Long-Time Dynamics of KdV Solitary Waves over a Variable Bottom*

S.I. Dejak[1‡†] and I.M. Sigal[1‡]

[1]University of Notre Dame, Notre Dame, U.S.A.

[1]University of Toronto, Toronto, Canada

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Abstract

We study the variable bottom generalized Korteweg-de Vries (bKdV) equation $\partial_t u = -\partial_x \left(\partial_x^2 u + f(u) - b(t,x)u\right)$, where f is a nonlinearity and b is a small, bounded and slowly varying function related to the varying depth of a channel of water. Many variable coefficient KdV-type equations, including the variable coefficient, variable bottom KdV equation, can be rescaled into the bKdV. We study the long time behaviour of solutions with initial conditions close to a stable, $b = 0$ solitary wave. We prove that for long time intervals, such solutions have the form of the solitary wave, whose centre and scale evolve according to a certain dynamical law involving the function $b(t,x)$, plus an $H^1(\mathbb{R})$-small fluctuation.

1 Introduction

We study the long time behaviour of solutions to a class of Korteweg-de Vries type equations, which we call the variable bottom generalized KdV equation (bKdV). These equations are of the form

$$\partial_t u = -\partial_x \left(\partial_x^2 u + f(u) - b(t,x)u\right), \tag{1}$$

where f is a nonlinearity and $b(t,x)$ is a real function. When $f(x) = x^2$, the bKdV is related to an equation for the bottom of the channel appearing in the derivation of the KdV from shallow water wave theory. Examples of possible choices for the nonlinearity are $f(u) = u^2$, the Kortweg-de Vries (KdV) from shallow water wave theory; $f(u) = u^3$, the modified KdV (mKdV) from plasma physics; and $f(u) = u^p$, the generalized power nonlinearity KdV (gKdV). When $b = 0$, (1) reduces to the generalized Korteweg-de Vries equation (GKdV)

$$\partial_t u = -\partial_x(\partial_x^2 u + f(u)). \tag{2}$$

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The KdV is obtain by unidirectionalizing the small amplitude, long wave/shallow water limit of the two dimensional water wave system with a constant bottom. The first such derivation was given by Korteweg and de Vries [32] over a century ago in an attempt to explain the existence of solitary waves of permanent form in a shallow channel. Numerous authors have improved the formal derivation using either asymptotic expansions [46] or Hamiltonian methods [17, 18]. Schneider and Wayne [38] have given a rigorous proof of the validity of the KdV in approximating the water wave system in the KdV regime over time intervals of $\mathrm{O}(1)$. The KdV also appears in algebraic geometry. A nice survey of the KdV and it's relation to algebraic geometry is given by Arbarello [1].

A remarkable property of the GKdV is the existence of spatially localized solitary (or travelling) waves, i.e. solutions of the form $u = Q_c(x - a - ct)$, where $a \in \mathbb{R}$ and c in some interval I. When $f(u) = u^p$ and $p \geq 2$, solitary waves are explicitly computed to be

$$Q_c(x) = c^{\frac{1}{p-1}} Q(c^{\frac{1}{2}}(x - ct)),$$

where

$$Q(x) = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \left(\cosh\left(\frac{p-1}{2}x\right)\right)^2.$$

It is generally believed that an arbitrary, say $H^1(\mathbb{R})$, solution to equation (2) eventually breaks up into a collection of solitary waves and radiation. A discussion of this phenomenon for the generalized KdV appears in Bona [8]. For the general, but integrable, case see Deift and Zhou [19].

As channels with constant bottom do not exist, it is of interest to know how solutions initially close to a solitary wave behave as the wave propagates over channels with a variable bottom. Derivations of KdV-type equations when the bottom varies slowly have been presented by numerous authors. See, for example [16, 44, 27, 33, 47]. The resulting equations are the KdV with variable coefficients depending on the variable bottom. These derivations are non-rigorous and agree to the leading order in the bottom length, i.e. in order $\sup |\partial_x h(x)|$. We assume a depth $h(x)$ of $\mathrm{O}(1)$ with length scale l_h. Consider solutions of the water wave system with wavelength scale l_λ, wave amplitude scale l_η and fluid velocity scale l_u. Then, if these scales are related as $l_\lambda = \mathrm{O}\left(\varepsilon^{\frac{1}{2}}\right)$, $l_\eta = \mathrm{O}(\varepsilon)$, $l_u = \mathrm{O}(\varepsilon)$ and $l_h = o(\varepsilon^{\frac{3}{2}})$, the leading equation for the wave amplitude, after an additional rescaling of the time variable, is (see equation (74) in [44])

$$\partial_t \eta = -\Gamma\left(\frac{\eta}{\epsilon} + \frac{3}{4h}\eta^2 + \frac{1}{6}\partial_x\left(h^2 \partial_x \eta\right)\right).$$

Here Γ is the anti-symmetric operator $\Gamma := \frac{1}{2}(c(x)\partial_x + \partial_x c(x))$, with $c(x) = \sqrt{gh(x)}$ (g is gravitational acceleration), and $\epsilon\eta(\epsilon^{-\frac{1}{2}}x, \epsilon^{-\frac{3}{2}}t)$ is the surface elevation measured from the flat interface $y = 0$. We assume $h = h_0 + h_1$ with $|h_1| << h_0$ and h_0 a nonzero constant. Dropping terms of $\mathrm{O}(\partial_x h)$ in the above equation for η, and changing variables as $\eta(x,t) = v(y,t)$, where $y = x - \frac{c_0}{\epsilon}t$ and $c_0 = \sqrt{gh_0}$, leads to an equation for v:

$$\partial_t v = -\partial_y\left(\frac{c - c_0}{\epsilon}v + \frac{3}{4h}cv^2 + \frac{1}{6}ch^2\partial_y^2 v\right).$$

To order $\mathrm{O}(h_1)$, solutions of this equation and solutions $u(x,t)$ to the bKdV with nonlinearity $f(u) = u^2$ and

$$b(x,t) = \frac{1}{\epsilon}\left(c\left(\frac{1}{\sqrt{6}}c_0^{\frac{1}{2}}h_0\left(x + \frac{c_0}{\epsilon}t\right)\right) - c_0\right)$$

are related by the transformation

$$v(y,t) = \frac{4h_0}{3c_0} u\left(\frac{\sqrt{6}}{c_0^{\frac{1}{2}} h_0} y, \frac{\sqrt{6}}{c_0^{\frac{1}{2}} h_0} t\right).$$

In a wider range of parameters one should add more complicated, in particular, nonlocal terms to (1). We expect that the modified equation can still be treated by the methods developed in this paper.

Similarly, in many other instances in mathematics and the sciences where the GKdV arises from an approximation of more complicated systems, the effects of higher order processes can often be collected into a term of the form $b(t,x)u$. Our main result stated at the end of the next section gives, for long time, an explicit, leading order description of a solution initially close to a solitary wave.

We assume that the coefficient b and nonlinearity f are such that (1) has global solutions for $H^1(\mathbb{R})$ data and that (1) with $b = 0$ possesses solitary wave solutions. We discuss the latter assumption in Section 2. Here we mention that the literature regarding well-posedness of the KdV ($b = 0, f(u) = u^2$) is extensive and well developed. Bona and Smith [7] proved global wellposedness of the KdV in $H^2(\mathbb{R})$. See also [28]. More recently, Kenig, Ponce, and Vega [30] have proved local wellposedness in $H^s(\mathbb{R})$ for $s \geq -\frac{3}{4}$ and global wellposedness in $H^1(\mathbb{R})$ for $s \geq 1$. Similar results are available for the gKdV [29]. More recently, local wellposedness results in negative Sobolev spaces for the KdV have been extended to global wellposedness results. See [15, 14]. We are not aware of a wellposedness result for the bKdV in $H^1(\mathbb{R})$. Hence, in the next section we give a global wellposedness result, whose proof (see Appendix A) uses results of [29], and perturbation and energy arguments. We conjecture that global wellposedness remains true for $b \in C^1$ bounded and subcritical nonlinearities.

Soliton solutions of the KdV equation are known to be orbitally stable. Although the linearized analysis of Jeffrey and Kakutani [26] suggested orbital stability, the first nonlinear stability result was given by Benjamin [2]. He assumed smooth solutions and used Lyapunov stability and spectral theory to prove his results. Bona [4] later corrected and improved Benjamin's result to solutions in $H^2(\mathbb{R})$. Weinstein [45] used variational methods, avoiding the use of an explicit spectral representation, and extended the orbital stability result to the GKdV. More recently, Grillakis, Shatah, and Strauss [24] extended the Lyapunov method to abstract Hamiltonian systems with symmetry. Numerical simulations of the soliton dynamics for the KdV were performed Bona et al. See [9, 10, 5, 6].

For nonlinear Schrödinger and Hartree equations, long-time dynamics of solitary waves were studied by Bronski and Gerrard [11], Fröhlich, Tsai and Yau [22], Keraani [31], and Fröhlich, Gustafson, Jonsson, and Sigal [21]. For related results and techniques for the NLS see also [12, 13, 23, 37, 36, 43, 42, 41, 39].

In our approach we use the fact that the bKdV is a (non-autonomous, if b depends on time) Hamiltonian system. As in the case of the nonlinear Schrödinger equation (see [21]), we construct a Hamiltonian reduction of this original, infinite dimensional dynamical system to a two dimensional dynamical system on a manifold of soliton configurations. The analysis of the general KdV immediately runs into the problem that the natural symplectic form ω is not defined on the tangent space of the soliton manifold. In the case of the mKdV ($f(u) = u^3$), the sympletic form is well defined on the tangent space because of the special structure of the solitary wave Q_c, and hence Dejak and Jonsson [20] were able to prove long time dynamics of solitary waves in this special case.

To address the problem regarding the symplectic form, we introduce a family of symplectic forms ω_α parametrized by a small parameter $\alpha > 0$, and approximating ω. We use the small parameter to control the errors generated by this approximation. This approach works, except at one crucial step: the resulting lower (coercivity) bound on the Hessian of the energy functional is too weak to close our energy estimates. To remedy this we show that the weak bound comes from the directions in which we regularized ω; on the orthogonal complement the lower bound is sufficiently good. Hence, we decompose a general tangent vector into "bad" and "good" directions, and use precise information about the "bad" directions to considerably improve the upper bound (involving the nonlinearity) and close the energy estimates.

In the next section we formulate our assumptions, state the main result and describe the organization of the paper. All $L^2(\mathbb{R})$ and Sobolev spaces used in this paper, except those in Section 4, are real.

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2 Preliminaries, Assumptions, and the Main Result

We begin with the following global wellposedness result proven in Appendix A. See the appendix also for the definitions of the norms of b used in the following theorem.

Theorem 1. *Let $u_0 \in H^1(\mathbb{R})$. For small enough*

$$\|b\|_{XT} := \|b\|_{L^2_T W^{4,\infty}_X} + \|b\|_{L^2_X L^\infty_T} + \|\widehat{b'}\|_{L^2_T L^1_X},$$

there is a unique, global solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ to (1) with $f = u^2$. With modification of the norm $\|b\|_{XT}$, the result continues to hold for $f = u^3$ and $f = u^4$.

The bKdV can be written in Hamiltonian form as

$$\partial_t u = \partial_x H_b'(u), \tag{3}$$

where H_b' is the $L^2(\mathbb{R})$ function corresponding to the Fréchet derivative ∂H_b in the $L^2(\mathbb{R})$ pairing. Here the Hamiltonian H_b is

$$H_b(u) := \int_{-\infty}^{\infty} \frac{1}{2}(\partial_x u)^2 - F(u) + \frac{1}{2} b(t,x) u^2 \, dx,$$

where the function F is the antiderivative of f with $F(0) = 0$. The operator ∂_x is the anti-self-adjoint operator (symplectic operator) generating the Poisson bracket

$$\{F, G\} = \frac{1}{2} \int_{-\infty}^{\infty} F'(u)\partial_x G'(u) - G'(u)\partial_x F'(u) \, dx,$$

defined for any F, G such that $F', G' \in H^{\frac{1}{2}}(\mathbb{R})$. The corresponding symplectic form is

$$\omega(v_1, v_2) = \frac{1}{2} \int_{-\infty}^{\infty} v_1(x)\partial_x^{-1} v_2(x) - v_2(x)\partial_x^{-1} v_1(x) \, dx,$$

defined for any $v_1, v_2 \in L^1(\mathbb{R})$. Here the operator ∂_x^{-1} is defined as

$$\partial_x^{-1} v(x) := \int_{-\infty}^{x} v(y)\,dy.$$

Note that $\partial_x^{-1} \cdot \partial_x = I$ and, on the space $\{u \in L^2(\mathbb{R}) \mid \int_{-\infty}^{\infty} u\,dx = 0\}$, ∂_x^{-1} is formally anti-self-adjoint with inverse ∂_x. Hence, if $\int_{-\infty}^{\infty} v_1(x)\,dx = 0$, then $\omega(v_1,\, v_2) = \int_{-\infty}^{\infty} v_1(x)\partial_x^{-1} v_2(x)\,dx$.

Note that if b depends on time t, then equation (3) is non-autonomous. It is, however, in the form of a conservation law, and hence the integral of the solution u is conserved provided u and its derivatives decay to zero at infinity:

$$\frac{d}{dt}\int_{-\infty}^{\infty} u\,dx = 0.$$

There are also conserved quantities associated to symmetries of (1) with $b = 0$. The simplest such corresponds to time translation invariance and is the Hamiltonian itself. This is also true if b is non-zero but time independent. If the potential $b = 0$, then (1) is also spatially translation invariant. Noether's theorem then implies that the flow preserves the momentum

$$P(u) := \frac{1}{2}\left\|u\right\|_{L^2}^2.$$

In general, when $b \neq 0$ the temporal and spatial translation symmetries are broken, and hence, the Hamiltonian and momentum are no longer conserved. Instead, one has the relations

$$\frac{d}{dt}H_b(u) = \frac{1}{2}\int_{-\infty}^{\infty} (\partial_t b)u^2\,dx, \tag{4}$$

$$\frac{d}{dt}P(u) = \frac{1}{2}\int_{-\infty}^{\infty} b' u^2\,dx, \tag{5}$$

where $b'(t,x) := \partial_x b(t,x)$. For later use, we also state the relation

$$\frac{d}{dt}\frac{1}{2}\int_{-\infty}^{\infty} bu^2\,dx = \int_{-\infty}^{\infty} \frac{1}{2}u^2\partial_t b + b'\left(uf(u) - \frac{3}{2}(\partial_x u)^2 - F(u)\right) - b'' u\partial_x u\,dx. \tag{6}$$

Assuming (1) is well-posed in $H^2(\mathbb{R})$, the above equalities are obtained after multiple integration by parts. Then, by density of $H^2(\mathbb{R})$ in $H^1(\mathbb{R})$, the equalities continue to hold for solutions in $H^1(\mathbb{R})$. To avoid these technical details, we assume the Hamiltonian flow on $H^1(\mathbb{R})$ enjoys (4), (5) and (6).

Consider the GKdV, i.e. equation (2). Under certain conditions on f, this equation has travelling wave solutions of the form $Q_c(x - ct)$, where Q_c a positive $H^2(\mathbb{R})$ function. Substituting $u = Q_c(x - ct)$ into the GKdV gives the scalar field equation

$$-\partial_x^2 Q_c + cQ_c - f(Q_c) = 0. \tag{7}$$

Existence of solutions to this equation has been studied by numerous authors. See [40, 3]. In particular, in [3], Berestyki and Lions give sufficient and necessary conditions for a positive and smooth solution Q_c to exist. We assume $g := -cu + f(u)$ satisfies the following conditions:

1. g is locally Lipschitz and $g(0) = 0$,

2. $x^* := \inf\{x > 0 \,|\, \int_0^x g(y)\, dy\}$ exists with $x^* > 0$ and $g(x^*) > 0$, and
3. $\lim_{s\to 0} \frac{g(s)}{s} \leq -m < 0$.

Then, as shown by Berestycki and Lions, (7) has a unique (modulo translations) solution $Q_c \in C^2$, which is positive, even (when centred at the origin), and with Q_c, $\partial_x Q_c$, and $\partial_x^2 Q_c$ exponentially decaying to zero at infinity ($\partial_x Q_c < 0$ for $x > 0$). Furthermore, if f is C^2, then the implicit function theorem implies that Q_c is C^2 with respect to the parameter c on some interval $I_0 \subset \mathbb{R}_+$. We assume that $x^m \partial_c^n Q_c \in L^1(\mathbb{R})$ for $n = 1, 2, 3$, $m = 0, 1, 2$ and that $\int_{-\infty}^{\infty} \partial_c Q_c\, dx \neq 0$. The first assumptions are needed for continuity and differentiability with respect to c of integrals containing $\partial_c^n Q_c$, and the last assumption is made for convenience. When $\int_{-\infty}^{\infty} \zeta_c^n\, dx = 0$, unboundedness of ∂_x does not present problems (see [20]).

The solitary waves Q_c are orbitally stable if $\delta'(c) > 0$, where $\delta(c) = P(Q_c)$. See Weinstein [45] for historically the first proof for general nonlinearities. Moreover, in [24], Grillakis, Shatah and Strauss proved that $\delta'(c) > 0$ is a necessary and sufficient condition for Q_c to be orbitally stable. In this paper, we assume that Q_c is stable for all c in some compact interval $I \subset I_0$, or equivalently that $\delta'(c) > 0$ on I. For $f(u) = u^p$, we have $\delta'(c) = \frac{5-p}{4(p-1)} \|Q_{c=1}\|_{L^2}^2$, which implies the well known stability criterion $p < 5$ corresponding to subcritical power nonlinearities.

The scalar field equation for the solitary wave can be viewed as an Euler-Lagrange equation for the extremals of the Hamiltonian $H_{b=0}$ subject to constant momentum $P(u)$. Moreover, Q_c is a stable solitary wave if and only if it is a minimizer of $H_{b=0}$ subject to constant momentum P. Thus, if c is the Lagrange multiplier associated to the momentum constraint, then Q_c is an extremal of

$$\begin{aligned}\Lambda_{ca}(u) &:= H_{b=0}(u) + cP(u) \\ &= \int_{-\infty}^{\infty} \frac{1}{2}(\partial_x u)^2 + \frac{1}{2}cu^2 - F(u)\, dx,\end{aligned} \tag{8}$$

and hence $\Lambda'_{ca}(Q_c) = 0$.

The functional Λ_{ca} is translationally invariant. Therefore, $Q_{ca}(x) := Q_c(x-a)$ is also an extremal of Λ_{ca}, and $Q_c(x - ct - a)$ is a solitary wave solution of (1) with $b = 0$. All such solutions form the two dimensional C^∞ manifold of solitary waves

$$M_s := \{Q_{ca} \,|\, c \in I, a \in \mathbb{R}\},$$

with tangent space $T_{Q_{ca}} M_s$ spanned by the vectors

$$\zeta_{ca}^{tr} := \partial_a Q_{ca} = -\partial_x Q_{ca} \text{ and } \zeta_{ca}^n := \partial_c Q_{ca}, \tag{9}$$

which we call the translation and normalization vectors. Notice that the two tangent vectors are orthogonal.

In addition to the requirements on b that (1) be globally wellposed, we assume the potential b is bounded, twice differentiable, and small in the sense that

$$|\partial_t^n \partial_x^m b| \leq \epsilon_a \epsilon_t^n \epsilon_x^m, \tag{10}$$

for $n = 0, 1$, $m = 0, 1, 2$, and $n + m \leq 2$. The positive constants ϵ_a, ϵ_x, and ϵ_t are amplitude, length, and time scales of the function b. We assume all are less than or equal to one.

Lastly, we make some explicit assumptions on the local nonlinearity f. We require the nonlinearity to be k times continuously differentiable, with $f^{(k)}$ bounded for some $k \geq 3$ and $f(0) = f'(0) = 0$. These assumptions ensure the Hamiltonian is finite on the space $H^1(\mathbb{R})$ and, since Q_c decays exponentially (see [3]), exponential decay of $f(Q_c)$ and $f'(Q_c)$.

We are ready to state our main result.

Theorem 2. *Let the above assumptions hold and assume $\delta'(c) > 0$ for all c in a compact set $I \subset I_0$. Let $0 < s < \frac{1}{2}$. Then, if $\epsilon_a \leq 1$, $\epsilon_x \leq 1$, and $\epsilon_0 << (\epsilon_a \epsilon_x)^{2s}$ are small enough, there is a positive constant C such that the solution to (1) with an initial condition u_0 satisfying $\inf_{Q_{ca} \in M_s} \|u_0 - Q_{ca}\|_{H^1} \leq \epsilon_0$ can be written as*

$$u(x,t) = Q_{c(t)}(x - a(t)) + \xi(x,t),$$

where $\|\xi(t)\|_{H^1} \leq \mathrm{O}\left(\epsilon_a^s \epsilon_x^s\right)$ for all times $t \leq C(\epsilon_t + \epsilon_x + \epsilon_a^s \epsilon_x^s)^{-1}$. Moreover, during this time interval the parameters $a(t)$ and $c(t)$ satisfy the equations

$$\begin{pmatrix} \dot{a} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} c - b(t,a) \\ 0 \end{pmatrix} + \mathrm{O}\left(\epsilon_a^{2s} \epsilon_x^{2s} + \epsilon_a \epsilon_x^2\right),$$

where c is assumed to lie in the compact set I.

Sketch of Proof and Paper Organization. To realize the Hamiltonian reduction we decompose functions in a neighbourhood of the soliton manifold M_s as

$$u = Q_{ca} + \xi$$

with ξ symplectically orthogonal to $T_{Q_{ca}} M_s$, i.e. $\xi \perp \partial_x^{-1} T_{Q_{ca}} M_s$. Unfortunately, since $\partial_x^{-1} T_{Q_{ca}} M_s \not\subseteq L^2(\mathbb{R})$, such a decomposition is ill-defined for $\xi \in H^1(\mathbb{R})$. To overcome this difficulty we construct in Section 3 an approximate symplectic form

$$\omega(v_1, v_2) = \frac{1}{2} \int_{-\infty}^{\infty} v_1(x) \left(\mathcal{K}_{Q\alpha} - \mathcal{K}_{Q\alpha}{}^*\right) v_2(x)\, dx,$$

where $\alpha > 0$ and $\mathcal{K}_{Q\alpha}$ is a bounded operator regularizing the unbounded operator ∂_x^{-1} in certain directions. We show that there is an $\epsilon_0 > 0$ such that if the solution u satisifes the estimate $\inf_{Q_{ca}} \|u - Q_{ca}\|_{H^1} < \epsilon_0$, then there are unique C^1 functions $a(u)$ and $c(u)$ such that $u = Q_{c(u)a(u)} + \xi$ with $\xi \in (\mathcal{K}_{Q\alpha} T_{Q_{ca}} M_s)^{\perp}$.

With the knowledge that the symplectic decomposition exists, we substitute $u = Q_{ca} + \xi$ into the bKdV (1) and split the resulting equation according to the decomposition

$$L^2(\mathbb{R}) = \mathcal{K}_{Q\alpha} T_{Q_{ca}} M_s \oplus (\mathcal{K}_{Q\alpha} T_{Q_{ca}} M_s)^{\perp}$$

to obtain equations for the parameters c and a, and an equation for the (infinite dimensional) fluctuation ξ. In Sections 4 and 5, we establish spectral properties and an anisotropic lower bound of the Hessian Λ''_{ca} on the space $(\mathcal{K}_{Q\alpha} T_{Q_{ca}} M_s)^{\perp}$. Using these properties we orthogonally decompose ξ into a "bad" ξ_b part and a "good" part ξ_g, where ξ_b is colinear with the minimizer η of $\|\xi\|_{L^2}^{-2} \langle \xi, \mathcal{L}_Q \xi \rangle$. In Section 6 we isolate the leading order terms in the equations for a and c and estimate the remainder, including all terms containing $\xi = \xi_b + \xi_g$. We use the special properties of the minimizer η to obtain a better estimate on the nonlinear terms containing ξ_b.

The proof that $\|\xi\|_{H^1}$ is sufficiently small is the final ingredient in the proof of the main theorem. The remaining sections concentrate on proving this crucial result. We employ a Lyapunov method and in Section 7 we construct the Lyapunov function M_c and prove an estimate on its time derivative. This estimate is later time maximized over an interval $[0, T]$, and integrated to obtain an upper bound on M_c involving the time T and the norms of ξ_b and ξ_g. This anisotropic upper bound is considerably better than an isotropic bound. We combine this upper bound with the anisotropic lower bound $M_c \gtrsim C\alpha \|\xi_b\|_{H^1} + C \|\xi_g\|_{H^1}$, which follows from the results of Section 5, and obtain and inequality involving the norms of ξ_b and ξ_g. Here α is the small regularization parameter mentioned in the introduction and will be taken small, in fact $\alpha = (\epsilon_a \epsilon_x)^s$. This inequality implies upper bounds on $\|\xi_b\|_{H^1}$ and $\|\xi_g\|_{H^1}$, provided $\|\xi(0)\|_{H^1}$ is small enough, via the standard argument given Section 8. We substitute this bound into the bound appearing in the dynamical equation for a and c, and take $\epsilon_a \epsilon_x$ and ϵ_0 small enough so that all intermediate results hold to complete the proof. $\square$

3 Modulation of Solutions

As stated in the previous section, we begin the proof by decomposing the solution of (1) into a modulated solitary wave and a fluctuation ξ:

$$u(x, t) = Q_{c(t)a(t)}(x) + \xi(x, t), \tag{11}$$

with a, c, and ξ fixed by an orthogonality condition, which we now describe. Ideally we would like to take ξ orthogonal to $\mathcal{K} T_{Q_{ca}} M_s$, where $\mathcal{K}$ is the symplectic operator defined on absolutely continuous functions g as

$$\mathcal{K} : g \mapsto \int_{-\infty}^{x} g(y)\, dy.$$

It is easy to see that $\partial_x \mathcal{K} = I$ and if $\lim_{x\to-\infty} g(x) = 0$, then $\mathcal{K}\partial_x = I$. The problem here is that $\mathcal{K} T_{Q_{ca}} M_s \nsubseteq L^2(\mathbb{R})$. More precisely, while $\mathcal{K}\zeta_{ca}^{tr} = -Q_{ca} \in L^2(\mathbb{R})$ we have that in general $\mathcal{K}\zeta_{ca}^{n} \notin L^2(\mathbb{R})$. In fact, if $f(u) = u^p$, then

$$\mathcal{K}\zeta_{ca}^{n}|_{a=0} = \frac{xQ_c}{2c} + \frac{3-p}{2c(p-1)} \int_{-\infty}^{y} Q_c(y)\, dy$$

and therefore

$$\lim_{x\to\infty} \mathcal{K}\zeta_{ca}^{n} = \frac{3-p}{2c(p-1)} \int_{-\infty}^{\infty} Q_c(y)\, dy.$$

Since Q_c is positive, $\mathcal{K}\zeta_{ca}^{n}$ is not an $L^2(\mathbb{R})$ function if $p \neq 3$. We remark that if $p = 3$, then there are no problems. This case is in the special class of nonlinearities considered in [20].

Our remedy to the above problem is to "regularize" the symplectic operator $\mathcal{K}$. Let P_Q be the $L^2(\mathbb{R})$ orthogonal projection onto the subspace spanned by the translation vector ζ_{ca}^{tr}, and let $\bar{P}_Q$ be its orthogonal complement. Then we define the anisotropic regularization $\mathcal{K}_{Q\alpha}$ of $\mathcal{K}$ as

$$\mathcal{K}_{Q\alpha} := \mathcal{K} P_Q + \partial_\alpha^{-1} \bar{P}_Q,$$

where $\partial_\alpha^{-1} := (\partial_x + \alpha)^{-1}$ is the regularization of $\mathcal{K}$. We do not regularize in the direction of ζ_{ca}^{tr} since $\mathcal{K}$ is well behaved on this vector.

For ∂_α^{-1} to exist, the parameter α must lie in the resolvent set $\rho(\partial_x) = \mathbb{C} \setminus i\mathbb{R}$, and in such a case ∂_α^{-1} acts explicitly as

$$\partial_\alpha^{-1} : g \mapsto \int_{-\infty}^{x} g(y) e^{\alpha(y-x)}\, dy$$

on all $L^2(\mathbb{R})$ functions g. The lemma below, proven in Appendix B, collects some properties of ∂_α^{-1}, which will be used in the course of proving the main result.

Lemma 3. *Let $\phi, \psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\alpha \in \mathbb{R}_+$. Then we have*

1. *The operator ∂_α^{-1} commutes with ∂_x and spatial translation $\mathcal{S}_a : f(x) \to f(x-a)$; that is, $\partial_x \partial_\alpha^{-1} = \partial_\alpha^{-1}\partial_x = I - \alpha\partial_\alpha^{-1}$ and $\mathcal{S}_a \partial_\alpha^{-1} = \partial_\alpha^{-1}\mathcal{S}_a$.*
2. *$\left\|\partial_\alpha^{-1}\phi\right\|_{L^\infty} \leq \|\phi\|_{L^1}$.*
3. *There is a constant C such that $\left\|\partial_\alpha^{-1}\phi\right\|_{L^2} \leq C\alpha^{-\frac{1}{2}} \|\phi\|_{L^1}$.*
4. *If $x\phi \in L^1(\mathbb{R})$, then $\left\|x\partial_\alpha^{-1}\phi\right\|_{L^2} \leq C\left(\alpha^{-\frac{3}{2}} \|\phi\|_{L^1} + \alpha^{-\frac{1}{2}} \|x\phi\|_{L^1}\right)$.*
5. *If $x\phi, x\psi \in L^1(\mathbb{R})$, then $\left|\langle \phi, \partial_\alpha^{-1}\psi\rangle - \langle \phi, \mathcal{K}\psi\rangle\right| \leq \alpha\left(\|\phi\|_{L^1}\|x\psi\|_{L^1} + \|x\phi\|_{L^1}\|\psi\|_{L^1}\right)$ and, in particular,*
$$\left|\langle \phi, \partial_\alpha^{-1}\phi\rangle - \frac{1}{2}\left(\int_{-\infty}^{\infty} \phi\, dx\right)^2\right| \leq 2\alpha \|\phi\|_{L^1}\|x\phi\|_{L^1}.$$
6. *If $x\phi, x^2\phi \in L^1(\mathbb{R})$, then*
$$\left\|\partial_\alpha^{-1}\phi\right\|_{L^2} = \frac{\pi}{\alpha}\left(\int_{-\infty}^{\infty} \phi\, dx\right)^2 + \mathrm{O}(1).$$

For α small, the above lemma implies that the properties of $\mathcal{K}$ and ∂_α^{-1} are similar. Thus, we require in (11) that

$$\xi \perp \mathcal{K}_{Q\alpha} T_{Q_{ca}} M_s. \tag{12}$$

The existence and uniqueness of parameters a and c such that $\xi = u - Q_{ca}$ satisfies (12) follows from the next lemma concerning a restriction of $\mathcal{K}_{Q\alpha}$ and the implicit function theorem.

The restriction $K_{Q\alpha}$ of $\mathcal{K}_{Q\alpha}$ to the tangent space $T_{Q_{ca}}M_s$ is defined by the equation $K_{Q\alpha}\mathcal{P}_Q = \mathcal{P}_Q \mathcal{K}_{Q\alpha}\mathcal{P}_Q$, where $\mathcal{P}_Q$ is the orthogonal projection onto $T_{Q_{ca}}M_s$. In the natural basis $\{\zeta_{ca}^{tr}, \zeta_{ca}^{n}\}$ of the tangent space $T_{Q_{ca}}M_s$, the matrix representation of $K_{Q\alpha}$ is $N^{-1}\Omega_{Q\alpha}$, where

$$N := \begin{pmatrix} \|\zeta_{ca}^{tr}\|_{L^2}^2 & 0 \\ 0 & \|\zeta_{ca}^{n}\|_{L^2}^2 \end{pmatrix}$$

and

$$\Omega_{Q\alpha} := \begin{pmatrix} \langle \zeta_{ca}^{tr}, \mathcal{K}_{Q\alpha}\zeta_{ca}^{tr}\rangle & \langle \zeta_{ca}^{n}, \mathcal{K}_{Q\alpha}\zeta_{ca}^{tr}\rangle \\ \langle \zeta_{ca}^{tr}, \mathcal{K}_{Q\alpha}\zeta_{ca}^{n}\rangle & \langle \zeta_{ca}^{n}, \mathcal{K}_{Q\alpha}\zeta_{ca}^{n}\rangle \end{pmatrix}. \tag{13}$$

Notice that the matrix $\Omega_{Q\alpha}$ depends on the base point Q_{ca}, and hence on the parameters a and c.

Lemma 4. *If $\delta'(c) > 0$ on a compact set $I \subset \mathbb{R}_+$ and $\alpha << \lfloor \delta' \rfloor := \inf_I \delta'$, then $\Omega_{Q\alpha}$ is invertible for all $c \in I$ and $a \in \mathbb{R}$, and*

$$\Omega_{Q\alpha}^{-1} = \frac{1}{\delta'(c)^2} \begin{pmatrix} \frac{1}{2}\left(\int_{-\infty}^{\infty} \zeta_{ca}^n \, dx\right)^2 & \delta'(c) \\ -\delta'(c) & 0 \end{pmatrix} + \mathrm{O}\left(\frac{\alpha}{\lfloor \delta' \rfloor}\right). \tag{14}$$

Hence, $\|\Omega_{Q\alpha}^{-1}\| = \mathrm{O}\left(\lfloor \delta' \rfloor^{-2}\right)$ for $\lfloor \delta' \rfloor$ small.

Proof. We use the relations $\zeta_{ca}^{tr} = -\partial_x Q_{ca}$, $\mathcal{K}_{Q\alpha}\zeta_{ca}^{tr} = \mathcal{K}\zeta_{ca}^{tr}$, $\mathcal{K}_{Q\alpha}\zeta_{ca}^n = \partial_\alpha^{-1}\zeta_{ca}^n$, anti-self-adjointness of ∂_x, and $\mathcal{K}\partial_x = I$ to simplify the matrix $\Omega_{Q\alpha}$ into

$$\Omega_{Q\alpha} = \begin{pmatrix} 0 & -\langle \zeta_{ca}^n, Q_{ca} \rangle \\ \langle Q_{ca}, \partial_x \partial_\alpha^{-1} \zeta_{ca}^n \rangle & \langle \zeta_{ca}^n, \partial_\alpha^{-1} \zeta_{ca}^n \rangle \end{pmatrix}.$$

Next, using statements 1 and 5 of the previous Lemma, we separate the leading order part of $\Omega_{Q\alpha}$ from the higher order parts, and use the relation $\delta'(c) = \langle Q_{ca}, \zeta_{ca}^n \rangle$ to obtain that

$$\Omega_{Q\alpha} = \begin{pmatrix} 0 & -\delta'(c) \\ \delta'(c) & \frac{1}{2}\left(\int_{-\infty}^{\infty} \zeta_{ca}^n \, dx\right)^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha \langle Q_{ca}, \partial_\alpha^{-1} \zeta_{ca}^n \rangle & R \end{pmatrix},$$

where $|R| \leq 2\alpha \sup_I \|\zeta_{ca}^n\|_{L^1} \|x\zeta_{ca}^n\|_{L^1}$. With Q_{ca} and ζ_{ca}^n exponentially decaying, the estimate $\left|\langle Q_{ca}, \partial_\alpha^{-1}\zeta_{ca}^n \rangle\right| \leq \|Q_{ca}\|_{L^1} \|\zeta_{ca}^n\|_{L^1}$ is clear from the properties of ∂_α^{-1}. Thus, if $\alpha \leq \frac{1}{2}\lfloor \delta' \rfloor \left(\sup_I \|Q_{ca}\|_{L^1} \|\zeta_{ca}^n\|_{L^1}\right)^{-1}$, then the determinant

$$\det \Omega_{Q\alpha} = \delta'(c)^2 + \alpha\delta'(c) \langle Q_{ca}, \partial_\alpha^{-1} \zeta_{ca}^n \rangle \geq \frac{1}{2} \lfloor \delta' \rfloor^2,$$

and hence it is nonzero for all $c \in I$ and $a \in \mathbb{R}$. The coadjoint formula and the above estimate give (14). The estimate of $\|\Omega_{Q\alpha}^{-1}\|$ follows from (14) and the assumption that $\int_{-\infty}^{\infty} \zeta_{ca}^n \, dx \neq 0$. ∎

Given $\varepsilon > 0$, define the tubular neighbourhood $U_\varepsilon := \{u \in L^2(\mathbb{R}) \mid \inf_{(c,a) \in I \times \mathbb{R}} \|u - Q_{ca}\|_{L^2} < \varepsilon\}$ of the solitary wave manifold M_s in $L^2(\mathbb{R})$.

Proposition 5. *Let $I \subset \mathbb{R}_+$ be a compact interval such that $c \mapsto Q_{ca}$ is $C^1(I)$. Then, if $\alpha << \lfloor \delta' \rfloor$, there exists a positive number $\varepsilon = \varepsilon(I) = \mathrm{O}\left(\alpha^{\frac{1}{2}} \lfloor \delta' \rfloor^4\right)$ and unique C^1 functions $a : U_\varepsilon \to \mathbb{R}_+$ and $c : U_\varepsilon \to I$, dependent on α and I, such that*

$$\left\langle Q_{c(u)a(u)} - u, \mathcal{K}_{Q\alpha}\zeta_{c(u)a(u)}^{tr} \right\rangle = 0 \text{ and } \left\langle Q_{c(u)a(u)} - u, \mathcal{K}_{Q\alpha}\zeta_{c(u)a(u)}^{n} \right\rangle = 0$$

for all $u \in U_\varepsilon$. Moreover, there is a positive real number $C = C(I)$ such that

$$\left\|u - Q_{c(u)a(u)}\right\|_{H^1} \leq C\alpha^{-\frac{1}{2}} \inf_{Q_{ca} \in M_s} \|u - Q_{ca}\|_{H^1} \tag{15}$$

for all $u \in U_\varepsilon \cap H^1(\mathbb{R})$.

Proof. Let $\mu := (a\ c)^T$ and define $F : L^2(\mathbb{R}) \times \mathbb{R}_+ \times I \to \mathbb{R}^2$ by

$$F : (u, \mu) \mapsto \begin{pmatrix} \langle Q_{ca} - u, \mathcal{K}_{Q\alpha} \zeta^{tr}_{ca} \rangle \\ \langle Q_{ca} - u, \mathcal{K}_{Q\alpha} \zeta^{n}_{ca} \rangle \end{pmatrix}.$$

The proposition is equivalent to solving $F(u, g(u)) = 0$ for a C^1 function g. Observe that F is C^1 and $F(Q_{ca}, \mu)$=0. To apply the implicit function theorem it suffices to check that $\partial_\mu F(Q_{ca}, \mu)$ is invertible. Then the there exists an open ball $B_\varepsilon(Q_{ca})$ of radius ε with centre Q_{ca}, and a unique function $g_{\alpha Q_{ca}} : B_\varepsilon(Q_{ca}) \to \mathbb{R}_+ \times I$, such that $F(u, g_{\alpha Q_{ca}}(u)) = 0$ for all $u \in B_\varepsilon(Q_{ca})$. Since $\partial_\mu F(Q_{ca}, \mu) = \Omega_{Q\alpha}$, the invertibility of $\partial_\mu F(Q_{ca}, \mu)$ follows from Lemma 4 provided α is small enough. The radius of the balls $B_\varepsilon(Q_{ca})$ depend on the parameters c, a and α. To obtain an estimate of the radius, and to show that we can take ε independent of the parameters c and a, we give a proof of the existence of the functions $g_{\alpha Q_{ca}}$ using the contraction mapping principle (just as in the proof of the implicit function theorem).

Expand $F(u, \mu)$ to linear order in μ around $\mu_0 = (a\,c)^T$:

$$F(u, \mu) = F(u, \mu_0) + \partial_\mu F(u, \mu_0)(\mu - \mu_0) + R(u, \mu), \tag{16}$$

where $R(u, \mu) = \frac{1}{2}\partial^2_\mu F(u, (1-\lambda)\mu_0 + \lambda\mu)(\mu - \mu_0)^2$ for some $\lambda \in [0, 1]$. The operator $\partial_\mu F(u, \mu_0)$ is computed to be $\partial_\mu F(u, \mu_0) = \Omega_{Q\alpha} + A$, where

$$A := \begin{pmatrix} \langle Q_{ca} - u, \partial_a Q_{ca} \rangle & \langle Q_{ca} - u, \partial_c Q_{ca} \rangle \\ \langle Q_{ca} - u, \partial_a \partial_\alpha^{-1} \zeta^n_{ca} \rangle & \langle Q_{ca} - u, \partial_c \partial_\alpha^{-1} \zeta^n_{ca} \rangle \,. \end{pmatrix}$$

If $u \in B_\varepsilon(Q_{ca})$, where ε remains to be chosen, then the properties of ∂_α^{-1} imply that $\|A\| \le C\alpha^{-\frac{1}{2}}\varepsilon$. Thus, since $\Omega_{Q\alpha}$ is invertible, if $\varepsilon << (C \sup_I \|\Omega^{-1}_{Q\alpha}\|)^{-1}\alpha^{\frac{1}{2}}$, then $\partial_\mu F(u, \mu_0)$ is invertible and $\|[\partial_\mu F(u, \mu_0)]^{-1}\| \le C \sup_I \|\Omega^{-1}_{Q\alpha}\|$. Hence, given u, $F(u, \mu) = 0$ has a solution μ if and only if

$$\mu = H(\mu) := \mu_0 - [\partial_\mu F(u, \mu_0)]^{-1} \left(F(u, \mu_0) + R(u, \mu)\right)$$

has a solution μ. The latter is equivalent to the function H having a fixed point. This is guaranteed by the contraction mapping principle if H is a strict contraction from some ball $B_\rho(\mu_0)$ to $B_\rho(\mu_0)$.

Say $\mu \in B_\rho(\mu_0)$, where ρ remains to be chosen, and consider the bound

$$\|H(\mu) - \mu_0\| \le C \sup_I \|\Omega^{-1}_{Q\alpha}\| \|F(u, \mu_0) + R(u, \mu)\|.$$

After subtracting $F(Q_{ca}, \mu_0) = 0$ from $F(u, \mu_0)$ and using the mean value theorem, the above becomes

$$\begin{aligned} \|H(\mu) - \mu_0\| \le C \sup_I \|\Omega^{-1}_{Q\alpha}\| \, (\|\partial_u F((1-\lambda_1)Q_{ca} + \lambda_1 u, \mu_0)\| \, \|u - Q_{ca}\| \\ + \frac{1}{2} \, \|\partial^2_\mu F(u, (1-\lambda_2)\mu_0 + \lambda_2\mu)\| \, \|\mu - \mu_0\|^2) \end{aligned}$$

for some λ_1, $\lambda_2 \in [0, 1]$. Again, using the properties of ∂_α^{-1} we find that

$$\begin{aligned} \|\partial_u F((1-\lambda_1)Q_{ca} + \lambda_1 u, \mu_0)\| &\le C\alpha^{-\frac{1}{2}} \\ \|\partial^2_\mu F(u, (1-\lambda_2)\mu_0 + \lambda_2\mu\| &\le C(1 + \alpha^{-\frac{1}{2}}) \end{aligned} \tag{17}$$

for all $\mu_0, \mu \in \mathbb{R}_+ \times I$ and $u \in B_\varepsilon(Q_{ca})$. Thus, if $\rho < 1$, then $\|H(\mu) - \mu_0\| \leq C \sup_I \|\Omega_{Q\alpha}^{-1}\| \left(\alpha^{-\frac{1}{2}}\varepsilon + \rho^2\right)$. Taking $\varepsilon << (C \sup_I \|\Omega_{Q\alpha}^{-1}\|)^{-1}\alpha^{\frac{1}{2}}\rho$ and $\rho << (C \sup_I \|\Omega_{Q\alpha}^{-1}\|)^{-1}$ implies H maps $B_\rho(\mu_0)$ into $B_\rho(\mu_0)$.

Let $\mu_1, \mu_2 \in B_\rho(\mu_0)$ and consider the bound

$$\|H(\mu_2) - H(\mu_1)\| \leq C \sup_I \|\Omega_{Q\alpha}^{-1}\| \, \|R(u, \mu_2) - R(u, \mu_1)\|$$

or, using the mean value theorem, the derivative of (16) with respect to μ, and the mean value theorem again,

$$\begin{aligned} \|H(\mu_2) - H(\mu_1)\| \leq C \sup_I \|\Omega_{Q\alpha}^{-1}\| \, \|\partial_\mu^2 F(u, (1-\lambda_2)[(1-\lambda_1)\mu_1 + \lambda_1\mu_2] + \lambda_2\mu_0)\| \\ \times \|(1-\lambda_1)\mu_1 + \lambda_1\mu_2 - \mu_0\| \, \|\mu_2 - \mu_1\| \end{aligned}$$

for some $\lambda_1, \lambda_2 \in [0, 1]$. Using (17) and $\|(1-\lambda_1)\mu_1 + \lambda_1\mu_2 - \mu_0\| < \rho$ then gives

$$\|H(\mu_2) - H(\mu_1)\| \leq C \sup_I \|\Omega_{Q\alpha}^{-1}\|(1 + \alpha^{-\frac{1}{2}}\varepsilon)\rho\|\mu_2 - \mu_1\|.$$

Thus, with the above choices of ε and ρ, H is a strict contraction. We conclude that the radii of the balls $B_\varepsilon(Q_{ca})$ can be taken independent of a, c (but dependent on I) and $\varepsilon = \mathrm{O}\left(\alpha^{\frac{1}{2}} \lfloor \delta'(c) \rfloor^4\right)$.

The above argument shows that there exists balls $\{B_\varepsilon(Q_{ca}) \,|\, a \in \mathbb{R}_+, c \in I\}$ with radius ε dependent only on the parameter α and the compact set I. Notice that $U_\varepsilon = \bigcup\{B_\varepsilon(Q_{ca}) \,|\, a \in \mathbb{R}_+, c \in I\}$. Pasting the C^1 functions $g_{\alpha Q_{ca}}$ together into a C^1 function $g_{\alpha I} : U_\varepsilon \to \mathbb{R}_+ \times I$ gives the required C^1 functions $a(u)$ and $c(u)$. Uniqueness follows from the uniqueness of each of the functions $g_{\alpha Q_{ca}}$.

Let $u \in U_\varepsilon$, $c \in I$ and $a \in \mathbb{R}$, and consider the equation

$$u - Q_{c(u)a(u)} = u - Q_{ca} + Q_{ca} - Q_{c(u)a(u)}.$$

Clearly, inequality (15) will follow if $\left\|Q_{ca} - Q_{c(u)a(u)}\right\|_{H^1} \leq C \left\|u - Q_{ca}\right\|_{H^1}$ for some positive constant C. Since the derivatives $\partial_c Q_{ca}$ and $\partial_a Q_{ca}$ are uniformly bounded in $H^1(\mathbb{R})$ over $I \times \mathbb{R}$, the mean value theorem gives that $\left\|Q_{ca} - Q_{c(u)a(u)}\right\|_{H^1} \leq C\|(a\, c)^T - (a(u)\, c(u))^T\|$, where the constant C does not depend on c, a, or α. The relations $g_{\alpha I}(Q_{ca}) = (a\, c)^T$ and $g_{\alpha I}(u) = (a(u)\, c(u))^T$ then imply $\left\|Q_{ca} - Q_{c(u)a(u)}\right\|_{H^1} \leq C\|g_{\alpha I}(Q_{ca}) - g_{\alpha I}(u)\|$. Again, we appeal to the mean value theorem and obtain (15), using the properties of ∂_α^{-1} and that $\partial_u g_{\alpha I} = -\partial_\mu F^{-1}\partial_u F$ is uniformly bounded in the parameters $c \in I$, $a \in \mathbb{R}_+$ and $u \in U_\varepsilon$. $\square$

4 Spectral Properties of the Hessian $\partial^2\Lambda_{ca}$

The Hessian $\partial^2\Lambda_{ca}$ at Q_{ca} in the $L^2(\mathbb{R})$ pairing is computed to be the unbounded operator

$$\mathcal{L}_Q := -\partial_x^2 + c - f'(Q_{ca}), \tag{18}$$

defined on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. We extend this operator to the corresponding complex spaces.

Proposition 6. *The self-adjoint operator $\mathcal{L}_Q$ has the following properties.*

1. $\mathcal{L}_Q \zeta_{ca}^{tr} = 0$ *and* $\mathcal{L}_Q \zeta_{ca}^{n} = -Q_{ca}$.
2. *All eigenvalues of* $\mathcal{L}_Q$ *are simple, and* $\mathrm{Null}\ \mathcal{L}_Q = \mathrm{Span}\ \{\zeta_{ca}^{tr}\}$.
3. $\mathcal{L}_Q$ *has exactly one negative eigenvalue.*
4. *The essential spectrum is* $[c, \infty) \subset \mathbb{R}_+$.
5. $\mathcal{L}_Q$ *has a finite number of eigenvalues in* $(-\infty, c)$.

Proof. Recall that the vectors $\zeta_{ca}^{tr} := -\partial_x Q_{ca}$ and $\zeta_{ca}^{n} := \partial_c Q_{ca}$ are in the Sobolev space $H^2(\mathbb{R})$. Thus, relations $\mathcal{L}_Q \zeta_{ca}^{tr} = 0$ and $\mathcal{L}_Q \zeta_{ca}^{n} = -Q_{ca}$ make sense, and are obtained by differentiating $\Lambda'_{ca}(Q_{ca}) = 0$ with respect to a and c. The first relation above proves that ζ_{ca}^{tr} is a null vector.

Say $\zeta, \eta \in H^2(\mathbb{R})$ are linearly independent eigenvectors of $\mathcal{L}_Q$ with the same eigenvalue. Then, since $\mathcal{L}_Q$ is a second order linear differential operator without a first order derivative, the Wronskian

$$W(\eta, \zeta) = \zeta \partial_x \eta - \eta \partial_x \zeta$$

is a non-zero constant. With η and ζ both in $H^2(\mathbb{R})$ however, the limit $\lim_{x\to\infty} W(\eta, \zeta)$ is zero. This contradicts the non vanishing of the Wronskian, and hence all eigenvalues of $\mathcal{L}_Q$ are simple and, in particular, $\mathrm{Null}\ \mathcal{L}_Q = \mathrm{Span}\ \{\zeta_{ca}^{tr}\}$.

Next we prove that the operator $\mathcal{L}_Q$ has exactly one negative eigenvalue using Sturm-Liouville theory on an infinite interval. Recall that the solitary wave $Q_{ca}(x)$ is a differentiable function, symmetric about $x = a$ and monotonically decreasing if $x > a$. This implies that the null vector ζ_{ca}^{tr}, or equivalently, the derivative of Q_{ca} with respect to x, has exactly one root at $x = a$. Therefore, by Sturm-Liouville theory, zero is the second eigenvalue and there is exactly one negative eigenvalue.

We use standard methods to compute the essential spectrum. Since the function $f'(Q_{ca}(x))$ is continuous and decays to zero at infinity, the bottom of the essential spectrum begins at $\lim_{x\to\infty}(c - f'(Q_{ca}(x))) = c$ and extends to infinity: $\sigma_{ess}(\mathcal{L}_Q) = [c, \infty)$. Furthermore, the bottom of the essential spectrum is not an accumulation point of the discrete spectrum since $f'(Q_{ca}(x))$ decays faster than x^{-2} at infinity. Hence, there is at most a finite number of eigenvalues in the interval $(-\infty, c)$. For details see [34, 35, 25]. □

5 Anisotropic Coercivity of the Hessian $\mathcal{L}_Q$ on $(\mathcal{K}_{Q\alpha} T_{Q_{ca}} M_s)^{\perp}$

In this section we prove strict positivity of the Hessian $\mathcal{L}_Q$ on the orthogonal complement of the 2-dimensional space $\mathcal{K}_{Q\alpha} T_{Q_{ca}} M_s = \mathrm{Span}\ \left\{Q_{ca},\, \partial_\alpha^{-1} \zeta_{ca}^{n}\right\}$. This result is a crucial ingredient in the proof of the bound on the fluctuation ξ.

Proposition 7. *Assume* $\delta'(c) > 0$ *on the compact set* $I \subset I_0$. *The following statements hold if* $\alpha > 0$ *is small enough and* $\xi \perp \mathcal{K}_{Q\alpha} T_{Q_{ca}} M_s$.

1. *There are positive real numbers* C_1 *and* C_2, *independent of* α, *and a function* $\varrho(\alpha)$ *satisfying* $C_1 \alpha \leq \varrho(\alpha) \leq C_2 \alpha$ *such that* $\langle \mathcal{L}_Q \xi, \xi \rangle \geq \varrho(\alpha) \|\xi\|_{H^1}^2$ *for all* $c \in I$ *and* $a \in \mathbb{R}$.
2. *The infimum* $\inf\{\langle \mathcal{L}_Q \xi, \xi \rangle \mid \xi \perp \mathcal{K}_{Q\alpha} T_{Q_{ca}} M_s$ *and* $\|\xi\|_{L^2} = 1\}$ *is attained and the unique minimizer* η *is of the form* $\eta = \gamma \zeta_{ca}^{tr} + \eta_\perp$, *where* $\|\eta_\perp\|_{H^1} = \mathrm{O}\left(\alpha^{\frac{1}{2}}\right)$ *and* $\gamma = \mathrm{O}\,(1)$.

3. *Let η be as above and let $\xi_g := \xi - \langle \xi, \eta \rangle \eta$. There exists a positive real number C_3 independent of α, such that $\langle \mathcal{L}_Q \xi_g, \xi_g \rangle \geq C_3 \|\xi_g\|_{H^1}^2$ (notice that α enters ξ_g through the minimizer η).*
4. *The Hessian $\mathcal{L}_Q$ is anisotropically coercive on $(\mathcal{K}_{Q\alpha} T_{Q_{ca}} M_s)^\perp$; that is,*

$$\langle \mathcal{L}_Q \xi, \xi \rangle \geq C_3 \|\xi_g\|_{H^1}^2 + C\alpha \|\xi_b\|_{H^1}^2 .$$

Proof. Define the set $X := \{\xi \in H^1(\mathbb{R}) \,|\, \xi \perp \mathcal{K}_{Q\alpha} T_{Q_{ca}} M_s \text{ and } \|\xi\|_{L^2} = 1\}$. Our first step is to prove an upper bound on $\inf_X \langle \mathcal{L}_Q \xi, \xi \rangle$. We do this by computing $\langle \mathcal{L}_Q \xi, \xi \rangle$ for the test function

$$\xi := \lambda_1 \zeta_{ca}^{tr} + \lambda_2 \partial_\alpha^{-1} \zeta_{ca}^n + \lambda_3 Q_{ca},$$

where λ_1, λ_2, and λ_3 are chosen to satisfy $\langle \xi, Q_{ca} \rangle = 0$, $\left\langle \xi, \partial_\alpha^{-1} \zeta_{ca}^n \right\rangle = 0$, and $\|\xi\|_{L^2}^2 = 1$. These conditions imply $\xi \in X$, and, after substituting ξ with its definition, have the form

$$\lambda_2 \left\langle Q_{ca}, \partial_\alpha^{-1} \zeta_{ca}^n \right\rangle + \lambda_3 \|Q_{ca}\|_{L^2}^2 = 0, \tag{19}$$

$$\lambda_1 \left\langle \zeta_{ca}^{tr}, \partial_\alpha^{-1} \zeta_{ca}^n \right\rangle + \lambda_2 \left\|\partial_\alpha^{-1} \zeta_{ca}^n\right\|_{L^2}^2 + \lambda_3 \left\langle Q_{ca}, \partial_\alpha^{-1} \zeta_{ca}^n \right\rangle = 0, \tag{20}$$

and

$$\begin{aligned} &\lambda_1^2 \left\|\zeta_{ca}^{tr}\right\|_{L^2}^2 + \lambda_2^2 \left\|\partial_\alpha^{-1} \zeta_{ca}^n\right\|_{L^2}^2 + \lambda_3^2 \|Q_{ca}\|_{L^2}^2 \\ &\qquad + 2\lambda_1 \lambda_2 \left\langle \zeta_{ca}^{tr}, \partial_\alpha^{-1} \zeta_{ca}^n \right\rangle + 2\lambda_2 \lambda_3 \left\langle Q_{ca}, \partial_\alpha^{-1} \zeta_{ca}^n \right\rangle = 1. \end{aligned} \tag{21}$$

Equation (20) can be solved for λ_1 when $\left\langle \zeta_{ca}^{tr}, \partial_\alpha^{-1} \zeta_{ca}^n \right\rangle$ is not zero. A straightforward computation using antisymmetry of ∂_x and statements 1 and 5 of Lemma 3 gives that

$$\left\langle \zeta_{ca}^{tr}, \partial_\alpha^{-1} \zeta_{ca}^n \right\rangle = \delta'(c) + \mathrm{O}(\alpha). \tag{22}$$

Thus, if α is small enough, then $\left\langle \zeta_{ca}^{tr}, \partial_\alpha^{-1} \zeta_{ca}^n \right\rangle$ is non-zero. We substitute for λ_1 in (21) using (20) and then use (19) to substitute for λ_3 to obtain

$$\left[\frac{\|\zeta_{ca}^{tr}\|_{L^2}^2}{\left\langle \zeta_{ca}^{tr}, \partial_\alpha^{-1} \zeta_{ca}^n \right\rangle^2} \left(\left\|\partial_\alpha^{-1} \zeta_{ca}^n\right\|_{L^2}^2 - \frac{\left\langle Q_{ca}, \partial_\alpha^{-1} \zeta_{ca}^n \right\rangle^2}{\|Q_{ca}\|_{L^2}^2} \right)^2 - \left\|\partial_\alpha^{-1} \zeta_{ca}^n\right\|_{L^2}^2 + \frac{\left\langle \partial_\alpha^{-1} \zeta_{ca}^n, Q_{ca} \right\rangle^2}{\|Q_c\|_{L^2}^2} \right] \lambda_2^2 = 1. \tag{23}$$

This equation, the relations $\left\|\partial_\alpha^{-1} \zeta_{ca}^n\right\|_{L^2}^2 = \frac{\pi}{\alpha} \left(\int_{-\infty}^{\infty} \zeta_{ca}^n \, dx \right)^2 + \mathrm{O}(1)$, $\left\langle Q_{ca}, \partial_\alpha^{-1} \zeta_{ca}^n \right\rangle = \mathrm{O}(1)$, and (22) imply that $\lambda_2 = \mathrm{O}(\alpha)$. Equation (19) then implies $\lambda_3 = \mathrm{O}(\alpha)$. Evaluating the quadratic form $\langle \mathcal{L}_Q \cdot, \cdot \rangle$ at the test function ξ and bounding with Hölder's inequality implies

$$\begin{aligned} \langle \mathcal{L}_Q \xi, \xi \rangle &\leq C \left(\lambda_2^2 \left\|\partial_\alpha^{-1} \zeta_{ca}^n\right\|_{H^1}^2 + \lambda_2 \lambda_3 \|Q_{ca}\|_{H^1} \left\|\partial_\alpha^{-1} \zeta_{ca}^n\right\|_{H^1} + \lambda_3^2 \|Q_{ca}\|_{H^1}^2 \right) \\ &\leq C(c)\alpha. \end{aligned}$$

In the last inequality we have used the bounds on λ_1 and λ_2, and the above estimate of $\left\|\partial_\alpha^{-1} \zeta_{ca}^n\right\|_{H^1}$. The constant $C(c)$ does not depend on the parameter a since $H^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ norms are translation invariant.

To prove the first part of the proposition, we first prove that $\inf_X \langle \mathcal{L}_Q \xi, \xi \rangle > 0$, or equivalently, that $\inf_{X \cap H^2(\mathbb{R})} \langle \mathcal{L}_Q \xi, \xi \rangle > 0$. By the max-min principle, $\inf_{X \cap H^2(\mathbb{R})} \langle \mathcal{L}_Q \xi, \xi \rangle$ is attained or is equal to the bottom of the essential spectrum. We take α small enough so that the above upper bound is below the essential spectrum. Let η be the minimizer.

We claim the set of vectors $\{\zeta_{ca}^{tr},\,\zeta_{ca}^{n},\,\eta\}$ is a linearly independent set. If they were dependent, then, since ζ_{ca}^{tr} and ζ_{ca}^{n} are orthogonal, there are non-zero constants γ_1 and γ_2 such that $\eta = \gamma_1\zeta_{ca}^{tr} + \gamma_2\zeta_{ca}^{n}$. Projecting this equation onto Q_{ca} and $\partial_\alpha^{-1}\zeta_{ca}^{n}$ gives the equations $\gamma_2\delta'(c) = 0$ and $\gamma_1\left\langle\zeta_{ca}^{tr}, \partial_\alpha^{-1}\zeta_{ca}^{n}\right\rangle + \gamma_2\left\langle\partial_\alpha^{-1}\zeta_{ca}^{n}, \zeta_{ca}^{tr}\right\rangle = 0$. Thus, if α is sufficiently small, then both constants are zero (we have used (22)). This is a contradiction since the zero function does not lie in X.

The above argument proves that Span $\{\zeta_{ca}^{tr},\,\zeta_{ca}^{n},\,\eta\}$ is three dimensional. The min-max principle states that if

$$\begin{aligned} E_3 &:= \inf_{V\subset H^2(\mathbb{R}),\,\dim V=3}\ \sup_{\xi\in V,\,\|\xi\|_{L^2}=1} \langle \mathcal{L}_Q\xi,\xi\rangle \\ &\leq \max_{\xi\in \operatorname{Span}\{\zeta_{ca}^{tr},\,\zeta_{ca}^{n},\,\eta\},\,\|\xi\|_{L^2}=1} \langle \mathcal{L}_Q\xi,\xi\rangle \end{aligned}$$

is below the essential spectrum, then it is the third eigenvalue counting multiplicity. Let $\xi = \gamma_1\eta+\gamma_2\zeta_{ca}^{n}+\gamma_3\zeta_{ca}^{tr}$ be the maximizer in the second line above. By Proposition 6 there are exactly two non-positive eigenvalues. Hence

$$0 < E_3 \leq \langle \mathcal{L}_Q\xi,\xi\rangle = \gamma_1^2\,\langle \mathcal{L}_Q\eta,\eta\rangle - \gamma_2^2\delta'(c).$$

Thus, since $\delta'(c) > 0$, we must have $\langle \mathcal{L}_Q\eta,\eta\rangle > 0$. The function $\sigma(c,\alpha) := \langle \mathcal{L}_Q\eta,\eta\rangle$ is continuous with respect to c since both Q_{ca} and $\partial_\alpha^{-1}\zeta_{ca}^{n}$ are continuous as mappings taking c to elements of $H^1(\mathbb{R})$. Taking the infimum of $\sigma(c,\alpha)$ over I implies $\langle \mathcal{L}_Q\xi,\xi\rangle \geq \rho(\alpha)\,\|\xi\|_{L^2}^2$ for all $\xi\perp\mathcal{K}_{Q\alpha}T_{Q_{ca}}M_s$, where $\rho(\alpha) := \inf_I \sigma(c,\alpha)$.

To complete the proof of the first statement (modulo the lower bound on $\varrho(\alpha)$), we improve the above lower bound to one involving $H^1(\mathbb{R})$ norms. If we define $K := \sup_I \|c - f'(Q_{ca})\|_{L^\infty}$, then $\langle \mathcal{L}_Q\xi,\xi\rangle \geq \|\partial_x\xi\|_{L^2}^2 - K\,\|\xi\|_{L^2}^2$ for all $\xi \in H^1(\mathbb{R})$. Adding a factor $\frac{K+1}{\rho(\alpha)}$ of this bound to the above bound gives the required result with

$$\varrho(\alpha) = \frac{\rho(\alpha)}{\rho(\alpha)+K+1}.$$

Notice that the upper bound $\sigma(c,\alpha) \leq C(c)\alpha$ derived above gives, after maximizing constants over $c \in I$, the uniform upper bound on ϱ.

As already shown, the minimizer η of $\inf_X \langle \mathcal{L}_Q\xi,\xi\rangle$ exists for α small enough. We prove the properties of η by manipulating its Euler-Lagrange equation

$$\mathcal{L}_Q\eta = \beta\eta + \beta_1 Q_{ca} + \beta_2\partial_\alpha^{-1}\zeta_{ca}^{n}, \tag{24}$$

where β, β_1, and β_2 are Lagrange multipliers for the constraints $\|\eta\|_{L^2} = 1$, $\langle \eta, Q_{ca}\rangle = 0$, and $\left\langle \eta, \partial_\alpha^{-1}\zeta_{ca}^{n}\right\rangle = 0$. The inner product of this equation with η shows that $\beta = \sigma$. Take α small enough so that $\beta = \sigma$ is not an eigenvalue of $\mathcal{L}_Q$. Then the minimizer is unique. Indeed, the difference ζ between two minimizers is a solution to $\mathcal{L}_Q\zeta = \beta\zeta$. Since β is not an eigenvalue, $\zeta = 0$ is the only solution to this equation.

We now decompose η orthogonally as $\eta = \gamma\zeta_{ca}^{tr} + \eta_\perp$, substitute this decomposition into (24), and use $\mathcal{L}_Q\zeta_{ca}^{tr} = 0$ to obtain the equation

$$(\mathcal{L}_Q - \sigma)\,\eta_\perp = \gamma\sigma\zeta_{ca}^{tr} + \beta_1 Q_{ca} + \beta_2\partial_\alpha^{-1}\zeta_{ca}^{n} \tag{25}$$

for $\eta_\perp$. To solve for $\eta_\perp$, we first project this equation by $\bar{P}_Q$, where $\bar{P}_Q = 1 - P_Q$ and P_Q is the orthogonal projection onto the nullspace of the operator $\mathcal{L}_Q$:

$$[L_Q - \sigma]\eta_\perp = \beta_1 Q_{ca} + \beta_2 \bar{P}_Q \partial_\alpha^{-1} \zeta_{ca}^n,$$

where L_Q is restriction of $\mathcal{L}_Q$ onto the orthogonal complement of the null space of $\mathcal{L}_Q$. The spectrum of $\mathcal{L}_Q$ has essential spectrum $[c, \infty)$ and a finite number of eigenvalues in $(-\infty, c)$. Moreover, $\mathcal{L}_Q$ is independent of α, and therefore, for α small enough, we conclude that the interval $[0, \sigma]$ is disjoint from the spectrum of L_Q, with the distance to the spectrum of L_Q bounded below by a positive number C independent of α. Hence, we can solve (25) for $\eta_\perp$ to obtain

$$\eta_\perp = (L_Q - \sigma)^{-1} \left(\beta_1 Q_{ca} + \beta_2 \bar{P}_Q \partial_\alpha^{-1} \zeta_{ca}^n\right). \tag{26}$$

To prove the $L^2(\mathbb{R})$ the estimate of $\eta_\perp$, we require estimates on the Lagrange multipliers β_1 and β_2. We take the inner product of (25) with ζ_{ca}^{tr} and use (22) to obtain that

$$\beta_2[\delta'(c) + \mathrm{O}(\alpha)] = -\gamma\sigma \left\|\zeta_{ca}^{tr}\right\|_{L^2}^2.$$

Thus, $\beta_2 = \mathrm{O}(\sigma)$ since the constraint $\|\eta\|_{L^2} = 1$ implies $\gamma = \mathrm{O}(1)$. Similarly, since η is orthogonal to Q_{ca}, the inner product of (24) with ζ_{ca}^n and statement 5 of Lemma 3 gives the relation

$$\beta_2 \left(\left(\int_{-\infty}^{\infty} \zeta_{ca}^n \, dx \right)^2 + \mathrm{O}(\alpha) \right) = -\sigma \langle \eta, \zeta_{ca}^n \rangle - \beta_1 \delta'(c).$$

The estimate $\beta_1 = \mathrm{O}(\sigma)$ is immediate using the estimate of β_2 and the assumption $\delta'(c) > 0$. We substitute the estimates of β_1 and β_2, and the estimate $\left\|\partial_\alpha^{-1}\zeta_{ca}^n\right\|_{L^2} = \mathrm{O}(\alpha^{-\frac{1}{2}})$ into (26), and use the above fact that σ is at least a distance C away from the spectrum of L_Q to conclude that

$$\|\eta_\perp\|_{L^2} = \mathrm{O}\left(\alpha^{-\frac{1}{2}}\sigma\right). \tag{27}$$

Replacing σ with its upper bound gives the third statement of the proposition.

We now prove a lower bound on the infimum $\sigma = \inf_X \langle \mathcal{L}_Q \xi, \xi \rangle$. We again need to take α small enough so that a minimizer exists. The orthogonal decomposition $\eta = \gamma\zeta_{ca}^{tr} + \eta_\perp$ of the minimizer implies $\sigma = \langle \mathcal{L}_Q \eta_\perp, \eta_\perp \rangle$. Substituting for $\mathcal{L}_Q \eta_\perp$ using (25) gives

$$\sigma = \sigma \|\eta_\perp\|_{L^2}^2 + \beta_2 \left\langle \eta_\perp, \partial_\alpha^{-1} \zeta_{ca}^n \right\rangle,$$

where we have used that $\langle \zeta_{ca}^{tr}, \eta_\perp \rangle = 0$ and, since the minimizer η is orthogonal to Q_{ca}, $\langle \eta_\perp, Q_{ca} \rangle = 0$. Thus, (27), $\beta_2 = \mathrm{O}(\sigma)$, and $\left\|\partial_\alpha^{-1}\zeta_{ca}^n\right\|_{L^2} = \mathrm{O}\left(\alpha^{-1}\right)$ imply

$$\sigma \leq C_1 \frac{\sigma^3}{\alpha} + C_2 \frac{\sigma^2}{\alpha}$$

or, since σ is positive, $C_1\sigma^2 + C_2\sigma - \alpha \geq 0$, where the constants C_1 and C_2 depend continuously on c. The positive root of the quadratic is a lower bound on σ. After rationalizing, we obtain

$$\sigma \geq \frac{2\alpha}{C_2 + \sqrt{C_2^2 + 4C_1\alpha}} \geq K_1\alpha,$$

for some constant K_1. Minimizing the constant over I completes the proof of the lower bound.

Our proof of statement three in the proposition requires that $\inf_Y \langle \mathcal{L}_Q \xi, \xi \rangle$ is positive, where $Y := \{\xi \in H^1(\mathbb{R}) \,|\, \xi \perp Q_c, \zeta_{ca}^{tr} \text{and } \|\xi\|_{L^2} = 1\}$. The argument is similar to the proof of $\inf_X \langle \mathcal{L}_Q \xi, \xi \rangle$. By the min-max principle either $\inf_{Y \cap H^2(\mathbb{R})} = \inf \sigma_{ess}(\mathcal{L}_Q)$ or the minimizer is attained. There is nothing to prove in the former case since $\inf \sigma_{ess}(\mathcal{L}_Q) = c > 0$. Thus, we assume η is a minimizer. As above, the set $\{\eta, \zeta_{ca}^{tr}, \zeta_{ca}^{n}\}$ is a linearly independent set due to the assumption $\delta'(c) > 0$, and the min-max principle implies the third eigenvalue E_3 satisfies

$$0 < E_3 \leq \gamma_3^2 \langle \mathcal{L}_Q \eta, \eta \rangle$$

for some constant γ_3. Thus, we must have $\langle \mathcal{L}_Q \xi, \xi \rangle > K_3$ for all $\xi \in Y$, where K_3 is a positive constant independent of α. As with the infimum over the set X, this inequality can be improved to the $H^1(\mathbb{R})$ estimate $\langle \mathcal{L}_Q \xi, \xi \rangle > C_3 \|\xi\|_{H^1}^2$ for all $\xi \perp Q_{ca}, \zeta_{ca}^{tr}$.

We now decompose ξ_g orthogonally as $\xi_g = \beta \zeta_{ca}^{tr} + \psi$. Since ξ_g is orthogonal to η and $\eta = \gamma \zeta_{ca}^{tr} + \eta_\perp$,

$$\beta = \left\|\zeta_{ca}^{tr}\right\|_{L^2}^{-2} \gamma^{-1} \langle \xi_g, \eta - \eta_\perp \rangle = \mathrm{O}\left(\|\eta_\perp\|_{L^2} \|\xi_g\|_{L^2}\right) = \mathrm{O}\left(\alpha^{\frac{1}{2}} \|\xi_g\|_{L^2}\right).$$

Substituting this bound into $\|\psi\|_{H^1}^2 \geq \|\xi_g\|_{H^1}^2 - \beta^2 \|\zeta_{ca}^{tr}\|_{H^1}^2$ gives that $\|\psi\|_{H^1}^2 \geq \|\xi_g\|_{H^1}^2 (1 - \alpha \|\zeta_{ca}^{tr}\|_{H^1}^2)$. Thus, if $\alpha < \frac{1}{2} \|\zeta_{ca}^{tr}\|_{H^1}^2$, then $\|\psi\|_{H^1}^2 \geq \frac{1}{2} \|\xi_g\|_{H^1}^2$. Substituting this into the inequality

$$\langle \mathcal{L}_Q \xi_g, \xi_g \rangle = \langle \mathcal{L}_Q \psi, \psi \rangle \geq C_3 \|\xi_g\|_{H^1}^2$$

(which follows from the fact that $\psi \perp Q_{ca}, \zeta_{ca}^{tr}$) completes the proof.

To prove the last statement we define $\xi_b := \xi - \xi_g$. Since the vectors ξ and η are both symplectically orthogonal to the tangent space, so is ξ_b. Thus, using the above inequalities,

$$\langle \mathcal{L}_Q \xi, \xi \rangle \geq C_3 \|\xi_g\|_{H^1}^2 + C\alpha \|\xi_b\|_{H^1}^2 + 2 \langle \xi, \eta \rangle \langle \mathcal{L}_Q \eta, \xi_g \rangle.$$

The cross term $\langle \mathcal{L}_Q \eta, \xi_g \rangle$ is zero; indeed, substitute for $\mathcal{L}_Q \eta$ using equation (24) and use $\langle \xi_g, \eta \rangle = 0$ to obtain

$$\langle \mathcal{L}_Q \eta, \xi_g \rangle = \beta_1 \langle Q_{ca}, \xi_g \rangle + \beta_2 \left\langle \partial_\alpha^{-1} \zeta_{ca}^n, \xi_g \right\rangle.$$

This expression, however, is zero since both ξ and η are orthogonal to Q_{ca} and $\partial_\alpha^{-1} \zeta_{ca}^n$, completing the proof. $\square$

6 Evolution Equations for the Fluctuation ξ and the Parameters a and c

In Section 3 we proved that if u remains close enough to the solitary wave manifold M_s, then we can write a solution u to (1) uniquely as a sum of a modulated solitary wave Q_{ca} and a fluctuation ξ satisfying orthogonality condition (12). Thus, as u evolves according to the initial value problem (1), the parameters $a(t)$ and $c(t)$ trace out a path in $\mathbb{R}^2$. The goal of this section is to derive the dynamical equations for the parameters a and c, and the fluctuation ξ. We obtain such equations by substituting the decomposition

$u = Q_{ca} + \xi$ into (1) and then projecting the resulting equation onto appropriate directions, with the intent of using the orthogonality condition on ξ.

From now on, u is the solution of (1) with initial condition u_0 satisfying $\inf_{Q_{ca}\in M_s} \|u_0 - Q_{ca}\|_{H^1} << \varepsilon$, and $T_0 = T_0(u_0)$ is the maximal time such that $u(t) \in U_\varepsilon$ for $0 \le t \le T_0$. Then for $0 \le t \le T_0$, u can be decomposed as in (11) and (12).

The majority of the work involves estimating the higher order terms of the resulting equation for the modulation parameters. It turns out that a naive attempt at bounding ξ directly with the Lyapunov method does not give good results. As will be seen later, the component of ξ in the direction of ζ_{ca}^{tr} is particularly problematic. On the other hand, $\zeta_{ca}^{tr} = -\partial_x Q_{ca}$ is the derivative of a function and the null vector of $\mathcal{L}_Q$. This can be used to improve the bound. Thus, in order to obtain better estimates on ξ, we orthogonally decompose the fluctuation as

$$\xi = \xi_b + \xi_g$$

where $\xi_b = \langle \xi, \eta \rangle \eta$. Recall that η is approximately ζ_{ca}^{tr} and is given in Section 5: $\eta = \gamma\zeta_{ca}^{tr} + \eta_\perp$, with $\|\eta\|_{L^2} = 1$, $\gamma = \mathrm{O}(1)$, and $\|\eta_\perp\|_{L^2} = \mathrm{O}\left(\alpha^{\frac{1}{2}}\right)$. We use the above decomposition to prove the following proposition regarding the dynamical equations for a and c.

Proposition 8. *Assume $\delta'(c) > 0$ on the compact set $I \subset \mathbb{R}_+$. Say $u = Q_{ca} + \xi$ is a solution to (1), where ξ satisfies (12) and $\xi = \xi_b + \xi_g$ as above. If $\alpha^{-\frac{1}{2}} \|\xi_g\|_{H^1} + \|\xi\|_{H^1}$ is small enough and $\alpha, \epsilon_x \le 1$, then, provided $c \in I$,*

$$\begin{pmatrix} \dot{a} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} c - b(t,a) \\ 0 \end{pmatrix} + b'(t,a)\frac{\delta(c)}{\delta'(c)^2} \begin{pmatrix} -\frac{1}{2}\left(\int_{-\infty}^{\infty} \zeta_{ca}^n \, dx\right)^2 \\ \delta'(c) \end{pmatrix} + Z(a,c,\xi), \tag{28}$$

where $|Z(a,c,\xi)| \le C\left(\alpha\epsilon_a\epsilon_x + \epsilon_a\epsilon_x^2 + \left(\alpha^{\frac{1}{2}} + \alpha^{-\frac{1}{2}}\epsilon_a\epsilon_x\right)\|\xi_g\|_{H^1} + (\alpha + \epsilon_a\epsilon_x)\|\xi\|_{H^1} + \|\xi\|_{H^1}^2\right)$, for some positive constant $C = C(I)$.

Proof. Recall that the solitary wave Q_{ca} is an extremal of the functional Λ_{ca}. To use this fact we rearrange definition (8) of Λ_{ca} to write the Hamiltonian H_b as

$$H_b(u) = \Lambda_{ca}(u) - cP(u) + \frac{1}{2}\int_{-\infty}^{\infty} bu^2(x)\, dx,$$

where for notational simplicity we have suppressed the space and time dependency of b. Substituting $Q_{ca} + \xi$ for u in (3) and using the above expression for H_b gives the equation

$$\dot{a}\zeta_{ca}^{tr} + \dot{c}\zeta_{ca}^n + \dot{\xi} = \partial_x\Lambda'_{ca}(Q_{ca} + \xi) - c\partial_x[Q_{ca} + \xi] + \partial_x[(Q_{ca} + \xi)b],$$

where dots indicate time differentiation. Taylor expanding $\Lambda'_{ca}(Q_{ca} + \xi)$ to linear order in ξ and using that Q_{ca} is an extremal of Λ_{ca} gives

$$\begin{aligned} \dot{\xi} = \partial_x\left[(\mathcal{L}_Q + \delta b + b(a) - c)\xi\right] + \partial_x N'(\xi) - [\dot{a} - c + b(a)]\zeta_{ca}^{tr} - \dot{c}\zeta_{ca}^n \\ + b'(a)\partial_x[(x-a)Q_{ca}] + \partial_x[\delta^2 b Q_{ca}]. \end{aligned} \tag{29}$$

We have used the relation $\zeta_{ca}^{tr} = -\partial_x Q_{ca}$, definition (18) of $\mathcal{L}_Q$, the definitions

$$\delta b := b(x) - b(a)$$

and

$$\delta^2 b := b(x) - b(a) - b'(a)(x-a),$$

and definition (50) of $N'(\xi)$ given in Appendix C to write the above equation in a convenient form. Define the vectors $\zeta_1 := \zeta_{ca}^{tr}$ and $\zeta_2 := \zeta_{ca}^{n}$. Projecting (29) onto $\mathcal{K}_{Q\alpha}\zeta_i$, for $i = 1$ and 2, and using the anti-self-adjointness of ∂_x gives the two equations

$$\begin{aligned} [\dot a - c + b(a)]\left[\left\langle \zeta_{ca}^{tr}, \mathcal{K}_{Q\alpha}\zeta_i\right\rangle + \left\langle \xi, \partial_x \mathcal{K}_{Q\alpha}\zeta_i\right\rangle\right] + \dot c\left\langle \zeta_{ca}^{n}, \mathcal{K}_{Q\alpha}\zeta_i\right\rangle + \left\langle \dot\xi, \mathcal{K}_{Q\alpha}\zeta_i\right\rangle - \dot a\left\langle \xi, \partial_x\mathcal{K}_{Q\alpha}\zeta_i\right\rangle = \\ - b'(a)\left\langle (x-a)Q_{ca}, \partial_x\mathcal{K}_{Q\alpha}\zeta_i\right\rangle - \left\langle \delta^2 b Q_{ca}, \partial_x\mathcal{K}_{Q\alpha}\zeta_i\right\rangle \\ - \left\langle \mathcal{L}_Q\xi, \partial_x\mathcal{K}_{Q\alpha}\zeta_i\right\rangle - \left\langle \delta b\,\xi, \partial_x\mathcal{K}_{Q\alpha}\zeta_i\right\rangle - \left\langle N'(\xi), \partial_x\mathcal{K}_{Q\alpha}\zeta_i\right\rangle. \end{aligned} \tag{30}$$

We can replace the term containing $\dot\xi$ since the time derivative of the orthogonality condition $\langle \xi, \mathcal{K}_{Q\alpha}\zeta_i\rangle = 0$ implies $\langle \dot\xi, \mathcal{K}_{Q\alpha}\zeta_i\rangle = \dot a\langle \xi, \partial_x\mathcal{K}_{Q\alpha}\zeta_i\rangle - \dot c\langle \xi, \partial_c\mathcal{K}_{Q\alpha}\zeta_i\rangle$. Note that we have used the relation $\partial_a\zeta_i = -\partial_x\zeta_i$. Thus, equations (30) in matrix form are

$$(I+B)\Omega_{Q\alpha}\begin{pmatrix} \dot a - c + b(a) \\ \dot c \end{pmatrix} = X + Y, \tag{31}$$

where

$$\begin{aligned} X &:= -b'(a)\delta'(c)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b'(a)\begin{pmatrix} 0 \\ \alpha\left\langle (x-a)Q_{ca}, \partial_\alpha^{-1}\zeta_{ca}^n\right\rangle \end{pmatrix} - \begin{pmatrix} \left\langle \delta^2 b Q_{ca}, \zeta_{ca}^{tr}\right\rangle \\ \left\langle \delta^2 b Q_{ca}, \zeta_{ca}^n - \alpha\partial_\alpha^{-1}\zeta_{ca}^n\right\rangle \end{pmatrix}, \\ Y &:= -\begin{pmatrix} \left\langle \mathcal{L}_Q\xi + \delta b\,\xi + N'(\xi), \zeta_{ca}^{tr}\right\rangle \\ \left\langle \mathcal{L}_Q\xi + \delta b\,\xi + N'(\xi), \zeta_{ca}^n - \alpha\partial_\alpha^{-1}\zeta_{ca}^n\right\rangle \end{pmatrix}, \end{aligned}$$

and

$$B := \begin{pmatrix} \langle \xi, \zeta_{ca}^{tr}\rangle & \langle \xi, \zeta_{ca}^{n}\rangle \\ \langle \xi, \zeta_{ca}^{n}\rangle & -\left\langle \xi, \partial_c\partial_\alpha^{-1}\zeta_{ca}^n\right\rangle \end{pmatrix}.$$

We have explicitly computed $\langle (x-a)Q_{ca}, \zeta_i\rangle$ and used the relations $\partial_x\mathcal{K}_{Q\alpha}\zeta_{ca}^{tr} = \zeta_{ca}^{tr}$ and $\partial_x\mathcal{K}_{Q\alpha}\zeta_{ca}^{n} = \zeta_{ca}^n - \alpha\partial_\alpha^{-1}\zeta_{ca}^n$ to simplify the above expressions.

We now estimate the error terms and solve for $\dot a$ and $\dot c$. The assumptions we made on the potential imply that

$$|\delta b| \le \epsilon_a\epsilon_x(x-a) \text{ and } |\delta^2 b| \le \epsilon_a\epsilon_x^2(x-a)^2. \tag{32}$$

Thus, by Hölder's inequality and the $L^\infty(\mathbb{R})$ estimate $\left\|\partial_\alpha^{-1}\zeta_i\right\|_{L^\infty} \le \|\zeta_i\|_{L^1}$,

$$\begin{aligned} \|X\| &= -b'(a)\delta(c)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathrm{O}\left(\alpha\epsilon_a\epsilon_x + \epsilon_a\epsilon_x^2\right) \\ &= \mathrm{O}\left(\epsilon_a\epsilon_x\right). \end{aligned}$$

In the last equality we have used $\alpha \le 1$ to bound $\alpha\epsilon_a\epsilon_x$ by $\epsilon_a\epsilon_x$. We now estimate the vector Y using the properties of $\mathcal{L}_Q$ given in Appendix 7. Indeed, the generalized nullspace relations $\mathcal{L}_Q\zeta_{ca}^{tr} = 0$ and

$\mathcal{L}_Q\zeta_{ca}^n = \partial_x\zeta_{ca}^{tr}$ imply $\langle \mathcal{L}_Q\xi, \zeta_i\rangle = 0$, where the orthogonality condition $\langle \xi, \mathcal{K}_{Q\alpha}\zeta_{ca}^{tr}\rangle = 0$ is used when $i = 2$. Although the same is not true of the inner product $\alpha\left\langle \mathcal{L}_Q\xi, \partial_\alpha^{-1}\zeta_{ca}^n\right\rangle$, we still use the relation $\mathcal{L}_Q\zeta_{ca}^{tr} = 0$ to obtain a bound. If we use $\eta = \gamma\zeta_{ca}^{tr} + \eta_\perp$, then the orthogonal decomposition $\xi_b + \xi_g$ of ξ becomes

$$\gamma\left\langle \xi, \eta\right\rangle \zeta_{ca}^{tr} + \xi_g + \left\langle \xi, \eta\right\rangle \eta_\perp,$$

where $\gamma = \mathrm{O}\left(1\right)$, $\left\|\eta_\perp\right\|_{L^2} = \mathrm{O}\left(\alpha^{\frac{1}{2}}\right)$ and $\left\|\eta\right\|_{L^2} = 1$. Thus,

$$\begin{aligned}
\alpha\left\langle \mathcal{L}_Q\xi, \partial_\alpha^{-1}\zeta_{ca}^n\right\rangle &= \alpha\left\langle \mathcal{L}_Q\xi_g, \partial_\alpha^{-1}\zeta_{ca}^n\right\rangle + \alpha\left\langle \xi, \eta\right\rangle\left\langle \mathcal{L}_Q\eta_\perp, \partial_\alpha^{-1}\zeta_{ca}^n\right\rangle \\
&= \mathrm{O}\left(\alpha^{\frac{1}{2}}\left\|\xi_g\right\|_{H^1} + \alpha\left\|\xi\right\|_{H^1}\right).
\end{aligned}$$

The terms containing the potential, but not the term $\partial_\alpha^{-1}\zeta_{ca}^n$, are easily estimated using the bound on $\eth$ and exponential decay of ζ_{ca}^{tr} and ζ_{ca}^n. The resulting estimates are

$$\left\langle \eth\xi, \zeta_{ca}^{tr}\right\rangle = \mathrm{O}\left(\epsilon_a\epsilon_x\left\|\xi\right\|_{L^2}\right) \text{ and } \left\langle \eth\xi, \zeta_{ca}^{n}\right\rangle = \mathrm{O}\left(\epsilon_a\epsilon_x\left\|\xi\right\|_{L^2}\right).$$

To obtain a bound of $\alpha\left\langle \eth\xi, \partial_\alpha^{-1}\zeta_{ca}^n\right\rangle$, we decompose ξ as above to obtain

$$\alpha\left\langle \eth\xi, \partial_\alpha^{-1}\zeta_{ca}^n\right\rangle = \gamma\alpha\left\langle \xi, \eta\right\rangle\left\langle \zeta_{ca}^{tr}, \eth\partial_\alpha^{-1}\zeta_{ca}^n\right\rangle + \alpha\left\langle \xi, \eta\right\rangle\left\langle \eta_\perp, \eth\partial_\alpha^{-1}\zeta_{ca}^n\right\rangle + \alpha\left\langle \xi_g, \eth\partial_\alpha^{-1}\zeta_{ca}^n\right\rangle.$$

Then the estimates $\left\|\partial_\alpha^{-1}\zeta_{ca}^n\right\|_\infty = \mathrm{O}\left(1\right)$ and $\left\|x\partial_\alpha^{-1}\zeta_{ca}^n\right\|_{L^2} = \mathrm{O}\left(\alpha^{-\frac{3}{2}}\right)$ imply

$$\alpha\left\langle \eth\xi, \partial_\alpha^{-1}\zeta_{ca}^n\right\rangle = \mathrm{O}\left(\alpha^{-\frac{1}{2}}\epsilon_a\epsilon_x\left\|\xi_g\right\|_{H^1} + \epsilon_a\epsilon_x\left\|\xi\right\|_{H^1}\right).$$

Lastly, Hölder's inequality and Lemma 12 imply $\left\langle N'(\xi), \zeta_i\right\rangle = \mathrm{O}\left(\left\|\xi\right\|_{H^1}^2\right)$ and $\alpha\left\langle N'(\xi), \partial_\alpha^{-1}\zeta_{ca}^n\right\rangle = \mathrm{O}\left(\alpha^{\frac{1}{2}}\left\|\xi\right\|_{H^1}^2\right)$. Adding the above estimates gives

$$\left\|Y\right\| = \left(\alpha^{\frac{1}{2}} + \alpha^{-\frac{1}{2}}\epsilon_a\epsilon_x\right)\left\|\xi_g\right\|_{H^1} + \left(\alpha + \epsilon_a\epsilon_x\right)\left\|\xi\right\|_{H^1} + \left\|\xi\right\|_{H^1}^2.$$

The inner products $\left\langle \xi, \zeta_{ca}^{tr}\right\rangle$ and $\left\langle \xi, \zeta_{ca}^{n}\right\rangle$ are clearly of order $\left\|\xi\right\|_{H^1}$. We estimate the remaining entry $\left\langle \xi, \partial_c\partial_\alpha^{-1}\zeta_{ca}^n\right\rangle = \left\langle \xi, \partial_\alpha^{-1}\partial_c^2 Q_{ca}\right\rangle$ of B by the same technique as above: we replace ξ by $\gamma\left\langle \xi, \eta\right\rangle\zeta_{ca}^{tr} + \xi_g + \left\langle \xi, \eta\right\rangle\eta_\perp$ and estimate the resulting expression to obtain

$$\left\|B\right\| = \mathrm{O}\left(\alpha^{-\frac{1}{2}}\left\|\xi_g\right\|_{H^1} + \left\|\xi\right\|_{H^1}\right). \tag{33}$$

We take $\left\|B\right\|$ smaller than one so that $I + B$ is invertible and $\left\|(I + B)^{-1}\right\| = \mathrm{O}\left(1\right)$. Acting on (31) by $(I + B)^{-1} = I - B(I + B)^{-1}$ and then $\Omega_{Q\alpha}$ gives that

$$\begin{pmatrix} \dot{a} - c + b(a) \\ \dot{c} \end{pmatrix} = \Omega_{Q\alpha}X - \Omega_{Q\alpha}B(I + B)^{-1}X + \Omega_{Q\alpha}(I + B)^{-1}Y.$$

We use the leading order expressions for $\Omega_{Q\alpha}$ and X, and the bounds on $\left\|X\right\|$, $\left\|Y\right\|$, $\left\|B\right\|$ and $\left\|(I + B)^{-1}\right\|$ to obtain the estimate

$$\begin{aligned}
\begin{pmatrix} \dot{a} - c + b(a) \\ \dot{c} \end{pmatrix} &= b'(a)\frac{\delta(c)}{\delta'(c)^2}\begin{pmatrix} -\frac{1}{2}\left(\int_{-\infty}^{\infty}\zeta_{ca}^n\,dx\right)^2 \\ \delta'(c) \end{pmatrix} \\
&\quad + \mathrm{O}\left(\alpha\epsilon_a\epsilon_x + \epsilon_a\epsilon_x^2 + \left(\alpha^{\frac{1}{2}} + \alpha^{-\frac{1}{2}}\epsilon_a\epsilon_x\right)\left\|\xi_g\right\|_{H^1} + \left(\alpha + \epsilon_a\epsilon_x\right)\left\|\xi\right\|_{H^1} + \left\|\xi\right\|_{H^1}^2\right).
\end{aligned}$$

In the order notation used above, the implicit constants are continuous with respect to the parameter c and independent of the parameter a. Maximizing these constants over the compact set I completes the proof. $\square$

7 The Lyapunov Function

In the last section we derived dynamical equations for the modulation parameters. These equations contain the $H^1(\mathbb{R})$ norm of the fluctuation. In this section we begin to prove a bound on ξ. Recall that the latter bound is needed to ensure that u remains close to the manifold of solitary waves M_s for long time.

We employ a Lyapunov argument with Lyapunov function

$$M_c(t) := \Lambda_{ca}(Q_{ca} + \xi) - \Lambda_{ca}(Q_{ca}) + b'(a) \langle (x-a)Q_{ca}, \xi \rangle . \tag{34}$$

Remark: if $f(u) = u^3$, the last term in the Lyapunov functional is not needed; however, apart from computational complexity, there is no disadvantage in using the above function for this special case as well.

Lemma 9. *Assume $\delta'(c) > 0$ on the set I. Let $u = Q_{ca} + \xi$ be as in Proposition 8. Let α, ϵ_a, ϵ_x, $\|\xi\|_{H^1} \le 1$. Then, if $\alpha^{-\frac{1}{2}} \|\xi_g\|_{H^1} + \|\xi\|_{H^1}$ is small enough and $c \in I$, there is a constant $C = C(I)$ such that*

$$\frac{d}{dt} M_c(t) \le C[(\epsilon_a \epsilon_x + \|\xi\|_{H^1}^2)\Xi], \tag{35}$$

where $\Xi := \epsilon_a \epsilon_x + \left(\alpha^{\frac{1}{2}} + \alpha^{-\frac{1}{2}} \epsilon_a \epsilon_x\right) \|\xi_g\|_{H^1} + (\alpha + \epsilon_x + \epsilon_t) \|\xi\|_{H^1} + \|\xi\|_{H^1}^2$.

Proof. Suppressing explicit dependence on x and t, we have by definition

$$\Lambda_{ca}(u) := H_b(u) - \frac{1}{2} \int_{-\infty}^{\infty} u^2 b \, dx + cP(u).$$

Thus, relations (4), (5) and (6) imply that the time derivative of Λ_{ca} along the solution u is

$$\frac{d}{dt} \Lambda_{ca}(u) = \int_{-\infty}^{\infty} \frac{1}{2} \dot{c} u^2 + b' \left[\frac{1}{2} c u^2 - u f(u) + \frac{3}{2} (\partial_x u)^2 + F(u) \right] + b'' \, u \partial_x u \, dx.$$

Substituting $Q_{ca} + \xi$ for u, manipulating the result using antisymmetry of ∂_x, and collecting appropriate terms into $b'(a) \langle \mathcal{L}_Q \xi, \partial_x((x-a)Q_{ca}) \rangle$, $\langle N'(\xi), \partial_x[\delta b(Q_{ca} + \xi)] \rangle$, and $\langle \Lambda'_{ca}(Q_{ca}), \partial_x(\delta b(Q_{ca} + \xi)) \rangle$ gives the relation

$$\begin{aligned} \frac{d}{dt} [\Lambda_{ca}(Q_{ca} + \xi) - \Lambda_{ca}(Q_{ca})] =& b'(a) \langle \mathcal{L}_Q \xi, \partial_x((x-a)Q_{ca}) \rangle + \dot{c} \langle Q_{ca}, \xi \rangle + \left\langle \mathcal{L}_Q \xi, \partial_x \left(\delta^2 b Q_{ca} \right) \right\rangle + \dot{c} \frac{1}{2} \|\xi\|_{L^2}^2 \\ &+ c \frac{1}{2} \langle b' \xi, \xi \rangle + \frac{3}{2} \langle b' \partial_x \xi, \partial_x \xi \rangle - \langle f'(Q_{ca}) \xi, \partial_x(\delta b \xi) \rangle \\ &+ \langle N'(\xi), \partial_x[\delta b(Q_{ca} + \xi)] \rangle + \langle b'' \xi, \partial_x \xi \rangle + \langle \Lambda'_{ca}(Q_{ca}), \partial_x[\delta b(Q_{ca} + \xi)] \rangle . \end{aligned}$$

The last term is zero because $\Lambda'_{ca}(Q_{ca}) = 0$. The inner product $\langle \xi, Q_c \rangle$ is equal to $-\alpha \left\langle \xi, \partial_\alpha^{-1} Q_{ca} \right\rangle = \mathrm{O}\left(\alpha^{\frac{1}{2}} \|\xi\|_{H^1} \right)$ since $-\partial_\alpha^{-1} \zeta_{ca}^{tr} = Q_{ca} - \alpha \partial_\alpha^{-1} Q_{ca}$ and $\xi \perp Q_{ca}$. We use Lemma 12, assumptions (10) on the potential, estimates (32), and

$$|\delta b'| \leq \epsilon_a \epsilon_x^2 x$$

to estimate the size of the time derivative. We also use that Q_{ca}, $\partial_x Q_{ca}$, $\partial_x^2 Q_{ca}$ and $f'(Q_{ca})$ are exponentially decaying. When $\epsilon_x \leq 1$, higher order terms like $\langle b''\xi, \partial_x \xi\rangle$ are bounded above by lower order terms like $\langle b'\xi, \xi\rangle$. Similarly, if $\|\xi\|_{H^1} \leq 1$, then $\epsilon_a\epsilon_x \|\xi\|_{H^1}^2 \leq \epsilon_a \epsilon_x \|\xi\|_{H^1}$. This procedure gives the estimate

$$\begin{aligned}\frac{d}{dt}[\Lambda_{ca}(Q_{ca}+\xi) - \Lambda_{ca}(Q_{ca})] =& b'(a)\langle \xi, \mathcal{L}_Q \partial_x((x-a)Q_{ca})\rangle + \langle N'(\xi), \delta b \partial_x \xi\rangle \\ &+ \mathrm{O}\left(|\dot{c}|\,\|\xi\|_{H^1}^2 + \epsilon_a\epsilon_x^2 \|\xi\|_{H^1} + \epsilon_a \epsilon_x \|\xi\|_{H^1}^2\right).\end{aligned}$$

We compute

$$\begin{aligned}\langle N'(\xi), \delta b\partial_x\xi\rangle =& \left\langle N'(\xi) + \frac{1}{2} f''(Q_c)\xi^2, \delta b \partial_x Q_c\right\rangle \\ &- \int_{-\infty}^{\infty} \left(F(Q_{ca}+\xi) - F(Q_{ca}) - f(Q_{ca})\xi - \frac{1}{2}f'(Q_{ca})\xi^2\right) b'\, dx,\end{aligned}$$

and use the second estimate and the proof of the third estimate of Lemma 12 to obtain $\langle N'(\xi), \delta b\partial_x\xi\rangle = \mathrm{O}\left(\epsilon_a\epsilon_x \|\xi\|_{H^1}^3\right)$. Thus, since $\epsilon_a\epsilon_x\|\xi\|_{H^1}^3 \leq \epsilon_a\epsilon_x \|\xi\|_{H^1}^2$ when $\|\xi\|_{H^1} \leq 1$, we have

$$\begin{aligned}\frac{d}{dt}[\Lambda_{ca}(Q_{ca}+\xi) - \Lambda_{ca}(Q_{ca})] =& b'(a)\langle \xi, \mathcal{L}_Q\partial_x((x-a)Q_{ca})\rangle \\ &+ \mathrm{O}\left(|\dot{c}|\,\|\xi\|_{H^1}^2 + \epsilon_a\epsilon_x^2\|\xi\|_{H^1} + \epsilon_a\epsilon_x\|\xi\|_{H^1}^2\right). \quad (36)\end{aligned}$$

When $f(u) = u^3$, $\langle \xi, \mathcal{L}_Q \partial_x((x-a)Q_{ca})\rangle = 0$ since $\zeta_{ca}^n = \partial_x[(x-a)Q_{ca}]$. In this special case the above estimate is sufficient for our purposes, but in general, we need to use the corrected Lyapunov functional. When $\xi \in C(\mathbb{R}, H^1(\mathbb{R})) \cap C^1(\mathbb{R}, H^{-2}(\mathbb{R}))$, $b'(a)\langle\xi, (x-a)Q_{ca}\rangle$ is continuously differentiable with respect to time;

$$\begin{aligned}\frac{d}{dt}\left[b'(a)\langle \xi, (x-a)Q_{ca}\rangle\right] =& \partial_t b'\langle\xi, (x-a)Q_{ca}\rangle + b'(a)\left\langle \dot{\xi}, (x-a)Q_{ca}\right\rangle + \dot{c}b'(a)\langle \xi, (x-a)\zeta_{ca}^n\rangle \\ &+ \dot{a}b'(a)\left\langle\xi, (x-a)\zeta_{ca}^{tr}\right\rangle + \dot{a}b''(a)\langle\xi, (x-a)Q_{ca}\rangle,\end{aligned}$$

where $\langle \xi, Q_{ca}\rangle = 0$ has been used to simplify the derivative. Substituting for $\partial_t\xi$ using (29) gives

$$\begin{aligned}\frac{d}{dt}[b'(a)\langle\xi, (x-a)Q_{ca}\rangle] =& -b'(a)\langle\xi, \mathcal{L}_Q\partial_x((x-a)Q_{ca})\rangle - [\dot{a} - c + b(a)]b'(a)\frac{1}{2}\|Q_{ca}\|_{L^2}^2 + \partial_t b'\langle \xi, (x-a)Q_{ca}\rangle \\ &+ [\dot{a} - c + b(a)]b'(a)\langle\partial_x\xi, (x-a)Q_{ca}\rangle + [\dot{a} - c + b(a)]b''(a)\langle\xi, (x-a)Q_{ca}\rangle \\ &+ \dot{c}b'(a)\langle\xi, (x-a)\zeta_{ca}^n\rangle - b'(a)\langle\xi, \delta b\partial_x((x-a)Q_{ca})\rangle - b'(a)\langle N'(\xi), \partial_x((x-a)Q_{ca})\rangle \\ &- b'(a)\left\langle\delta^2 b Q_{ca}, \partial_x((x-a)Q_{ca})\right\rangle + [c - b(a)]b''(a)\langle\xi, (x-a)Q_{ca}\rangle.\end{aligned}$$

We estimate using the same assumptions used to derive (36). If $\|\xi\|_{H^1}$ and ϵ_x are less than 1, then

$$\begin{aligned}\frac{d}{dt}[b'(a)\langle\xi, (x-a)Q_{ca}\rangle] =& -b'(a)\langle\xi, \mathcal{L}_Q\partial_x((x-a)Q_{ca})\rangle + \mathrm{O}\left(|\dot{a} - c + b(a)|\epsilon_a\epsilon_x + |\dot{c}|\epsilon_a\epsilon_x\|\xi\|_{H^1}\right) \\ &+ \mathrm{O}\left(\epsilon_a^2\epsilon_x^3 + ((1+\epsilon_a)\epsilon_x^2 + \epsilon_x\epsilon_t)\epsilon_a\|\xi\|_{H^1} + \epsilon_a\epsilon_x\|\xi\|_{H^1}^2\right).\end{aligned}$$

Adding the above expression to (36) gives an upper bound containing $|\dot{c}|$ and $|\dot{a}-c+b(a)|$. Replacing these quantities using the bound

$$|\dot{c}|+|\dot{a}-c+b(a)| = \mathrm{O}\left(\epsilon_a\epsilon_x + \left(\alpha^{\frac{1}{2}}+\alpha^{-\frac{1}{2}}\epsilon_a\epsilon_x\right)\|\xi_g\|_{H^1} + (\alpha+\epsilon_a\epsilon_x)\|\xi\|_{H^1} + \|\xi\|_{H^1}^2\right)$$

from Proposition 8, and bounding higher order terms by lower order terms gives (35). To use the above bounds on $|\dot{c}|$ and $|\dot{a}-c+b(a)|$ we must assume $\alpha^{-\frac{1}{2}}\|\xi_g\|_{H^1}+\|\xi\|_{H^1}$ is small enough so that Proposition 8 holds. □

8 Bound on the Fluctuation and Proof of Main Theorem

We are now in a position to prove the bound on ξ.

Proposition 10. *Say $u = Q_{ca}+\xi$ is a solution to (1), where ξ satisfies (12). Let ϵ_a, $\epsilon_x \le 1$ and $0 < s < \frac{1}{2}$. Then, if $\epsilon_a\epsilon_x$ is small enough, there are constants C_1, C_2, and C_3, such that if the initial condition u_0 satisfies $\inf_{Q_{ca}\in M_s}\|u_0 - Q_{ca}\|_{H^1} << (\epsilon_a\epsilon_x)^{2s}$, then*

$$\|\xi(t)\|_{H^1} \le C_1(\epsilon_a\epsilon_x)^s \text{ and } \|\xi_g(t)\|_{H^1} \le C_2(\epsilon_a\epsilon_x)^{\frac{3s}{2}},$$

for all times $t \le T_1 := C_3\left((\epsilon_a\epsilon_x)^s+\epsilon_t+\epsilon_x\right)^{-1}$.

Proof. We choose $\epsilon_0 := \inf_{Q_{ca}\in M_s}\|u_0-Q_{ca}\|_{H^1}$ small enough so that $\|\xi(0)\|_{H^1} \le C\alpha^{-\frac{1}{2}}\epsilon_0 < \varepsilon$ (see (15)) is small enough to satisfy the conditions of Proposition 5 and Lemma 9. Then, continuity of the solution $u = Q_{ca}+\xi$ in $H^1(\mathbb{R})$ with respect to time implies the conditions continue to be satisfied over a non-empty time interval $[0,T]$. We will obtain an estimate of $\|\xi(t)\|_{H^1}$ over a time interval $[0,T]$ by deriving an equality for $\|\xi(t)\|_{H^1}$ from upper and lower bounds on the Lyapunov functional. We suppress dependence on t for notational convenience.

Define $\|\xi\|_T := \sup_{[0,T]}\|\xi(t)\|_{H^1}$ and $|\Xi|_T := \sup_{[0,T]}|\Xi|$. Integrating the time maximized upper bound in Lemma 9 gives

$$M_c(t) \le |M_c(0)| + C(\epsilon_a\epsilon_x + \|\xi\|_T^2)|\Xi|_T T$$

for all $t\in[0,T]$. A lower bound is obtained by expanding the $\Lambda_{ca}(Q_{ca}+\xi)$ term in $M_c(t)$ to quadratic order and using $\Lambda'_{ca}(Q_{ca}) = 0$ to obtain

$$M_c(t) = \langle \mathcal{L}_Q\xi,\xi\rangle + N(\xi) + b'(a)\langle \xi,(x-a)Q_{ca}\rangle,$$

where the nonlinear remainder $N(\xi)$ is defined in Appendix C. Estimating $N(\xi)$ with Lemma 12 and using anisotropic coercivity of the Hessian $\mathcal{L}_Q$ (Proposition 7) gives

$$M_c(t) \ge C_3\|\xi_g\|_{H^1}^2 + C\alpha\|\xi_b\|_{H^1}^2 - C(\epsilon_a\epsilon_x\|\xi\|_{H^1} + \|\xi\|_{H^1}^3).$$

Together with the upper bound on $M_c(t)$, this bound implies

$$\|\xi_g\|_{H^1}^2,\ \alpha\|\xi_b\|_{H^1}^2 \le \|\xi_g\|_{H^1}^2 + \alpha\|\xi_b\|_{H^1}^2 \le |M_c(0)| + (\epsilon_a\epsilon_x + \|\xi\|_T^2)|\Xi|_T T + \epsilon_a\epsilon_x\|\xi\|_T + \|\xi\|_T^3. \tag{37}$$

Note that we have set non-essential constants to unity. Since the above inequalities hold for all $t \in [0,T]$, we can replace $\|\xi_b\|_{H^1}^2$ with $\|\xi_b\|_T^2$ and $\|\xi_g\|_{H^1}^2$ with $\|\xi_g\|_T^2$. Multiplying the resulting inequality for $\|\xi_g\|_T^2$ by α and adding to the inequality for $\alpha\|\xi_b\|_{H^1}^2$ gives

$$\alpha\|\xi\|_T^2 \le |M_c(0)| + (\epsilon_a\epsilon_x + \|\xi\|_T^2)|\Xi|_T T + \epsilon_a\epsilon_x\|\xi\|_T + \|\xi\|_T^3 .$$

Next, we take $|\Xi|_T T = \mathrm{O}(\alpha)$ and $\|\xi\|_T = \mathrm{O}(\alpha)$ to obtain the bound

$$\|\xi\|_T \le \alpha^{-\frac{1}{2}}|M_c(0)|^{\frac{1}{2}} + (\epsilon_a\epsilon_x)^{\frac{1}{2}} + \epsilon_a\epsilon_x\alpha^{-1}.$$

The initial value of the Lyapunov functional $M_c(0)$ can be bounded by the $H^1(\mathbb{R})$ norm of the initial fluctuation $\|\xi(0)\|_{H^1} \le C\alpha^{-\frac{1}{2}}\epsilon_0$. Indeed, Taylor expanding $\Lambda_{ca}(Q_{ca}+\xi)$ to second order in ξ, and using the third estimate in Lemma 12 gives $|M(0)| = \mathrm{O}\left(\alpha^{-1}\epsilon_0^2 + \alpha^{-\frac{1}{2}}\epsilon_a\epsilon_x\epsilon_0\right)$ if $\epsilon_0 \le 1$.

By order considerations, if we choose $\alpha = (\epsilon_a\epsilon_x)^s$, with $0 < s < \frac{1}{2}$, and if $\epsilon_0 << (\epsilon_a\epsilon_x)^{2s}$, then the bound $\|\xi\|_T = \mathrm{O}((\epsilon_a\epsilon_x)^s)$ holds for $T = \mathrm{O}(\alpha|\Xi|_T)$. The bound $\|\xi_g\|_T = \mathrm{O}\left((\epsilon_a\epsilon_x)^{\frac{3s}{2}}\right)$ is obtained by substituting the bound for $\|\xi\|_{H^1}$ into second inequality of (37). We obtain the conservative estimate $T_1 = \mathrm{O}\left([(\epsilon_a\epsilon_x)^s + \epsilon_t + \epsilon_x]^{-1}\right)$ by substituting the bounds on $\|\xi\|_T$ and $\|\xi_g\|_T$ into the expression for $|\Xi|_T$. To complete the proof, we take ϵ_0, ϵ_a, and ϵ_x sufficiently small so that the smallness assumptions in all the propositions and lemmas hold. □

We now prove the main theorem.

Proof of Theorem 2. By our choice $\epsilon_0 < \varepsilon$, there is a (maximal) time T_0 such that the solution u in (1) is in U_ε for time $t \le T_0$. Hence decomposition (11) with (12) and Proposition 10 are valid for u and imply the statements of the main theorem and in particular $\|\xi(t)\|_{H^1} \le C_1(\epsilon_a\epsilon_x)^s$ for times $t \le \min\{T_0, T_1\}$. If we take $\epsilon_a\epsilon_x$ such that $C_1(\epsilon_a\epsilon_x)^s < \varepsilon = \mathrm{O}\left((\epsilon_a\epsilon_x)^{\frac{s}{2}}\right)$, then the above bound holds for $t \le T_1$ by maximality of the time T_0. □

Appendices

A Global Wellposedness of the bKdV

In this appendix we prove Theorem 1, global wellposedness of the bKdV equation

$$\partial_t u = -\partial_x(\partial_x^2 u + u^2 + b(t,x)u) \tag{38}$$

in $H^1(\mathbb{R})$, with an appropriate condition on b. We extend the local wellposedness proof of Kenig-Ponce-Vega [29] in the case of $b = 0$, and use an energy argument to extend local wellposedness to global wellposedness. Define

$$\|f\|_{L_X^p L_T^q} := \left(\int_{-\infty}^{\infty}\left(\int_{-T}^{T} f(x,t)^q\,dx\,dt\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}$$

and similarly the $L_T^p L_X^q$, $L_T^p H_X^1$ and $L_T^p W_X^{k,p}$ norms (recall $W^{k,p}$ is the Sobolev space based on $L^p(\mathbb{R})$). Let $\hat{f}$ denote the Fourier transform of f.

We begin with a lemma.

Lemma 11. *Let $g \in C(\mathbb{C})$ and $g' \in L^\infty(\mathbb{C})$ (in distribution). If $b \in L^2(\mathbb{R})$ with $\widehat{b'} \in L^1(\mathbb{R})$, then*

$$\|[g(\partial_x), b]\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \le C \|g'\|_{L^\infty} \left\|\widehat{b'}\right\|_{L^1}.$$

Proof. Writting b as a Fourier integral and interchanging the integral and commutator gives

$$[g(\partial_x), b] = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \hat{b}(k) \left[g(\partial_x), e^{ikx}\right] dk. \tag{39}$$

We require an estimate of $\left[g(\partial_x), e^{ikx}\right] = e^{ikx}\left(e^{-ikx} g(\partial_x) e^{ikx} - g(\partial_x)\right)$. The term enclosed in brackets can be written as an integral:

$$\left[g(\partial_x), e^{ikx}\right] = e^{ikx} \int_0^1 \frac{d}{ds}\left(e^{-ikxs} g(\partial_x) e^{ikxs}\right) ds,$$

or on differentiating and computing in Fourier space,

$$\left[g(\partial_x), e^{ikx}\right] = ike^{ikx} \int_0^1 g'(\partial_x + iks)\, ds.$$

Hence $\left\|\left[g(\partial_x), e^{ikx}\right] f\right\|_{L^2} \le |k| \|g'\|_{L^\infty} \|f\|_{L^2}$ for all $f \in L^2(\mathbb{R})$. Substituting this bound into (39) proves the lemma. □

Proof of Theorem 1. The proof is p power specific ($f = u^p$); we only present the proof for p=2. We begin by proving local wellposedness. Let W be the unitary group generated by $-\partial_x^3$; that is, $W(t) := e^{-\partial_x^3 t}$. Consider the bKdV without the term bu. We rewrite this equation as a fixed point problem $u = \Phi(u)$, where the map Φ is defined by

$$\Phi : v \mapsto W(t)u_0 - \int_0^t W(t-\tau) v \partial_x v \, d\tau.$$

Given an initial condition u_0, Kenig, Ponce and Vega [29] proved that Φ is a strict contraction on a ball $B_{X_T^s}(a)$ of radius $a = a(\|u_0\|_{H^s})$ and centre $v = 0$ in the space X_T^s for all $s > \frac{3}{4}$, where

$$X_T^s := \{v \in C([-T,T], H^s(\mathbb{R})) \,|\, \Lambda^T(v) \le \infty\},$$

and

$$\Lambda_T^s(v) := \|v\|_{L_T^\infty H_X^s} + \|\partial_x v\|_{L_T^4 L_X^\infty} + \||\partial_x|^s \partial_x v\|_{L_X^\infty L_T^2} + (1+T)^{-\rho} \|v\|_{L_X^2 L_T^\infty}.$$

Here $\rho > \frac{3}{2}$. More precisely, given $\|u_0\|_{H^s}$ and $0 < \epsilon \le 1$, there is an a and T such that $u_0 \in B_{X_T^s}(a)$, $\Phi : B_{X_T^s}(a) \to B_{X_T^s}(\epsilon a)$ and $\Lambda_T^s(\Phi(v) - \Phi(\tilde{v})) \le \epsilon \Lambda_T^s(v - \tilde{v})$.

We now formulate (38) as a fixed point problem. Define

$$\Psi : v \mapsto -\int_0^t W(t-\tau)\partial_x(bv)\,d\tau.$$

Then, the bKdV is equivalent to solving $u = \Phi(u) + \Psi(u)$. Thus, if $\Phi + \Psi$ is a strict contraction on some ball $B_{X_T^s}(a)$, then equation (38) has a solution in the same class as (38) with $b = 0$. To prove this we need estimates of $\Lambda_T^s(\Psi(v))$ and $\Lambda_T^s(\Psi(v) - \Psi(\tilde{v}))$ for $v \in B_{X_T^s}(a)$. We will make use of the estimates

$$\left\| |\partial_x|^{\frac{1}{4}} W(t)g(x) \right\|_{L_T^4 L_X^\infty} \leq C \left\| g \right\|_{L_X^2}, \tag{40}$$

$$\left\| \partial_x W(t)g(x) \right\|_{L_X^\infty L_T^2} \leq C \left\| g \right\|_{L_X^2}, \tag{41}$$

$$\left\| W(t)g(x) \right\|_{L_X^2 L_T^\infty} \leq C(1+T)^\rho \left\| g \right\|_{H_X^1}, \tag{42}$$

the proofs of which are given in [29].

We estimate $\Lambda_T^s(\Psi(v))$ for $s = 1$, beginning with $\left\|\Psi(v)\right\|_{L_T^\infty H_X^1}$. The inequality

$$\begin{aligned}\left\| \int_0^t W(t-\tau)\partial_x(bv)\,d\tau \right\|_{L_T^\infty H_X^1} &\leq C \left\| \int_0^t W(t-\tau)\partial_x(bv)\,d\tau \right\|_{L_T^\infty L_X^2} \\ &\quad + C \left\| \int_0^t W(t-\tau)|\partial_x|\partial_x(bv)\,d\tau \right\|_{L_T^\infty L_X^2}\end{aligned}$$

follows from the commutativity of $|\partial_x| = -i\partial_x$ with $W(t-\tau)$. Taking the L_X^2 norm inside the integrals via Minkowski's inequality, and using that $W(t-\tau)$ is unitary gives that

$$\left\|\Psi(v)\right\|_{L_T^\infty H_X^1} \leq C \left\| \int_0^t \left\|\partial_x(bv)\right\|_{L_X^2} + \left\| |\partial_x|\partial_x(bv)\right\|_{L_X^2}\; d\tau \right\|_{L_T^\infty}.$$

We increase the integration domain to $[-T, T]$ and use Hölder's inequality to obtain the estimate

$$\left\|\Psi(v)\right\|_{L_T^\infty H_X^1} \leq C \left(\left\|\partial_x(bv)\right\|_{L_T^2 L_X^2} + \left\| |\partial_x|\partial_x(bv)\right\|_{L_T^2 L_X^2} \right).$$

The bound

$$\left\|\partial_x \Psi(v)\right\|_{L_T^4 L_X^\infty} \leq \left\| \int_{-T}^T \left\|\partial_x W(t-\tau)\partial_x(bv)\right\|_{L_X^\infty}\; d\tau \right\|_{L_T^4}$$

is obtained from $\left\|\partial_x\Psi(v)\right\|_{L_T^4 L_X^\infty}$ by moving the derivative ∂_x and $L^\infty(\mathbb{R})$ norm into the integral over τ, and increasing the domain of integration to $[-T, T]$. Minkowski's inequality then implies

$$\left\|\partial_x \Psi(v)\right\|_{L_T^4 L_X^\infty} \leq \int_{-T}^T \left\|\partial_x W(t-\tau)\partial_x(bv)\right\|_{L_T^4 L_X^\infty}\; d\tau.$$

We use that ∂_x commutes with W and the group properties of W to rewrite this inequality:

$$\left\|\partial_x \Psi(v)\right\|_{L_T^4 L_X^\infty} \leq \int_{-T}^T \left\| |\partial_x|^{\frac{1}{4}} W(t) W(-\tau)\sigma |\partial_x|^{\frac{3}{4}} \partial_x(bv) \right\|_{L_T^4 L_X^\infty}\; d\tau,$$

where σ is multiplication by $i\,sgn(k)$ in Fourier space. The quantity $W(-\tau)\sigma|\partial_x|^{\frac{3}{4}}\partial_x(bv)$ does not depend on time t. Thus, we use estimate (40) and that W and σ preserve the $L^2(\mathbb{R})$ norm to obtain the bound

$$\|\partial_x\Psi(v)\|_{L^4_T L^\infty_X} \le \int_{-T}^{T} \left\| |\partial_x|^{\frac{3}{4}}\partial_x(bv)\right\|_{L^2_X} \, d\tau$$

or $\|\partial_x\Psi(v)\|_{L^4_T L^\infty_X} \le C\left(\|\partial_x(bv)\|_{L^2_T L^2_X} + \||\partial_x|\partial_x(bv)\|_{L^2_T L^2_X}\right)$.

Again, Minkowski's inequality and the properties of W give the bound

$$\||\partial_x|\partial_x v\|_{L^\infty_X L^2_T} \le \int_0^t \left\||\partial_x|W(t)W(-\tau)\partial_x^2(bv)\right\|_{L^\infty_X L^2_T} \, d\tau.$$

Since the same holds with integration over $[-T,T]$ and $W(-\tau)\partial_x^2(bv)$ is independent of time t, (41) implies

$$\||\partial_x|\partial_x v\|_{L^\infty_X L^2_T} \le \|W(-\tau)|\partial_x|\partial_x(bv)\|_{L^1_T L^2_X} \le \||\partial_x|\partial_x(bv)\|_{L^2_T L^2_X}.$$

As above, we find that

$$(1+T)^{-\rho}\|\Psi(v)\|_{L^2_X L^\infty_T} \le (1+T)^{-\rho}\int_{-T}^{T} \|W(t)W(-\tau)\partial_x(bv)\|_{L^2_X L^\infty_T} \, d\tau.$$

Estimate (42) then implies

$$(1+T)^{-\rho}\|\Psi(v)\|_{L^2_X L^\infty_T} \le \|W(-\tau)\partial_x(bv)\|_{L^1_T H^1_X} \le C\left(\|\partial_x(bv)\|_{L^2_T L^2_X} + \||\partial_x|\partial_x(bv)\|_{L^2_T L^2_X}\right).$$

Combining all estimates gives that

$$\Lambda^s_T(\Psi(v)) \le C\left(\|\partial_x(bv)\|_{L^2_T L^2_X} + \||\partial_x|\partial_x(bv)\|_{L^2_T L^2_X}\right), \tag{43}$$

and since $\Psi(v)-\Psi(\tilde v) = \Psi(v-\tilde v)$,

$$\Lambda^s_T(\Psi(v)-\Psi(\tilde v)) \le C\left(\|\partial_x(b(v-\tilde v))\|_{L^2_T L^2_X} + \||\partial_x|\partial_x(b(v-\tilde v))\|_{L^2_T L^2_X}\right). \tag{44}$$

Thus, to prove $\Phi+\Psi$ is a strict contraction we need estimates of the quantities appearing on the right hand side of (43) and (44).

Hölder's inequality gives

$$\begin{aligned}\|\partial_x(bv)\|_{L^2_T L^2_X} &\le \|b'\|_{L^2_T L^\infty_X}\|v\|_{L^\infty_T L^2_X} + \|b\|_{L^2_T L^\infty_X}\|\partial_x v\|_{L^\infty_T L^2_X},\\ \||\partial_x|\partial_x(bv)\|_{L^2_T L^2_X} &\le \|b''\|_{L^2_T L^\infty_X}\|v\|_{L^\infty_T L^2_X} + \|b'\|_{L^2_T L^\infty_X}\|\partial_x v\|_{L^\infty_T L^2_X} + \||\partial_x|(b\partial_x v)\|_{L^2_T L^2_X}.\end{aligned}$$

We need to estimate

$$\||\partial_x|(b\partial_x v)\|_{L^2_T L^2_X} \le \|[|\partial_x|,\, b]\,\partial_x v\|_{L^2_T L^2_X} + \|b|\partial_x|\partial_x v\|_{L^2_T L^2_X}. \tag{45}$$

The first term on the right hand side is bounded using Lemma 11. We obtain that

$$\|[|\partial_x|,\, b]\,\partial_x v\|_{L^2_T L^2_X} \le C\|\widehat{b'}\|_{L^2_T L^1_X}\|\partial_x v\|_{L^\infty_T L^2_X},$$

where $\|\widehat{b'}\|_{L^1_X}$ is the L^1 norm of $\widehat{b'}$ in the frequency variable. Using Hölder's inequality, the second term in (45) is bounded as

$$\|b|\partial_x|\partial_x v\|_{L^2_T L^2_X} = \|b|\partial_x|\partial_x v\|_{L^2_X L^2_T} \leq \|b\|_{L^2_X L^\infty_T} \||\partial_x|\partial_x v\|_{L^\infty_X L^2_T}.$$

Combining all the estimates implies $\Lambda^1_T(\Psi(v)) \leq C\|b\|_{XT}\Lambda^1_T(v)$ and $\Lambda^T(\Psi(v) - \Psi(\tilde{v})) \leq C\|b\|_{XT}\Lambda^1_T(v - \tilde{v})$, where

$$\|b\|_{XT,1} := \|b\|_{L^2_T W^{2,\infty}_X} + \|b\|_{L^2_X L^\infty_T} + \|\widehat{b'}\|_{L^2_T L^1_X}.$$

If $\varepsilon = C\|b\|_{XT,1}$, then the above estimates imply that $\Psi : B_{X^s_T}(a) \to B_{X^s_T}(\varepsilon a)$ and

$$\Lambda^1_T(\Psi(v) - \Psi(\tilde{v})) \leq \varepsilon \Lambda^1_T(v - \tilde{v}). \tag{46}$$

Thus, $\Phi + \Psi : B_{X^s_T}(a) \to B_{X^s_T}((\varepsilon + \epsilon)\, a)$ and $\Lambda^1_T((\Phi+\Psi)(v) - (\Phi+\Psi)(\tilde{v})) \leq (\varepsilon + \epsilon)\Lambda^1_T(v - \tilde{v})$. Furthermore, if we take $\varepsilon + \epsilon < 1$, then $\Phi + \Psi$ is a strict contraction on $B_{X^s_T}(a)$. Invoking the fixed point theorem completes the proof of the local existence and uniqueness.

Kenig, Ponce and Vega [29] also proved that for all $0 < T_1 < T$,

$$\Lambda^1_{T_1}(\Phi_{v_0}(v) - \Phi_{\tilde{v}_0}(\tilde{v})) \leq C\left(\|v_0 - \tilde{v}_0\|_{H^1} + f(T_1)(\Lambda^1_{T_1}(v) + \Lambda^1_{T_1}(\tilde{v}))\Lambda^1_{T_1}(v - \tilde{v})\right),$$

where $f(T_1) \to 0$ as $T_1 \to 0$ and Φ_{v_0} is the map associated to the fixed point problem with initial condition v_0. The map Ψ is independent of initial condition; therefore, the triangle inequality and estimate (46) imply

$$\Lambda^1_{T_1}((\Phi_{v_0} + \Psi_{v_0})(v) - (\Phi_{\tilde{v}_0} + \Psi_{\tilde{v}_0})(\tilde{v})) \leq C\left(\|v_0 - \tilde{v}_0\|_{H^1} + [f(T_1)(\Lambda^1_{T_1}(v) + \Lambda^1_{T_1}(\tilde{v})) + \|b\|_{XT}]\Lambda^1_{T_1}(v - \tilde{v})\right).$$

Let v be a solution to (38) with initial condition v_0 and similarly for $\tilde{v}$. Then, if T_1 and $\|b\|_{XT}$ are small enough, $\Lambda^1_{T_1}(v - \tilde{v}) \leq C\|v_0 - \tilde{v}_0\|_{H^1}$. This proves continuity of the solution with respect to the initial condition and completes the proof of local wellposedness of (38) in $H^1(\mathbb{R})$.

To extend local wellposedness to global wellposedness we require the identities

$$\partial_t H_b(u) = \frac{1}{2}\int_{-\infty}^{\infty} (\partial_t b) u^2 \, dx \text{ and } \partial_t \|u\|^2_{L^2} = \int_{-\infty}^{\infty} b' u^2 \, dx \tag{47}$$

to hold for all $H^1(\mathbb{R})$ solutions to (38). When $u \in H^3(\mathbb{R})$ both of these follow trivially by integration by parts. We appeal to a density argument to prove that the identities continue to hold in $H^1(\mathbb{R})$. Let $\{u_{0n}\}$ be a sequence of initial values in $H^3(\mathbb{R})$ converging in $H^1(\mathbb{R})$ to u_0. Then, if (38) is locally wellposed in $H^3(\mathbb{R})$, there are corresponding solutions $u_n \in C([-T, T], H^3(\mathbb{R}))$ with $u_n(0) = u_{0n}$ and

$$\lim_{n\to 0} \sup_{[-T,T]} \|u_n - u\|_{H^1_X} = 0.$$

We have used that the time interval appearing in the local wellposedness result depends continuously only on the $H^1(\mathbb{R})$ norm of the initial condition. Hence,

$$\lim_{n\to\infty} \partial_t H_b(u_n) = \lim_{n\to\infty} \frac{1}{2}\int_{-\infty}^{\infty} (\partial_t b) u_n^2 \, dx = \frac{1}{2}\int_{-\infty}^{\infty} (\partial_t b) u^2 \, dx.$$

Since $\partial_t H(u_n) \to \partial_t H(u)$ in distribution, the first identity of (47) holds. Similarly,

$$\lim_{n\to\infty} \partial_t \|u_n\|^2_{L^2_X} = \lim_{n\to\infty} \int_{-\infty}^{\infty} b' u_n^2 \, dx = \int_{-\infty}^{\infty} b' u^2 \, dx,$$

and since $\partial_t \|u_n\|^2_{L^2_X} \to \partial_t \|u\|^2_{L^2_X}$ in distribution, the second identity in (47) also holds. The above assumed local wellposedness in $H^3(\mathbb{R})$. The proof of this fact proceeds as above and one finds that (38) is locally wellposed in $H^3(\mathbb{R})$ if

$$\|b\|_{XT,3} := \|b\|_{L^2_T W^{4,\infty}_X} + \|b\|_{L^2_X L^\infty_T} + \|\widehat{b'}\|_{L^2_T L^1_X}$$

is small enough.

We now extend the local result to a global result. The identities of (47) imply

$$\frac{d}{dt}\|u\|^2_{L^2} = \int_{-\infty}^{\infty} b' u^2\,dx \le \epsilon_a\epsilon_x \|u\|^2_{L^2} \text{ and } \partial_t H(u) = \frac{1}{2}\int_{-\infty}^{\infty} (\partial_t b)u^2\,dx \le \frac{\epsilon_a\epsilon_x}{2}\|u\|^2_{L^2}.$$

Integrating the first by Gronwall's inequality implies $\|u\|_{L^2} \le \|u_0\|_{L^2}\exp(\epsilon_a\epsilon_x t)$. Substituting this bound into the above bound on the time derivative of the Hamiltonian and integrating gives

$$\frac{1}{2}\|\partial_x u\|^2_{L^2} \le |H(u_0)| + \frac{\epsilon_t}{2\epsilon_x}\|u_0\|^2_{L^2}\exp(\epsilon_a\epsilon_x t) + \|u\|^2_{L^2} + \frac{1}{2}\left|\int_{-\infty}^{\infty} bu^2\,dx\right|.$$

Using the bound on $\|u\|^2_{L^2}$ again then gives

$$\frac{1}{2}\|\partial_x u\|^2_{L^2} \le |H(u_0)| + \left(1 + \frac{\epsilon_t}{2\epsilon_x} + \frac{1}{2}\epsilon_a\right)\|u_0\|^2_{L^2}\exp(\epsilon_a\epsilon_x t) \tag{48}$$

This inequality implies global existence. Indeed, say there is a time T such that $\lim_{t\to T}\|u\|_{H^1} = \infty$. This clearly contradicts (48). Uniqueness follows from uniqueness of local solutions. □

B Proof of Lemma 3

Commutativity and the relation $\partial_x\partial_\alpha^{-1} = I - \alpha\partial_\alpha^{-1}$ are direct consequences of $(\partial_x + \alpha)\partial_\alpha^{-1} = I$. Commutativity with $\mathcal{S}_a$ is proved using that

$$(\partial_{x-a} + \alpha)^{-1} : g \mapsto e^{-\alpha(x-a)}\int_{-\infty}^{x-a} g(y)e^{\alpha(y-a)}\,d(y-a)$$

and $\partial_x + \alpha = \partial_{x-a} + \alpha$. We prove statements two and five using the above explicit formula with $a = 0$. Indeed, the inequality

$$|\partial_\alpha^{-1}\phi| \le \int_{-\infty}^{\infty} |\phi(x)|\,dx = \|\phi\|_{L^1}$$

gives statement two, and since $e^{\alpha(x-y)} - 1 \le \alpha|x-y|e^{\alpha(x-y)}$, the inequality

$$\left|\langle\phi, \partial_\alpha^{-1}\psi\rangle - \langle\phi, \partial_x^{-1}\psi\rangle\right| \le \alpha\int_{-\infty}^{\infty}|\phi(x)|\int_{-\infty}^{\infty}|\psi(y)|(|x| + |y|)\,dy\,dx.$$

gives statement five if $x\phi$ and $x\psi$ are integrable.

We prove the remaining statements in Fourier space. Let $\hat{\phi}$ be the Fourier transform of ϕ. Plancherel's theorem implies

$$\begin{aligned}\left\|\partial_\alpha^{-1}\phi\right\|_{L^2}^2 &= \int_{-\infty}^{\infty}(k^2+\alpha^2)^{-1}|\hat{\phi}(k)|^2\,dk \\ &= \int_{-\infty}^{\infty}(k^2+\alpha^2)^{-1}|\hat{\phi}(0)|^2\,dk + \int_{-\infty}^{\infty}(k^2+\alpha^2)^{-1}(|\hat{\phi}(k)|^2-|\hat{\phi}(0)|^2)\,dk. \end{aligned} \tag{49}$$

The first equality immediately gives the third statement since $\left\|\hat{\phi}\right\|_{L^\infty} \leq \|\phi\|_{L^1}$. A similar argument gives statement four. To prove the last statement, we concentrate on the second integral of (49) since the first is easily computed to be

$$\frac{\pi}{\alpha}\left(\int_{-\infty}^{\infty}\phi\,dx^2\right).$$

When ϕ, $x\phi$, and $x^2\phi$ are integrable, $\hat{\phi}$ and $|\hat{\phi}|^2$ are twice differentiable. Furthermore, since $|\hat{\phi}|^2$ is even, Taylor's theorem implies $|\hat{\phi}(k)|^2 - |\hat{\phi}(0)|^2 = \mathrm{O}\left(k^2\right)$. Thus, $(k^2+\alpha^2)^{-1}(|\hat{\phi}(k)|^2-|\hat{\phi}(0)|^2)$ is integrable for all $\alpha \in \mathbb{R}_+$ and

$$\int_{-\infty}^{\infty}(k^2+\alpha^2)^{-1}(|\hat{\phi}(k)|^2-|\hat{\phi}(0)|^2)\,dk = \mathrm{O}\left(1\right).$$

This completes the proof.

C Estimates of Nonlinear Remainders

Define

$$N(\xi) := -\int_{-\infty}^{\infty} F(Q_{ca}+\xi) - F(Q_{ca}) - F'(Q_{ca})\xi - \frac{1}{2}F''(Q_{ca})\xi^2\,dx$$

and

$$N'(\xi) := -\left(f(Q_{ca}+\xi) - f(Q_{ca}) - f'(Q_{ca})\xi\right). \tag{50}$$

Note that $N'(\xi) = \partial_\xi N(\xi)$ under the $L^2(\mathbb{R})$ pairing.

Lemma 12. *If $\|\xi\|_{H^1} \leq 1$ and $f \in C^k(\mathbb{R})$ for some $k \geq 3$, with $f^{(k)} \in L^\infty(\mathbb{R})$, then there are positive constants C_1, C_2, and C_3 such that*

1. $\|N'(\xi)\|_{L^2} \leq C_1\|\xi\|_{H^1}^2$,
2. $\left\|N'(\xi) + \frac{1}{2}f''(Q_{ca})\xi^2\right\|_{L^2} \leq C_2\|\xi\|_{H^1}^3$,
3. $|N(\xi)| \leq C_3\|\xi\|_{H^1}^3$.

Proof. Taylor's remainder theorem implies

$$N'(\xi) = -\sum_{n=2}^{k-1}\frac{1}{n!}f^{(n)}(Q_{ca})\xi^n - R(Q_{ca},\xi),$$

where, since $f^{(k)} \in L^\infty(\mathbb{R})$, $|R(Q_{ca}, \xi)| \leq C|\xi|^k$. Recall that Q_{ca} is continuous and decays exponentially to zero. Together with the assumption that $f \in C^k(\mathbb{R})$, this implies $f^{(n)}(Q_{ca}) \in L^\infty(\mathbb{R})$ for $2 \leq n \leq k-1$. Thus, after pulling out the largest constant,

$$\|N'(\xi)\|_{L^2} \leq C \sum_{n=2}^{k} \|\xi^n\|_{L^2}\,.$$

To obtain statement 1 we use the bound $\|\xi^n\|_{L^2} \leq C\,\|\xi\|_{H^1}^n$, which is obtained from the inequality $\|\xi\|_{L^\infty} \leq C\,\|\xi\|_{H^1}$ and the assumption that $\|\xi\|_{H^1} \leq 1$.

Clearly, slight modification of the above proof gives items 2 and 3. For the latter we use that the assumptions on f imply $F \in C^{k+1}(\mathbb{R})$ with $F^{(k+1)} \in L^\infty(\mathbb{R})$. □

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