About a result of S.M. Kozlov

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Abstract

We give an alternative proof and improve upon a result of S.M. Kozlov [7]. It deals with the asymptotic of the integrated density of states of the acoustic operator $H_\omega = -\nabla \rho_\omega \nabla$, at the bottom of the spectrum.

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1 Introduction

Let H_ω, be the self adjoint operator on $L^2(\mathbb{R}^d)$ formally defined by:

$$H_\omega = H(\rho_\omega) = -\nabla \cdot \rho_\omega \cdot \nabla, \tag{1.1}$$

where ρ_ω is a positive and bounded function.
H_ω is called the acoustic operator, see [1] for the physical interpretations. Let us start by defining the main object of our study: the integrated density of states. For this, we consider Λ a cube of $\mathbb{R}^d$. We note by $H_{\omega,\Lambda}$ the restriction of H_ω to Λ with self-adjoint boundary conditions. As H_ω is elliptic, the resolvent of $H_{\omega,\Lambda}$ is compact and, consequently, the spectrum of $H_{\omega,\Lambda}$

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is discrete and is made of isolated eigenvalues of finite multiplicity [13]. We define

$$N_\Lambda(E) = \frac{1}{\mathrm{vol}(\Lambda)} \cdot \#\{\text{eigenvalues of } \mathrm{A}_{\omega,\Lambda} \leq \mathrm{E}\}. \tag{1.2}$$

Here $\mathrm{vol}(\Lambda)$ is the volume of Λ in the Lebesgue sense and $\#E$ is the cardinal of E.

It is shown that the limit of $N_\Lambda(E)$ when Λ tends to $\mathbb{R}^d$ exists almost surely and is independent of the boundary conditions. It is called the **integrated density of states** of A_ω (IDS as acronym). See [12].

The question we are interested in here regards the behavior of N at the bottom of the spectrum of H_ω. In previous works [8, 9, 10, 11], the author gives the behavior of N at the internal band edges of the spectrum of (1.1). It was a Lifshitz behavior (N decreases exponentially fast). In the present situation, for the bottom of the spectrum, it is known that it can't decrease more than polynomially fast, [7]. Here we compare the behavior of N to the behavior of the IDS of some periodic operator with exponentially precision.

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1.1 The model

Consider the random Schrödinger operator

$$H_\omega = -\nabla \frac{1}{\rho_\omega} \nabla. \tag{1.3}$$

Where ρ_ω is a bounded, $\mathbb{Z}^d$-ergodic random field such that there exists some constant $\rho_* > 1$, satisfying

$$\rho_* \leq \rho_\omega \leq \rho_*. \tag{1.4}$$

We assume that ρ_ω is of Anderson type i.e. it has the form

$$\rho_\omega(x) = \rho^+(x) + \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma \rho^0(x - \gamma) \tag{1.5}$$

where

- ρ^+ is a-$\mathbb{Z}^d$-periodic measurable function,
- ρ^0 is a compactly supported measurable function,
- $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are non trivial, i.i.d. random variables.

The choice of our model ensures that A_ω is a measurable family of self-adjoint operators and ergodic [4, 12]. Indeed, if τ_γ refers to the translation by γ, then $(\tau_\gamma)_{\gamma\in\mathbb{Z}^d}$ is a group of unitary operators on $L^2(\mathbb{R}^d)$ and for $\gamma\in\mathbb{Z}^d$ we have

$$\tau_\gamma A_\omega \tau_{-\gamma} = A_{\tau_\gamma\omega}.$$

According to [4, 12], we know that there exists $\Sigma, \Sigma_{pp}, \Sigma_{ac}$ and Σ_{sc} closed and non-random sets of $\mathbb{R}$ such that Σ is the spectrum of A_ω with probability one and such that if σ_{pp} (respectively σ_{ac} and σ_{sc}) design the pure point spectrum (respectively the absolutely continuous and singular continuous spectrum) of A_ω, then $\Sigma_{pp}=\sigma_{pp}, \Sigma_{ac}=\sigma_{ac}$ and $\Sigma_{sc}=\sigma_{sc}$ with probability one. H_ω is defined as the Friedrichs extension of the following positive quadratic form

$$\mathcal{H}_\omega[\psi] = \langle \rho_\omega \nabla\psi, \nabla\psi\rangle \quad , \ \psi\in H^1_{\mathrm{loc}}(\mathbb{R}^d).$$

1.2 The result

We shall prove

Theorem 1.1. *There exists $\alpha,\ \tau>0$ and $C>1$ such that when $E\to 0^+$ we have,*

$$\overline{\mathcal{N}}(E-E^\alpha) - Ce^{-E^{-\tau}} \le \mathcal{N}(E) \le \overline{\mathcal{N}}(E+E^\alpha) + Ce^{-E^{-\tau}} \tag{1.6}$$

where $\overline{\mathcal{N}}$ is the integrated density of states of the following periodic operator

$$\overline{H} = -\nabla\overline{\rho}\nabla; \tag{1.7}$$

and $\overline{\rho} = \mathbb{E}(\rho_\omega)$.

Remark 1.2. *1)The improvement over Kozlov's result essentially consists in the estimate of the remainder term and the exponential precision .*

$$\mathcal{N}(E) \to \overline{\mathcal{N}}(E+o(E)) \quad \text{exponentially as E} \to 0^+.$$

2)We don't believe this estimate to be optimal: namely, we expect the exponent α to be larger than the one we found in the present study.

2 Proof of Theorem 1.1

2.1 The periodic approximation

Pick $n \in \mathbb{N} \setminus \{0\}$ and define the following periodic Schrödinger operator

$$H_\omega^n = -\nabla \rho_\omega^n \nabla.$$

Here,

$$\rho_\omega^n = \rho^{+,n} + \rho_\omega^{0,n} = \rho^+(x) + \sum_{\gamma \in \Lambda_n \cap \mathbb{Z}^d} \omega_\gamma \sum_{\beta \in (2n+1)\mathbb{Z}^d} \rho^0(x - \gamma + \beta).$$

Where Λ_k is the cube

$$\Lambda_n = \{x \in \mathbb{R}^d; \forall 1 \leq j \leq d,\ -\frac{2k+1}{2} < x_j \leq \frac{2k+1}{2}\}.$$

For ω fixed and $n \in \mathbb{N}^*$, H_ω^n is a $(2n+1)\mathbb{Z}^d$-periodic self-adjoint Schrödinger operator.
Let $\overline{\omega} = \mathbb{E}(\omega_0)$ and $\overline{\rho^{0,n}} = \sum_{\gamma \in \Lambda_n \cap \mathbb{Z}^d} \overline{\omega} \sum_{\beta \in (2n+1)\mathbb{Z}^d} \rho^0(x - \gamma + \beta)$.

2.2 Some Floquet Theory

Now we review some standard facts from the Floquet theory for periodic operators. Basic references of this material are in [13].
Let the torus $\mathbb{T}_{2n+1}^* = \mathbb{R}^d / 2\pi(2n+1)\mathbb{Z}^d$. We define $\mathcal{H}_n$ by

$$\mathcal{H}_n = \{u(x,\theta) \in L^2_{loc}(\mathbb{R}^d) \otimes L^2(\mathbb{T}_{2n+1}^*); \\ \forall (x,\theta,\gamma) \in \mathbb{R}^d \times \mathbb{T}_{2n+1}^* \times (2n+1)\mathbb{Z}^d;\ u(x+\gamma,\theta) = e^{i\gamma\theta} u(x,\theta)\}.$$

There exists U a unitary isometry from $L^2(\mathbb{R}^d)$ to $\mathcal{H}_n$ such that H_ω^n admits the following Floquet decomposition [13]

$$U H_\omega^n U^* = \int_{\mathbb{T}_{2n+1}^*}^{\oplus} H_\omega^n(\theta) d\theta.$$

Here $H_\omega^n(\theta)$ is the self adjoint operator on $\mathcal{H}_{n,\theta}$ defined as the operator H_ω^n acting on $\mathcal{H}_{n,\theta}^1$ with

$$\mathcal{H}_{n,\theta} = \{u \in L^2_{loc}(\mathbb{R}^d); \forall \gamma \in (2n+1)\mathbb{Z}^d, u(x+\gamma) = e^{i\gamma\theta} u(x)\},$$

and

$$\mathcal{H}^1_{n,\theta} = \{u \in \mathcal{H}_{n,\theta};\ \partial^\alpha_x u \in \mathcal{H}_{n,\theta};\ |\alpha| = 1\}.$$

As H^n_ω is elliptic, we know that, $H^n_\omega(\theta)$ has a compact resolvent; hence its spectrum is discrete [13]. We denote its eigenvalues, called Floquet eigenvalues of $H^n_\omega(\theta)$, by

$$E_0(n,\omega,\theta) \le E_1(n,\omega,\theta) \le \cdots \le E_k(n,\omega,\theta) \le \cdots.$$

The corresponding eigenfunctions are denoted by $(w(x,\cdot)_k)_{k\in\mathbb{N}}$. The functions $(\theta \to E_k(n,\omega,\theta))_{k\in\mathbb{N}}$ are Lipshitz-continuous, and we have

$$E_k(n,\omega,\theta) \to +\infty \ \text{ as } k \to +\infty \ \text{ uniformly in } \ \theta.$$

The spectrum $\sigma(A^n_\omega)$ of A^n_ω has a band structure. (i.e $\sigma(A^n_\omega) = \bigcup_{k\in\mathbb{N}} E_k(n,\omega,\mathbb{T}^*)$). Let $\mathcal{N}^n_\omega$ be the integrated density of states of H^n_ω; it satisfies

$$\mathcal{N}^n_\omega(E) = \sum_{k\in\mathbb{N}} \frac{1}{(2\pi)^d} \int_{\{\theta\in\mathbb{T}^*_{2n+1};\ E_k(n,\omega,\theta)\le E\}} d\theta = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^*_{2n+1}} \mathcal{V}(H^n_\omega(\theta),E)d\theta. \tag{2.8}$$

Here $\mathcal{V}(B,E)$ is the number of eigenvalues of B less or equal to E. Let $d\mathcal{N}^n_\omega$ be the derivative of $\mathcal{N}^n_\omega$ in the distribution sense. As $\mathcal{N}^n_\omega$ is increasing, $d\mathcal{N}^n_\omega$ is a positive measure; it is the density of states of H^n_ω. We denote by $d\mathcal{N}$ the density of states of H_ω. For all $\varphi \in C^\infty_0(\mathbb{R}), d\mathcal{N}^n_\omega$ verifies [6],

$$\begin{aligned}\langle \varphi, d\mathcal{N}^n_\omega\rangle &= \frac{1}{(2\pi)^d}\int_{\theta\in\mathbb{T}^*_n} \mathrm{tr}_{\mathcal{H}_\theta}\Big(\varphi(H^n_\omega(\theta))\Big)d\theta,\\ &= \frac{1}{\mathrm{vol}(C_k)}\mathrm{tr}\Big(\chi_{C_k}\varphi(H^n_\omega)\chi_{C_k}\Big),\end{aligned} \tag{2.9}$$

where for $\Lambda \subset \mathbb{R}^d$, χ_Λ will design the characteristic function of Λ and $\mathrm{tr}(A)$ is the trace of A (we index by $\mathcal{H}_\theta$ if the trace is taken in $\mathcal{H}_\theta$).

Lemma 2.1 ([6]). *For any $\varphi \in C^\infty_0(\mathbb{R})$ and for almost all $\omega \in \Omega$ we have*

$$\lim_{n\to\infty} \mathbb{E}(\langle\varphi, d\mathcal{N}^n_\omega\rangle) = \langle\varphi, d\mathcal{N}\rangle.$$

Moreover, we have that the IDS of H_ω is exponentially well-approximated by the expectation of the IDS of the periodic operators H^n_ω when n is polynomial in ε^{-1}. More precisely we have

Theorem 2.2 ([6]). *Pick $\eta > 0$ and $I \subset \mathbb{R}$, a compact interval. There exists $\varepsilon_0 > 0$ and $\rho > 0$ such that, for $E \in I$, $\varepsilon \in (0, \varepsilon_0)$ and $n \geq \varepsilon^{-\rho}$, one has*

$$\begin{aligned} &\mathbb{E}(\mathcal{N}^n_\omega(E + \varepsilon/2)) - \mathbb{E}(\mathcal{N}^n_\omega(E - \varepsilon/2)) - e^{-\varepsilon^{-\eta}} \\ &\qquad\qquad \leq \mathcal{N}(E + \varepsilon) - \mathcal{N}(E - \varepsilon) \leq \\ &\qquad\qquad \mathbb{E}(\mathcal{N}^n_\omega(E + 2\varepsilon)) - \mathbb{E}(\mathcal{N}^n_\omega(E - 2\varepsilon)) + e^{-\varepsilon^{-\eta}}. \end{aligned} \tag{2.10}$$

Remark 2.3. *This lemma is proven in [6] for the Schrödinger case. It is still true for our case. The proof is based on the Helffer-Sjöstrand formula and the resolvent equation with the exponential decay of the resolvent kernels (the Combes-Thomas argument).*

Now we study the periodic approximations. For a vector space E, we note by $dim(E)$ the dimension of E. We have,

$$\begin{aligned} \mathcal{V}(H^n_\omega(\theta), E) &= \sup dim\{\mathcal{E} \subset \mathcal{H}^n_{n,\theta}, \text{ such that}, \forall u \in \mathcal{E};\ \langle H^n_\omega(\theta)u, u\rangle \leq E\|u\|^2\} \\ &= \sup dim\{\mathcal{E} \subset \mathcal{H}^n_{n,\theta}, \text{ such that}, \forall u \in \mathcal{E}; \\ &\qquad \langle \Big(H^n_\omega(\theta) - \overline{H}^n(\theta)\Big)u, u\rangle + \langle \overline{H}^n(\theta)u, u\rangle \leq E\|u\|^2\}. \end{aligned}$$

Let, $\alpha > 0$ and

$$\mathcal{E}^\alpha_1(\theta) = \{u \in \mathcal{H}^1_{n,\theta};\ \Big|\langle \Big(H^n_\omega(\theta) - \overline{H}^n(\theta)\Big)u, u\rangle\Big| \leq E^\alpha\|u\|^2\},$$

and

$$\mathcal{E}^\alpha_2(\theta) = \{u \in \mathcal{H}^1_{n,\theta};\ \Big|\langle \Big(H^n_\omega(\theta) - \overline{H}^n(\theta)\Big)u, u\rangle\Big| \geq E^\alpha\|u\|^2\}.$$

Then we have

$$\begin{aligned} \mathcal{V}(H^n_\omega(\theta), E) &\leq \sup dim\{\mathcal{E} \subset \mathcal{E}^\alpha_1(\theta), \text{ such that}, \forall u \in \mathcal{E};\ \langle H^n_\omega(\theta)u, u\rangle \leq E\|u\|^2\} \\ &+ \sup dim\{\mathcal{E} \subset \mathcal{E}^\alpha_2(\theta), \text{ such that}, \forall u \in \mathcal{E};\ \langle H^n_\omega(\theta)u, u\rangle \leq E\|u\|^2\} \\ &\leq \sup dim\{\mathcal{E} \subset \mathcal{E}^\alpha_1(\theta), \text{ such that}, \forall u \in \mathcal{E};\ \langle \overline{H}^n(\theta)u, u\rangle \leq (E + E^\alpha)\|u\|^2\} \\ &+ \sup dim\{\mathcal{E} \subset \mathcal{E}^\alpha_2(\theta), \text{ such that}, \forall u \in \mathcal{E}; \langle H^n_\omega(\theta)u, u\rangle \leq E\|u\|^2\}. \end{aligned}$$

So, we get

$$\begin{aligned} &\mathcal{V}(H^n_\omega(\theta), E) \leq \mathcal{V}(\overline{H}^n(\theta), (E + E^\alpha)) \\ &\quad + \sup dim\{\mathcal{E} \subset \mathcal{E}^\alpha_2(\theta), \text{ such that}, \forall u \in \mathcal{E}; \langle H^n_\omega(\theta)u, u\rangle \leq E\|u\|^2\}. \end{aligned} \tag{2.11}$$

Now integrating both sides of (2.11) over $\mathbb{T}^*_{2n+1}$ and taking into account (2.8), we get that

$$\begin{aligned}&\mathcal{N}^n_\omega(E) \leq \overline{\mathcal{N}}^n((E+E^\alpha))+\\ &\frac{1}{(2\pi)^d}\int_{\mathbb{T}^*_{2n+1}} \dim\{\mathcal{E}\subset\mathcal{E}^\alpha_2(\theta),\ \text{such that}, \forall u\in\mathcal{E};\ \langle H^n_\omega(\theta)u,u\rangle \leq E\|u\|^2\}d\theta.\end{aligned} \tag{2.12}$$

Where $\overline{\mathcal{N}}^n$ is the IDS of $\overline{H}^n$.
Notice that $\dim\{\mathcal{E}\subset\mathcal{E}^\alpha_2(\theta),\ \text{such that}, \forall u\in\mathcal{E};\langle H^n_\omega(\theta)u,u\rangle \leq E\|u\|^2\}$ is bounded by the number of eigenvalues of $-\Delta^n(\theta)$ less than $E\rho*$ which is it self bounded by Cn^d (C depends only on E). As the volume of $\mathbb{T}^*_{2n+1}$ is $(2\pi(2n+1))^{-d}$ we get that for some $C>0$ we have

$$\mathbb{E}(\mathcal{N}^n_\omega(E)) \leq \overline{\mathcal{N}}^n\Big((E+E^\alpha)\Big) + C\mathbb{P}(\Omega_{n,E,\alpha}). \tag{2.13}$$

With

$$\Omega_{n,E,\alpha} = \Big\{\omega;\ \exists u\in\mathcal{E}^\alpha_2(\theta); \|u\|_{L^2(\mathbb{R}^d)}=1;\ \langle\nabla u, u\nabla\rangle \leq E\rho^*\|u\|^2\Big\}.$$

Now let us consider

$$\begin{aligned}&\mathcal{V}(\overline{H}^n(\theta),(E-E^\alpha)) =\\ &\quad \sup dim\{\mathcal{E}\subset\mathcal{H}^1_{n,\theta},\ \text{such that}, \forall u\in\mathcal{E};\ \langle\overline{H}^n(\theta)u,u\rangle \leq (E-E^\alpha)\|u\|^2\}.\end{aligned}$$

$$\begin{aligned}&= \sup dim\{\mathcal{E}\subset\mathcal{H}^1_{n,\theta},\ \text{such that}, \forall u\in\mathcal{E};\\ &\qquad\qquad \langle\Big(\overline{H}^n(\theta)-H^n_\omega(\theta)\Big)u,u\rangle + \langle H^n_\omega(\theta)u,u\rangle \leq (E-E^\alpha)\|u\|^2\}\end{aligned}$$

$$\begin{aligned}&\leq \sup dim\{\mathcal{E}\subset\mathcal{E}^\alpha_1(\theta),\ \text{such that}, \forall u\in\mathcal{E};\ \langle\langle H^n_\omega(\theta)u,u\rangle \leq E\|u\|^2\}\\ &+\sup dim\{\mathcal{E}\subset\mathcal{E}^\alpha_2(\theta),\ \text{such that}, \forall u\in\mathcal{E};\ \langle\overline{H}^n(\theta)u,u\rangle \leq (E-E^\alpha)\|u\|^2\}.\end{aligned}$$

Using the same computation carried out from (2.11) to (2.13), we get that

$$\overline{\mathcal{N}}((E-E^\alpha)) - C\mathbb{P}(\Omega_{n,E,\alpha}) \leq \mathcal{N}(E). \tag{2.14}$$

Now we have to estimate the following probability, $\mathbb{P}(\Omega_{n,E,\alpha})$. It is the purpose of the following Lemma. It is a large deviation argument.

Lemma 2.4. *There exists $\tau > 0$ such that for E sufficiently small and n large, we have*

$$\mathbb{P}(\Omega_{n,E,\alpha}) \leq e^{-E^{-\tau}}.$$

Now the proof of Theorem 1.1 is just to take into account Theorem 2.2 and Lemma 2.4

The proof of Lemma 2.4

We prove this Lemma using techniques of [5]. We have $\Omega_{n,E,\alpha} \subset \Omega'_{n,E,\alpha}$. With

$$\Omega'_{n,E,\alpha} = \Big\{\omega;\ \exists u \in H^1(\mathbb{R}); \|u\|_{L^2(\mathbb{R}^d)} = 1;\ \|\nabla u\|^2 \leq E\rho_*; \\ \text{and } \left|\langle\Big(H^n_\omega(\theta) - \overline{H}^n(\theta)\Big)u, u\rangle\right| \geq E^\alpha \|u\|^2\ \Big\}.$$

Let us estimate the probability of the latest events. Notice that we asked that

$$\left|\langle\Big(H^n_\omega(\theta) - \overline{H}^n(\theta)\Big)u, u\rangle\right| = \sum_{i=1}^d \left|\langle\Big(\rho^n_\omega(\theta) - \overline{\rho}^n(\theta)\Big)\partial_{x_i} u, \partial_{x_i} u\rangle\right| \geq E^\alpha \|u\|^2. \tag{2.15}$$

Let $u \in H^1(\mathbb{R}^d)$, then u can be written using the Floquet decomposition as:

$$u = \sum_{k\in\mathbb{N}} \int_{\mathbb{T}^*_{2n+1}} \chi_k(\theta) w_k(x,\theta) d\theta.$$

Where $(w(\cdot,\theta)_k)_{k\in\mathbb{N}}$ are the Floquet eigenfunctions of $-\Delta^\theta_n$ associated to $(E_k(\theta))_{k\in\mathbb{N}}$.

By this, for u such that $\langle -\Delta u, u\rangle \leq E\rho_*$ we have:

$$\Big(\sum_{k\geq 0} \int_{\mathbb{T}^*_{2n+1}} |E_k(\theta)|^2 |\chi_k(\theta)|^2 d\theta\Big) \leq CE^2. \tag{2.16}$$

0 is the bottom of the spectrum of $-\Delta$. It is a simple non-degenerate Floquet eigenvalue [13]. Hence there exists $C > 0$ such that

- For $k \neq 0, \forall \theta \in \mathbb{T}^*_{2n+1}$

$$|E_k(\theta)| \geq 1/C, \tag{2.17}$$

- and $\exists Z = \{\theta_j \in \mathbb{T}^*_{2n+1};\ 1 \leq j \leq n_0\}$ such that $E_0(\theta_j) = 0$.

$$|E_0(\theta)| \geq 1/C \inf_{1 \leq j \leq n_0} |\theta - \theta_j|^2. \tag{2.18}$$

Let $(2l+1) = [E^{-1/2+2\rho'}]_\circ \cdot [E^{-\rho'}]_\circ$ and $(2k+1) = [E^{-\eta}]_\circ$, where $\alpha < \rho' < \frac{d}{4(d+1)}$ and $\eta > 0$ such that $(2l+1) \cdot (2k+1) = 2n+1$. Here $[\cdot]_\circ$ denotes the largest odd integer smaller than $\cdot$.
From (2.16), (2.17) and (2.18) we get that

$$\sum_{k \geq 1} \int_{\mathbb{T}^*_{2n+1}} |\chi_k(\theta)|^2 d\theta + \sum_{j=1}^{n_0} \int_{|\theta - \theta_j| > \frac{1}{l}} |\chi_0(\theta)|^2 d\theta \leq CE^2 l^2 \leq CE^{2\rho'}. \tag{2.19}$$

Hence we write

$$u = \sum_{j=1}^{n_0} u_j + u^e, \quad \text{where} \quad u_j = \int_{|\theta - \theta_j| \leq \frac{1}{l}} \chi_0(\theta) w_0(\cdot, \theta_j) d\theta; \quad \|u^e\| \leq CE^{2\rho'}, \tag{2.20}$$

and we have

$$\sum_{j=1}^{n_0} \|u_j\|^2 = \|u\| - CE^{\rho'} = 1 - CE^{2\rho'}.$$

Now using (2.20) in (2.15), we get that for E small we have

$$\sum_{1 \leq j, j' \leq n_0} \left| \langle \left(\rho^n_\omega - \overline{\rho}^n \right) \nabla u_j, \nabla u_{j'} \rangle \right| \geq E^\alpha / 4. \tag{2.21}$$

So, for some $1 \leq j, j' \leq n_0$, one has

$$\left| \langle \left(\rho^n_\omega - \overline{\rho}^n \right) \nabla u_j, \nabla u_{j'} \rangle \right| \geq E^\alpha / (2n_0)^2. \tag{2.22}$$

Now we state a Lemma based on the Uncertainly principle and proved in [5].

Lemma 2.5. *[5]Fix $1 \leq j \leq n_0$. For $1 \leq l' \leq l$ there exists $\widetilde{u}_j \in L^2(\mathbb{R}^d)$ such that*
1) $\widetilde{u}_j$ is constant on each cube

$$\Lambda_{\gamma, l'} = \{x = (x_1, \cdots, x_d);\ \forall 1 \leq i \leq d - l' - \frac{1}{2} \leq x_i - (2l'+1)\gamma_i < l' + \frac{1}{2}\}$$

where $\gamma = (\gamma_1, \cdots, \gamma_d) \in \mathbb{Z}^d$.
2) $\exists C > 0$ *depending only on* $w_0(\cdot, \theta)$ *such that*

$$\|u_j - \widetilde{u}_j \cdot \overline{w}_0(\cdot, \theta_j)\|_{L^2(\mathbb{R}^d)} \leq Cl'/l, \tag{2.23}$$

where $\overline{w}_0(\cdot, \theta)$ *is the periodic component of* $w_0(\cdot, \theta)$ *i.e* $w_0(\cdot, \theta) = e^{ix\theta}\overline{w}_0(\cdot, \theta)$.

Let

$$\psi_j(x) = \widetilde{u}_j(x)\overline{w}_0(x, \theta_j) = \overline{w}_0(x, \theta_j) \sum_{\beta \in \mathbb{Z}^d} (2l'+1)^{-d/2} a_j(\beta) \mathbf{1}_{(2l'+1)\beta + \Lambda_{0,l'}}.$$

$\widetilde{u}_j(x)\overline{w}_0(x, \theta_j) \in L^2(\mathbb{R}^d)$ using the periodicity of $\overline{w}_0(x, \theta_j)$ we get,

$$\|\psi_j\|_{L^2(\mathbb{R}^d)} = \|\widetilde{u}_j(x)\overline{w}_0(x, \theta_j)\|_{L^2(\mathbb{R}^d)} = \sum_{\beta \in \mathbb{Z}^d} |a_j(\beta)|^2 \int_{\Lambda_{0,0}} |\overline{w}_0(\cdot, \theta_j)|^2 dx. \tag{2.24}$$

Then using (2.23) and the fact that $\int_{\Lambda_{0,0}} |w_0(x, \theta_j)|^2 dx = \int_{\Lambda_{0,0}} |\overline{w}_0(x, \theta_j)|^2 dx$; we get that there exists $C > 0$ such that

$$\sum_{\beta \in \mathbb{Z}^d} |a_j(\beta)|^2 \leq C\|u_j\|_{L^2(\mathbb{R}^d)} < +\infty. \tag{2.25}$$

We set $2l'+1 = [E^{-1/2+2\rho'}]_\circ$ and $2k'+1 = [E^{-\rho'}]_\circ \cdot [E^{-\eta}]_\circ$, for $\alpha < \rho' < \frac{d}{4(d+1)}$ and $\eta > 0$ so that $(2n+1) = (2l'+1) \cdot (2k'+1)$. So taking into account (2.22), (2.23) and the choice of l and l', we get

$$\sum_{i=1}^{d} \left| \langle (\rho_\omega^{0,n} - \overline{\rho}_\omega^{0,n}) \partial_{x_i} \psi_j, \partial_{x_i} \psi_{j'} \rangle \right| \geq E^\alpha/(2n_0)^2 - CE^{\rho'} \geq E^\alpha/(4n_0)^2. \tag{2.26}$$

We set

$$\sum_{i=1}^{d} \left| \langle (\rho_\omega^{0,n} - \overline{\rho^{0,n}}) \partial_{x_i} \psi_j, \partial_{x_i} \psi_{j'} \rangle \right| = \sum_{1 \leq i \leq d} |A_{j,j'}^i|.$$

With

$$A_{j,j'}^i = \langle (\rho_\omega^{0,n} - \overline{\rho^{0,n}}) \partial_{x_i} \psi_j, \partial_{x_i} \psi_{j'} \rangle.$$

We have

$$A^i_{j,j'} = \sum_{\beta\in\mathbb{Z}^d}\sum_{\gamma\in\mathbb{Z}^d_{2n+1}} (2l'+1)^{-d} a_j(\beta)\cdot\overline{a_{j'}}(\beta)\cdot$$
$$\cdot\int_{(2l'+1)\beta+\Lambda_{0,l'}} (\rho^{0,n}_\omega - \overline{\rho^{0,n}})(x-\gamma)\partial_{x_i}\overline{w}_0(x,\theta_j)\cdot\overline{\partial_{x_i}\overline{w}_0(x,\theta'_j)}dx$$

$$= \sum_{\beta\in\mathbb{Z}^d}\sum_{\gamma\in\mathbb{Z}^d_{2n+1}} a_j(\beta)\cdot\overline{a_{j'}(\beta)}\cdot$$
$$\frac{1}{(2l'+1)^d}\int_{\Lambda_{0,l'}} (\rho^{0,n}_\omega - \overline{\rho^{0,n}})(x-\gamma+(2l'+1)\beta)\partial_{x_i}\overline{w}_0(x,\theta_j)\cdot\overline{\partial_{x_i}\overline{w}_0(x,\theta_{j'})}dx. \tag{2.27}$$

As ρ^n_ω is $(2n+1)\mathbb{Z}^d$-periodic and $(2l'+1)(2k'+1) = (2n+1)$ we get that

$$A^i_{j,j'} = \sum_{\beta\in\mathbb{Z}^d_{2k'+1}}\sum_{\beta'\in\mathbb{Z}^d}\sum_{\gamma\in\mathbb{Z}^d_{2n+1}} a_j(\beta+(2k'+1)\beta')\cdot\overline{a_{j'}(\beta+(2k'+1)\beta')}\cdot$$
$$\frac{1}{(2l'+1)^d}\int_{\Lambda_{0,l'}} (\rho^{0,n}_\omega - \overline{\rho^{0,n}})(x-\gamma+(2l'+1)\beta)\partial_{x_i}\overline{w}_0(x,\theta_j)\overline{\partial_{x_i}\overline{w}_0(x,\theta_{j'})}dx. \tag{2.28}$$

Using the expression of ρ_ω we get that.

$$A^i_{j,j'} = \sum_{\beta\in\mathbb{Z}^d_{2k'+1}}\sum_{\gamma\in\mathbb{Z}^d_{2n+1}}\sum_{\beta'\in\mathbb{Z}^d} (\omega_\gamma-\overline{\omega})a_j(\beta+(2k'+1)\beta')\cdot\overline{a_{j'}(\beta+(2k'+1)\beta')}\cdot$$
$$\frac{1}{(2l'+1)^d}\int_{\Lambda_{0,l'}} \rho^0(x-\gamma+(2l'+1)\beta)\partial_{x_i}\overline{w}_0(x,\theta_j)\cdot\overline{\partial_{x_i}\overline{w}(x,\theta_{j'})}dx. \tag{2.29}$$

We set

$$B^i_{j,j'}(\beta) = \sum_{\beta'\in\mathbb{Z}^d} a_j(\beta+(2k'+1)\beta')\cdot\overline{a_{j'}(\beta+(2k'+1)\beta')}. \tag{2.30}$$

Then we get that

$$A^i_{j,j'} = (2l'+1)^{-d}\sum_{\gamma\in\mathbb{Z}^d_{2n+1}} (\omega_\gamma-\overline{\omega})\Big(\sum_{\beta\in\mathbb{Z}_{2k'+1}} B^i_{j,j'}(\beta)\cdot$$
$$\int_{\Lambda_{0,l'}} \rho^0(x-\gamma+(2l'+1)\beta)\overline{w}_0(x,\theta_j)\cdot\overline{\overline{w}(x,\theta_{j'})}dx\Big)$$

$$= (2l'+1)^{-d} \sum_{\gamma\in\mathbb{Z}^d_{2l'+1}} \Big[\sum_{\gamma'\in\mathbb{Z}^d_{2k'+1}} (\omega_{\gamma+(2l'+1)\gamma'} - \overline{\omega}) \Big(\sum_{\beta\in\mathbb{Z}_{2k'+1}} B^i_{j,j'}(\beta) \cdot$$

$$\int_{\Lambda_{0,l'}} \rho^0(x-\gamma+(2l'+1)(\beta-\gamma'))\partial_{x_i}\overline{w}_0(x,\theta_j)\cdot \overline{\partial_{x_i}\overline{w}(x,\theta_{j'})dx}\Big)\Big]. \quad (2.31)$$

We set

$$C^i_{j,j'}(\gamma,\gamma') = \sum_{\beta\in\mathbb{Z}^d_{2k'+1}} B^i_{j,j'}(\beta) \int_{\Lambda_{0,l'}} \rho^0(x-\gamma+(2l'+1)(\beta-\gamma'))\partial_{x_i}\overline{w}_0(x,\theta_j)\cdot\overline{\partial_{x_i}\overline{w}_0(x,\theta_{j'})}dx$$

and

$$Y^i_{j,j'}(\gamma) = \sum_{\gamma'\in\mathbb{Z}^d_{2k'+1}} (\omega_{\gamma+(2l'+1)\gamma'} - \overline{\omega}) C^i_{j,j'}(\gamma,\gamma').$$

Then

$$A^i_{j,j'} = \frac{1}{(2l'+1)^d} \sum_{\gamma\in\mathbb{Z}^d_{2l'+1}} Y^i_{j,j'}(\gamma). \quad (2.32)$$

Notice that $(Y^i_{j,j'}(\gamma))_{\gamma\in\mathbb{Z}^d_{2l'+1}}$ are bounded random and independent variables with $\mathbb{E}(Y^i_{j,j'}(\gamma)) = 0$. Indeed, using the fact that ρ^0 is compactly supported and (2.25) we get that

$$|Y_{j,j'}(\gamma)| \le ||u^0||^2.$$

So, to estimate the probability of $\Omega(n,E,\alpha)$ it suffices to estimate the probability that

$$E^\alpha/(4n_0)^2 \le \frac{1}{(2l'+1)^d} \sum_{\gamma\in\mathbb{Z}^d_{2l'+1}} Y^i_{j,j'}(\gamma). \quad (2.33)$$

This probability is given by the the large deviation principle which gives that [2]

$$\mathbb{P}\Big(E^\alpha/(4n_0)^2 \le \frac{1}{(2l'+1)^d} \cdot \sum_{\gamma\in\mathbb{Z}^d_{2l'+1}} Y^i_{j,j'}(\gamma)\Big) \le e^{-c(l')^d E^{2\alpha}} \le e^{-cE^{-d/2+2d\rho'+2\alpha}}.$$

Here we have used the expression of l'. Using the fact that for our choice of ρ' we have $-d/2+2d\rho'+2\alpha<0$, so for some $\tau>0$ and E sufficiently small, one has

$$\mathbb{P}\Big(E^{\eta'} \le \frac{1}{(2l'+1)^d} \cdot \sum_{\gamma\in\mathbb{Z}^d_{2l'+1}} Y^i_{j,j'}(\gamma)\Big) \le e^{-E^{-\alpha}}.$$

As the probability of $\Omega(n, E, \alpha)$ is bounded by the sum over $1 \leq i \leq d$ and $1 \leq j, j' \leq n_0$ of the probability estimate previously, we get the result of the Lemma 2.4 $\square$

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