QUALITATIVE ANALYSIS OF PHASE–PORTRAIT FOR A CLASS OF PLANAR VECTOR FIELDS VIA THE COMPARISON METHOD.

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Abstract. The phase–portrait of the second order differential equation:

$$\ddot{x} + \sum_{l=0}^{n} f_l(x)\dot{x}^l = 0\,,$$

is studied. Some results concerning existence, non–existence and uniqueness of limit cycles are presented. Among these, a generalization of the classical Massera uniqueness result is proved.

1. Introduction

In this paper we investigate the qualitative behavior of the differential equation:

$$\ddot{x} + P(x,\dot{x}) = 0\,, \tag{1.1}$$

where $P(x,y) = \sum_{l=0}^{n} f_l(x)y^l$, for some fixed $n \in \mathbb{N}$, which in the phase–plane can be rewritten as the differential autonomous system:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -P(x,y)\,. \end{cases} \tag{1.2}$$

Such systems have been proposed by H.I. Freedman and Y. Kuang [1] in the study of a Gause–type predator–prey model.

If we assume some standard regularity assumptions on the functions $f_l(x)$, $l \in \{0,\dots,n\}$, for the uniqueness of Cauchy's problem, for instance let $f_0(x) = g(x)$:

A1) $g \in Lip(\mathbb{R})$;
A2) $f_l \in \mathcal{C}(\mathbb{R})$, $l \in \{1,\dots,n\}$;

and the hypothesis:

B) $xg(x) > 0$ for all $x \neq 0$;

then the origin is the only singular point, and trajectories turn clockwise around it.

The problem of existence and uniqueness of limit cycles for system (1.1) will be considered. All examples and applications presented will involve polynomials, because this is, from our point of view, the more interesting situation; we nevertheless observe that all our results hold in a more general setting than polynomial functions.

Clearly equation (1.1) is a particular case of the classical generalized Liénard equation:

$$\ddot{x} + f(x,\dot{x})\dot{x} + g(x) = 0\,,$$

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as well as the equivalent system:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -f(x,y)y - g(x)\,. \end{cases}$$

is a generalization of system (1.2).

We observe that a large number of results, in the huge literature concerning these generalized Liénard equations, are obtained with some assumptions on the growth of $f(x,y)$, for instance:

$$f(x,y) > -M\,,$$

being M some suitable positive constant. This because otherwise trajectories might be not continuable and the uniform boundedness of the trajectories is a crucial step in order to apply the classical Poincaré–Bendixon Theory.

This is not the case of system (1.2) where in general we will consider functions $f_l(x)$, $l \in \{0,\dots,n\}$ to be polynomials, hence most of the classical result cannot be used here.

As far as we know system (1.2) has been studied only in two particular cases (the classical Liénard equation):

$$P_1(x,y) = g(x) + f_1(x)y\,,$$

and

$$P_2(x,y) = g(x) + f_1(x)y + f_2(x)y^2\,,$$

largely studied by several authors in the last decade (see for instance [2, 3, 6, 5, 4]).

We believe that the main reason for this, is the existence of a well known transformation, discussed in section 2, bringing equation:

$$\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = 0\,, \tag{1.3}$$

into some Liénard system. In this way all classical results of existence of limit cycles for the Liénard equation may be used. A second reason, for considering systems (1.3), lies in the fact that its trajectories are always continuable. Of course this can be obtained via the transformation into a Liénard equation, but even in a direct way, considering the slope of the trajectories in the phase–plane.

When one is interested in general cases, these properties are no longer valid.

In this paper we will exploit some geometrical properties of the phase–portrait of system (1.2) and when $P(x,y) = P_2(x,y)$, we will not make use of the transformation of the system into a Liénard one, but we will use a slight different system.

The plan of the paper is the following. In section 2 we will study system (1.2), splitting P into two parts: one exhibiting some suitable symmetries and considering the other as a "perturbation", then some properties will be enlighten using a comparison method. The main result of this section is the following Theorem, which gives a necessary condition for the existence of limit cycles.

Theorem 1.1 (Non–existence)**.** *Let us consider the second order differential equation:*

$$\ddot{x} + \sum_{l=1}^{N} f_{2l-1}(x)\dot{x}^{2l-1} + \sum_{l=1}^{M} f_{2l}(x)\dot{x}^{2l} + g(x) = 0\,, \tag{1.4}$$

where N and M are positive integer. Assume regularity hypotheses A and B to be satisfied by the $f_l(x)$'s and $g(x)$, then systems (1.4) *has not periodic orbits provided all the $f_{2l-1}(x)$'s, $l \in \{1,\dots,N\}$, never change sign.*

Section 3 deal with system (1.3). As already mentioned the study is performed using a new differential system rather than transforming system (1.3) into some Liénard equation. The main result of this section is the following:

Theorem 1.2 (Existence). *Assume hypotheses A and B to hold, let* $F_1(x) = \int_0^x f_1(s)\,ds$, $G(x) = \int_0^x g(s)\,ds$ *and assume:*

C) *there exists* $\delta > 0$ *s.t.* $f_1(x) < 0$ *for* $|x| < \delta$*, but* f_1 *is not always negative;*

D1) *there exists* $c > 0$ *s.t.:*

$$F_1(x) > -c \quad \textit{if } x > 0 \quad \textit{and} \quad F_1(x) < c \quad \textit{if } x < 0\,;$$

D2) $\limsup_{x\to\pm\infty} [G(x) \pm F_1(x)] = +\infty$;

D3) $\lim_{x\to\pm\infty} \int_0^x f_2(s)\,ds = l_\pm > -\infty$,

E) *there exists* $\Delta' > 0$ *s.t. either:*

E1) $f_2(x) > 0$ *for* $x \le -\Delta'$,

E2) $\limsup_{x\to-\infty} \left[F_1(x) + \sqrt{\frac{-g(x)}{f_2(x)}}\right] = L_- < +\infty$;

or

E1') $f_2(x) < 0$ *for* $x \ge \Delta'$,

E2') $\limsup_{x\to+\infty} \left[F_1(x) - \sqrt{\frac{-g(x)}{f_2(x)}}\right] = L_+ > -\infty$.

Then system (1.3) *has at least a periodic orbit.*

Certainly this result can be somehow, directly obtained after transforming the initial equation into a Liénard's one, but however we point out that in our formulation based on a geometric study of the phase–plane, assumptions keep their geometrical meaning, as has been used in [11]. Moreover in some applications, as for instance Theorem 3.3, they can be easily verified.

Some examples of limit cycles for the general case, when there is not global boundedness of solutions, are also presented.

In Section 4 we consider the problem of uniqueness of limit cycles. A classical result due to Massera is generalized as to include the following equation:

$$\ddot{x} + \sum_{l=0}^{N} f_{2l+1}(x)\dot{x}^{2l+1} + x = 0\,, \tag{1.5}$$

for some positive integer N. The Classical Massera result is recovered when $N = 0$.

Theorem 1.3 (Existence and Uniqueness). *Let* $f_{2l+1}(x)$, $l \in \{0,\ldots,N\}$, $N \ge 1$,*verify assumption A and moreover:*

L1) *there exists* $\delta > 0$ *s.t.* $f_1(x) < 0$ *for* $|x| < \delta$;

L2) $f_{2l+1}(x) \ge 0$, $l \in \{1,\ldots,N\}$, *for all* x;

L3) $f_{2l+1}(x)$, $l \in \{0,\ldots,N\}$, *is monotone increasing (respectively decreasing) for* $x > 0$ *(respectively* $x < 0$*).*

Then (1.5) *has a unique globally attracting limit cycle.*

As far as we know there are no other generalizations of classical Massera's result and the problem of adapting his geometrical ideas to a more general situation, still remain an open question. For this reason we believe that this particular result is in some way significant.

Remark 1.4. *Assumptions L1 and L3 for* $f_1(x)$ *don't ensure the existence of the limit cycle in the Massera's case* ($N = 0$)*; on the contrary our assumptions guarantee that* (1.5) *always has a limit cycle. For instance if* $f_1(x) = (x^2 - 1)e^{-x^2}$

in Massera's case the limit cycle doesn't exist [1] *but adding a "perturbation", small as we want, say* $f_3(x) = \epsilon x^2/(x^2+1)$ *with* $\epsilon > 0$*, we ensure existence, and thus uniqueness of the limit cycle.*

2. A non–existence result

The aim of this section is to prove the *non–existence* result Theorem 1.1. Throughout this section regularity hypotheses to guarantee existence and uniqueness of the Cauchy initial problem, will be assumed; also sign assumption B will hold.

We will prove this result by considering firstly the simplest case of (1.3) dealing with only two terms, and secondly the general situation involving several terms.

In the phase–plane equation (1.3) can be rewritten as a first order differential system:

$$\begin{cases} \dot{u} &= v \\ \dot{v} &= -f_1(u)v - f_2(u)v^2 - g(u)\,, \end{cases} \tag{2.1}$$

thus using the well known transformation [6]:

$$\begin{cases} x &= u \\ y &= \Big(v - \int_0^u f_1(s) exp\Big\{ \int_0^s f_2(r)\,dr \Big\}\,ds \Big)\, exp\Big\{ - \int_0^u f_2(r)\,dr \Big\}\,, \end{cases} \tag{2.2}$$

with the rescaling of time:

$$d\tau = exp\Big\{ - \int_0^x f_2(s)\,ds \Big\}\,dt\,, \tag{2.3}$$

system (2.1) can be brought into a Liénard one (still denoting by $\dot{x} = dx/d\tau$):

$$\begin{cases} \dot{x} &= y - \tilde{F}(x) \\ \dot{y} &= -\tilde{g}(x)\,, \end{cases} \tag{2.4}$$

where:

$$\tilde{F}(x) = \int_0^x f_1(s) exp\Big\{ \int_0^s f_2(r)\,dr \Big\}\,ds \quad \text{and} \quad \tilde{g}(x) = g(x) exp\Big\{ 2\int_0^x f_2(r)\,dr \Big\}\,.$$

A large number of results are based on determine hypotheses on $\tilde{F}$ and $\tilde{g}$ to ensure existence, uniqueness or non–existence of limit cycles for system (2.4), and then to transport them to the original system. We will adopt a different point of view, considering system (2.1) as a "perturbation" of the system:

$$\begin{cases} \dot{u} &= v \\ \dot{v} &= -f_2(u)v^2 - g(u)\,, \end{cases}$$

and exploiting the symmetry of the last one w.r.t. $v \mapsto -v$.

The "unperturbed" system, i.e. with $f_1(x) \equiv 0$, is transformed into an Hamiltonian one:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -\tilde{g}(x)\,, \end{cases} \tag{2.5}$$

[1] This claim follows easily studying the Massera case in the associate Liénard plane, where the flow through circles $\{x^2+y^2=r^2\}$ is $-x\int_0^x f_1(s)\,ds$, which doesn't change sign with this choice of $f_1(x)$.

whose Hamilton function is: $H(x, y) = y^2/2 + \tilde{G}(x)$, where $\tilde{G}(x) = \int_0^x \tilde{g}(s)\,ds$. We remark that the origin is a *local center*. It can be a global one according to the behavior of $\tilde{G}(x)$ for large $|x|$.

Remark 2.1. *Restricting to the class of polynomials the origin is a* global center *if and only if $f_2(x)$ is polynomial of odd degree with positive leading coefficient. In all the remaining cases the origin is a local center and at least a separatrix appears.*

The transformation (2.2) is invertible, hence one can bring back to the original plane the energy curves $\{H = const\}$: they will keep the same topology being only slightly distorted in the phase–plane (see for example Figure 1).

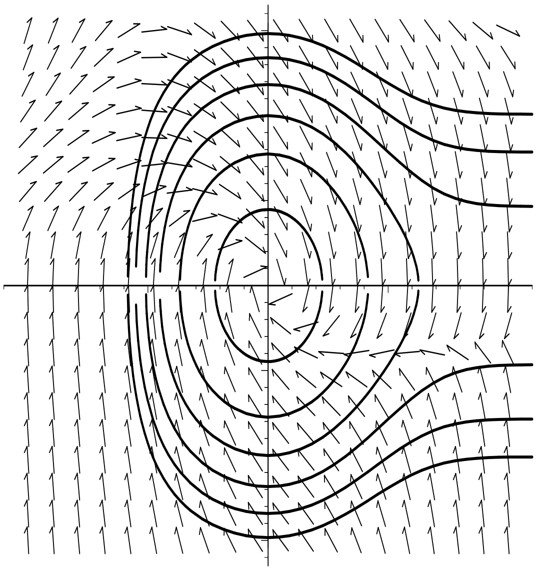

FIGURE 1. $f_1(x) = x^2 + 1$, $f_2(x) = -x^2$, $g(x) = x$, energy levels $\{H = \lambda\}$ with $\lambda \in \{0.1, 0.3, 0.5, 0.7, 0.9, 1.1\}$.

Let us now suppose the perturbation $f_1(x) \not\equiv 0$. We will show, using simple comparison arguments, that system (2.1) cannot have limit cycles if f_1 never changes sign. Let us suppose $f_1(x) > 0$ for all x, and let us compare the *slopes* of system (2.1), when $f_1 \geq 0$, but not identically zero, say case A, and $f_1 \equiv 0$ say case B:

$$\frac{dv}{du}\Big|_A = -f_1(u) + \frac{dv}{du}\Big|_B \leq \frac{dv}{du}\Big|_B .$$

Hence trajectories in case A enter trajectories of case B and no limit cycle can exist. This concludes the proof for (1.3).

We point out that this situation occurs both if (2.5) has a global or local center, namely this situation is not the *perturbation* of a global center.

We remark that this is not a new result, in fact if f_1 keep constant sign, then $\tilde{F}$ vanishes only at $x = 0$, and it is positive (respectively negative) for $x > 0$ (respectively $x < 0$), thus Liénard system (2.4) has not limit cycle and so does the initial system. But our generalization, obtained from Remark 2.2 and Remark 2.3, is to the best of our knowledge new, and it cannot be obtained "passing through some Liénard system" because the previous transformation cannot be anymore extended.

Remark 2.2. *We can allow terms $f_{2l}(x)\dot{x}^{2l}$ into* (1.3)*:*

$$\ddot{x} + f_1(x)\dot{x} + \sum_{l=1}^{M} f_{2l}(x)\dot{x}^{2l} + g(x) = 0\,. \tag{2.6}$$

For such kind of systems there is not generically a transformation mapping them into some Liénard systems. Nevertheless we can prove our non–existence result. A sketch of the proof of this claim is as follows: assume $f_1 \equiv 0$ and consider systems (2.6) *in the phase–plane. Then there is a trivial symmetry $y \mapsto -y$ which allows to prove the existence of a local center. Thus, once we add e perturbation $f_1(x) > 0$ we can use the same reasoning as before, comparing slopes, to prove the non–existence of limit cycles.*

The following remark concludes the proof of Theorem 1.1.

Remark 2.3. *Provided $f_{2l-1}(x) \geq 0$, $l \in \{1,\dots,N\}$ for all x, we can generalize further* (2.6) *by considering:*

$$\ddot{x} + \sum_{l=1}^{N} f_{2l-1}(x)\dot{x}^{2l-1} + \sum_{l=1}^{M} f_{2l}(x)\dot{x}^{2l} + g(x) = 0\,.$$

The proof is straightforward and we omit it.

3. Existence of limit cycles

In this part we will prove two existence results; the first one, Theorem 1.2, is more general for the involved functions but deals with system (1.3), the second one, Theorem 3.3, holds only for polynomials but for a slightly more general equation (3.6).

Theorem 1.2 (Existence)**.** *Assume hypotheses A and B to hold, let $F_1(x) = \int_0^x f_1(s)\,ds$, $G(x) = \int_0^x g(s)\,ds$ and assume:*

- C) *there exists $\delta > 0$ s.t. $f_1(x) < 0$ for $|x| < \delta$, but f_1 is not always negative;*
- D1) *there exists $c > 0$ s.t.:*

$$F_1(x) > -c \quad \text{if } x > 0 \quad \text{and} \quad F_1(x) < c \quad \text{if } x < 0\,;$$

- D2) $\limsup_{x\to\pm\infty}\left[G(x) \pm F_1(x)\right] = +\infty$*;*
- D3) $\lim_{x\to\pm\infty}\int_0^x f_2(s)\,ds = l_\pm > -\infty$*,*
- E) *there exists $\Delta' > 0$ s.t. either:*
  - E1) *$f_2(x) > 0$ for $x \leq -\Delta'$,*
  - E2) $\limsup_{x\to-\infty}\left[F_1(x) + \sqrt{\frac{-g(x)}{f_2(x)}}\right] = L_- < +\infty$*;*

  or
  - E1') *$f_2(x) < 0$ for $x \geq \Delta'$,*
  - E2') $\limsup_{x\to+\infty}\left[F_1(x) - \sqrt{\frac{-g(x)}{f_2(x)}}\right] = L_+ > -\infty$.

Then system (1.3) *has at least a periodic orbit.*

Remark 3.1 (About hypothesis D3)**.** *We point out that hypothesis D3 can be removed, but in that case some growth conditions on $f_1(x)$ and/or $g(x)$ have to be assumed. The exact ones would be clear in the proof of Theorem 1.2. Roughly* *speaking if* $\int_0^x f_2(s)\,ds$ *diverges to* $-\infty$ *as some odd power,* x^{2k+1}*, for* $x \to -\infty$*, then* $f_1(x)$ *has to grow faster than* $e^{x^{2k+1}}$ *to "compensate" this divergence. If not the intersection property with the* ∞*–isocline is not guarantee, thus there can be non–winding trajectories.*

Remark 3.2 (Polynomial case)**.** *The sign assumptions on* f_2 *(conditions E1 or E1') and hypothesis D3 cannot be simultaneously verified if* $f_2(x)$ *is a polynomial. Hence Theorem 1.2 cannot be applied to the important class of polynomials. Nevertheless, we will provide in § 3.2 a result (Theorem 3.3) for the following second order differential equation with polynomial coefficients:*

$$p(x)\ddot{x} + p(x)q_1(x)\dot{x} + q_2(x)\dot{x}^2 + r(x) = 0\,,$$

which is strictly related to (1.3)*.*

3.1. **Proof of Theorem 1.2.** As already remarked in the previous section, system (1.3) rewrites, in the phase–plane:

$$\begin{cases} \dot{u} &= v \\ \dot{v} &= -f_1(u)v - f_2(u)v^2 - g(u)\,. \end{cases} \tag{3.1}$$

Let us introduce [8] a new systems of coordinates $x = u$ and $y = v + F_1(u)$, such that the previous phase–plane system rewrites as:

$$\begin{cases} \dot{x} = y - F_1(x) \\ \dot{y} = -f_2(x)(y - F_1(x))^2 - g(x)\,. \end{cases} \tag{3.2}$$

Hypothesis C ensures that the origin is a *repellor*, in fact looking at the flow through the ovals $\mathcal{O}_r = \{y^2/2 + G(x) = r^2\}$ and r is a small positive number, we get:

$$\frac{d}{dt}\mathcal{O}_r\Big|_{flow} = -g(x)F_1(x) - yf_2(x)\left(y - F_1(x)\right)^2\,,$$

and the claim follows remarking that the last term in the right hand side is of higher order than $-g(x)F_1(x)$ close enough to the origin, thus the sign is determined by the first one.

Thus to prove the existence of a closed trajectory is enough to prove the existence of a solution spiraling toward the origin.

First of all we want to guarantee that all trajectories intersect the curve $y = F_1(x)$. For Liénard system one has a necessary and sufficient condition to guarantee such property, moreover as already remarked in the previous section, system (3.2) is equivalent, through the transformation (2.2) and the rescaling of time (2.3), to a Liénard one :

$$\begin{cases} \dot{\xi} &= \eta - \tilde{F}(\xi) \\ \dot{\eta} &= -\tilde{g}(\xi)\,, \end{cases} \tag{3.3}$$

where:

$$\tilde{F}(\xi) = \int_0^\xi f_1(s)exp\Big\{\int_0^s f_2(r)\,dr\Big\}\,ds \quad \text{and} \quad \tilde{g}(\xi) = g(\xi)exp\Big\{2\int_0^\xi f_2(r)\,dr\Big\}\,.$$

We observe that the curve $y = F_1(x)$ corresponds in the phase–plane (3.1) to the curve $v = 0$, which in the Liénard plane (3.3) transform into $\eta = \tilde{F}(\xi)$, hence intersection with $y = F_1(x)$ occurs if and only if intersection with $\eta = \tilde{F}(\xi)$ occurs.

Let $\tilde{G}(\xi) = \int_0^\xi \tilde{g}(s)\, ds$. Assuming conditions B and D1 one can apply the following result [10]:

- for any $\xi_0 \geq 0$ and $\eta_0 > \tilde{F}(\xi_0)$, the solution of (3.3) passing through (ξ_0, η_0) will intersect $\eta = \tilde{F}(\xi)$ at some $\left(\xi', \tilde{F}(\xi')\right)$, with $\xi' > \xi_0$, if and only if:

$$\limsup_{\xi \to +\infty} \left[\tilde{G}(\xi) + \tilde{F}(\xi)\right] = +\infty\,; \tag{3.4}$$

- for any $\xi_0 < 0$ and $\eta_0 < \tilde{F}(\xi_0)$, the solution of (3.3) passing through (ξ_0, η_0) will intersect $\eta = \tilde{F}(\xi)$ at some $\left(\xi', \tilde{F}(\xi')\right)$, with $\xi' < \xi_0$, if and only if:

$$\limsup_{\xi \to -\infty} \left[\tilde{G}(\xi) - \tilde{F}(\xi)\right] = +\infty\,. \tag{3.5}$$

We claim that from hypotheses D2 and D3 conditions (3.4) and (3.5) follow. Let us look for example to $\tilde{G}(\xi)$ assuming $\int_0^{+\infty} f_2(s)\, ds = l_+$ to be finite and positive, then by definition of limit there exists $M > 0$ s.t.:

$$\frac{l_+}{2} \leq \int_0^x f_2(s)\, ds \leq \frac{3l_+}{2}\,,$$

for all $x \geq M$. Let $\xi > M$, by definition:

$$\tilde{G}(\xi) = \int_0^M g(x) e^{-2\int_0^x f_2(s)\, ds}\, dx + \int_M^\xi g(x) e^{-2\int_0^x f_2(s)\, ds}\, dx\,,$$

thus the first term is a constant, depending on M, whereas the second can be estimated by:

$$e^{-3l_+} \int_M^\xi g(x)\, dx \leq \int_M^\xi g(x) e^{-2\int_0^x f_2(s)\, ds}\, dx \leq e^{-l_+} \int_M^\xi g(x)\, dx\,.$$

Finally putting everything together we obtain, for all $\xi > M$, the bound for $\tilde{G}(\xi)$:

$$\tilde{G}(M) + e^{-3l_+}\left(G(\xi) - G(M)\right) \leq \tilde{G}(\xi) \leq \tilde{G}(M) + e^{-l_+}\left(G(\xi) - G(M)\right)\,,$$

which implies that $G(\xi)$ and $\tilde{G}(\xi)$ have the same behavior for large ξ.

The other cases can be handled similarly and we omit them.

In this way we have the intersection property: trajectory of (3.2) passing through (x_0, y_0), where $x_0 \geq 0$ and $y_0 > F_1(x_0)$, will intersect the curve $y = F_1(x)$ at some $(x', F_1(x'))$ with $x' > x_0$. And similarly for the one starting at (x_0, y_0), where $x_0 < 0$ and $y_0 < F_1(x_0)$.

To conclude the proof we must control that not all trajectories escape. To do this we will study the *zero–isocline* curves which are explicitly given by:

$$y^{\pm}_{isoc}(x) = F_1(x) \pm \sqrt{-\frac{g(x)}{f_2(x)}}\,.$$

Let us assume hypothesis E1 to hold (the case for E1' can be handle similarly and we will omit it). Because of the sign assumptions on $f_2(x)$ and $g(x)$, $y^{\pm}_{isoc}(x)$ are well defined for x belonging to $(-\infty, -\Delta') \cup \mathcal{I}$, where $\mathcal{I}$ is the union of open disjoint intervals, whose endpoints are zeros [2] of $f_2(x)$. This gives rise to (possible several) branches for the zero–isocline curves. Let $y^{+}_{left}(x)$ be the leftmost one above

[2] If $f_2(0) = 0$, because also $g(0) = 0$, the point 0 will be an endpoint of some interval in $\mathcal{I}$ according to the behavior of $g(x)/f_2(x)$ close to $x = 0$.

$y = F_1(x)$, defined for $x < -\Delta'$. Such branch can be used, assuming hypothesis E2, to prove the existence of a spiraling orbit toward the origin.

Before to prove this fact, we have to control the continuability of solutions. We claim that for all $\bar{\alpha} < \bar{\beta} < 0$ (respectively $0 < \bar{\alpha}' < \bar{\beta}'$) and for all $\bar{y} > 0$ (respectively $\bar{y}' < 0$), the solution of (3.2) with initial datum $(\bar{\alpha}, \bar{y})$ will intersect the line $x = \bar{\beta}$ in finite time (respectively the solution of (3.2) with initial datum $(\bar{\beta}', \bar{y}')$ will intersect the line $x = \bar{\alpha}'$). To see this just consider the system in the phase–plane and recall that in this transformation vertical lines are mapped on vertical lines. The slope of such a solution is $dv/du = -f_1(u) - f_2(u)v - g(u)/v$, hence if $|v|$ is large enough, this slope is sub–linear, giving rise to the continuability from $\bar{\alpha}$ to $\bar{\beta}$. Moreover, the explicit solution is bounded by some exponential function which is the responsible for the "strong" distortion of the limit cycles shape (see Figure 3).

We are now able to conclude our proof. Let us consider an orbit starting at $x_0 < -\Delta'$, $y_0 = y^+_{left}(x_0)$, recall that we are assuming conditions E1 and E2 to hold, if instead of those we were assuming hypotheses E1' and E2' we should start from the rightmost branch of the zero–isocline below $y = F_1(x)$ from some $x_0 > \Delta'$. Such trajectory will intersect the y axis by the previous discussion about continuability (that's arc AB in Figure 2). Then, thanks to condition (3.4) it will reach the curve $y = F_1(x)$ passing eventually through branches of zero–isocline (arcs BC and CD in Figure 2). On $y = F_1(x)$, $\dot{x}$ changes sign and the solution goes back till intersecting again the y axis at some negative ordinate (arcs DE and EF in Figure 2). Using now (3.5) we are able to prove that this trajectory will reach again the curve $y = F_1(x)$ at some $x < 0$, where again $\dot{x}$ changes sign. Before reaching this curve, it has to intersect a branch of zero–isocline below the curve $y = F_1(x)$, hence $\dot{y} > 0$. Because this trajectory cannot intersect the leftmost branch of $y^+_{left}(x)$ upon which the vector field points to the right, it will reach anew the quadrant from which it started spiraling toward the origin when doing a tour around it. Thus it can be used as outer boundary of a Poincaré–Bendixon domain and this concludes the proof.

3.2. **The polynomial case.** As already remarked Theorem 1.2 cannot be applied to the class of polynomials, in fact the sign assumptions (conditions E1 or E1') and hypothesis D3 cannot be simultaneously verified by polynomials. Nevertheless we can prove a similar result for a slightly different second order differential equation with polynomial coefficients.

Theorem 3.3. *The second order differential equation:*

$$p(x)\ddot{x} + p(x)q_1(x)\dot{x} + q_2(x)\dot{x}^2 + r(x) = 0\,, \tag{3.6}$$

where p, $(q)_{j=1,2}$ and r are real polynomials *of the x variable, has at least a limit cycle provided the following conditions hold:*

H1) *$p(x) > 0$ for all x, $q_1(0) < 0$ and $xr(x) > 0$ for $x \neq 0$;*
H2) *$q_1(x)$ has even degree and the leading coefficient is positive;*
H3) $\deg p(x) \geq \deg q_2(x) + 2$*;*
H4) *if $q_2(x)$ has odd degree then the leading coefficient must be negative;*
H5) $\deg r(x) \leq 2\deg q_1(x) + \deg q_2(x) + 1$.

We observe that the special form for the coefficient of $\dot{x}$ is not a restrictive one, we choose it only to state hypotheses of Theorem 3.3 in a simplest and more clear

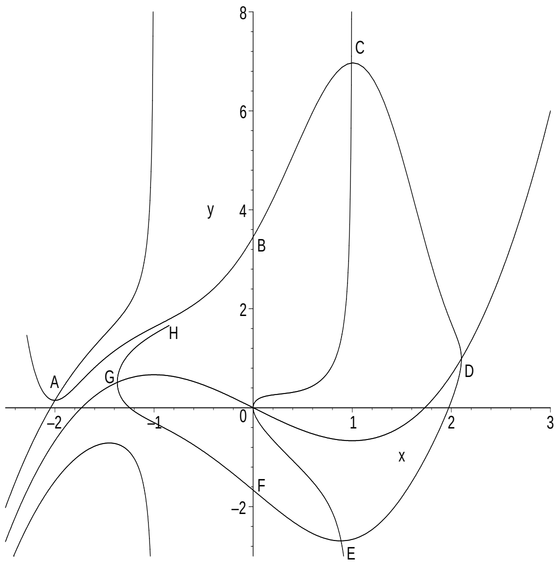

FIGURE 2. The zero–isocline curves and the ∞–isocline curve $y = F_1(x)$, with a trajectory spiraling toward the origin.

way than using some polynomial $\tilde{q}_1(x)$ instead of the factorization $p(x)q_1(x)$. In this way, the applicability of the present result is straightforward: just look at the polynomials and compare their degrees.

Proof. Because $p(x)$ never vanishes we can divide (3.6) by it, then we set $f_1(x) = q_1(x)$, $f_2(x) = \frac{q_2(x)}{p(x)}$ and $g(x) = \frac{r(x)}{p(x)}$. The proof will be complete once we show that f_1, f_2 and g verify hypotheses of Theorem 1.2.

Existence and uniqueness of the Cauchy initial value problem associates to (3.6) is trivially guaranteed because we are dealing with polynomials.

Signs assumption H1 ensures that hypothesis B holds and by continuity of $q_1(x)$ also the first part of assumption C is verified. Also the second part of it holds, because q_1 has even degree and positive leading coefficient.

Hypotheses D1 and D2 are verified because the primitive of $q_1(x)$ is an odd degree polynomial with positive leading coefficient and moreover the primitive of $g(x)$ vanishing at $x = 0$ is not negative.

By H3 it follows that the integral $\int_0^x q_2(s)/p(s)\,ds$ is well defined and finite for all x.

Let α (respectively β) be the smallest (respectively largest) zero of $q_2(x)$. If q_2 has even degree and positive leading coefficient, then $q_2(x)$, hence $f_2(x)$, is positive for all $x < \alpha$. Moreover let $Q_1(x) = \int_0^x q_1(s)\,ds$, then for $x \to -\infty$:

$$Q_1(x) + \sqrt{\frac{-r(x)}{q_2(x)}} \sim x^{\deg q_1 + 1}\left(a + \sqrt{\frac{b}{c}}|x|^{\frac{\deg r - \deg q_2}{2} - \deg q_1 - 1}\right),$$

where a, b and c are the positive [3] leading coefficient of respectively Q_1, r and q_2. Hence assuming H5 hypothesis E2 is verified.

If q_2 has even degree and negative leading coefficient, then $q_2(x)$, hence $f_2(x)$, is negative for all $x > \beta$. Moreover for $x \to +\infty$ one has:

$$Q_1(x) - \sqrt{\frac{-r(x)}{q_2(x)}} \sim x^{\deg q_1 + 1} \left(a - \sqrt{\frac{b}{c}} |x|^{\frac{\deg r - \deg q_2}{2} - \deg q_1 - 1} \right) ,$$

and again H5 ensures hypothesis E2' to hold.

Finally if q_2 has odd degree and negative leading coefficient, then both couples of hypotheses E1, E2 and E1', E2' hold with $\Delta' = \max\{|\alpha|, |\beta|\}$. $\square$

In the following Figure 3 we show some "exotic" limit cycle whose existence is guaranteed by this last Theorem.

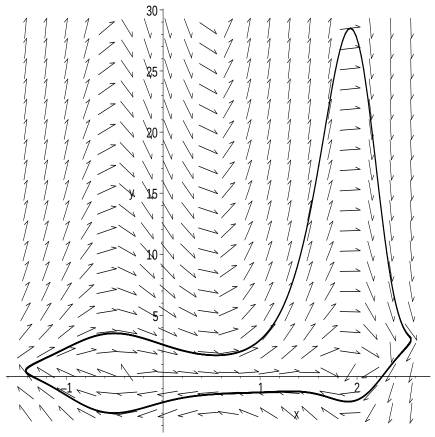

FIGURE 3. We show an exotic limit cycle, whose existence is proved by Theorem 3.3. $p(x) = (x/3)^6 + 1$, $q_1(x) = x^2 - 1$, $q_2(x) = x^4 - 4x^2 + 1$ and $r(x) = x$.

To conclude this section let us discuss briefly the general situation where there are few results about existence of limit cycles. This is because, as already mentioned, one cannot always have continuability of solutions, hence the Poincaré–Bendixon Theorem, as usually done in Literature, cannot be applied "in large". Nevertheless, using standard Hopf's bifurcation argument, we are able to produce examples with small amplitude limit cycles, even if global continuability is not allowed.

To fix the ideas, let us consider the following example:

$$\ddot{x} + (ax^2 - b)\dot{x} + (x^2 + 1)\dot{x}^2 + x^3\dot{x}^3 + x = 0 ;$$

[3]Assumption on the sign of r implies that r cannot have even degree nor negative leading coefficient.

phase–space investigation shows that the slope is superlinear, hence not all trajectories are continuable. Near the origin the dynamics is dominated by the $\dot{x}$ term, if $a > 0$ and $b = 0$ the origin is a local attractor. For positive, suitable small b a Hopf's bifurcation takes place: the origin changes its stability and a small amplitude limit cycle appears, while "in the large" orbits behavior doesn't change too much.

The idea can be generalized as to include other situations, where always small amplitude limit cycles exist and there is not global continuability of solutions.

4. A generalization of a Theorem by Massera

4.1. **Introduction.** Massera's Theorem (see [7] but also [9] pages 307–308) concerns the problem of existence and uniqueness of limit cycles for the second order differential equation (1.5), which in phase–space can be rewritten as:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -f_1(x)y - x \,. \end{cases} \tag{4.1}$$

Under the assumptions: $f_1(x) < 0$ for $|x| < \delta$, for some positive δ, and f_1 increasing (respectively decreasing) for $x > 0$ (respectively $x < 0$), Massera proved that if a limit cycle exists, then it is globally attracting, hence unique. The proof is based on the "geometrical" construction of the so called *geodesic system*, according to Z.–F. Zhang [12].

The previous assumptions on $f_1(x)$ do not guarantee the existence of the limit cycle; an easy calculation (see for instance footnote 1 at page 4) shows that system (4.1) has not limit cycles at all, if for all $\delta_1 < \delta_2$ one has: $\int_{\delta_1}^{\delta_2} f_1(x)\,dx < 0$. We will see in a while that this situation cannot occur in our case. See Remark 1.4 and Figure 5 for an explicit example.

Generalizations of this Theorem, for instance by replacing $\dot{y} = -f_1(x)y - x$ with $\dot{y} = -f_1(x)y - g(x)$, are not available. Here we propose a new point of view considering the following system:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -f_1(x)y - \sum_{l=1}^{N} f_{2l+1}(x)y^{2l+1} - x \,, \end{cases} \tag{4.2}$$

under the additional hypothesis: $f_{2l+1}(x) \geq 0$ for all x and $f_{2l+1}(x)$ is increasing (respectively decreasing) for $x > 0$ (respectively $x < 0$).

We point out that system (4.2) admits a limit cycle even if system (4.1) doesn't, moreover if the classical Massera system has a unique limit cycle then it must contain in its interior the one of (4.2).

4.2. **Existence of limit cycles.** Let us consider the circles $\mathcal{C}_r = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = r^2\}$, we claim that the flow of (4.2) is transversal (more precisely it points outward) to $\mathcal{C}_r$ for r small enough, hence the origin is a *repellor*.

To prove this claim let us consider the angle between $\mathcal{C}_r$ and the vector field $X = (y, -f_1(x)y - \sum_{l=1}^{N} f_{2l+1}(x)y^{2l+1} - x)$ valued at $\mathcal{C}_r$:

$$\alpha_r =< \nabla\mathcal{C}_r, X > \Big|_{\mathcal{C}_r} = -2y^2 \left(f_1(x) + \sum_{l=1}^{N} f_{2l+1}(x)y^{2l} \right) ,$$

thus if $r > 0$ is small enough, f_1 is negative and the remaining positive terms cannot compensate it.

On the other hand if r is large enough then $\alpha_r < 0$, in fact its leading term, $-2y^{2N+2} f_{2N+1}(x)$, is negative because of the positiveness of $f_{2N+1}(x)$. Hence by Poincaré–Bendixon's Theorem we have constructed the inner and outer boundaries of an invariant domain which must contain at least one limit cycle.

4.3. **The cycle is star–shaped.** A fundamental assumption for the proof of Massera's Theorem is that the limit cycle must be *star–shaped* w.r.t. the origin. Z.–F. Zhang (see [12] pages 246–253) pointed out, that generalizations of Massera's Theorem following the geometrical ideas of geodetic systems, must rely on this assumptions.

The aim of this section is to prove that this condition is verified for system (4.2).

Let assume the limit cycle to be not star–shaped w.r.t. the origin, thus there is at least a half–line from the origin intersecting the cycle at least at two points. Starting from the point A on the y–axis the vector field points right–down, $\dot{x} > 0$ and $\dot{y} < 0$ if x is small enough, hence the geometry must be as in Figure 4.

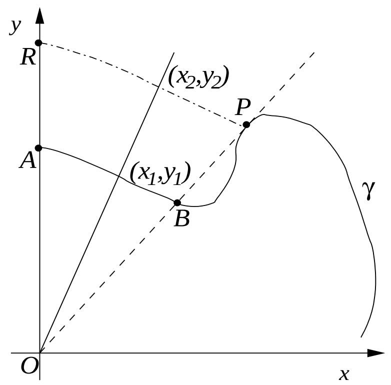

FIGURE 4. Part of a non star–shaped limit cycle.

Let $\lambda = \overline{PO}/\overline{BO}$ and let us construct the arc RP homothetic to AB, with ratio λ. Comparing the slopes of the vector field on these two arcs we will show a contradiction, hence the limit cycle must be star–shaped. Take any point (x_1, y_1) on the arc AB and its homothetic one (x_2, y_2), let $y = mx$ be the straight line passing through these two points. We can assume $x_2 > x_1 > 0$. Let us compare the slopes at (x_1, mx_1) and (x_2, mx_2):

$$\begin{aligned} \frac{dy}{dx}(x_2, mx_2) &= -f_1(x_2) - \sum_{l=1}^{N} f_{2l+1}(x_2)(mx_2)^{2l} - m \\ &< -f_1(x_1) - \sum_{l=1}^{N} f_{2l+1}(x_1)(mx_1)^{2l} - m = \frac{dy}{dx}(x_1, mx_1)\,, \end{aligned} \tag{4.3}$$

where we used the fact that the functions $x \mapsto -(mx)^{2l} f_{2l+1}(x)$, $l \in \{0, \ldots, N\}$ are decreasing for $x > 0$. This implies that orbits starting on the arc RP move toward the arc AB. But this gives rise to a contradiction, in fact for these orbits we have $\dot{x} > 0$, they cannot accumulate to a point somewhere in the generalized quadrilateral with vertices (x_1, y_1), B, P and (x_2, y_2) because the origin is the only fixed point, thus they must intersect the limit cycle somewhere between points A and P, which is impossible because the system is autonomous and Cauchy's uniqueness and existence assumptions are guaranteed.

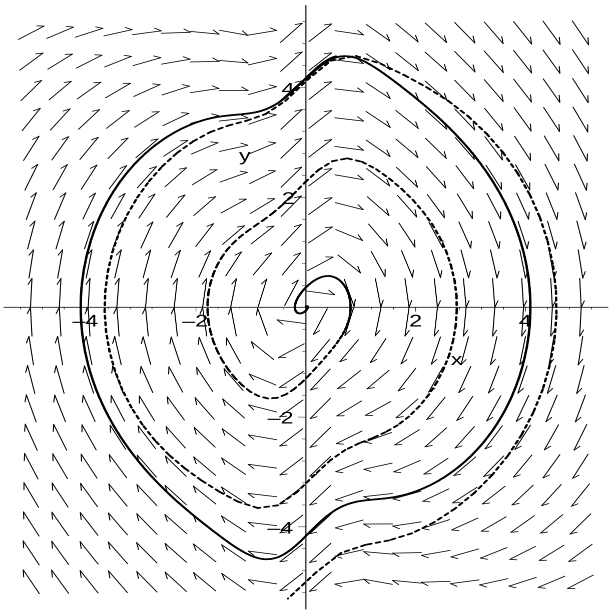

FIGURE 5. The limit cycle for system (4.2) with: $f_1(x) = (x^2 - 1)e^{-x^2}$, $f_3(x) = x^2/50(x^2+1)$ (solid line), and the spiraling orbit for the classical Massera (4.1) with initial datum $(0, 0.01)$ (dashed line).

4.4. **Uniqueness of limit cycle.** In this paragraph we will prove uniqueness of the limit cycle, by showing its hyperbolicity. In our proof we will follow closely the original ideas of Massera.

Let us take a limit cycle, γ_1, and consider its homothetic by a factor $\mu > 1$. Take a point $(x_1, y_1) \in \gamma_1$ and its homothetic one $(x_2, y_2) \in \gamma_2$, let $y = mx$ be the straight line passing through these two points. Comparing again the slopes, as we did in (4.3), we get:

$$\frac{dy}{dx}(x_2, mx_2) \leq \frac{dy}{dx}(x_1, mx_1)\,,$$

for all x_1 and x_2. Here the equal sign appears because we are considering also $x_1 = 0$. Hence trajectories starting on γ_2 will go inward and accumulating to γ_1.

Similarly, considering γ_3 homothetic of γ_1 by a factor $0 < \mu < 1$, and comparing again slopes we will obtain that γ_1 is internally attracting. This proves that γ_1 is globally hyperbolic, hence unique.

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