H_\pm \text{ are bounded below and } \sigma_{\text{ess}}(H_\pm) \subset [0,\infty). \tag{Σ_{ess}}$$

Assuming (Σ_{ess}), we can list the negative eigenvalues of H_+ and H_- together as $-E_1 \le -E_2 \le \ldots$, with $E_n > 0$. The list is either finite (or even empty) or $E_n \to 0$. The E_n's of course depend on the boundary condition (1.2). However, we will usually be interested in situations where $\sum E_n^p < \infty$, with $p \ge 0$, and, by the interlacing property, this condition is independent of α.

We deal with the one-dimensional case only in this paper, but our methods are not limited to this situation. We plan to explore higher-dimensional operators in a future project.

Our original motivation for this work came from the following result, which completely clarifies the situation when $\{E_n\}$ is a finite set.

Theorem 1.1 (Damanik-Killip [5] (see also [6]))**.** *Assume (Σ_{ess}). Moreover, assume that $\{E_n\}$ is finite. Then $\sigma_{\text{ess}} = [0,\infty)$, and the spectrum is purely absolutely continuous on $[0,\infty)$ for any boundary condition at $x = 0$.*

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Here the statements refer to H_+, say. Of course, since $-V$ satisfies the same hypotheses as V, we automatically obtain the same assertions for H_- as well, so this distinction is actually irrelevant. We keep this convention, however, because it will help to slightly simplify the formulation of our results.

In [5], it is also assumed that $V \in \ell_\infty(L_2)$, that is, $\sup_{x\ge 0}\int_x^{x+1} V^2(t)\,dt < \infty$. Our treatment below, especially the material from Sect. 3, will show that this technical assumption is unnecessary.

The aim of this paper is to develop tools for handling arbitrary discrete spectra $\{E_n\}$, not necessarily finite. At the heart of the matter is a new method for estimating the potential in terms of the E_n's in general situations. We defer the exact description of this to Sect. 2 and limit ourselves to a few general remarks in this introduction (refer to Theorem 2.1 below for the full picture).

The most important aspect of our method is this: The rate of convergence with which E_n tends to zero determines the *geometry* of the situation. More precisely, we obtain intervals I_n whose lengths obey the scaling relation $|I_n| \sim E_n^{-1/2}$. We will also write V as $W' + W^2$ plus a remainder, with $\|W\|_{L_2(I_n)} \lesssim |I_n|^{-1/2}$. This representation of V is natural in this context because $-d^2/dx^2 + q(x)$ with Dirichlet boundary conditions ($y = 0$) has no negative spectrum if and only if $q = w' + w^2$ for some w. More importantly, it is also very useful in the applications we want to make in this paper. Note also that any relation between the potential and the eigenvalues must respect the invariance of the problem under the rescaling $V(x) \to g^2 V(gx)$, $E \to g^2 E$.

To give a more specific impression of what we can do with our techniques, we mention the following:

Corollary 1.2. *Assume* (Σ_{ess}). *Moreover, assume that* $\sum E_n^{1/2} < \infty$. *Then there exists* $V_0 \in L_1(0,\infty)$ *so that* $H_+ + V_0$ *with Dirichlet boundary conditions has no negative spectrum.*

This is indeed immediate from Theorem 2.1, which says that $V - (W' + W^2) \in L_1$ for a suitable W in this situation. So small eigenvalues can be removed by a small perturbation. The exponent $1/2$ in the hypothesis is not essential. For instance, it is also true that if, more generally, $\sum E_n^p < \infty$ with $p \ge 1/2$, then there exists $V_0 \in \ell_{2p}(L_1)$, that is,

$$\sum_{n=0}^{\infty} \left(\int_n^{n+1} |V_0(x)|\,dx \right)^{2p} < \infty,$$

so that $H_+ + V_0 \ge 0$.

In some instances, our basic problem of obtaining information on the potential from a knowledge of its eigenvalues can be attacked with completely different tools, called *sum rules* (aka trace formulae). While this approach is elegant and leads to very satisfactory results where it works, it is indirect and less systematic and one is restricted to those combinations of the E_n's that happen to show up in the sum rules one can produce. See, for instance, [12, 14, 15, 19, 26] for recent work on sum rules.

The converse problem, that is, the problem of estimating the eigenvalues in terms of the potential, is classical and has received considerable attention over the years. We mention, in particular, the topic of *Lieb-Thirring inequalities* (see, e.g., [16] for further information on this). For sign-definite potentials, we obtain *inverse*

Lieb-Thirring inequalities as a by-product of our general method; see Theorem 1.8 below.

We now turn to discussing the consequences of our method concerning the spectral properties of $H_{\pm}$. Taking related results into account (see [4, Theorem 1] and Theorem 1.1 above), the following does not come as a surprise.

Theorem 1.3. *Assume* (Σ_{ess}). *Then* $\sigma_{\text{ess}} = [0, \infty)$.

It is now natural to inquire about the structure of the spectrum on $(0, \infty)$.

Theorem 1.4. *Assume* (Σ_{ess}). *Moreover, assume that* $\sum E_n^{1/2} < \infty$. *Then there exists absolutely continuous spectrum essentially supported by* $(0, \infty)$.

Killip and Simon have proved earlier the discrete analog of this. The essential ingredient in their analysis is a sum rule, and they in fact establish the so-called Szegő condition. See [12] for these statements, especially Theorems 3 and 7; compare also [26]. Under a slightly stronger assumption, we will also prove that on large sets of energies E, the solutions asymptotically look like plane waves. By this we mean that

$$y(x) = e^{i\sqrt{E}x} + o(1) \qquad (x \to \infty). \tag{1.3}$$

We let S be the exceptional set where we do *not* have solutions of this asymptotic form. So we define

$$S = \{E > 0 : \text{ no solution of (1.1) satisfies (1.3)}\}.$$

Note that if $E \in (0, \infty) \setminus S$, then the complex conjugate of the solution y from (1.3) is a linearly independent solution of the same equation, so we have complete control over the solution space for such E's. In particular, there is no subordinate solution then, so that the singular part of the spectral measure on $(0, \infty)$ must be supported by S for any boundary condition (1.2).

Theorem 1.5. *Assume* (Σ_{ess}). *Moreover, assume that* $\sum E_n^p < \infty$ *for some* $p < 1/2$. *Then* $|S| = 0$.

In particular, this again implies that H_+ has absolutely continuous spectrum essentially supported by $(0, \infty)$. We obtain a more detailed statement here, giving asymptotic formulae for the solutions, but are unable to treat the borderline case $p = 1/2$. The situation is completely analogous to the known results on operators with L_q potentials V. This is no coincidence, because the techniques are the same: Theorem 1.5 crucially depends on work of Christ and Kiselev [1, 2], and the proof of Theorem 1.4 follows ideas of Deift and Killip [7].

Perhaps somewhat surprisingly, the statement of Theorem 1.5 can be sharpened if p can be taken smaller than $1/4$.

Theorem 1.6. *Assume* (Σ_{ess}). *Moreover, assume that* $\sum E_n^p < \infty$, *with* $0 \le p < 1/4$. *Then* $\dim S \le 4p$.

Note that the corresponding statement on L_q potentials is false: There are potentials $V \in \bigcap_{q>1} L_q$ with $\dim S = 1$ (see [22, Theorem 4.2b)]).

As explained above, Theorem 1.6 implies that the singular part of the spectral measure is supported on a set of dimension $\le 4p$. A related consequence is the fact that the spectrum is *purely* absolutely continuous on $[0, \infty)$ for all boundary conditions not from an exceptional set $B \subset [0, \pi)$, where again $\dim B \le 4p$ (see [23, Theorem 5.1] for this conclusion).

If, on the other hand, $\sum E_n^p < \infty$ with $1/4 \le p < 1/2$, no strengthening of the statement of Theorem 1.5 is obtained, in spite of the stronger hypothesis. The following theorem shows that no such improvement is possible:

Theorem 1.7. *Let $e_n > 0$ be a non-increasing sequence with $\sum e_n^{1/4} = \infty$. Then there exists a potential V so that (Σ_{ess}) holds, $E_n \le e_n$, and $\dim S = 1$.*

Thus the bound from Theorem 1.6 is correct at the extreme values $p = 0$ and $p = 1/4$. We make the obvious conjecture that it is optimal throughout its range of validity. The examples used in the proof of Theorem 1.7 also show that given e_n's with $\sum e_n^{1/2} = \infty$, there exists a potential so that $E_n \le e_n$ and $\sigma_{\text{ac}} = \emptyset$. Hence Theorem 1.4 is optimal, too, and Theorem 1.5 fails to address only the borderline value $p = 1/2$. In this context, the work of Muscalu, Tao, and Thiele [17, 18] is also relevant.

One of the main difficulties, when estimating V in terms of the E_n's, comes from the fact that V can take both signs. It is therefore interesting to compare the above results with the situation for sign-definite potentials, say $V \le 0$. Then only H_+ can have negative eigenvalues. In this situation, the control on V exerted by the eigenvalues gets more explicit.

Theorem 1.8. *Assume (Σ_{ess}). Moreover, assume that $V \le 0$, and consider Neumann boundary conditions ($\alpha = \pi/2$ in (1.2)).*

a) *For $0 < p \le 1/2$, there exists a constant C_p so that*

$$\int_0^\infty |V(x)|^{p+1/2}\, dx \le C_p \sum E_n^p.$$

b) *For $p \ge 1/2$ and $E_0 > 0$, there exists a constant $C_p(E_0)$ so that*

$$\sum_{n=0}^\infty \left(\int_n^{n+1} |V(x)|\, dx \right)^{2p} \le C_p(E_0) \sum E_n^p,$$

provided that $E_1 \le E_0$.

This is reassuring, but we emphasize again that these inequalities do not really catch the essence of our method. As outlined above, the behavior of E_n governs the geometry of the situation, and this part of the information gets lost when we pass to global bounds as in Theorem 1.8. This effect is also responsible for the additional assumption that $\sup E_n \le E_0$ from part b): The intervals $(n, n+1)$ are not adapted to the underlying geometry. The need for such a restriction is also apparent from the fact that part b) is not invariant under the rescaling $V(x) \to g^2 V(gx)$, $E \to g^2 E$.

The estimates from part a) might be called *inverse Lieb-Thirring inequalities.* Lieb-Thirring inequalities are bounds of the type of part a) but with the opposite sign. They hold for $p \ge 1/2$. In particular, for $p = 1/2$, we have inequalities in both directions, so $\int |V|$ and $\sum E_n^{1/2}$ are comparable. This is not a new result; on the contrary, $p = 1/2$ is essentially a sum rule, and the best constant is known ([10], see also [25]).

It is clear that we cannot have inverse Lieb-Thirring inequalities for $p > 1/2$ because V can have local singularities so that $V \notin L_q$ for any $q > 1$. It is also important to work with *Neumann* boundary conditions as there are non-zero potentials $V \leq 0$ with no Dirichlet eigenvalues. The whole-line analog of Theorem 1.8 also holds and is perhaps more natural for precisely this reason.

As for the spectral properties, Theorem 1.8 has the following consequences:

Corollary 1.9. *Assume* (Σ_{ess})*. Moreover, assume that* $V \leq 0$*.*

a) *If* $\sum E_n^{1/2} < \infty$*, then the spectrum is purely absolutely continuous on* $(0, \infty)$ *for all boundary conditions.*

b) *If* $\sum E_n < \infty$*, then there is absolutely continuous spectrum essentially supported by* $(0, \infty)$*.*

c) *If* $\sum E_n^p < \infty$ *for some* $p < 1$*, then the solutions satisfy WKB-type asymptotic formulae for Lebesgue almost all energies* $E > 0$*.*

Part a) follows because Theorem 1.8a) says that $V \in L_1$. As pointed out above, this part of the corollary has been known before. Rybkin [24, Theorem 1] has proved that the assertion of part b) holds if $V \in \ell_2(L_1)$, that is, if

$$\sum \left(\int_n^{n+1} |V(x)|\, dx \right)^2 < \infty.$$

This is the ultimate form of a well-known theorem of Deift and Killip [7] which states that there is absolutely continuous spectrum essentially supported by $(0, \infty)$ if $V \in L_1 + L_2$. So part b) of the corollary follows from Theorem 1.8b).

The asymptotic formula alluded to in part c) reads

$$y(x, E) = \exp\left(i\sqrt{E}x - \frac{i}{2\sqrt{E}} \int_0^x V(t)\, dt \right) + o(1) \qquad (x \to \infty).$$

Christ and Kiselev [1] prove that this holds at almost all energies if $V \in \ell_p(L_1)$ for some $p < 2$, so Theorem 1.8b) also implies part c) of the corollary.

As above, these results are complemented by the following:

Theorem 1.10. *Let* $e_n > 0$ *be a non-increasing sequence with* $\sum e_n = \infty$*. Then there exists a potential* $V \leq 0$ *so that* (Σ_{ess}) *holds,* $E_n \leq e_n$*, and* $\sigma_{\text{ac}} = \emptyset$*.*

We organize this paper in the obvious way: Section 2 gives a detailed discussion of our general method. The subsequent sections are concerned with the applications of this to the spectral theory of $H_\pm$, in the order suggested by this introduction. In the final section, we present the examples announced in Theorems 1.7 and 1.10.

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2. A Method for Estimating V

As in the previous section, let $H_\pm = -d^2/dx^2 \pm V(x)$. We write H_σ if we work with one of these operators, but do not want to specify which one. Boundary conditions (where necessary) will *always* be Dirichlet boundary conditions ($y = 0$).

The following theorem may be viewed as the principal result of this paper. Things become slightly easier in the whole-line setting because we avoid the somewhat artificial technical problems associated with the effect that a boundary condition can screen part of the potential. So we discuss this case first. The modifications needed to handle half-line problems will be described after having completed the treatment of the whole-line case.

Theorem 2.1. *Consider $H_\pm$ on $L_2(\mathbb{R})$. Assume (Σ_{ess}). Then there exist a partition of $\mathbb{R}$ into intervals $J_n^{(k)}$ with disjoint interiors and a decomposition $V = W' + Q$ with the following properties:*

a) (basic properties of W, Q) *W is absolutely continuous. If $\sum E_n^{1/2} < \infty$, then $W \in L_2(\mathbb{R})$ and $Q \in L_1(\mathbb{R})$.*

b) (geometry of the intervals) *The indices k, n vary over the following sets: $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, $-N_n - 1 < k < N_n' + 1$, where $N_n, N_n' \in \mathbb{N}_0 \cup \{\infty\}$. We choose the natural numbering with respect to k, that is, $J_n^{(k)}$ lies to the left of $J_n^{(k+1)}$. Then, if we denote the length of $J_n^{(0)}$ by ℓ_n, we have that*

$$2^{|k|-4}\ell_n \le \left| J_n^{(k)} \right| \le 2^{|k|-2}\ell_n.$$

In particular, we can have $N_n = \infty$ for at most one index n and also $N_n' = \infty$ for at most one n.

c) (detailed estimates on W, Q) *If $J = J_n^{(k)}$ for some n, k, then*

$$\int_J W^2(x)\, dx \le \frac{10^3}{|J|}, \quad \int_J |Q(x)|\, dx \le \frac{10^3}{|J|}.$$

d) (growth of the lengths) $\ell_n \ge 4E_n^{-1/2}$.

We give explicit constants in these estimates; this will avoid any misgivings one might possibly have about hidden dependencies of our constants. However, no attempt will be made to optimize these constants because their values do not matter to us here.

We start with some preparations. Lemmas 2.2 and 2.4 will be the basic ingredients to the proof of Theorem 2.1.

Lemma 2.2. *Suppose that $H_\pm \ge -\epsilon$ on $L_2(a, b)$ for some $\epsilon \ge 0$. Then, on (a, b), we can write $V = W' + Q$ where W, Q satisfy the following estimates:*

$$\int_a^b (\varphi(x)W(x))^2\, dx \le \epsilon \int_a^b \varphi^2(x)\, dx + \int_a^b \varphi'^2(x)\, dx$$

for all $\varphi \in H_1(a, b)$ with $\varphi(a) = \varphi(b) = 0$. Moreover,

$$|Q(x)| \le 2\left(\epsilon^{1/2} + \frac{1}{\operatorname{dist}(x, (a,b)^c)}\right)|W(x)|$$

for all $x \in (a, b)$.

Proof. The hypothesis that $H_\pm \ge -\epsilon$ implies that there are zero-free (on (a, b)) solutions u, v of $-u'' + Vu = -\epsilon u$ and $-v'' - Vv = -\epsilon v$, respectively. (We really have to take the open interval; if $\epsilon > 0$ has been taken as small as possible, at least one of the functions u, v has zeros at both endpoints. This explains why the estimates on Q get worse as we approach the endpoints.)

As in [5], define

$$\gamma = \frac{1}{2}\left(\frac{u'}{u} + \frac{v'}{v}\right), \quad W = \frac{1}{2}\left(\frac{u'}{u} - \frac{v'}{v}\right). \tag{2.1}$$

Then

$$\begin{aligned} \gamma' &= -\gamma^2 + \epsilon - W^2 \\ V &= W' + 2\gamma W \equiv W' + Q \end{aligned} \tag{2.2}$$

(2.2) is the Schrödinger equation, with potential $\epsilon - W^2$ and zero energy, in Riccati form. Since (2.1) explicitly displays a solution γ on all of (a,b) and since $\gamma = y'/y$ (or rather $y = \exp(\int^x \gamma))$ transforms back to the Schrödinger form of (2.2), it is clear that $-d^2/dx^2 + \epsilon - W^2 \geq 0$ on (a,b). The estimate on W now follows at once by using φ as a test function in the quadratic form of this operator.

To analyze γ, we compare with solutions γ_0 of $\gamma_0' = \epsilon - \gamma_0^2$. We begin with the case when $\epsilon > 0$. The following holds: If $\gamma(x) = \gamma_0(x)$ for some $x \in (a,b)$, then $\gamma_0(t) \leq \gamma(t)$ for all $t \leq x$ from this interval and $\gamma_0(t) \geq \gamma(t)$ if $t \geq x$. Now fix $x \in (a,b)$. Then, by what has just been noted, the value of γ at this x must have the property that the solution γ_0 with $\gamma_0(x) = \gamma(x)$ satisfies $\sup_{a+\delta \leq t \leq x} \gamma_0(t) < \infty$ and $\inf_{x \leq t \leq b-\delta} \gamma_0(t) > -\infty$ for all $\delta > 0$. It follows from these conditions that

$$-\epsilon^{1/2} \coth \epsilon^{1/2}(b-x) \leq \gamma(x) \leq \epsilon^{1/2} \coth \epsilon^{1/2}(x-a).$$

This can be seen by again writing the equation for γ_0 in Schrödinger form ($\gamma_0 = y'/y$, $y'' = \epsilon y$) and by noting that the largest (smallest) solution γ_0 comes from a y that has a zero at a (at b). Since $\coth x \leq 1 + x^{-1}$ for $x > 0$, we see that

$$|\gamma(x)| \leq \epsilon^{1/2} + \frac{1}{\operatorname{dist}(x,(a,b)^c)}. \tag{2.3}$$

This bound in fact works for all cases ($\epsilon \geq 0$). One can either take suitable limits or make slight adjustments in the above arguments. As $Q = 2\gamma W$, the asserted estimate on Q is an immediate consequence of (2.3). □

Next, we show that the eigenvalue $-\epsilon$ can already be seen on a length scale $L \sim \epsilon^{-1/2}$. We need the following calculation with quadratic forms (which appears to go back to Jacobi [11]; see also Courant-Hilbert [3, p. 458]):

Lemma 2.3. *Suppose $-f'' + Vf = Ef$, $\varphi \in H_1(a,b)$, $(\varphi^2 ff')(a) = (\varphi^2 ff')(b)$. Then*

$$\int_a^b \left[(\varphi f)'^2 + V(\varphi f)^2\right] = \int_a^b \varphi'^2 f^2 + E \int_a^b \varphi^2 f^2.$$

Proof. An integration by parts shows that

$$\begin{aligned}\int_a^b \varphi^2 f'^2 &= \varphi^2 ff'\Big|_a^b - \int_a^b f\left(\varphi^2 f'\right)' = -2\int_a^b \varphi\varphi' ff' - \int_a^b \varphi^2 ff'' \\ &= -2\int_a^b \varphi\varphi' ff' + \int_a^b \varphi^2(E-V)f^2.\end{aligned}$$

Plug this into

$$\int_a^b (\varphi f)'^2 = \int_a^b \varphi'^2 f^2 + 2\int_a^b \varphi\varphi' ff' + \int_a^b \varphi^2 f'^2$$

to obtain the lemma. □

Lemma 2.4. *Assume that the smallest eigenvalue of H_σ on (a,b) (call it $-\epsilon$) is negative. If $b - a \geq 6\epsilon^{-1/2}$, then there exists a subinterval $I \subset (a,b)$ of length $|I| = 6\epsilon^{-1/2}$ so that H_σ on I has an eigenvalue $\leq -\epsilon/2$.*

In contrast to the previous two lemmas, we now allow unbounded intervals (a,b) as well.

Proof. Let f be the corresponding eigenfunction, so $-f''+\sigma V f = -\epsilon f$ and $f=0$ at the finite endpoints of (a,b). Let $L=\epsilon^{-1/2}$, and pick a c that maximizes $\int_{c-L}^{c+L} f^2$. Define

$$\varphi(x) = \begin{cases} 1 & |x-c|\le L, \\ 3/2 - |x-c|/(2L) & L < |x-c| < 3L, \\ 0 & |x-c| \ge 3L. \end{cases}$$

We now take I as the support of φ, intersected with (a,b). It can of course happen that $|I| < 6\epsilon^{-1/2}$, but it certainly suffices to show that H_σ on I has an eigenvalue $\le -\epsilon/2$. We can then simply replace I by a larger interval $I' \supset I$ of the desired length. By the min-max principle, the eigenvalues will only go down.

The set $\{x\in I : \varphi'(x) \ne 0\}$ consists of at most two intervals of length $\le 2L$ each, so by the choice of c, we have that $\int_I \varphi'^2 f^2 \le (1/2L^2)\int_I \varphi^2 f^2$. The function φf is in the form domain of the operator H_σ on I (call this form Q_I), and Lemma 2.3 shows that

$$Q_I(\varphi f) = \int_I \varphi'^2 f^2 - \epsilon \int_I \varphi^2 f^2 \le \left(\frac{1}{2L^2} - \epsilon \right) \int_I \varphi^2 f^2 = -\frac{\epsilon}{2} \int_I \varphi^2 f^2.$$

Hence H_σ on I indeed has an eigenvalue $\le -\epsilon/2$. □

We are now ready for the proof of Theorem 2.1. It is probably advisable not to pay too much attention to the specific evaluation of the constants in the first reading of the following proof. In most cases, it is almost immediate that an inequality of the type $a \le Cb$ holds while finding a concrete value for such a C usually requires an additional (but elementary) calculation.

Proof of Theorem 2.1. We will present a method for inductively finding intervals with the required properties. In the main part of this proof, the symbols I_n, J_k will be used for these intervals and we will write $L_n = |I_n|$ for their lengths. We avoid using the letters from the statement of the theorem from the beginning, because our initial choices will be modified later on. However, it is true that the $J_n^{(0)}$'s from the theorem are essentially the I_n's, and the $J_n^{(k)}$'s for $k\ne 0$ correspond to suitable J_k's. Here is an outline of the method. Put $\epsilon_1 = E_1$. The basic idea is to use Lemma 2.2 to write $V = W_1' + Q_1$ with W_1, Q_1 satisfying certain inequalities. Then we remove an interval I_1, $|I_1| = 6\epsilon_1^{-1/2}$ with the properties stated in Lemma 2.4. On I_1, we keep the W_1, Q_1 just constructed. In the next step, we consider $H_\pm$ on $S_2 = \mathbb{R}\setminus I_1$. We update ϵ and obtain a new value ϵ_2. Note that $\epsilon_2 \le \epsilon_1$; in fact, after some steps the improvement must be substantial, because the min-max principle says that if $N(E)$ denotes the number of eigenvalues below $-E$ of the original operators, then there can be at most $N(E)$ disjoint intervals with ground state energies $\le -E$. On S_2, we construct a new $V = W_2' + Q_2$ decomposition with the help of Lemma 2.2. The idea is that since $\epsilon_2 \le \epsilon_1$, these new functions admit improved bounds. We again remove an interval of length $|I_2| = 6\epsilon_2^{-1/2}$ by using Lemma 2.4, and we continue on $S_3 = S_2 \setminus I_2$. On I_2, the functions W_2, Q_2 are our final choices. In reality, the procedure does not work quite as smoothly because the estimates coming from Lemma 2.2 get worse if I_n lies close to the boundaries of the current S_n.

We now give the details. Put $S_1 = \mathbb{R}$ and write the smallest eigenvalue of $H_\pm$ on S_1 as $-\epsilon_1$ ($\epsilon_1 > 0$). This first step is easier than the general step because most of the technical problems come from the influence of the complement of S_n, and $S_1^c = \emptyset$.

Apply Lemma 2.4 to obtain an interval $I_1 \subset S_1$ of length $L_1 \equiv 6\epsilon_1^{-1/2}$. H_σ for suitable $\sigma = \pm$ has an eigenvalue $\le -\epsilon_1/2$ on I_1. We also use Lemma 2.2 to write V as $V = W_1' + Q_1$. Define a test function φ as follows: $\varphi \equiv 1$ on $3I_1$, $\varphi \equiv 0$ outside $5I_1$, and φ is linear on the remaining two intervals. Here, we denote by kI the interval of length $k|I|$ with the same center as I. By using this φ, we obtain from Lemma 2.2 that

$$\int_{3I_1} W_1^2 \le \int \varphi^2 W_1^2 \le \epsilon_1 \int \varphi^2 + \int \varphi'^2.$$

Evaluate the integrals and recall that $\epsilon_1 = 36/L_1^2$. This gives the bound

$$\int_{3I_1} W_1^2 \le \frac{134}{L_1}. \tag{2.4}$$

To estimate Q, we again use Lemma 2.2. Observe that the second term in parentheses from the bound on Q is $\le L_1^{-1}$ on $3I_1$ if we take $(a,b) = 5I_1$. Hence the Cauchy-Schwarz inequality together with the bound on $\|W\|_{L_2(3I_1)}$ already proved show that

$$\int_{3I_1} |Q_1| \le \frac{281}{L_1}. \tag{2.5}$$

We put $S_2 = \mathbb{R} \setminus I_1$; this concludes the first step. Let us summarize what we have achieved: First, H_+ or H_- has an eigenvalue $\le -\epsilon_1/2$ on I_1. Second, we also have defined W, Q on I_1 and estimated these functions there. In fact, we have done this on the larger interval $3I_1$; this additional information will be useful later.

We now move on to the general step. The situation is as follows: Our set S_n is a collection of finitely many disjoint intervals, up to two of them being half lines (usually there will be exactly two half lines). For a bounded component $(a,b) \subset S_n$, there are intervals immediately to the left and right, respectively, of (a,b) that were generated in earlier steps. Call them I_- and I_+, respectively. By what has just been observed, they are also equal to I_j, I_k for suitable indices $j, k < n$. The construction we are about to describe makes sure that $b-a \ge 2L_-, 2L_+$, where the $\pm$ notation is used with the same meaning as above, that is, $L_- = L_j$ and $L_+ = L_k$. Moreover, in such a situation, we have $W' + Q$ decompositions with control of the type (2.4), (2.5) on $3I_\pm \cap (a,b)$. This follows from the fact that these intervals were generated in previous steps.

To run step n, first update ϵ. More specifically, write $-\epsilon_n$ for the smallest eigenvalue of the operators $H_\pm$ on S_n. Then, by the min-max principle, $S_n \subset S_{n-1}$ implies that $\epsilon_n \le \epsilon_{n-1}$. Choose a component (a,b) of S_n so that the eigenvalue $-\epsilon_n$ occurs there. We will assume that $b - a < \infty$; in the half-line case, the discussion is similar but easier. Put $L_n = 6\epsilon_n^{-1/2}$ and use Lemma 2.2 with $\epsilon = \epsilon_n$ to define W_n, Q_n on (a,b). Several cases arise:

a) *No conflict with the boundary:* This is the easy case because the machine based on Lemmas 2.2 and 2.4 works smoothly. The precise condition we will use is that $7\widetilde{I}_n \subset (a,b)$, with $\widetilde{I}_n$ chosen according to Lemma 2.4.

Let $I_n = \widetilde{I}_n$. Lemma 2.2 with a tent-shaped test function supported on $5I_n$ shows that we have the following analogs of (2.4), (2.5):

$$\int_{3I_n} W_n^2 \le \frac{134}{L_n}, \quad \int_{3I_n} |Q_n| \le \frac{281}{L_n}. \tag{2.6}$$

Let $S_{n+1} = S_n \setminus I_n$ and proceed with step $n+1$. Note that the two newly generated intervals $(a,b)\setminus I_n$ satisfy the condition that their length is at least twice the length of their neighbors. This obviously holds if this neighbor is taken to be I_n, and it also holds for their other neighbors because these intervals come from earlier steps and $L_j \le L_n$ if $j \le n$. This inequality in turn follows from the definition $L_i = 6\epsilon_i^{-1/2}$ and the fact that ϵ_i is non-increasing.

b) *The extreme opposite of case a):* Assume now that $b - a < L_n$, so that applying Lemma 2.4 is out of the question. We will just remove the whole interval (a,b) and obtain control on W_n, Q_n by what we call the *boundary method.* To (slightly) simplify the notation, we relabel $a \to 0$, $b \to 2L$ for the time being. Then $L_\pm \le L$. We also drop the index n for the functions W_n, Q_n.

Lemma 2.5 (The boundary method)**.** *In the situation described above, determine $L_0 \in (L_-/4, L_-/2]$ so that $2^N L_0 = L$ for some $N \in \mathbb{N}$. Define $J_k = [2^{k-1}L_0, 2^k L_0]$ $(k = 1, \ldots, N)$. Then*

$$\int_{J_k} W^2 \le \frac{6}{|J_k|}, \quad \int_{J_k} |Q| \le \frac{13}{|J_k|}. \tag{2.7}$$

Proof of Lemma 2.5. This follows as above (compare (2.6)) from a straightforward application of Lemma 2.2 with test functions of tent form supported by $3J_k$ and equal to 1 on J_k. □

The estimates (2.7) are worse than (2.6) in that we need to cut the interval $[0, L]$ into smaller pieces. The reason for this is that we are close to possibly large eigenvalues (on I_-).

We can now apply Lemma 2.5 and the analog of this from the right to get estimates on W_n, Q_n on $(a + L_-/2, b - L_+/2)$. An additional issue needs to be addressed: We must match W_n with the W's coming from $I_\pm$.

Lemma 2.6. *Suppose that $V = w' + q$ on $[c,d]$. It is then possible to define new functions W, Q so that we still have $V = W' + Q$, but also $W(c) = 0$. Moreover, the following estimates can be achieved:*

$$|W(x)| \le |w(x)| \quad (c \le x \le d), \qquad \int_c^d |Q(x)|\,dx \le \int_c^d |q(x)|\,dx + 2|w(c)|.$$

Proof of Lemma 2.6. Define, for $t > 0$ (and typically small),

$$\chi(x) = \begin{cases} 1 & x > c + t, \\ (1/t)(x - c) & c \le x \le c + t, \end{cases}$$

and let $W = \chi w$, so $Q = q + ((1-\chi)w)'$. Clearly, $|W| \le |w|$, and

$$\int_c^d |Q| - \int_c^d |q| \le \int_c^{c+t} |w'| + \int_c^d |\chi' w| \le \int_c^{c+t} (|V| + |q|) + (1/t)\int_c^{c+t} |w| \to |w(c)|$$

as $t \to 0+$. So if $w(c) \ne 0$, a sufficiently small $t > 0$ will give the desired estimates. □

The lemma is useful in our situation because a weak-type estimate shows that W is small most of the time. More precisely, consider the interval $J = (L_-/2, L_-)$. Then, writing W_- for the W obtained in the step in which the interval I_- was removed, it follows from (2.6) that

$$\left|\{x \in J : |W_-(x)| > CL_-^{-1}\}\right| \le \frac{134}{C^2}\, L_-. \tag{2.8}$$

Since $L \ge 2L_-$, this estimate also holds with W_n in place of W_-. In particular, since $134/24^2 < 1/4$, there exists an $x_0 \in J$ so that $|W_-(x_0)|, |W_n(x_0)| \le 24L_-^{-1}$. We can now apply Lemma 2.6 (and its mirror version to the left) with $c = x_0$ to obtain a modification of W_-, W_n, Q_-, Q_n in a neighborhood of x_0 with both W's now vanishing at x_0. In particular, we can now change from W_- to W_n at x_0 without destroying the absolute continuity of W.

Since $L_0 \le x_0 < 4L_0$, we have that $x_0 \in J_k$ either for $k = 1$ or for $k = 2$. Fix this k. Then

$$\int_{J_k} |Q| \le \frac{13}{|J_k|} + \frac{281}{L_-} + \frac{4 \cdot 24}{L_-}.$$

Indeed, the first two terms on the right-hand side are the old bounds on $\int |Q_n|$ and $\int |Q_-|$, respectively, while the last term collects the contributions $2|W_n(x_0)|$, $2|W_-(x_0)|$, which come from Lemma 2.6. Since $L_- \ge 2L_0 \ge |J_k|$, we finally obtain $\int_{J_k} |Q| \le 390/|J_k|$. Similarly, $\int_{J_k} W^2 \le 140/|J_k|$. We do not want to bother about whether or not different W's have been matched on a given interval, so we simply change the constants for all k to these new values.

So far, we have treated the left half of (a, b) (temporarily denoted by $(0, 2L)$ for convenience). Now apply the mirror version of this to the right half of (a, b).

We summarize: We have basically subdivided (a, b) into two series of intervals J_k (one coming from the left, the other from the right) with geometrically increasing lengths $\approx 2^k L_\pm$. We qualify this by saying "basically" because we also modify the neighboring intervals $I_\pm$: We add a piece of length L_0, where L_0 has the same meaning as above (and of course depends on whether we are considering I_- or I_+). This is exactly that part of (a, b) that has not been covered by the J_k's. For later use, we record this modification and call the enlarged intervals $I_\pm^{(1)}$. Note, however, that the original intervals might also be used again. More specifically, this happens if the boundary method is also applied to the left of I_- (or to the right of I_+) at a later stage. If, conversely, this has happened before step n, then we already have modifications of these intervals and we then denote the new modification by $I_\pm^{(2)}$. (In fact, these conventions are a bit pedantic and we could also discard the original intervals right away except that then the condition that components of S_{n+1} are at least twice the size of the neighboring I_k's may be violated.)

We have extended the $V = W' + Q$ representation to the J_k's, with the following estimates on W and Q:

$$\int_{J_k} W^2(x)\, dx \le \frac{140}{|J_k|}, \quad \int_{J_k} |Q(x)|\, dx \le \frac{390}{|J_k|}. \tag{2.9}$$

We put $S_{n+1} = S_n \setminus [a, b]$ and proceed with step $n + 1$.

c) $\widetilde{I}_n$ *close to precisely one boundary point:* This case arises when $a \in 7\widetilde{I}_n$, $b \notin 7\widetilde{I}_n$ or conversely. Let us assume the first situation. Also recall that $\widetilde{I}_n$, as always in this proof, is a subinterval of $[a, b]$ of length $L_n = 6\epsilon_n^{-1/2}$ chosen according to Lemma 2.4. We let $I_n = L_n + \widetilde{I}_n$, that is, we take the copy of $\widetilde{I}_n$ immediately to the right of the original choice. We apply the boundary method (Lemma 2.5) to obtain control of the type (2.7) on $[a+L_0, \inf I_n]$, where $L_-/4 < L_0 \le L_-/2$. In other words, we cover this interval, which has a length between $L_n/2$ and $4L_n$, by a series of intervals J_k of geometrically increasing lengths $|J_k| = 2^{k-1}L_0$. Recall also in this context that $L_- \le L_n$. Next, we again match W_- and W_n on $(a+L_-/2, a+L_-)$ with the help of Lemma 2.6. As explained above, this leads to the estimates (2.9). Finally, in analogy to case b), we also modify the left neighbor I_- by adding $[a, a+L_0]$ and call the new interval $I_-^{(k)}$ if this was the kth modification (so $k=1$ or $k=2$).

On the remaining part of $3I_n$, we use Lemma 2.2 with a suitable tent-shaped test function φ. More precisely, $\varphi = 1$ on this set, that is, on the right two-thirds of $3I_n$. Moreover, $\varphi = 0$ outside the interval of size $4L_n$ that is centered at the right endpoint of I_n. We thus see that

$$\int W_n^2(x)\, dx \le \frac{98}{L_n}, \quad \int |Q_n(x)|\, dx \le \frac{196}{L_n}, \tag{2.10}$$

where the integrations are over I_n and the interval to the right of I_n of the same size (in other words, over the part of $3I_n$ not already treated by the boundary method). The constants are smaller here because this set is also smaller than $3I_n$. However, we will not insist on this; rather, we use the values from (2.6) in this case as well.

We delete I_n and everything to the left of I_n (inside $[a,b]$, that is) to obtain our new set S_{n+1}. The final choice of W, Q has also been described above (W_n, Q_n except on an initial piece of size $\approx L_-$). Note that S_{n+1} now has the same number of components as S_n. The interval (a,b) was replaced by (a_1, b) with $a_1 > a$. However, since $b \notin 7\widetilde{I}_n$, the new component (a_1, b) still satisfies our condition that its length is at least twice the lengths corresponding to the neighboring intervals. For the right neighbor, this again follows from the fact that this right neighbor equals I_j for a suitable $j < n$ and $L_j \le L_n$.

d) $\widetilde{I}_n$ *close to both boundary points:* In other words, $a, b \in 7\widetilde{I}_n$, but, in contrast to case b), $L_n \le b - a$. This case does not require new ideas; indeed, d) and b) could have been subsumed under one case. Use the boundary method from both a and b, and match W_n with W_- and W_+. We obtain two series of intervals J_k and the estimates from (2.9) on W, Q on the intervals J_k. We define, as usual, $S_{n+1} = S_n \setminus [a,b]$. So we have in fact just removed the component (a,b).

All cases have been covered now and we can run the algorithm. One final adjustment is necessary, however. It may happen that S_n for some n contains components (a,b) with $H_\pm \ge 0$ there. As long as at least one of $H_\pm$ has negative eigenvalues on S_n, such an interval will not be dealt with by the algorithm. If $b - a < \infty$, we treat these intervals with the method of case b) as soon as they arise. This is in fact the obvious thing to do because formally $L = 6\epsilon^{-1/2} = \infty$ (since $\epsilon = 0$ by assumption). So the boundary method gives us two sequences of intervals J_k with the usual properties (see Lemma 2.5 and the discussion of case b)). We then continue the algorithm with a new S_n from which these components (a,b) have been removed.

Similarly, if $H_\pm \ge 0$ on a half line (a,b), the boundary method produces an infinite series of intervals J_k that cover (a,b). The usual estimates on W and Q hold, and $|J_k| \approx 2^{|k|}L$, where L is the length of the neighbor of (a,b). We again remove (a,b) from S_n and continue.

We can now be sure that our inductive construction produces intervals that cover all of $\mathbb{R}$. Indeed, intervals (a,b) on which $H_\pm \geq 0$ are treated immediately, and if H_σ has a negative eigenvalue $-\epsilon$ on (a,b), the min-max principle guarantees that there are only finitely many other intervals with eigenvalues $< -\epsilon$, and after the corresponding number of steps, at the latest, the algorithm will take care of (a,b). It may of course happen that only part of (a,b) is removed then, but the length of this removed part is at least $\min\{L_n, b-a\}$, and $L_n = 6\epsilon_n^{-1/2}$ increases, so a finite number of such steps will suffice to cover all of (a,b).

To finally verify the assertions of Theorem 2.1, we first need to relabel our intervals. The $J_n^{(0)}$'s are essentially the intervals I_n that arise in cases a) and c), but with possible later modifications taken into account. Recall that these modifications occur if the boundary method is applied at a later stage with I_n taking the role of a left or right neighbor of the interval currently under consideration. In other words, we let $J_n^{(0)} = I_n^{(j)}$, with j (which counts the number of modifications) maximal. Recall also that each of these modifications consists of adding an interval of length L, with $L_n/4 < L \leq L_n/2$, to I_n. Moreover, in such an application of the boundary method, a new series of intervals is generated, with geometrically increasing lengths. These new intervals lie to the left, respectively right, of $J_n^{(0)}$. They are now called $J_n^{(k)}$, with $k \leq -1$ in the first case and $k \geq 1$ in the second case. If an I_n is modified twice, we also obtain two series of intervals $J_n^{(k)}$, one for $k \leq -1$ and another one for $k \geq 1$.

We define ℓ_n as the length of the final interval $J_n^{(0)}$. Then $J_n^{(0)}$ is contained in the larger interval on which the estimates on W, Q were originally established – compare (2.6), (2.10); for example, $J_n^{(0)} \subset 3I_n$ if we were in case a) at that step. Since also $L_n \leq \ell_n \leq 2L_n$, we obtain the following final estimates on the intervals $J_n^{(0)}$:

$$\int_{J_n^{(0)}} W^2(x)\, dx \leq \frac{268}{\ell_n}, \quad \int_{J_n^{(0)}} |Q(x)|\, dx \leq \frac{562}{\ell_n}.$$

Finally, we renumber everything so that $\ell_1 \leq \ell_2 \leq \cdots$. In particular, there are no gaps in this sequence now while in the original numbering, it can happen that there are no $J_n^{(k)}$'s for a given $n \in \mathbb{N}$ (namely, if we were in case b) or d) at step n).

These intervals $J_n^{(k)}$ certainly have the properties stated in part b) of Theorem 2.1. Moreover, c) just records the estimates we have obtained above, with all constants generously replaced by 10^3. It is also clear that d) holds: Indeed, H_+ or H_- on $J_n^{(0)}$ has an eigenvalue $-E$ satisfying

$$E \geq \frac{\epsilon_n}{2} = \frac{18}{L_n^2} \geq \frac{18}{\ell_n^2}.$$

By the min-max principle, the eigenvalues can only go up if Dirichlet boundary conditions are introduced. Thus, since the $J_n^{(0)}$'s are disjoint and since the sequences ℓ_n and $E_n^{-1/2}$ are non-decreasing, we must have that $\ell_n \geq 3\sqrt{2}E_n^{-1/2} > 4E_n^{-1/2}$, as claimed.

It is clear from the construction that W is absolutely continuous, and the remaining assertions of part a) now follow by summing the bounds on $\int W^2$, $\int |Q|$. We of course use the now established part d) here. $\square$

We now discuss half-line problems with Dirichlet boundary conditions at the origin. Later, in Section 6, we will also discuss a variant of the procedure we are about to describe for Neumann boundary conditions in a related but simpler situation.

The additional issue that needs to be addressed here is the question of how an initial interval $[0, L]$ should be handled. We start with a tent φ supported by $[0, 3L_1]$ and equal to one on $[L_1, 2L_1]$, where $L_1 = \epsilon_1^{-1/2}$. This gives the bounds

$$\int_{L_1}^{2L_1} W^2(x)\, dx \leq \frac{4}{L_1}, \quad \int_{L_1}^{2L_1} |Q(x)|\, dx \leq \frac{8}{L_1}. \tag{2.11}$$

We now simply admit that we do not have such estimates on $[0, L_1]$, remove this interval nevertheless, and continue as in the proof of Theorem 2.1. There is also no guarantee that H_+ or H_- will have a small eigenvalue on $[0, L_1]$, so we really remove this interval without having achieved anything there. However, now that this has been done, we are exactly in the situation from the proof of Theorem 2.1. If, at some point, one of our intervals $\widetilde{I}_n$ lies close to L_1, we now can apply the boundary method without any modifications because (2.11) gives us the required a priori control in a neighborhood of the boundary point L_1.

3. Prüfer Variables

Some of our results will depend on an analysis of the solutions to the Schrödinger equation (1.1). We are interested in positive energies E, and we write $E = k^2$, with $k > 0$. In order to study solutions $y(x, k)$ of (1.1), we use the following Prüfer-type variables; these are particularly well adapted to the situation where $V = W' + Q$. One may also say that we will not study (1.1) directly, but rather an associated Dirac system. In any event, introduce the solution vector Y as

$$Y(x, k) = \begin{pmatrix} y(x, k) \\ (y'(x, k) - W(x)y(x, k))/k \end{pmatrix},$$

and write

$$Y(x, k) = R(x, k) \begin{pmatrix} \sin(\psi(x, k)/2) \\ \cos(\psi(x, k)/2) \end{pmatrix},$$

with $R(x, k) > 0$ and $\psi(x, k)$ continuous in x. A computation shows that R, ψ obey the following equations:

$$(\ln R(x, k))' = -W(x) \cos \psi(x, k) + \frac{Q(x) - W^2(x)}{2k} \sin \psi(x, k), \tag{3.1}$$

$$\psi'(x, k) = 2k + 2W(x) \sin \psi(x, k) + \frac{Q(x) - W^2(x)}{k} (\cos \psi(x, k) - 1). \tag{3.2}$$

The last term in both equations is integrable if $\sum E_n^{1/2} < \infty$ and thus does not change the asymptotics.

For later use, we also note that Y solves the first-order Dirac system

$$Y'(x, k) = \left[\begin{pmatrix} W(x) & k \\ -k & -W(x) \end{pmatrix} + \frac{Q(x) - W^2(x)}{k} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] Y(x, k).$$

Again, the L_1 term will be treated as a perturbation, and thus we will also study the solutions Y_0 of the unperturbed system

$$Y_0'(x, k) = \begin{pmatrix} W(x) & k \\ -k & -W(x) \end{pmatrix} Y_0(x, k). \tag{3.3}$$

If we change the independent variable as follows,

$$Y_0(x,k) = \begin{pmatrix} e^{ikx} & e^{-ikx} \\ ie^{ikx} & -ie^{-ikx} \end{pmatrix} Z(x,k),$$

this becomes

$$Z'(x,k) = W(x) \begin{pmatrix} 0 & e^{-2ikx} \\ e^{2ikx} & 0 \end{pmatrix} Z(x,k). \tag{3.4}$$

4. Proof of Theorem 1.3

Theorem 2.1 shows that we can find disjoint intervals I_n with lengths $L_n \to \infty$, so that on I_n, we can write $V = W' + Q$ with $\int_{I_n} W^2, \int_{I_n} |Q| \lesssim L_n^{-1}$. Actually, this only requires Lemmas 2.2 and 2.4 and not the full-fledged method from the proof of Theorem 2.1. The easiest way to obtain such I_n's is to work on the remaining half line at each step.

The Prüfer equation (3.2) shows that

$$\psi(a_n,k) - \psi(a_{n-1},k) = 2kL_n + 2\int_{a_{n-1}}^{a_n} W(x)\sin\psi(x,k)\,dx + O(L_n^{-1}).$$

Here $(a_{n-1},a_n) = I_n$ is one of the intervals from the preceding paragraph, and the constant implicit in $O(L_n^{-1})$ remains bounded if $k > 0$ stays away from zero.

Since $\|W\|_{L_1(I_n)} \le L_n^{1/2}\|W\|_{L_2(I_n)} \lesssim 1$, it follows that

$$\psi(a_n,k) - \psi(a_{n-1},k) = 2kL_n + O(1), \tag{4.1}$$

with uniform control on the error term for $k \ge k_0 > 0$.

Since $y(x) = 0$ precisely if $\psi(x) = 2n\pi$ with $n \in \mathbb{Z}$ and thus $\psi'(x) = 2k > 0$ at such a point, the Prüfer angle ψ may be used to count the zeros of y. Now suppose that $0 < k_1 < k_2$. Then (4.1) and the above remarks show that for sufficiently large n, every solution $y(\cdot,k_2)$ of the equation with $E = k_2^2$ has more zeros on I_n than any non-trivial solution $y(\cdot,k_1)$ for $E = k_1^2$. By Sturm comparison, $y(\cdot,k_2)$ then also has more zeros than $y(\cdot,k_1)$ on $(0,x)$ for all large x. Hence, by oscillation theory again, $[k_1,k_2] \cap \sigma \ne \emptyset$. This holds whenever $0 < k_1 < k_2$, so $\sigma \supset [0,\infty)$. □

5. Spectral Properties

In this section, we prove Theorems 1.4, 1.5, and 1.6. The first two proofs follow ideas of Deift-Killip [7] and Christ-Kiselev [1, 2], respectively, rather closely. Our presentation will be very sketchy in these cases.

Let us begin with the *proof of Theorem 1.4.* Theorem 2.1 shows that $V = W' + W^2 + Q - W^2 \equiv W' + W^2 + V_0$, with $V_0 \in L_1$. This is a perturbation that is of relative trace class in the form sense, so it suffices to consider the modified potential $\widetilde{V} = W' + W^2$. Note that the form of $\widetilde{V}$ guarantees that $-d^2/dx^2 + \widetilde{V}$ on $L_2(\mathbb{R})$ has no negative eigenvalues. Let us now temporarily assume that W is also of compact support $\subset (0,\infty)$. Then, as $\int W' = 0$, the first Faddeev-Zakharov trace formula [27] reads

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \ln(1 - |r(k)|^2)\,dk = -\int_0^{\infty} W^2(x)\,dx. \tag{5.1}$$

Here, the reflection coefficient r is defined as $r = b/a$, where $f = ae^{ikx} + be^{-ikx}$ is the expansion of the solution f close to zero that is equal to e^{ikx} to the right of the support of W. Now, given (5.1), one can prove Theorem 1.4 by following the arguments of [7]: Approximate the actual W in L_2 by compactly supported W_n's. The sum rule (5.1) provides a uniform lower bound on the spectral densities of these problems. One then also needs weak $*$ convergence of the spectral measures. We can again (as in [7]) deduce this from the locally uniform convergence of the m functions on the upper half plane. This can most conveniently be established in the framework of the associated Dirac system (3.3). Since a compactly supported perturbation does not change the absolutely continuous spectrum, we may also assume that $0 \notin \operatorname{supp} W$. This is helpful here because then $Y(0) = (y(0), y'(0)/k)^t$, and hence the m functions of the original Schrödinger equation and the associated Dirac system are directly related. We conclude our sketch of the proof with these remarks. The reader may also wish to consult the introduction of [18] for further background information. □

We now want to analyze the situation under the stronger hypotheses of Theorems 1.5 and 1.6. The notation from Theorem 2.1 is too clumsy for this purpose. So we reorganize the intervals $J_n^{(k)}$ once more and now simply denote them by $I_n = (a_{n-1}, a_n)$, with $0 = a_0 < a_1 < \cdots$. We also write $L_n = |I_n|$. So Theorem 2.1c) (or rather the half-line version discussed at the end of Section 2) now says that

$$\int_{I_n} W^2 \lesssim L_n^{-1}, \qquad \int_{I_n} |Q| \lesssim L_n^{-1} \quad (n \geq 2). \tag{5.2}$$

The following observation is essential for what follows: If $\sum E_n^p < \infty$ ($p > 0$), then also $\sum L_n^{-2p} < \infty$. To prove this, note that by their definition, the lengths L_n are a rearrangement of $|J_n^{(k)}|$. Moreover, Theorem 2.1b), d) shows that $|J_n^{(k)}| \gtrsim 2^{|k|} E_n^{-1/2}$. Thus

$$\sum_{n=1}^{\infty} L_n^{-2p} \lesssim \sum_{n=1}^{\infty} \sum_{k=-N_n}^{N'_n} 2^{-2p|k|} E_n^p,$$

and our claim follows.

Now, given Theorem 2.1, the *proof of Theorem 1.5* consists of not much more than a quotation: Assume that $\sum E_n^p < \infty$ with $0 < p < 1/2$. Hölder's inequality shows that

$$\int_{I_n} |W|^{2p+1} \leq \left(\int_{I_n} W^2 \right)^{p+1/2} L_n^{1/2-p} \lesssim L_n^{-2p}.$$

Thus $W \in L_q$ for some $q < 2$. Now the machinery of Christ-Kiselev [1, 2] gives the desired asymptotics for the solutions Y_0 of the unperturbed system (3.3); see especially the system (2.2) from [1] and the analysis that follows. Here, we instead use (3.4) as our starting point. Actually, the situation is simpler than in [1] because we have Fourier transforms instead of WKB transforms. To also obtain the corresponding asymptotics for Y, one can finally use a standard perturbative argument based on Levinson's Theorem [8, Theorem 1.3.1] (compare again [1]). □

Proof of Theorem 1.6. We now assume that $\sum E_n^p < \infty$, with $0 < p < 1/4$. As observed above, this implies that $\sum L_n^{-2p} < \infty$. We will use ideas from [5, 21, 23]. Also, the following estimate will be a crucial ingredient.

Lemma 5.1. *Let $\alpha \in (0,1)$. Then there exists a constant C_α so that for all $f \in L_2(a,b)$, all finite Borel measures μ on $\mathbb{R}$, and all measurable functions c with $a \le c(k) \le b$, the following estimate holds ($L \equiv b-a$):*

$$\int d\mu(k) \left| \int_a^{c(k)} dx\, f(x) e^{2ikx} \right| \le C_\alpha \mathcal{E}_\alpha^{1/2}(\mu) L^{(1-\alpha)/2} \|f\|_{L_2(a,b)}.$$

Here $\mathcal{E}_\alpha(\mu) \equiv \iint d\mu(k)\, d\mu(l)(1+|k-l|^{-\alpha})$ denotes the α energy of μ.

This follows by slightly adjusting the calculation from [28, vol. II, pg. 196], so we will not give the proof here. Compare also [5, Lemma 7.6].

Note that if for some $E > 0$, we have two linearly independent solutions for which the limits $\lim_{x\to\infty} R(x,k)$, $\lim_{x\to\infty}(\psi(x,k) - 2kx)$ exist, then it follows (by taking a suitable linear combination) that $E \notin S$. Thus it suffices to show that for arbitrary but fixed initial values $R(0,k)$, $\psi(0,k)$, these limits exist off a set of dimension at most $4p$. We will split the proof of this into two parts. This is not really necessary, but it will help to make the presentation more transparent. In the first step, we will show that the limits exist on the subsequence a_n as long as we stay off an exceptional set. In the second step, we will extend this to sequences tending to infinity arbitrarily. Actually, rather similar arguments are applied in both steps, so the second step will not be very difficult once we have completed the first step.

So in this first step, we are concerned with the series

$$\sum_{n=1}^{\infty} \left| \int_{a_{n-1}}^{a_n} W(x) e^{i\psi(x,k)}\, dx \right|. \tag{5.3}$$

Indeed, real and imaginary parts of the integrals give us the leading terms from the equations for R and $\psi - 2kx$, respectively; see (3.1), (3.2). So it suffices to show that this series converges off an exceptional set of dimension $\le 4p$.

We will need control on the maximal function

$$M_n(k) \equiv \max_{a_{n-1}\le c \le a_n} \left| \int_{a_{n-1}}^{c} W(x) e^{2ikx}\, dx \right|.$$

Let μ be any (Borel) measure with finite $4p$ energy. Since $\|W\|_{L_2(a_{n-1},a_n)} \lesssim L_n^{-1/2}$ by (5.2), Lemma 5.1 says that $\|M_n\|_{L_1(\mu)} \lesssim L_n^{-2p}$. Now $\sum L_n^{-2p} < \infty$, so the Monotone Convergence Theorem shows that $M_n(k) \in \ell_1$ for μ almost every k. Since μ is only assumed to have finite $4p$ energy but is otherwise arbitrary, it follows that $\dim S_0 \le 4p$, where

$$S_0 = \left\{ k > 0 : \sum_{n=1}^{\infty} M_n(k) = \infty \right\}.$$

This conclusion is nothing but a standard relation between capacities and Hausdorff dimensions; for example, one can argue as follows: Suppose that, contrary to our claim, $\dim S_0 > 4p$, and fix $d \in (4p, \dim S_0)$. Then, since S_0 is a Borel set of infinite d-dimensional Hausdorff measure, there exists a finite measure $\mu \ne 0$ supported by S_0 with $\mu(I) \le C|I|^d$ for all intervals $I \subset \mathbb{R}$ [9, Theorem 5.6]. It is easily seen that $\mathcal{E}_{4p}(\mu) < \infty$ for such a μ. So what we have shown above now says that $M_n(k) \in \ell_1$ for μ almost every k, which clearly contradicts the fact that μ is supported by S_0.

We now claim, more specifically, that (5.3) converges if $k \notin S_0$. Write $\psi = 2kx + \varphi$, and consider one of the integrals $\int_{a_{n-1}}^{a_n} W(x)e^{i\psi(x,k)}\, dx$ from (5.3). Integration by parts gives

$$\int_{a_{n-1}}^{a_n} W(x)e^{i\psi(x,k)}\, dx = e^{i\varphi(a_n,k)} \int_{a_{n-1}}^{a_n} W(x)e^{2ikx}\, dx$$
$$- i \int_{a_{n-1}}^{a_n} dx\, e^{i\varphi(x,k)} (2W(x)\sin\psi(x,k) + \rho(x)) \int_{a_{n-1}}^{x} dt\, W(t)e^{2ikt}.$$

We have abbreviated the integrable term as $\rho = (1/k)(Q - W^2)(\cos\psi - 1)$. The first term on the right-hand side is summable for $k \notin S_0$, so we must now show that the series over the second term on the right-hand side is also absolutely convergent for these k. This, however, is immediate from the bound

$$\left(2\|W\|_{L_1(a_{n-1},a_n)} + \|\rho\|_{L_1(a_{n-1},a_n)}\right) M_n(k)$$

on this term since $\|W\|_{L_1(a_{n-1},a_n)} \lesssim 1$.

So we know now that $\lim_{n\to\infty} R(a_n,k)$, $\lim_{n\to\infty}(\psi(a_n,k) - 2ka_n)$ exist for all $k \notin S_0$. As the second step of the proof of Theorem 1.6, we need to extend this to sequences tending to infinity in an arbitrary way. This suggests that we look at

$$\widetilde{M}_n(k) \equiv \max_{a_{n-1}\le c\le a_n} \left| \int_{a_{n-1}}^{c} W(x)e^{i\psi(x,k)}\, dx \right| .$$

We claim that $\widetilde{M}_n(k) \to 0$ if $k \notin S_0$. This will complete the proof of Theorem 1.6 because it will then follow that $S \subset S_0$.

To prove our claim on $\widetilde{M}_n(k)$, we proceed as above and reduce matters to the corresponding statement on $M_n(k)$, which we know is true. This does not require new ideas. We just repeat the above computations, but with the upper limit a_n now replaced by $c = c(k)$. Everything goes through as before, and we have in fact proved the stronger statement that $\widetilde{M}_n(k) \in \ell_1$ if $k \notin S_0$. □

6. Sign-Definite Potentials

We prove Theorem 1.8 here. We use the strategy from Section 2. The treatment simplifies considerably because there is no need to resort to Lemma 2.2. Indeed, if $H_+ \ge -\epsilon$ on I, then

$$-\int V\varphi^2 \le \epsilon \int \varphi^2 + \int \varphi'^2 \tag{6.1}$$

for all test functions $\varphi \in H_1(I)$ that vanish at the finite endpoints of I. Since $V \le 0$ now, this may be used to bound the L_1 norm of V over suitable intervals. Therefore, the proof of Theorem 2.1 now produces intervals $J_n^{(k)}$ with the same geometry as before, and $\int_J |V| \lesssim 1/|J|$. In particular, if $\sum E_n^p < \infty$, then also $\sum |J_n^{(k)}|^{-2p} < \infty$, and this latter sum may be estimated by a multiple of the first sum. The constant only depends on p. This follows as usual from $|J_n^{(k)}| \gtrsim 2^{|k|} E_n^{-1/2}$ by first summing over k and then over n.

Theorem 1.8a) for the whole-line problem is now immediate from Hölder's inequality, which gives, for $0 < p \le 1/2$,

$$\int_J |V|^{p+1/2} \le \left(\int_J |V|\right)^{p+1/2} |J|^{1/2-p} \lesssim |J|^{-2p}.$$

The proof of part b) is similar. Since $|J_n^{(k)}| \gtrsim E_1^{-1/2} \geq E_0^{-1/2}$, we have that for $p \geq 1/2$,

$$\sum \left(\int_n^{n+1} |V| \right)^{2p} \lesssim \sum_{n,k} \left(\int_{J_n^{(k)}} |V| \right)^{2p}$$

with a constant that depends on E_0.

To prove Theorem 1.8 on the half line with Neumann boundary conditions at the origin, we use a variant of the argument from the end of Sect. 2. Namely, let again $L_1 = \epsilon_1^{-1/2}$. However, due to the Neumann boundary conditions (instead of Dirichlet), we can use a test function φ now defined as follows: $\varphi = 1$ on $(0, 2L_1)$, $\varphi = 0$ on $(3L_1, \infty)$, and φ is linear on the remaining piece. Then (6.1) yields

$$\int_0^{2L_1} |V(x)|\, dx \leq \frac{4}{L_1}.$$

As discussed at the end of Sect. 2, we can remove $(0, L_1)$ and run the algorithm from the proof of Theorem 2.1 on the remaining half line. (In particular, we impose *Dirichlet* boundary conditions at $x = L_1$.) In contrast to the situation there, we now have control on V on $(0, L_1)$ as well. So Theorem 1.8 for the half line with Neumann boundary conditions now follows as above. $\square$

7. Counterexamples

In this section, we prove Theorems 1.7 and 1.10. In both cases, we use sparse potentials and rely on previous work on the spectral properties of these models [13, 22]. We then need control on the discrete spectrum, but, fortunately, this is easy. We consider two different types of bumps. Let

$$V_g(x) = -g\chi_{(-1,1)}(x), \quad W_g(x) = g\left(\chi_{(-1,0)}(x) - \chi_{(0,1)}(x)\right).$$

Lemma 7.1. a) *For small $g > 0$, the operator $-d^2/dx^2 + V_g(x)$ on $L_2(\mathbb{R})$ has precisely one eigenvalue $-E$, and $E = E(g) = g^2 + O(g^3)$.*

b) *For small $g > 0$, the operator $-d^2/dx^2 + W_g(x)$ on $L_2(\mathbb{R})$ has precisely one eigenvalue $-E$, and $E = E(g) = g^4/9 + O(g^5)$.*

Sketch of the proof. It is clear that $-d^2/dx^2 + V_g(x)$ has at most one eigenvalue for small $g > 0$. This follows from an elementary analysis of the solutions at zero energy (the number of zeros on $(-N, \infty)$ of a solution y with $y(-N) = 0$ is the number of negative eigenvalues of the operator on that interval). Since $W_g \geq V_g$, $-d^2/dx^2 + W_g(x)$ has at most one eigenvalue for small g. That the operators have at least one eigenvalue for $g > 0$ is a classical result for sign-definite potentials [20, Theorem XIII.11] and follows from recent work [4, 6] (and, incidentally, also from our Lemma 2.2) for arbitrary potentials.

To approximately compute this eigenvalue $-E$, we note that the corresponding eigenfunction y must be a multiple of $e^{-E^{1/2}|x|}$ for $x \leq -1$ and $x \geq 1$; the constant factors may be different on these two half lines. So $-E$ is an eigenvalue precisely if there exists $c \in \mathbb{R}$ so that

$$c \begin{pmatrix} 1 \\ -E^{1/2} \end{pmatrix} = T_g(1, -1; -E) \begin{pmatrix} 1 \\ E^{1/2} \end{pmatrix}. \tag{7.1}$$

Here, T_g is the transfer matrix, that is, the matrix that takes solution vectors $(y, y')^t$ at $x = -1$ to their value at $x = 1$. We of course have explicit formulae for the transfer matrices for V_g and W_g, respectively, and a somewhat cumbersome but completely elementary discussion of (7.1) then establishes the asserted asymptotics of E. $\square$

Now consider a (half-line) potential V of the form

$$V(x) = \sum_{n=1}^{\infty} V_{g_n}(x - x_n), \tag{7.2}$$

with $g_n \to 0$. The x_n's are typically very rapidly increasing so that the individual bumps are well separated and thus almost independent of one another. To rigorously analyze V, we build it up successively. The following lemma describes this situation.

Lemma 7.2. *Consider $H_a = -d^2/dx^2 + Q(x) + V_g(x - a)$, where Q has compact support and g is sufficiently small so that Lemma 7.1 applies. Suppose that $-d^2/dx^2 + Q(x)$ has precisely N negative eigenvalues $-\widetilde{E}_1, \ldots, -\widetilde{E}_N$ on $L_2(0, \infty)$. Then, for any $\epsilon > 0$, there exists a_0 so that for all $a \geq a_0$, the following holds: H_a on $L_2(0, \infty)$ has precisely $N + 1$ negative eigenvalues $-E_1, \ldots, -E_{N+1}$, and*

$$\left| E_i - \widetilde{E}_i \right| < \epsilon, \quad |E_{N+1} - E(g)| < \epsilon.$$

An analogous statement holds for H_a on $L_2(0, 2a)$.

In other words, there is almost no interaction between Q and V_g and the eigenvalues approximately behave like those of the orthogonal sum of $-d^2/dx^2 + Q(x)$ and $-d^2/dx^2 + V_g(x)$.

Proof. Let y be the solution of $-y'' + (Q + V_g)y = 0$ with $y(0) = 0$, $y'(0) = 1$. By taking a large enough, we can make sure that y has precisely N zeros on $(0, a/2)$ (say) and $y(a/2)y'(a/2) \geq 0$, $y(a/2) \neq 0$ (if y, y' had different signs, there would be another zero on $(a/2, \infty)$ for zero potential). Here we again use oscillation theory; more precisely, the number of positive zeros of y is exactly the number of negative eigenvalues. An elementary discussion now shows that y has exactly one more zero on $(a/2, \infty)$ if a is large enough. Thus H_a has precisely $N + 1$ negative eigenvalues. Since this additional zero in fact lies in $(a+1, 2a)$, this also holds for H_a on $L_2(0, 2a)$.

It is now easy to approximately locate these eigenvalues: Cut off the eigenfunctions of $-d^2/dx^2 + Q(x)$ (on $L_2(0, \infty)$) and of $-d^2/dx^2 + V_g(x - a)$ (on $L_2(\mathbb{R})$) by multiplying by suitable smooth functions which are equal to one except in a neighborhood of, say, $a/2$ (and $2a$ for this latter eigenfunction). If a is large enough, we thus obtain functions φ_i with

$$\|(H_a + \widetilde{E}_i)\varphi_i\| < \epsilon\|\varphi_i\| \quad (i = 1, \ldots, N), \qquad \|(H_a + E(g))\varphi_{N+1}\| < \epsilon\|\varphi_{N+1}\|.$$

This completes the proof of the lemma. $\square$

We now claim that given any sequence $\epsilon_n > 0$, we can find x_n's in (7.2), so that $-d^2/dx^2 + V(x)$ has eigenvalues $-E_n$, satisfying $|E_n - E(g_n)| < \epsilon_n$, and no other negative eigenvalues. To prove this, we may assume that ϵ_n decreases. To simplify the book keeping, we in fact further assume that the intervals $(E(g_n) - \epsilon_n, E(g_n) + \epsilon_n)$ are disjoint. This is to say, we assume that the g_n's are distinct and then further decrease the ϵ_n's if necessary.

Now apply Lemma 7.2 with $Q = 0$, $g = g_1$, and $\epsilon = 2^{-2}\epsilon_1$ to find x_1. In the second step, apply the lemma with $Q = V_{g_1}(x - x_1)$, $g = g_2$, and $\epsilon = 2^{-3}\epsilon_2$ (which is also $\leq 2^{-3}\epsilon_1$ by our assumption). This gives x_2; we may also demand that $x_2 - 1 > 2x_1$. Continue in this way. The construction ensures that $-d^2/dx^2 + V(x)$ on $L_2(0, 2x_n)$ has precisely n negative eigenvalues $E_i^{(n)}$, and these satisfy $|E_i^{(n)} - E(g_i)| \leq \epsilon_i/2$. Moreover, for fixed i, $E_i^{(n)}$ is a Cauchy sequence and hence convergent. Since every eigenvalue of the half-line problem is an accumulation point of eigenvalues of the problems on $(0, 2x_n)$, only the limits $E_i \equiv \lim_{n\to\infty} E_i^{(n)}$ can be eigenvalues of $-d^2/dx^2 + V(x)$. On the other hand, the half-line problem has at least as many eigenvalues $< -E$ as the corresponding problem on $(0, 2x_n)$; thus, since the E_i's are from the disjoint intervals $(E(g_i) - \epsilon_i, E(g_i) + \epsilon_i)$ and the problems on $(0, 2x_n)$ have spectrum in these intervals for large n by construction and Lemma 7.2, the negative spectrum is precisely the set $\{E_i\}$.

Let us now prove Theorem 1.10. Given $e_n > 0$ with $e_n \to 0$, $\sum e_n = \infty$, pick $g_n > 0$ so that $E(g_n) = e_n/2$, say. By slightly decreasing the e_n's if necessary, we may of course assume that the e_n's and hence also the g_n's are distinct. By Lemma 7.1a), the g_n's will satisfy $g_n \sim (e_n/2)^{1/2}$ in the sense that the ratio tends to one. Now choose x_n's as above so that the operator $-d^2/dx^2 + V(x)$ with V as in (7.2) has eigenvalues $E_n \leq e_n$. Since $V \leq 0$, $H_- = -d^2/dx^2 - V(x)$ has no negative spectrum. When choosing the x_n's, we can further require that $x_n/x_{n+1} \to 0$. Since also $\sum g_n^2 = \infty$, a well-known result on sparse potentials applies and the spectrum is purely singular continuous on $(0, \infty)$ (see [13, Theorem 1.6(2)]).

The proof of Theorem 1.7 uses W_g and [22, Theorem 4.2b)] instead of V_g and [13, Theorem 1.6(2)], respectively, but is otherwise analogous. First of all, the analog of Lemma 7.2 holds, with a similar proof. Let $e_n > 0$ with $\sum e_n^{1/4} = \infty$ be given. If e_n is non-increasing, we also have that $\sum e_{2n}^{1/4} = \infty$. Determine g_n's so that $E(g_n) = e_{2n}/9$, where $E(g)$ now refers to W_g. Lemma 7.1b) shows that then $g_n \sim e_{2n}^{1/4}$. Define V as in (7.2), but with W_g instead of V_g. We can then find x_n's so that $-d^2/dx^2 + V(x)$ has eigenvalues $E_n \leq e_{2n}$ in this case as well, following the same arguments as above. Moreover, since $-d^2/dx^2 - W_g(x)$ on $L_2(\mathbb{R})$ has the same eigenvalue $-E(g)$ as $-d^2/dx^2 + W_g$, we can also arrange that $-d^2/dx^2 - V(x)$ has eigenvalues $E_n' \leq e_{2n}$. Thus, after combining the eigenvalues of $H_\pm$ in one sequence E_n, we have that $E_n \leq e_n$, as desired. Finally, we can again require that $x_n/x_{n+1} \to 0$. Since $\sum g_n = \infty$, Theorem 4.2b) from [22] applies and shows that $\dim S = 1$.

References

[1] M. Christ and A. Kiselev, WKB asymptotic behavior of almost all generalized eigenfunctions for one-dimensional Schrödinger operators with slowly decaying potentials, *J. Funct. Anal.* **179** (2001), 426–447.

[2] M. Christ and A. Kiselev, WKB and spectral analysis of one-dimensional Schrödinger operators with slowly varying potentials, *Commun. Math. Phys.* **218** (2001), 245–262.

[3] R. Courant and D. Hilbert, *Methods of Mathematical Physics. Vol. I.* Interscience Publishers, Inc., New York, 1953.

[4] D. Damanik, D. Hundertmark, R. Killip, and B. Simon, Variational estimates for discrete Schödinger operators with potentials of indefinite sign, *Commun. Math. Phys.* **238** (2003), 545–562.

[5] D. Damanik and R. Killip, Half-line Schrödinger operators with no bound states, to appear in *Acta Math.*

[6] D. Damanik, R. Killip, and B. Simon, Schrödinger operators with few bound states, preprint (arXiv/math-ph/0409074)

[7] P. Deift and R. Killip, On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials, *Commun. Math. Phys.* **203** (1999), 341–347.

[8] M.S.P. Eastham, *The Asympotic Solution of Linear Differential Systems.* London Mathematical Society Monographs, New Series, vol. 4, Oxford University Press, Oxford 1989.

[9] K.J. Falconer, *The Geometry of Fractal Sets.* Cambridge University Press, Cambridge, 1985.

[10] V. Glaser, H. Grosse, and A. Martin, Bounds on the number of eigenvalues of the Schrödinger operator, *Commun. Math. Phys.* **59** (1978), 197–212.

[11] C. Jacobi, Zur Theorie der Variations-Rechnung und der Differential-Gleichungen, *J. Reine Angew. Math.* **17** (1837), 68–82.

[12] R. Killip and B. Simon, Sum rules for Jacobi matrices and their applications to spectral theory, *Ann. of Math.* **158** (2003), 253–321.

[13] A. Kiselev, Y. Last, and B. Simon, Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators, *Commun. Math. Phys.* **194** (1998), 1–45.

[14] S. Kupin, On a spectral property of Jacobi matrices, *Proc. Amer. Math. Soc.* **132** (2004), 1377–1383.

[15] A. Laptev, S. Naboko, and O. Safronov, On new relations between spectral properties of Jacobi matrices and their coefficients, *Commun. Math. Phys.* **241** (2003), 91–110.

[16] A. Laptev and T. Weidl, Recent results on Lieb-Thirring inequalities, *Journées "Équations aux Dérivées Partielles"* (*La Chapelle sur Erdre, 2000*), Exp. No. XX, 14 pp., Univ. Nantes, Nantes, 2000.

[17] C. Muscalu, T. Tao, and C. Thiele, A counterexample to a multilinear endpoint question of Christ and Kiselev, *Math. Res. Letters* **10** (2003), 237–246.

[18] C. Muscalu, T. Tao, and C. Thiele, A Carleson theorem for a Cantor group model of the scattering transform, *Nonlinearity* **16** (2003), 219–246.

[19] F. Nazarov, F. Peherstorfer, A. Volberg, and P. Yuditskii, On generalized sum rules for Jacobi matrices, to appear in *Int. Math. Res. Not.*

[20] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, IV. Analysis of Operators.* Academic Press, New York 1978.

[21] C. Remling, The absolutely continuous spectrum of one-dimensional Schrödinger operators with decaying potentials, *Commun. Math. Phys.* **193** (1998), 151–170.

[22] C. Remling, Embedded singular continuous spectrum for one-dimensional Schrödinger operators, *Trans. Amer. Math. Soc.* **351** (1999), 2479–2497.

[23] C. Remling, Bounds on embedded singular spectrum for one-dimensional Schrödinger operators, *Proc. Amer. Math. Soc.* **128** (2000), 161–171.

[24] A. Rybkin, On the absolutely continuous and negative discrete spectra of Schrödinger operators on the line with locally integrable globally square summable potentials, *J. Math. Phys.* **45** (2004), 1418–1425.

[25] U.-W. Schmincke, On Schrödinger's factorization method for Sturm-Liouville operators, *Proc. Roy. Soc. Edinburgh A* **80** (1978), 67–84.

[26] B. Simon and A. Zlatoš, Sum rules and the Szegő condition for orthogonal polynomials on the real line, *Commun. Math. Phys.* **242** (2003), 393–423.

[27] V.E. Zakharov and L.D. Faddeev, Korteweg de Vries equation: a completely integrable Hamiltonian system, *Funct. Anal. Appl.* **5** (1971), 280–287.

[28] A. Zygmund, *Trigonometric Series, Vol. I, II.* Cambridge University Press, Cambridge 1959.

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