Stationary van Hove limit

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Abstract

The weak coupling (van Hove) limit of one parameter groups of contractions is studied by the stationary approach. We show that the resolvent of the properly renormalized and rescaled generator of a contractive semigroup has a limit as the coupling constant goes to zero. This limit is the resolvent of the generator of a certain contractive semigroup. Our results can be viewed as a stationary counterpart to the well known results about the weak coupling limit obtained by the time-dependent approach, due to E. B. Davies. We compare both approaches.

Keywords: strongly continuous semigroups of contractions, weak coupling limit, perturbation theory

1 Introduction

Let $\mathcal{X}$ be a Banach space with a distinguished bounded projection P. Suppose that U_t^λ is a one parameter strongly continuous group of isometries on $\mathcal{X}$ generated by $L_\lambda := L_0 + \lambda Q$. Assume that P commutes with the free dynamics $U_t^0 = \mathrm{e}^{tL_0}$ or equivalently P commutes with L_0. Our main object of interest is the reduced dynamics

$$\mathbb{R} \ni t \mapsto PU_t^\lambda P$$

as an operator on $\mathrm{Ran}P$.

The reduced dynamics was studied in a series of papers [Dav1, Dav2] and in the book [Dav3] by E.B. Davies. First he showed that the reduced dynamics after appropriately rescaling, i.e.

$$t \mapsto PU_{t/\lambda^2}^\lambda P,$$

can be approximated, as the coupling constant λ goes to zero, by a certain one parameter semigroup on $\mathrm{Ran}P$ depending on λ. The generator of this semigroup is a quadratic polynomial in λ^{-1}. By the convergence we mean that for each fixed time t the norm of the difference between the resulting semigroup and the reduced rescaled dynamics tends to zero as $|\lambda|$ becomes smaller. We will call this result, for the reasons that soon become clear, the pointwise (in time) van Hove limit with the 1st order term.

The second result obtained by Davies describes the case with $PQP = 0$. He proved that the reduced dynamics with a rescaled time renormalized by the free evolution, i.e.

$$t \mapsto U^0_{-t/\lambda^2} P U^\lambda_{t/\lambda^2} P, \tag{1.1}$$

has a limit, for each fixed time t, as the coupling constant λ goes to zero. The limit is a one parameter semigroup, independent of λ. The generator of the resulting semigroup is often called the Davies generator. We will call this limit the pointwise (in time) van Hove limit without the 1st order term.

One can distinguish two approaches to semigroups: the time-dependent and the stationary approach. The former concentrates on the study of semigroups themselves. The latter focuses at the resolvent of their generators. Davies used the time-dependent approach, both in the choice of the assumptions for his results and in their statements. In our paper we use mostly the stationary approach.

Our main results are contained in three theorems. First in the Theorem 3.1 we study, the stationary van Hove limit for the reduced dynamics as the coupling constant λ goes to zero. More precisely, we describe the asymptotics of the rescaled resolvent of L_λ reduced by P by the resolvent of a certain operator $A_{\lambda,0}$. This is the stationary counterpart to the first result of Davies.

Theorem 3.2 describes the case wthout the 1st order term ($PQP = 0$). We additionally assume that the spectrum of PL_0P consist of isolated points. We obtain a simple asymptotics of the Laplace transform of (1.1) given by the resolvent of a certain operator Γ independent of λ commuting with PL_0P. We prove that the operator Γ is the generator of a contractive semigroup. This is the stationary counterpart to the second result of Davies.

The main results about the stationary van Hove limit involve resolvents. They have, however, easy time dependent corollaries, which we call the smeared out weak coupling limit. By this we mean that the difference of the rescaled restricted dynamics and the approximating dynamics averaged over time with some continuous function vanishing at infinity compact support tends to zero as the coupling constant goes to zero. This version of the result is also contained in Theorems 3.1 and 3.2.

In Theorem 3.3 we show how one can obtain pointwise (in time) van Hove limit without the first order term starting from the stationary van Hove limit.

Theorem 3.5 gives conditions when one can apply both the time-dependent method of Davies and our stationary results. We also find out that the semigroup obtained in the van Hove limit is generated by the so-called Level Shift Operator, see [DJ2, DJ3, DJ4, DF].

In the physical literature one can trace back the weak coupling limit to works of Wigner-Weisskopf, Pauli and also van Hove [vH1, vH2, vH3]. First rigorous mathematical treatment of this issue comes from Davies [Dav1, Dav2, Dav3], who gave both its abstract theory and presented applications to open quantum systems (see also [LeSp]).

In his papers, Davies uses the time dependent approach, i.e, he works with the restricted dynamics. The key step in this approach is the construction of the resulting semigroup by the integral formula (e.g. Th.3.1.33 in [BR1]). The use of this formula induces technical assumptions which may be not easy to verify. For the convenience of the reader we describe the result of Davies in the Theorem 3.4.

In our approach to the weak coupling limit, instead of working with the perturbed dynamics, we investigate the resolvent of the perturbed generator. We use some regularity assumptions for resolvents which seem easier to verify in some circumstances. In particular our regularity assumptions are closely related to the so called Limiting Absorption Principle which can be investigated with help of the so called Mourre Theory [DJ1].

We end this introduction with a description of the main physical motivation of our work—an application of the van Hove limit to open quantum systems. We essentially follow [LeSp], see also [Dav1, Dav2]. For more information, especially concerning the relationship of the van Hove limit to applications of the method of the Level Shift Operator to the return to equilibrium, we refer the reader to [DJ4].

Let us consider a quantum mechanical system which consists of two parts—the small system $\mathcal{S}$ and the reservoir part $\mathcal{R}$. To describe $\mathcal{S}$ one chooses appropriate Hilbert space $\mathcal{H}_{\mathcal{S}}$. Then the states are given by density matrices (i.e. trace class normalized positive operators on $\mathcal{H}_{\mathcal{S}}$). The time evolution of the isolated small system $\mathcal{S}$ is implemented by the Hamiltonian $H_{\mathcal{S}}$. In a similar way we describe the reservoir part $\mathcal{R}$. We have Hilbert space $\mathcal{H}_{\mathcal{R}}$, the Liouvillean $L_{\mathcal{R}}$ (the generator of the time evolution) and density matrices on $\mathcal{H}_{\mathcal{R}}$. Let us additionally assume that there exists a stationary state, denoted $\omega_{\mathcal{R}}$, of $\mathcal{R}$ for the evolution implemented by $L_{\mathcal{R}}$.

The time evolution of whole system $\mathcal{S}+\mathcal{R}$ is given by self-adjoint operator

$$L_\lambda := H_{\mathcal{S}} \otimes \mathbf{1}_{\mathcal{R}} + \mathbf{1}_{\mathcal{S}} \otimes L_{\mathcal{R}} + \lambda V$$

acting on $\mathcal{H} := \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}$ where V is some interaction operator and $\lambda \in \mathbb{R}$. Hence for any density matrix χ on $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}$ its evolution is given by

$$U_t^\lambda \chi := \mathrm{e}^{-tiL_\lambda} \chi \mathrm{e}^{tiL_\lambda}.$$

Assume that the initial state of $\mathcal{S}+\mathcal{R}$ is $\rho \otimes \omega_{\mathcal{R}}$ for some density matrix ρ on $\mathcal{H}_{\mathcal{S}}$. Then after time t the state of $\mathcal{S}+\mathcal{R}$ is given by $U_t^\lambda(\rho \otimes \omega_{\mathcal{R}})$. If we treat $\mathcal{R}$ just as a device which induces changes of $\mathcal{S}$ and we want only to know what happens to $\mathcal{S}$, then to obtain the state of $\mathcal{S}$ at time t we take the partial trace over the degrees of freedom of $\mathcal{R}$

$$\rho(t) := \mathrm{tr}_{\mathcal{R}}\Big(U_t^\lambda(\rho \otimes \omega_{\mathcal{R}})\Big). \tag{1.2}$$

Note that the action of U_t^λ can be extended to the whole space of the trace class operators $B^1(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}})$. Recall that $B^1(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}})$ is a Banach space under the norm $\|\cdot\|_1 = \mathrm{tr}|\cdot|$ and U_t^λ is a one parameter strongly continuous group of isometries on $B^1(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}})$. If we introduce the operator

$$P:\ B^1(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}) \to B^1(\mathcal{H}_{\mathcal{S}}) \otimes \omega_{\mathcal{R}} \subset B^1(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}),$$
$$PW := \mathrm{tr}_{\mathcal{R}}(W) \otimes \omega_{\mathcal{R}},$$

then P is a projection of norm one and the equation (1.2) can be rewritten

$$\rho(t) \otimes \omega_{\mathcal{R}} := P U_t^\lambda P(\rho \otimes \omega_{\mathcal{R}}).$$

note also that $[U_t^0, P] = 0$. Therefore, we have a setup, where we can apply our results. In the weak coupling limit we obtain completely positive semigroup of contractions which is sometimes called a Quantum Dynamical Markov semigroup. Hence starting with a fully reversible dynamics for the whole system, we end up with an irreversible evolution of the small subsystem. Now, in the weak coupling approximation, when we study the small system we may exchange complicated object $PU_t^\lambda P$ for a semigroup and use it in order to determine physical quantities. However there is a price to be paid—the results that we get in this approximation are typically the lowest order nonvanishing corrections in the coupling constant to the real quantities.

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2 Preliminaries

2.1 Notation

Let $\mathcal{X}$ be a Banach space. For a linear operator L on $\mathcal{X}$, $\mathrm{sp}L$ denotes its spectrum and $\mathrm{Dom}L$ its domain. If Ξ is an isolated bounded subset of $\mathrm{sp}L$ then the spectral projection of L onto Ξ, defined by the usual integral formula [Kato], is denoted $\mathbf{1}_\Xi(L)$. If e is an isolated point of $\mathrm{sp}L$, then we will write $\mathbf{1}_e(L)$ for $\mathbf{1}_{\{e\}}(L)$

Let $\mathbf{1}^{\mathrm{vv}}$ be a distinguished bounded projection on $\mathcal{X}$. It will be convenient to denote $\mathbf{1}^{\overline{\mathrm{vv}}} := \mathbf{1} - \mathbf{1}^{\mathrm{vv}}$. We also introduce closed subspaces

$$\mathcal{X}^{\mathrm{v}} := \mathbf{1}^{\mathrm{vv}}\mathcal{X}, \qquad\qquad \mathcal{X}^{\overline{\mathrm{v}}} := \mathbf{1}^{\overline{\mathrm{vv}}}\mathcal{X} = \mathrm{Ker}\mathbf{1}^{\mathrm{vv}},$$

so that $\mathcal{X}$ is decomposed into a direct sum

$$\mathcal{X} = \mathcal{X}^{\mathrm{v}} \oplus \mathcal{X}^{\overline{\mathrm{v}}}. \tag{2.3}$$

Any operator H on $\mathcal{X}$ satisfying

$$\mathrm{Dom}(H) = \Big(\mathrm{Dom}(H) \cap \mathcal{X}^{\mathrm{v}}\Big) \oplus \Big(\mathrm{Dom}(H) \cap \mathcal{X}^{\overline{\mathrm{v}}}\Big)$$

can be written with respect to the decomposition (2.3) as

$$H = \begin{bmatrix} H^{\mathrm{vv}} & H^{\mathrm{v}\overline{\mathrm{v}}} \\ H^{\overline{\mathrm{v}}\mathrm{v}} & H^{\overline{\mathrm{vv}}} \end{bmatrix} = H^{\mathrm{vv}} + H^{\mathrm{v}\overline{\mathrm{v}}} + H^{\overline{\mathrm{v}}\mathrm{v}} + H^{\overline{\mathrm{vv}}}. \tag{2.4}$$

Obviously $H^{\mathrm{vv}} = \mathbf{1}^{\mathrm{vv}} H \mathbf{1}^{\mathrm{vv}}$, etc. In particular

$$\mathbf{1} := \begin{bmatrix} \mathbf{1}^{\mathrm{vv}} & 0 \\ 0 & \mathbf{1}^{\overline{\mathrm{vv}}} \end{bmatrix}.$$

For $e \in \mathrm{i}\mathbb{R}$ and for $\alpha \geq 0$ we denote

$$\mathbf{Wedge}(e, \alpha) := \{z \in \mathbb{C} \ : \ \mathrm{Re}z > 0\ ,\ |\mathrm{Im}(z - e)| \leq \alpha \mathrm{Re}z\}.$$

3 Main results

Let L_0 be the generator of a one parameter strongly continuous group of isometries $t \mapsto \mathrm{e}^{tL_0}$ on the Banach space $\mathcal{X}$. Recall that L_0 is norm closed, norm densely defined, conservative operator (i.e. both L_0 and $-L_0$ are dissipative) and $\mathrm{sp}L_0 \subset \mathrm{i}\mathbb{R}$ [BR1, Dav3].

Let $\mathbf{1}^{\mathrm{vv}}$ be a distinguished projection on $\mathcal{X}$ such that $\|\mathbf{1}^{\mathrm{vv}}\| = 1$. Assume that $\mathbf{1}^{\mathrm{vv}}$ commutes with L_0 or equivalently $[\mathbf{1}^{\mathrm{vv}}, \mathrm{e}^{tL_0}] = 0$ for all t. Then the operator L_0 written wrt the decomposition (2.3) has the form

$$L_0 = \begin{bmatrix} L_0^{\mathrm{vv}} & 0 \\ 0 & L_0^{\overline{\mathrm{vv}}} \end{bmatrix}. \tag{3.5}$$

We write for shortness $E := L_0^{\mathrm{vv}}$. Note that E generates a one parameter strongly continuous group of isometries on $\mathcal{X}^{\mathrm{v}}$.

Let Q with $\mathrm{Dom}L_0 \subset \mathrm{Dom}Q$ be another operator that we will treat as a perturbation of L_0. Fix $\lambda_0 > 0$. We assume that for $0 \leq \lambda < \lambda_0$ the operator

$$L_\lambda := L_0 + \lambda Q,$$

defined on $\mathrm{Dom}L_\lambda = \mathrm{Dom}L_0$ is the generator of a one parameter strongly continuous semigroup of contractions on $\mathcal{X}$.

We will assume that the off-diagonal elements of Q, i.e. $Q^{\overline{\mathrm{v}}\mathrm{v}}$ and $Q^{\mathrm{v}\overline{\mathrm{v}}}$ are bounded. We also assume that for all $0 \leq \lambda < \lambda_0$ operator $E + \lambda Q^{\mathrm{vv}}$ generates a group of isometries on $\mathcal{X}^{\mathrm{v}}$.

Note that if L_λ with bounded $Q^{\overline{\mathrm{v}}\mathrm{v}}$ and $Q^{\mathrm{v}\overline{\mathrm{v}}}$ generates a group of isometries then $E + \lambda Q^{\mathrm{vv}}$ generates a group of isometries on $\mathcal{X}^{\mathrm{v}}$.

3.1 Van Hove limit—stationary approach

In this subsection we discuss the van Hove limit under the assumptions involving the resolvent of L_λ. The statement of our result is similar to the statement of the results of Davies, which we recall later.

3.1.1 Van Hove limit with the 1st order term

Theorem 3.1 *Assume that for all* $0 \leq \lambda < \lambda_0$ *:*

(i) *For all* $\xi > 0$ *we have* $\xi \notin \mathrm{sp}(L_\lambda^{\overline{\mathrm{vv}}})$*,*

(ii) *There exists an operator* $\Gamma_0 \in B(\mathcal{X}^{\mathrm{v}})$ *such that, for any* $\xi > 0$*,*

$$\Gamma_0 := \lim_{\lambda \searrow 0} Q^{\mathrm{v}\overline{\mathrm{v}}}(\lambda^2 \xi \mathbf{1}^{\overline{\mathrm{vv}}} - L_\lambda^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}\mathrm{v}} \tag{3.6}$$

(Note that the RHS of (3.6) may depend on ξ*. We assume that it does not).*

Let

$$A_{\lambda,0} := E + \lambda Q^{\mathrm{vv}} + \lambda^2 \Gamma_0. \tag{3.7}$$

Then the following holds:

(1) Γ_0 *generates a semigroup of contractions on* $\mathcal{X}^{\mathrm{v}}$*.*

(2) $A_{\lambda,0}$ *generates a semigroup of contractions on* $\mathcal{X}^{\mathrm{v}}$.

(3) *For each* $\xi > 0$,

$$\lim_{\lambda\searrow 0}\Big(\mathbf{1}^{\mathrm{vv}}(\xi\mathbf{1} - \lambda^{-2}L_\lambda)^{-1}\mathbf{1}^{\mathrm{vv}} - (\xi\mathbf{1}^{\mathrm{vv}} - \lambda^{-2}A_{\lambda,0})^{-1}\Big) = 0, \tag{3.8}$$

(4) *For any* $f \in C_0([0,\infty[)$

$$\lim_{\lambda\searrow 0}\int_0^\infty f(\sigma)\Big(\mathbf{1}^{\mathrm{vv}}\mathrm{e}^{\sigma\lambda^{-2}L_\lambda}\mathbf{1}^{\mathrm{vv}} - \mathrm{e}^{\sigma\lambda^{-2}A_{\lambda,0}}\Big)\mathrm{d}\sigma = 0. \tag{3.9}$$

Note that above we use the notation $(z\mathbf{1}^{\overline{\mathrm{vv}}} - L_\lambda^{\overline{\mathrm{vv}}})^{-1}$ for the inverse of the operator $z\mathbf{1}^{\overline{\mathrm{vv}}} - L_\lambda^{\overline{\mathrm{vv}}}$ restricted to $\mathcal{X}^{\overline{\mathrm{v}}}$. We will often use a similar notation without a comment.

3.1.2 Van Hove limit without the 1st order term

In this subsection we describe two versions of the van Hove limit without the 1st order term. In the first we either work at the resolvents or smear out the dynamics in time. In the second, we work at the dynamics pointwise in time. The statement of the second result is essentialy the same as that of Davies, however assumptions are different.

We will need the following additional assumptions

Assumption 3.A $\mathrm{sp}E$ *is a finite set.*

Note that Assumption 3.A implies that we can write

$$\mathbf{1}^{\mathrm{vv}} = \sum_{e\in\mathrm{sp}E}\mathbf{1}_e(E).$$

Assumption 3.B $Q^{\mathrm{vv}} = 0$.

Theorem 3.2 *Let 3.A and 3.B hold. Assume additionally that:*

(i) *For* $0 \le \lambda < \lambda_0$, *for each* $e \in \mathrm{sp}E$ *and for all* $\xi > 0$ *we have* $e + \xi \notin \mathrm{sp}(L_\lambda^{\overline{\mathrm{vv}}})$,

(ii) *There exists an operator* $\Gamma \in B(\mathcal{X}^{\mathrm{v}})$ *such that, for any* $\xi > 0$,

$$\Gamma := \sum_{e\in\mathrm{sp}E}\lim_{\lambda\searrow 0}\mathbf{1}_e(E)Q^{\mathrm{v}\overline{\mathrm{v}}}\Big((e+\lambda^2\xi)\mathbf{1}^{\overline{\mathrm{vv}}} - L_\lambda^{\overline{\mathrm{vv}}}\Big)^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathbf{1}_e(E), \tag{3.10}$$

(Note that the RHS of (3.10) may depend on ξ*. We assume that it does not).*

(iii) *For any* $e, e' \in \mathrm{sp}E$, $e \neq e'$ *and* $\xi > 0$

$$\lim_{\lambda\searrow 0}\lambda\mathbf{1}_e(E)Q^{\mathrm{v}\overline{\mathrm{v}}}\Big((e+\lambda^2\xi)\mathbf{1}^{\overline{\mathrm{vv}}} - L_\lambda^{\overline{\mathrm{vv}}}\Big)^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathbf{1}_{e'}(E) = 0,$$
$$\lim_{\lambda\searrow 0}\lambda\mathbf{1}_{e'}(E)Q^{\mathrm{v}\overline{\mathrm{v}}}\Big((e+\lambda^2\xi)\mathbf{1}^{\overline{\mathrm{vv}}} - L_\lambda^{\overline{\mathrm{vv}}}\Big)^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathbf{1}_e(E) = 0;$$

Then the following holds:

(1) Γ *generates semigroup of contractions on* $\mathcal{X}^{\mathrm{v}}$*,*

(2) $[E,\Gamma]=0$*,*

(3) *For each* $\xi>0$ *we have*

$$\lim_{\lambda\searrow 0}\sum_{e\in\mathrm{sp}E}\mathbf{1}_e(E)\mathbf{1}^{\mathrm{vv}}\Big(\xi\mathbf{1}-\lambda^{-2}(L_\lambda-e\mathbf{1})\Big)^{-1}\mathbf{1}^{\mathrm{vv}}=\big(\xi\mathbf{1}^{\mathrm{vv}}-\Gamma\big)^{-1}, \tag{3.11}$$

(4) *For any* $f\in C_0([0,\infty[)$

$$\lim_{\lambda\searrow 0}\int_0^\infty f(\sigma)\Big(\mathrm{e}^{-\sigma\lambda^{-2}E}\mathbf{1}^{\mathrm{vv}}\mathrm{e}^{\sigma\lambda^{-2}L_\lambda}\mathbf{1}^{\mathrm{vv}}-\mathrm{e}^{\sigma\Gamma}\Big)\,\mathrm{d}\sigma=0. \tag{3.12}$$

Theorem 3.3 *Let 3.A and 3.B hold. Assume additionally that:*

(i) *For* $0\le\lambda<\lambda_0$ *and all* $\xi\in\mathbb{C}$*,* $\mathrm{Re}\xi>0$*, we have* $\xi\notin\mathrm{sp}(L_\lambda^{\overline{\mathrm{v}}\overline{\mathrm{v}}})$*.*

(ii) *For all* $\alpha_0\ge 0$ *and for any* $\xi\in\mathbf{Wedge}(0,\alpha_0)$*, there exists an operator* $\Gamma\in B(\mathcal{X}^{\mathrm{v}})$ *such that,*

$$\Gamma:=\sum_{e\in\mathrm{sp}E}\lim_{\lambda\searrow 0}\mathbf{1}_e(E)Q^{\mathrm{v}\overline{\mathrm{v}}}\Big((e+\lambda^2\xi)\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_\lambda^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\Big)^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathbf{1}_e(E), \tag{3.13}$$

(Note that the RHS of (3.13) may depend on ξ*. We assume that it does not).*

(iii) *For each* $e\in\mathrm{sp}E$ *we have*

$$\sup_{\mathrm{Re}\xi>0;\,0\le\lambda\le\lambda_0}\|Q^{\mathrm{v}\overline{\mathrm{v}}}\big((e+\lambda^2\xi)\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_\lambda^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\big)^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}\|<\infty$$

Then:

(1) *All statemants of the Theorem 3.2 hold. Besides (3) holds in a stronger form: for* $\alpha_0\ge 0$ *the formula (3.11) is valid for each* $\xi\in\mathbf{Wedge}(0,\alpha_0)$*.*

(2) *Let* $\psi\in\mathcal{X}^{\mathrm{v}}$*. For* $\sigma\ge 0$ *we have*

$$\lim_{\lambda\to 0}\Big(\mathrm{e}^{-\sigma\lambda^{-2}E}\mathbf{1}^{\mathrm{vv}}\mathrm{e}^{\sigma\lambda^{-2}L_\lambda}\mathbf{1}^{\mathrm{vv}}-\mathrm{e}^{\sigma\Gamma}\Big)\psi=0 \tag{3.14}$$

uniformly for $\sigma\in[\tau_0,\tau_1]$ *for any fixed* $0<\tau_0\le\tau_1<\infty$*.*

3.2 Van Hove limit—time dependend approach

In this section we discuss the van Hove limit under the assumptions involving the dynamics. In Theorem 3.4) we recall the original approach to the van Hove limit due to E.B. Davies [Dav1, Dav2, Dav3]. (Strictly speaking, Davies assumed that the perturbation Q is bounded. In Theorem 3.4 we impose slightly less restrictive assumptions, which can be handled by an essentially the same proof).

Let L_0, Q and $\mathbf{1}^{\mathrm{vv}}$ be the same as before. Clearly, $L_\lambda^{\overline{\mathrm{vv}}}$ generates a semigroup on $\mathcal{X}^{\overline{\mathrm{v}}}$. Therefore, we can define the operator

$$K(\lambda,\tau) := \int\limits_{x=0}^{\lambda^{-2}\tau} \mathrm{e}^{-xL_\lambda^{\mathrm{vv}}} Q^{\mathrm{v}\overline{\mathrm{v}}} \mathrm{e}^{xL_\lambda^{\overline{\mathrm{vv}}}} Q^{\overline{\mathrm{v}}\mathrm{v}} \mathrm{d}x. \tag{3.15}$$

The following theorem describes the van Hove limit for the dynamics in both cases—with and without the 1st order term.

Theorem 3.4 *Assume additionally that:*

(i) *For all $\tau_1 > 0$ there is a constant $C > 0$ such that $\|K(\lambda,\tau)\| < C$ for $|\lambda| \leq 1$ and $0 \leq \tau \leq \tau_1$,*

(ii) *There exists bounded operator K on $\mathcal{X}^{\mathrm{v}}$ such that if $0 < \tau_0 \leq \tau_1 < \infty$ then*

$$\lim_{\lambda\to 0}\Big(\sup_{\tau_0\leq\tau\leq\tau_1} \|K(\lambda,\tau) - K\|\Big) = 0.$$

Then the following holds:

(1)
$$\lim_{\lambda\to 0}\Big(\sup_{0\leq t\leq\tau_1} \|\mathbf{1}^{\mathrm{vv}}\mathrm{e}^{t\lambda^{-2}L_\lambda}\mathbf{1}^{\mathrm{vv}} - \mathrm{e}^{t\lambda^{-2}(E+\lambda Q^{\mathrm{vv}}+\lambda^2 K)}\|\Big) = 0. \tag{3.16}$$

(2) *If additionally 3.A and 3.B hold then*

$$\lim_{\lambda\to 0}\Big(\sup_{0\leq t\leq\tau_1} \|\mathrm{e}^{-t\lambda^{-2}E}\mathbf{1}^{\mathrm{vv}}\mathrm{e}^{t\lambda^{-2}L_\lambda}\mathbf{1}^{\mathrm{vv}} - \mathrm{e}^{tK^\natural}\|\Big) = 0 \tag{3.17}$$

where

$$K^\natural := \sum_{e\in\mathrm{sp}E} \mathbf{1}_e(E) K \mathbf{1}_e(E) = \lim_{a\to\infty}\frac{1}{2a}\int\limits_{-a}^{a} \mathrm{e}^{tE} K \mathrm{e}^{-tE}\mathrm{d}t.$$

Let us recall how the operators $K(\lambda,\tau)$ are motivated. If we treat the off-diagonal elements of L_λ as a (bounded) perturbation of the diagonal part of L_λ then, by a well known formula, we have

$$\mathrm{e}^{tL_\lambda} = \mathrm{e}^{t(L_\lambda^{\mathrm{vv}}+L_\lambda^{\overline{\mathrm{vv}}})} + \lambda\int\limits_{s=0}^{t} \mathrm{e}^{(t-s)(L_\lambda^{\mathrm{vv}}+L_\lambda^{\overline{\mathrm{vv}}})}(Q^{\mathrm{v}\overline{\mathrm{v}}} + Q^{\overline{\mathrm{v}}\mathrm{v}})\mathrm{e}^{sL_\lambda}\mathrm{d}s.$$

Using this formula one gets

$$\mathbf{1}^{\mathrm{vv}}\mathrm{e}^{\lambda^{-2}\tau L_\lambda}\mathbf{1}^{\mathrm{vv}} = \mathbf{1}^{\mathrm{vv}}\mathrm{e}^{\lambda^{-2}\tau L_\lambda^{\mathrm{vv}}} + \int\limits_{\sigma=0}^{\tau} \mathbf{1}^{\mathrm{vv}}\mathrm{e}^{\lambda^{-2}(\tau-\sigma)L_\lambda^{\mathrm{vv}}} K(\lambda,\tau-\sigma)\mathbf{1}^{\mathrm{vv}}\mathrm{e}^{\lambda^{-2}\sigma L_\lambda}\mathbf{1}^{\mathrm{vv}}\mathrm{d}\sigma$$

Now we discuss how one can obtain the van Hove limit for the resolvents under time dependent assumptions. In fact we show when one can use both stationary and time-dependent approaches. We will concentrate on the case without the 1st order term.

Theorem 3.5 *Let 3.A and 3.B hold. Assume additionally that:*

(i) *For* $0 \leq \lambda < \lambda_0$ *and all* $z \in \mathbb{C}$, $\mathrm{Re} z > 0$ *we have* $z \notin \mathrm{sp}(L_\lambda^{\overline{\mathrm{vv}}})$,

(ii) $\int\limits_0^\infty \sup\limits_{0\leq\lambda<\lambda_0} \|Q^{\mathrm{v}\overline{\mathrm{v}}}\mathrm{e}^{sL_\lambda^{\overline{\mathrm{vv}}}}Q^{\overline{\mathrm{v}}\mathrm{v}}\|\mathrm{d}s < \infty$,

(iii) *For any* $s > 0$, $\lim\limits_{\lambda\searrow 0} Q^{\mathrm{v}\overline{\mathrm{v}}}\mathrm{e}^{sL_\lambda^{\overline{\mathrm{vv}}}}Q^{\overline{\mathrm{v}}\mathrm{v}} = Q^{\mathrm{v}\overline{\mathrm{v}}}\mathrm{e}^{sL_0^{\overline{\mathrm{vv}}}}Q^{\overline{\mathrm{v}}\mathrm{v}}$.

Then:

(1) *The assumptions of both theorem 3.2 and theorem 3.4 hold. Moreover, we have*

$$K^{\natural} = \Gamma.$$

(2) *There following limits exist, coincide and equal to* Γ*:*

$$\begin{aligned} &\lim_{\epsilon\searrow 0}\sum_{e\in\mathrm{sp}E}\mathbf{1}_e(E)Q^{\mathrm{v}\overline{\mathrm{v}}}\Big((e+\epsilon)\mathbf{1}^{\overline{\mathrm{vv}}} - L_0^{\overline{\mathrm{vv}}}\Big)^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathbf{1}_e(E) \\ &= \lim_{\epsilon\searrow 0}\sum_{e\in\mathrm{sp}E}\int\limits_0^\infty \mathrm{e}^{-\epsilon s}\mathbf{1}_e(E)Q\mathrm{e}^{sL_0}Q\mathrm{e}^{-sL_0}\mathbf{1}_e(E)\mathrm{d}s. \end{aligned} \tag{3.18}$$

Note that the assumptions of Theorem 3.5 are stronger then that of Theorem 3.3 and Theorem 3.4 (2).

The operator (3.18) is often called the Level Shift Operator. It is used to describe the second order shift of eigenvalues of L_λ [DJ2, DJ3, DJ4, DF].

4 Proofs

Lemma 4.1 *For* $0 \leq \lambda < \lambda_0$ *the operator*

$$\widetilde{L}_\lambda = \left[\begin{array}{cc} 0 & \lambda Q^{\mathrm{v}\overline{\mathrm{v}}} \\ \lambda Q^{\overline{\mathrm{v}}\mathrm{v}} & L_0^{\overline{\mathrm{vv}}} + \lambda Q^{\overline{\mathrm{vv}}} \end{array}\right]$$

defined on $\mathrm{Dom}(\widetilde{L}_\lambda) = \mathcal{X}^{\mathrm{v}} \oplus \mathrm{Dom}(L_0^{\overline{\mathrm{vv}}})$ *generates a semigroup of contractions on* $\mathcal{X}$,

Proof. Let $0 \leq \lambda < \lambda_0$. The operator $\widetilde{L}_\lambda$ is densely defined. By the Lumer-Phillips theorem (Th. 3.1.16 in [BR1]) it is sufficient to show that : (I) $\widetilde{L}_\lambda$ is dissipative; (II) for some $\epsilon > 0$ we have $\epsilon \notin \mathrm{sp}(\widetilde{L}_\lambda)$.

Step (I): For $0 \leq \lambda$ the operator

$$Z := L_\lambda - L_\lambda^{\mathrm{vv}}$$

with the domain $\mathrm{Dom}(Z) = \mathrm{Dom}(E) \oplus \mathrm{Dom}(L_0^{\overline{\mathrm{vv}}})$ is densely defined and dissipative. Hence, by the Proposition 3.1.15 in [BR1], Z is closable and its closure is dissipative. But the closure of Z coincides with $\widetilde{L}_\lambda$.

Step (II): For $0 \leq \lambda < \lambda_0$ the operator $L_\lambda - \lambda(Q^{\mathrm{v}\overline{\mathrm{v}}} + Q^{\overline{\mathrm{v}}\mathrm{v}})$ generates a semigroup such that for $\mathrm{Re}z > \|\lambda(Q^{\mathrm{v}\overline{\mathrm{v}}} + Q^{\overline{\mathrm{v}}\mathrm{v}})\|$ we have $z \notin \mathrm{sp}(L_\lambda - \lambda(Q^{\mathrm{v}\overline{\mathrm{v}}} + Q^{\overline{\mathrm{v}}\mathrm{v}}))$ and

$$\left\| \left(z\mathbf{1} - \left(L_\lambda - \lambda(Q^{\mathrm{v}\overline{\mathrm{v}}} + Q^{\overline{\mathrm{v}}\mathrm{v}}) \right) \right)^{-1} \right\| \leq (\mathrm{Re}z - \|\lambda(Q^{\mathrm{v}\overline{\mathrm{v}}} + Q^{\overline{\mathrm{v}}\mathrm{v}})\|)^{-1}.$$

Hence, for all $\epsilon > \epsilon_0$ for some ϵ_0 large enough,

$$\left\| \left(\epsilon\mathbf{1} - \left(\widetilde{L}_\lambda - \lambda(Q^{\mathrm{v}\overline{\mathrm{v}}} + Q^{\overline{\mathrm{v}}\mathrm{v}}) \right) \right)^{-1} (\lambda(Q^{\mathrm{v}\overline{\mathrm{v}}} + Q^{\overline{\mathrm{v}}\mathrm{v}})) \right\| < 1.$$

Hence, by the Neumann theorem, $\epsilon\mathbf{1} - \widetilde{L}_\lambda$ is invertible for all $\epsilon > \epsilon_0$, and so $\epsilon \notin \mathrm{sp}(\widetilde{L}_\lambda)$. $\square$

By the Feshbach Projection Method [BFS1, BFS2, DJ1, DF], if $z \notin \mathrm{sp}(L_\lambda) \cup \mathrm{sp}(L_\lambda^{\overline{\mathrm{vv}}})$ then the restricted resolvent is given by

$$\mathbf{1}^{\mathrm{vv}}(z\mathbf{1} - L_\lambda)^{-1}\mathbf{1}^{\mathrm{vv}} = G_{\mathrm{v}}^{-1}(z)$$

where

$$G_{\mathrm{v}}(z) = z\mathbf{1}^{\mathrm{vv}} - E - \lambda Q^{\mathrm{vv}} - \lambda^2 Q^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - L_\lambda^{\overline{\mathrm{vv}}})^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}.$$

Hence

$$\mathbf{1}^{\mathrm{vv}}(\xi - \lambda^{-2}L_\lambda)^{-1}\mathbf{1}^{\mathrm{vv}} = \lambda^2 G_{\mathrm{v}}^{-1}(\lambda^2\xi).$$

In what follows we will use these facts without a comment.

Proof of the Theorem 3.1 (1) By Lemma 4.1, the operator $\widetilde{L}_\lambda$ generates a semigroup of contractions, which for $\xi > 0$ implies

$$\left\| \left(\xi\mathbf{1}^{\mathrm{vv}} - \lambda^2 Q^{\mathrm{v}\overline{\mathrm{v}}}\left(\xi\mathbf{1}^{\overline{\mathrm{vv}}} - L_\lambda^{\overline{\mathrm{vv}}}\right)^{-1} Q^{\overline{\mathrm{v}}\mathrm{v}} \right)^{-1} \right\| = \|\mathbf{1}^{\mathrm{vv}}(\xi\mathbf{1} - \widetilde{L}_\lambda)^{-1}\mathbf{1}^{\mathrm{vv}}\| \leq \xi^{-1}.$$

Hence, for all $\xi > \xi_0 > 0$, the operator $\left(\xi\mathbf{1}^{\mathrm{vv}} - Q^{\mathrm{v}\overline{\mathrm{v}}}(\lambda^2\xi\mathbf{1}^{\overline{\mathrm{vv}}} - L_\lambda^{\overline{\mathrm{vv}}})^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}\right)^{-1}$ is uniformly bounded. We know that for any $\xi > 0$

$$\lim_{\lambda \searrow 0} \left(\xi\mathbf{1}^{\overline{\mathrm{vv}}} - Q^{\mathrm{v}\overline{\mathrm{v}}}(\xi\lambda^2\mathbf{1}^{\overline{\mathrm{vv}}} - L_\lambda^{\overline{\mathrm{vv}}})^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}} \right) = \xi\mathbf{1}^{\overline{\mathrm{vv}}} - \Gamma_0 \tag{4.19}$$

Therefore, for all $\xi > \xi_0$, the operator $\xi\mathbf{1}^{\mathrm{vv}} - \Gamma_0$ is invertible on $\mathcal{X}^{\mathrm{v}}$ and

$$\|(\xi\mathbf{1}^{\mathrm{vv}} - \Gamma_0)^{-1}\| \leq \xi^{-1}.$$

Hence Γ_0 generates a semigroup of contractions on $\mathcal{X}^{\mathrm{v}}$ (Theorem 2.21 and Corollary 2.22 in [Dav3]).

(2) Let $0 \leq \lambda < \lambda_0$. Since $E + \lambda Q^{\mathrm{vv}}$ generates a group of isometries and Γ_0 is bounded and dissipative then the result follows from Th.3.1.32 in [BR1].

(3) Let $0 < \lambda < \lambda_0$. Recall that for $\xi > 0$

$$\|\lambda^2 G_{\mathrm{v}}^{-1}(\lambda^2\xi)\| \leq \xi^{-1}. \tag{4.20}$$

$$\|(\xi \mathbf{1}^{\mathrm{vv}} - \lambda^{-2}A_{\lambda,0})^{-1}\| \leq \xi^{-1} \tag{4.21}$$

$$\lim_{\lambda\searrow 0} Q^{\mathrm{v}\overline{\mathrm{v}}}(\lambda^2\xi\mathbf{1}^{\overline{\mathrm{vv}}} - L_\lambda^{\overline{\mathrm{vv}}})^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}} = \Gamma_0 \tag{4.22}$$

For any $\xi > 0$ we have

$$\begin{aligned} &\left\|\mathbf{1}^{\mathrm{vv}}(\xi\mathbf{1} - \lambda^{-2}L_\lambda)^{-1}\mathbf{1}^{\mathrm{vv}} - (\xi\mathbf{1}^{\mathrm{vv}} - \lambda^{-2}A_{\lambda,0})^{-1}\right\| \\ = &\left\|\lambda^2 G_{\mathrm{v}}^{-1}(\lambda^2\xi)\Big(Q^{\mathrm{v}\overline{\mathrm{v}}}(\lambda^2\xi\mathbf{1}^{\overline{\mathrm{vv}}} - L_\lambda^{\overline{\mathrm{vv}}})^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}} - \Gamma_0\Big)(\xi\mathbf{1}^{\mathrm{vv}} - \lambda^{-2}A_{\lambda,0})^{-1}\right\| \end{aligned}$$

Hence by (4.20), (4.21) and (4.22) the RHS of the above expression tends to zero as λ tends to zero.

(4) For $\xi > 0$, by the Laplace transform, we have

$$\mathbf{1}^{\mathrm{vv}}(\xi\mathbf{1} - \lambda^{-2}L_\lambda)^{-1}\mathbf{1}^{\mathrm{vv}} - (\xi\mathbf{1}^{\mathrm{vv}} - \lambda^{-2}A_{\lambda,0})^{-1} = \int_0^\infty \mathrm{e}^{-\xi\sigma}\Big(\mathbf{1}^{\mathrm{vv}}\mathrm{e}^{\lambda^{-2}\sigma L_\lambda}\mathbf{1}^{\mathrm{vv}} - \mathrm{e}^{\lambda^{-2}\sigma A_{\lambda,0}}\Big)\mathrm{d}\sigma$$

Hence by (3.8) we get

$$\lim_{\lambda\searrow 0}\int_0^\infty \mathrm{e}^{-\xi\sigma}\Big(\mathbf{1}^{\mathrm{vv}}\mathrm{e}^{\lambda^{-2}\sigma L_\lambda}\mathbf{1}^{\mathrm{vv}} - \mathrm{e}^{\lambda^{-2}\sigma A_{\lambda,0}}\Big)\mathrm{d}\sigma = 0. \tag{4.23}$$

By the Stone–Weierstrass theorem, the family of functions

$$[0,\infty[\ni \sigma \mapsto \mathrm{e}^{-\xi\sigma} \in \mathbb{R} \qquad\qquad \xi > 0$$

forms an algebra which is dense in continuous functions on $[0,\infty[$ vanishing at infinity. This fact together with (4.23) implies (3.9). $\square$

Lemma 4.2 *Let E be the generator of a group of isometries. Let e be an isolated point in* $\mathrm{sp}(E)$. *Then e is a semisimple eigenvalue which means $E\mathbf{1}_e(E) = e\mathbf{1}_e(E)$ and $\|\mathbf{1}_e(E)\| = 1$.*

Proof. Let e be an isolated point in $\mathrm{sp}(E)$. Then for any $\epsilon > 0$

$$\left\|\big((e+\epsilon)\mathbf{1}^{\mathrm{vv}} - E\big)^{-1}\right\| \leq \epsilon^{-1}. \tag{4.24}$$

So for $z \in \{z \in \mathbb{C} \ :\ \mathrm{dist}(e,z) < \delta\}\backslash\{e\}$ for some $\delta > 0$ we have

$$(z\mathbf{1}^{\mathrm{vv}} - E)^{-1} = \mathbf{1}_e(E)(z-e)^{-1} + h(z) \tag{4.25}$$

where h is an analytic function on $\{z \in \mathbb{C} \ : \ \mathrm{dist}(e,z) < \delta\}$. Hence e is semisimple. But (4.25) also implies that

$$\lim_{\epsilon \searrow 0} \epsilon\big((e+\epsilon)\mathbf{1}^{\mathrm{vv}} - E\big)^{-1} = \mathbf{1}_e(E).$$

and hence, by (4.24), we get $\|\mathbf{1}_e(E)\| = 1$. □

For an isolated point $e \in \mathrm{sp}E$ let us write for shortness

$$\mathbf{1}^{ee} := \mathbf{1}_e(E), \qquad \mathbf{1}^{\underline{ee}} := \mathbf{1}^{\mathrm{vv}} - \mathbf{1}_e(E)$$

$$\mathcal{X}^e := \mathrm{Ran}\mathbf{1}_e(E), \qquad \mathcal{X}^{\underline{e}} := \mathrm{Ran}\mathbf{1}^{\underline{ee}}$$

then

$$\mathcal{X}^{\mathrm{v}} = \mathcal{X}^e \oplus \mathcal{X}^{\underline{e}}. \tag{4.26}$$

If $e', e \in \mathrm{sp}E$ and $A \in B(\mathcal{X}^{\mathrm{v}})$ then we denote $A^{e'e} := \mathbf{1}^{e'e'}A\mathbf{1}^{ee}$.

Proof of the Theorem 3.2. (2) Follows immediately if we note that the Lemma 4.2 implies $E = \sum\limits_{e \in \mathrm{sp}E} e\mathbf{1}_e(E)$ and that we have $\Gamma := \sum\limits_{e \in \mathrm{sp}E} \mathbf{1}_e(E)\Gamma\mathbf{1}_e(E)$.

(3) Let $e \in \mathrm{sp}E$. Set

$$G_\lambda(\xi, e) := \xi\mathbf{1}^{\mathrm{vv}} + \lambda^{-2}(e\mathbf{1}^{\mathrm{vv}} - E) - Q^{\mathrm{v}\overline{\mathrm{v}}}\Big((\lambda^2\xi + e)\mathbf{1}^{\overline{\mathrm{vv}}} - L_\lambda^{\overline{\mathrm{vv}}}\Big)^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}.$$

For $\xi > 0$ we have

$$G_\lambda(\xi, e)^{-1} = \mathbf{1}^{\mathrm{vv}}\left(\xi + \lambda^{-2}(e - L_\lambda)\right)^{-1}\mathbf{1}^{\mathrm{vv}}$$

This and the dissipativity of L_λ implies the bound

$$\|G_\lambda(\xi, e)^{-1}\| \leq \xi^{-1}. \tag{4.27}$$

Write for shortness G instead of $G_\lambda(\xi, e)$.

Decompose $G = G_{\mathrm{diag}} + G_{\mathrm{off}}$ into its diagonal and off-diagonal part:

$$\begin{aligned} G_{\mathrm{diag}} &:= \sum_{e' \in \mathrm{sp}E} \mathbf{1}^{e'e'}G\mathbf{1}^{e'e'}, \\ G_{\mathrm{off}} &:= \sum_{e' \in \mathrm{sp}E} \mathbf{1}^{e'e'}G\mathbf{1}^{\underline{e'e'}} = \sum_{\mathrm{i}e' \in \mathrm{sp}E} \mathbf{1}^{\underline{e'e'}}G\mathbf{1}^{e'e'}. \end{aligned}$$

First we would like to show that for $\xi > 0$ and small enough λ, G_{diag} is invertible. By the Neumann theorem, it is easy to see that $\mathbf{1}^{\underline{ee}}G_{\mathrm{diag}}$ is invertible on $\mathrm{Ran}\mathbf{1}^{\underline{ee}}$ for small enough λ. Moreover, we have

$$\|\mathbf{1}^{\underline{ee}}G_{\mathrm{diag}}^{-1}\| \leq c\lambda^2. \tag{4.28}$$

It is more complicated to prove that $\mathbf{1}^{ee}G_{\mathrm{diag}}$ is invertible on $\mathrm{Ran}\mathbf{1}^{ee}$.

We fix $\xi > 0$. We know that G is invertible and $\|G^{-1}\| \leq \xi^{-1}$. Hence we can write

$$G_{\mathrm{diag}}G^{-1} = \mathbf{1} - G_{\mathrm{off}}G^{-1}.$$

Therefore

$$\begin{aligned} \mathbf{1}^{ee}G_{\mathrm{diag}}G^{-1} &= \mathbf{1}^{ee} - \mathbf{1}^{ee}G_{\mathrm{off}}\mathbf{1}^{\underline{ee}}G^{-1}, \\ \mathbf{1}^{\underline{ee}}G_{\mathrm{diag}}G^{-1} &= \mathbf{1}^{\underline{ee}} - \mathbf{1}^{\underline{ee}}G_{\mathrm{off}}G^{-1}. \end{aligned} \quad . \tag{4.29}$$

The latter identity can be for small enough λ transformed into

$$\mathbf{1}^{\underline{ee}}G^{-1} = G_{\text{diag}}^{-1}\mathbf{1}^{\underline{ee}} - G_{\text{diag}}^{-1}\mathbf{1}^{\underline{ee}}G_{\text{off}}G^{-1}. \tag{4.30}$$

We insert (4.30) into the first identity of (4.29) to obtain

$$\mathbf{1}^{ee}G_{\text{diag}}G^{-1} = \mathbf{1}^{ee} - \mathbf{1}^{ee}G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1} + \mathbf{1}^{ee}G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}G_{\text{off}}G^{-1}. \tag{4.31}$$

We multiply (4.31) from the right by $\mathbf{1}^{ee}$ to get

$$\mathbf{1}^{ee}G_{\text{diag}}\mathbf{1}^{ee}G^{-1}\mathbf{1}^{ee} = \mathbf{1}^{ee} + \mathbf{1}^{ee}G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}G_{\text{off}}G^{-1}\mathbf{1}^{ee}. \tag{4.32}$$

Now, using

$$\lim_{\lambda\searrow 0}\|\lambda G_{\text{off}}\| = 0, \tag{4.33}$$

(4.27) and (4.28) we obtain

$$\lim_{\lambda\searrow 0}\mathbf{1}^{ee}G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}G_{\text{off}}G^{-1}\mathbf{1}^{ee} = 0.$$

Thus, for small enough λ,

$$\mathbf{1}^{ee}G_{\text{diag}}B_1 = \mathbf{1}^{ee},$$

where

$$B_1 := \mathbf{1}^{ee}G^{-1}\mathbf{1}^{ee}\left(\mathbf{1}^{ee} + \mathbf{1}^{ee}G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}G_{\text{off}}G^{-1}\mathbf{1}^{ee}\right)^{-1}.$$

Similarly, for small enough λ, we find B_2 such that

$$B_2\mathbf{1}^{ee}G_{\text{diag}} = \mathbf{1}^{ee}.$$

This implies that $\mathbf{1}^{ee}G_{\text{diag}}$ is invertible on $\mathrm{Ran}\mathbf{1}^{ee}$.

Next, we can write:

$$G^{-1} = G_{\text{diag}}^{-1} - G_{\text{diag}}^{-1}G_{\text{off}}G_{\text{diag}}^{-1} + G_{\text{diag}}^{-1}G_{\text{off}}G_{\text{diag}}^{-1}G_{\text{off}}G^{-1}.$$

Hence

$$\mathbf{1}^{ee}G^{-1} = \mathbf{1}^{ee}G_{\text{diag}}^{-1}\left(\mathbf{1} - G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1} + G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}G_{\text{off}}G^{-1}\right). \tag{4.34}$$

Therefore, for a fixed ξ, by (4.27), (4.28) and (4.33) we see that as $\lambda\searrow 0$ we have

$$-G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1} + G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}G_{\text{off}}G^{-1} \to 0.$$

Therefore, for small enough λ, we can invert the expression in the bracket of (4.34) Consequently,

$$\begin{aligned}\mathbf{1}^{ee}(G_{\text{diag}}^{-1} - G^{-1}) = \ & \mathbf{1}^{ee}G^{-1}\left(1 - G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1} + G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}G_{\text{off}}G^{-1}\right)^{-1}\\ &\times\left(G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1} - G_{\text{off}}\mathbf{1}^{\underline{ee}}G_{\text{diag}}^{-1}G_{\text{off}}G^{-1}\right).\end{aligned} \tag{4.35}$$

Therefore, for a fixed ξ, by (4.27), (4.28) and (4.33) we see that as $\lambda\searrow 0$ we have

$$\mathbf{1}^{ee}(G_{\text{diag}}^{-1} - G^{-1}) \to 0. \tag{4.36}$$

(4.27) and (4.36) imply that $\mathbf{1}^{ee}G_{\rm diag}^{-1}$ is uniformly bounded as $\lambda \searrow 0$. We know that

$$\mathbf{1}^{ee}G_{\rm diag} \to \mathbf{1}^{ee}\xi - \mathbf{1}^{ee}\Gamma. \tag{4.37}$$

Therefore, $\xi\mathbf{1}^{ee} - \mathbf{1}^{ee}\Gamma$ is invertible on ${\rm Ran}\mathbf{1}^{ee}$ and

$$\mathbf{1}^{ee}G_{\rm diag}^{-1} \to (\mathbf{1}^{ee}\xi - \mathbf{1}^{ee}\Gamma)^{-1}.$$

Using again (4.36) we see that

$$\mathbf{1}^{ee}G^{-1} \to (\mathbf{1}^{ee}\xi - \mathbf{1}^{ee}\Gamma)^{-1}. \tag{4.38}$$

Summing up (4.38) over e we obtain

$$\sum_{e\in{\rm sp}E}\mathbf{1}^{ee}G_\lambda(\xi,{\rm i}e)^{-1} \to (\xi\mathbf{1}^{\rm vv} - \Gamma)^{-1}, \tag{4.39}$$

which ends the proof of (3).

(1) We have

$$\begin{aligned}\sum_{e\in{\rm sp}E}\mathbf{1}^{ee}G_\lambda(\xi,e)^{-1} &= \sum_{e\in{\rm sp}E}\int\limits_0^\infty {\rm e}^{-t(\xi+\lambda^{-2}e)}\mathbf{1}^{ee}{\rm e}^{tL_\lambda/\lambda^2}\mathbf{1}^{\rm vv}{\rm d}t\\ &= \int\limits_0^\infty {\rm e}^{-t\xi}{\rm e}^{-tE/\lambda^2}\mathbf{1}^{\rm vv}{\rm e}^{tL_\lambda/\lambda^2}\mathbf{1}^{\rm vv}{\rm d}t\end{aligned} \tag{4.40}$$

Clearly, $\|{\rm e}^{-tE/\lambda^2}\mathbf{1}^{\rm vv}{\rm e}^{tL_\lambda/\lambda^2}\mathbf{1}^{\rm vv}\| \leq 1$. Therefore,

$$\left\|\sum_{e\in{\rm sp}E}\mathbf{1}^{ee}G_\lambda(\xi,{\rm i}e)^{-1}\right\| \leq \xi^{-1}.$$

Hence, by (4.39),

$$\left\|(\xi\mathbf{1}^{\rm vv} - \Gamma)^{-1}\right\| \leq \xi^{-1},$$

which means that Γ is the generator of a semigroup of contractions.

(4) To prove this we repeat the argument used in the proof of the Theorem 3.1. □

Proof of the Theorem 3.3 (1) Follows by a simple modification of the argument used in the theorem 3.2

(2) For $e \in {\rm sp}E$ and for ${\rm Re}\xi > 0$ we denote

$$G_\lambda(\xi,e) := \xi\mathbf{1}^{\rm vv} + \lambda^{-2}(e\mathbf{1}^{\rm vv} - E) - Q^{\rm v\overline{v}}\Big((\lambda^2\xi + e)\mathbf{1}^{\overline{\rm vv}} - L_\lambda^{\overline{\rm vv}}\Big)^{-1}Q^{\overline{\rm v}{\rm v}}$$

Obviously

$$G_\lambda(\xi,e)^{-1} = \mathbf{1}^{\rm vv}\left(\xi + \lambda^{-2}(e - L_\lambda)\right)^{-1}\mathbf{1}^{\rm vv}$$

and

$$\|G_\lambda(\xi,e)^{-1}\| \leq {\rm Re}\xi^{-1} \tag{4.41}$$

Let $\psi \in \mathcal{X}^{\rm v}$. Let $\omega_0 > 0$ and $\sigma \geq 0$. By the inverse Laplace transform ([HiPh] chapter XI) and by the proof of the Theorem 3.2, uniformly for $0 < \tau_0 \leq \sigma \leq \tau_1$, we get

$$\begin{aligned}
&\big(\mathrm{e}^{-\sigma\lambda^{-2}E}\mathbf{1}^{\rm vv}\mathrm{e}^{\sigma\lambda^{-2}L_\lambda}\mathbf{1}^{\rm vv} - \mathrm{e}^{\sigma\Gamma}\big)\psi \\
&= \frac{1}{2\pi\mathrm{i}}\lim_{R\to\infty}\int\limits_{\omega_0+\mathrm{i}[-R,R]} \mathrm{e}^{\sigma\xi}\sum_{e\in\mathrm{sp}E}\Big(\mathbf{1}^{ee}\big((\xi+\lambda^{-2}e)\mathbf{1}-\lambda^{-2}L_\lambda\big)^{-1}\mathbf{1}^{\rm vv} - (\xi\mathbf{1}^{ee}-\Gamma^{ee})^{-1}\Big)\psi\mathrm{d}\xi \\
&= \frac{1}{2\pi\mathrm{i}}\sum_{e\in\mathrm{sp}E}\lim_{R\to\infty}\int\limits_{-R}^{R} f_\lambda(y,e)\mathrm{i}\mathrm{d}y
\end{aligned}$$

where

$$f_\lambda(y,e) := \mathrm{e}^{\sigma(\omega_0+\mathrm{i}y)}\Big(\mathbf{1}^{ee}G_\lambda(\omega_0+\mathrm{i}y,e)^{-1} - \big((\omega_0+\mathrm{i}y)\mathbf{1}^{ee}-\Gamma^{ee}\big)^{-1}\Big)\psi.$$

For each $e \in \mathrm{sp}E$

$$\Big\|\frac{d}{dy}f_\lambda(y,e)\Big\| \leq \mathrm{e}^{\sigma\omega_0}2(\sigma\omega_0^{-1}+\omega_0^{-2})\|\psi\|.$$

This shows that the family $f_\lambda(y,e)$ is equicontinuous as $\lambda\to 0$.

For any fixed $R > 0$ if only $\alpha_0 = (R+1)/\omega_0$ then $\xi = \omega_0 + \mathrm{i}y \in \mathbf{Wedge}(0,\alpha_0)$ hence, by (1), we get the pointwise convergence $f_\lambda(y,e)\to 0$ as $\lambda\searrow 0$ for $y\in[-R,R]$. Finally pointwise convergence together with equicontinuity implies uniform convergence on compacts, so for any fixed $R > 0$

$$\lim_{\lambda\searrow 0}\int\limits_{-R}^{R} f_\lambda(y,e)\psi\mathrm{i}\mathrm{d}y = \int\limits_{-R}^{R}\lim_{\lambda\searrow 0} f_\lambda(y,e)\psi\mathrm{i}\mathrm{d}y = 0.$$

To end the proof it is sufficient to show that for each $e\in\mathrm{sp}E$

$$\lim_{R\to\infty}\sup_{0\leq\lambda<\lambda_0}\Big\|\int\limits_{|y|>R} f_\lambda(y,e)\mathrm{i}\mathrm{d}y\Big\| = 0$$

Since Γ^{ee} is independent of λ we need only to show that

$$\lim_{R\to\infty}\sup_{0\leq\lambda<\lambda_0}\Big\|\int\limits_{|y|>R}\mathrm{e}^{\sigma(\omega_0+\mathrm{i}y)}\mathbf{1}^{ee}G_\lambda(\omega_0+\mathrm{i}y,e)^{-1}\psi\mathrm{d}y\Big\| = 0. \tag{4.42}$$

Decompose $G_\lambda(\xi,e)$ into its diagonal and off-diagonal part.

$$\begin{aligned}
G_{\rm diag} &:= \sum_{e'\in\mathrm{sp}E}\mathbf{1}^{e'e'}G_\lambda(\xi,e)\mathbf{1}^{e'e'} \\
&= \xi\mathbf{1}^{ee} - T_\lambda(\xi,e)^{ee} + \sum_{e'\in\mathrm{sp}E;\ e'\neq e}\Big(\big(\xi-\lambda^{-2}(e'-e)\big)\mathbf{1}^{e'e'} - T_\lambda(\xi,e)^{e'e'}\Big) \\
G_{\rm off} &:= \sum_{e'\in\mathrm{sp}E}\mathbf{1}^{e'e'}G_\lambda(\xi,e)\mathbf{1}^{\underline{e'e'}} = \sum_{\mathrm{i}e'\in\mathrm{sp}E}\mathbf{1}^{\underline{e'e'}}G_\lambda(\xi,e)\mathbf{1}^{e'e'} \\
&= \sum_{e'\in\mathrm{sp}E}T_\lambda(\xi,e)^{e'\underline{e'}}.
\end{aligned}$$

where

$$T_\lambda(\xi,e) := Q^{\mathrm{v}\overline{\mathrm{v}}}\Big((\lambda^2\xi+e)\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - \lambda Q^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\Big)^{-1} Q^{\overline{\mathrm{v}}\mathrm{v}}.$$

By the assumption for each $e \in \mathrm{sp}E$

$$\sup_{\mathrm{Re}\xi>0;\,0\le\lambda<\lambda_0} \|T_\lambda(\xi,e)\| < C < \infty. \tag{4.43}$$

In the rest of the proof we write for shortness $G = G_\lambda(\xi,e)$ and $T = T_\lambda(\xi,e)$.

Fix $e \in \mathrm{sp}E$ and let $\xi := \omega_0 + \mathrm{i}y$. Fix $\omega_0 > C+1$. Then, by the Neumann theorem, the operator $\mathbf{1}^{ee}G_{\mathrm{diag}} = \xi\mathbf{1}^{ee} - T^{ee}$ is invertible on $\mathcal{X}^e$ and we have

$$\|(\xi\mathbf{1}^{ee} - T^{ee})^{-1}\| \le (|y|-C)^{-1} \qquad\qquad \text{for } |y|>C \tag{4.44}$$

Note that for each $e' \in \mathrm{sp}E$, $e' \neq e$ the operator $\lambda^{-2}(e'-e)\mathbf{1}^{e'e'}$ generates a group of isometries on $\mathcal{X}^{e'}$. Hence the operator $\mathbf{1}^{e'e'}G_{\mathrm{diag}} = \big(\xi - \lambda^{-2}(e'-e)\big)\mathbf{1}^{e'e'} - T^{e'e'}$ is invertible on $\mathcal{X}^{e'}$ and we have

$$\Big\|\Big(\big(\xi-\lambda^{-2}(e'-e)\big)\mathbf{1}^{e'e'} - T^{e'e'}\Big)^{-1}\Big\| \le (\omega_0 - C)^{-1} < 1. \tag{4.45}$$

This shows that G_{diag} is invertible on $\mathcal{X}^{\mathrm{v}}$. We have

$$\begin{aligned}
\mathbf{1}^{ee}G^{-1} =\;& \mathbf{1}^{ee}G_{\mathrm{diag}}^{-1}(\mathbf{1}^{ee} - G_{\mathrm{off}}\mathbf{1}^{\underline{ee}}G^{-1})\\
=\;& \xi^{-1}\Big(\mathbf{1}^{ee} + T^{ee}(\xi\mathbf{1}^{ee}-T^{ee})^{-1}\Big)\Big(\mathbf{1}^{ee} - G_{\mathrm{off}}\mathbf{1}^{\underline{ee}}G^{-1}\Big)\\
=\;& \xi^{-1}\Big(\mathbf{1}^{ee} + T^{ee}(\xi\mathbf{1}^{ee}-T^{ee})^{-1}\Big)\Big(\mathbf{1}^{ee} - G_{\mathrm{off}}\mathbf{1}^{\underline{ee}}G_{\mathrm{diag}}^{-1} + G_{\mathrm{off}}\mathbf{1}^{\underline{ee}}G_{\mathrm{diag}}^{-1}G_{\mathrm{off}}\mathbf{1}^{ee}G^{-1}\Big)
\end{aligned}$$

Now, using (4.41), (4.43), (4.44) and (4.45), we get for $|y|>C$

$$\|\xi^{-1}T^{ee}(\xi\mathbf{1}^{ee}-T^{ee})^{-1}(\mathbf{1}^{ee} - G_{\mathrm{off}}\mathbf{1}^{\underline{ee}}G^{-1})\| \le D_1(\omega_0^2+y^2)^{-1/2}(|y|-C)^{-1}$$

$$\|\xi^{-1}G_{\mathrm{off}}\mathbf{1}^{\underline{ee}}G_{\mathrm{diag}}^{-1}G_{\mathrm{off}}\mathbf{1}^{ee}G^{-1}\| \le D_1(\omega_0^2+y^2)^{-1/2}(|y|-C)^{-1}$$

for some $D_1>0$ independent of λ. Hence to prove (4.42) if suffices to show that

$$\lim_{R\to\infty}\sup_{|\lambda|<\lambda_0}\Big\|\int\limits_{|y|>R} \mathrm{e}^{\sigma(\omega_0+\mathrm{i}y)}(\omega_0+\mathrm{i}y)^{-1}(\mathbf{1}^{ee} - G_{\mathrm{off}}\mathbf{1}^{\underline{ee}}G_{\mathrm{diag}}^{-1})\psi\mathrm{d}y\Big\| = 0.$$

The first term in the above expression is independent of λ hence we need only to consider the second term. We have

$$\mathbf{1}^{\underline{ee}}G_{\mathrm{diag}}^{-1} = \sum_{e'\in\mathrm{sp}E;\,e'\neq e}\big(\xi-\lambda^{-2}(e'-e)\big)^{-1}\Big(\mathbf{1}^{e'e'} + T^{e'e'}\Big(\big(\xi-\lambda^{-2}(e'-e)\big)\mathbf{1}^{e'e'} - T^{e'e'}\Big)^{-1}\Big)$$

Hence, by (4.43) and (4.45), we get

$$\|\mathrm{e}^{\sigma(\omega_0+\mathrm{i}y)}(\omega_0+\mathrm{i}y)^{-1}G_{\mathrm{off}}\mathbf{1}^{\underline{ee}}G_{\mathrm{diag}}^{-1}\| \le D_2 g(y)g(y-\lambda^{-1}\mathrm{Im}(e'-e)) \tag{4.46}$$

for some $D_2 > 0$ independent of λ, where

$$\mathbb{R} \ni y \mapsto g(y) := \left(\omega_0^2 + y^2\right)^{-1/2} \in \mathbb{R}.$$

By the Hölder inequality

$$\int\limits_{|y|>R} g(y)g(y-\lambda^{-1}\mathrm{Im}(e'-e))dy \leq \quad \|g\|_{L^2(]-\infty,-R]\cup[R,\infty[,\mathrm{d}x)}\|g\|_{L^2(\mathbb{R},\mathrm{d}x)} \to 0. \tag{4.47}$$

Now, by (4.46) and (4.47), we get

$$\Big\| \int\limits_{|y|>R} \mathrm{e}^{\sigma(\omega_0+\mathrm{i}y)}(\omega_0+\mathrm{i}y)^{-1}G_{\mathrm{off}}\mathbf{1}^{\underline{ee}}G_{\mathrm{diag}}^{-1}\psi\mathrm{d}y\Big\| \to 0$$

idependently of λ. This ends the proof of (3.14). □

Proof of Theorem 3.5. Set

$$f(s) := \sup_{|\lambda|\leq\lambda_0} \|Q^{\mathrm{v}\overline{\mathrm{v}}}\mathrm{e}^{sL_\lambda^{\overline{\mathrm{v}}\overline{\mathrm{v}}}}Q^{\overline{\mathrm{v}}\mathrm{v}}\|.$$

We know that $f(t)$ is integrable.

For any $e \in \mathrm{i}\mathbb{R}$ and $\xi \geq 0$ we can dominate the integrand in the integral

$$\begin{aligned} F_\lambda(e,\xi) \quad &:= \int\limits_0^\infty Q^{\mathrm{v}\overline{\mathrm{v}}}\mathrm{e}^{sL_\lambda^{\overline{\mathrm{v}}\overline{\mathrm{v}}}}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathrm{e}^{-(e+\lambda^2\xi)s}\mathrm{d}s \\ &= Q^{\mathrm{v}\overline{\mathrm{v}}}\Big(\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}(e+\lambda^2\xi) - L_\lambda^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\Big)^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}} \end{aligned} \tag{4.48}$$

by $f(s)$. Hence, using the dominated convergence theorem we see that $F_\lambda(e,\xi)$ is continuous at $\lambda = 0$ and $\xi \geq 0$. But

$$\begin{aligned} &\sum_{e\in\mathrm{sp}E} \mathbf{1}_e(E)F_0(e,0)\mathbf{1}_e(E) \\ = \quad &\sum_{e\in\mathrm{sp}E} \lim_{\lambda\to 0} \mathbf{1}_e(E)Q^{\mathrm{v}\overline{\mathrm{v}}}\left(\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}(e+\lambda^2\xi) - L_\lambda^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\right)^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathbf{1}_e(E) \quad = \Gamma. \end{aligned}$$

Recall (3.15), the definition o $K(\lambda,t)$:

$$K(\lambda,t) := \int\limits_0^{\lambda^{-2}t} \mathrm{e}^{-sE}Q^{\mathrm{v}\overline{\mathrm{v}}}\mathrm{e}^{sL_\lambda^{\overline{\mathrm{v}}\overline{\mathrm{v}}}}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathrm{d}s.$$

Its integrand can also be dominated by $f(s)$. Hence, using again the dominated convergence theorem, we see that, for $\lambda \to 0$, $K(\lambda,t)$ is convergent to

$$K = \int\limits_0^\infty \mathrm{e}^{-sE}Q^{\mathrm{v}\overline{\mathrm{v}}}\mathrm{e}^{sL_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}}}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathrm{d}s.$$

Therefore,

$$\begin{aligned} K^{\natural} &= \sum_{e \in \mathrm{sp} E} \mathbf{1}_e(E) \int\limits_0^\infty \mathrm{e}^{-es} Q^{\mathrm{v}\overline{\mathrm{v}}} \mathrm{e}^{sL_0^{\overline{\mathrm{v}\mathrm{v}}}} Q^{\mathrm{v}\overline{\mathrm{v}}} \mathrm{d}s \mathbf{1}_e(E) \\ &= \sum_{e \in \mathrm{sp} E} \mathbf{1}_e(E) F_0(e, 0) \mathbf{1}_e(E). \end{aligned}$$

□

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