Cantor families of periodic solutions for completely resonant nonlinear wave equations

Massimiliano Berti[*], Philippe Bolle[†]

Abstract: We prove existence of small amplitude, $2\pi/\omega$-periodic in time solutions of completely resonant nonlinear wave equations with Dirichlet boundary conditions, for any frequency ω belonging to a Cantor-like set of positive measure and for a new set of nonlinearities. The proof relies on a suitable Lyapunov-Schmidt decomposition and a variant of the Nash-Moser Implicit Function Theorem. In spite of the complete resonance of the equation we show that we can still reduce the problem to a *finite* dimensional bifurcation equation. Moreover, a new simple approach for the inversion of the linearized operators required by the Nash-Moser scheme is developed. It allows to deal also with nonlinearities which are not odd and with finite spatial regularity.

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1 Introduction and main result

We consider the *completely resonant* nonlinear wave equation

$$\begin{cases} u_{tt} - u_{xx} + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{1}$$

where the nonlinearity $f(x,u) = a_p(x)u^p + O(u^{p+1})$ with $p \geq 2$ is analytic with respect to u but has only finite regularity with respect to x. More precisely, we assume

(H) There is $\rho > 0$ such that $\forall (x,u) \in (0,\pi) \times (-\rho,\rho)$, $f(x,u) = \sum_{k=p}^{\infty} a_k(x)u^k$, $p \geq 2$, where $a_k(x) \in H^1((0,\pi), \mathbf{R})$ and $\sum_{k=p}^{\infty} ||a_k||_{H^1} r^k < \infty$ for any $r \in (0,\rho)$.

We look for small amplitude, $2\pi/\omega$-periodic in time solutions of equation (1) for all frequency ω close to 1 and in some set of *positive measure*.

Equation (1) is an infinite dimensional Hamiltonian system possessing an elliptic equilibrium at $u = 0$. The frequencies of the linear oscillations at 0 are $\omega_j = j$, $\forall j = 1, 2, \ldots$, and satisfy *infinitely many resonance* relations. Any solution $v = \sum_{j \geq 1} a_j \cos(jt + \theta_j) \sin(jx)$ of the linearized equation at $u = 0$,

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{2}$$

is 2π-periodic in time. For this reason equation (1) is called a completely resonant Hamiltonian PDE.

Existence of periodic solutions close to a completely resonant elliptic equilibrium for finite dimensional Hamiltonian systems has been proved by Weinstein [25], Moser [21] and Fadell-Rabinowitz [13]. The proofs are based on the classical Lyapunov-Schmidt decomposition which splits the problem into

[*]SISSA, Via Beirut 2-4, 34014, Trieste, Italy, berti@sissa.it.

[†]Département de mathématiques, Université d'Avignon, 33, rue Louis Pasteur, 84000 Avignon, France, philippe.bolle@univ-avignon.fr.

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two equations: the *range equation*, solved through the standard Implicit Function Theorem, and the *bifurcation equation*, solved via variational arguments.

For proving existence of small amplitude periodic solutions of completely resonant Hamiltonian PDEs like (1) two main difficulties must be overcome:

- (i) a "*small denominators*" problem which arises when solving the range equation;
- (ii) the presence of an *infinite dimensional* bifurcation equation: which solutions v of the linearized equation (2) can be continued to solutions of the nonlinear equation (1)?

The appearance of the small denominators problem (i) is easily explained: the eigenvalues of the operator $\partial_{tt} - \partial_{xx}$ in the spaces of functions $u(t,x)$, $2\pi/\omega$-periodic in time and such that, say, $u(t,.) \in H_0^1(0,\pi)$ for all t, are $-\omega^2 l^2 + j^2$, $l \in \mathbf{Z}$, $j \geq 1$. Therefore, for almost every $\omega \in \mathbf{R}$, the eigenvalues accumulate to 0. As a consequence, for most ω, the inverse operator of $\partial_{tt} - \partial_{xx}$ is unbounded and the standard Implicit Function Theorem is not applicable.

The first existence results for small amplitude periodic solutions of (1) have been obtained in[2] [19] for the nonlinearity $f(x,u) = u^3$, and in [2] for $f(x,u) = u^3 + O(u^5)$, imposing on the frequency ω the "strongly non-resonance" condition $|\omega l - j| \geq \gamma/l$, $\forall l \neq j$. For $\gamma > 0$ small enough, the frequencies ω satisfying such condition accumulate to $\omega = 1$ but form a set $\mathcal{W}_\gamma$ of zero measure. For such ω's the spectrum of $\partial_{tt} - \partial_{xx}$ does not accumulate to 0 and so the small divisor problem (i) is by-passed. Next, problem (ii) is solved by means of the Implicit Function Theorem, observing that the 0^{th}-order bifurcation equation (which is an approximation of the exact bifurcation equation) possesses, for $f(x,u) = u^3$, non-degenerate periodic solutions, see [3].

In [4]-[5], for the same set $\mathcal{W}_\gamma$ of strongly non-resonant frequencies, existence and multiplicity of periodic solutions has been proved for any nonlinearity $f(u)$. The novelty of [4]-[5] was to solve the bifurcation equation via a variational principle at fixed frequency which, jointly with min-max arguments, enables to find solutions of (1) as critical points of the Lagrangian action functional.

Unlike [2]-[4]-[5], a new feature of the results of this paper is that the set of frequencies ω for which we prove existence of $2\pi/\omega$-periodic in time solutions of (1) has positive measure, actually has full density at $\omega = 1$.

The existence of periodic solutions for a set of frequencies of positive measure has been proved in [9] in the case of periodic boundary conditions in x and for the nonlinearity $f(x,u) = u^3 + \sum_{4 \leq j \leq d} a_j(x) u^j$ where the $a_j(x)$ are trigonometric cosine polynomials in x. The nonlinear equation $u_{tt} - u_{xx} + u^3 = 0$ possesses a continuum of small amplitude, analytic and non-degenerate periodic solutions in the form of travelling waves $u(t,x) = \delta p_0(\omega t + x)$ where $\omega^2 = 1 + \delta^2$ and p_0 is a non-trivial 2π-periodic solution of the ordinary differential equation $p_0'' = -p_0^3$. With these properties at hand the small divisors problem (i) is solved via a Nash-Moser Implicit function Theorem adapting the estimates of Craig-Wayne [11].

Recently, the existence of periodic solutions of (1) for frequencies ω in a set of positive measure has been proved in [16] using the Lindstedt series method for odd analytic nonlinearities $f(u) = au^3 + O(u^5)$ with $a \neq 0$. The reason for which $f(u)$ must be odd is that the solutions are obtained as analytic sine-series in x, see comments at the end of the section.

We also quote the recent papers [17]-[18] on the standing wave problem for a perfect fluid under gravity and with infinite depth which leads to a nonlinear and completely resonant second order equation.

In this paper we prove the existence of periodic solutions of the completely resonant wave equation (1) with Dirichlet boundary conditions for a set of frequencies ω's of positive measure and with full density at $\omega = 1$ and for a new set of nonlinearities $f(x,u)$ satisfying $(\mathbf{H})$ (including for example $f(x,u) = u^2$); we do *not* require that $f(x,u)$ can be extended on $(-\pi,\pi) \times \mathbf{R}$ to an analytic function $g(x,u)$ satisfying the oddness assumption $g(-x,-u) = -g(x,u)$, and we assume only finite regularity in the spatial variable x, see *Theorem 1.1*.

Let us describe accurately our result. Normalizing the period to 2π, we look for solutions $u(t,x)$,

[2]Actually [19] deals with the case of periodic boundary conditions in x, i.e. $u(t, x + 2\pi) = u(t,x)$.

2π-periodic in time, of the equation

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{3}$$

in the real Hilbert space

$$X_{\sigma,s} := \Big\{ u(t,x) = \sum_{l\in\mathbf{Z}} \exp(\mathrm{i}lt)\ u_l(x) \ \Big| \ u_l \in H_0^1((0,\pi),\mathbf{C}),\ \ \overline{u_l}(x) = u_{-l}(x)\ \forall l \in \mathbf{Z},$$
$$\text{and}\ \ ||u||^2_{\sigma,s} := \sum_{l\in\mathbf{Z}} \exp(2\sigma|l|)(l^{2s}+1)||u_l||^2_{H^1} < +\infty \Big\}.$$

For $\sigma > 0, s \geq 0$, the space $X_{\sigma,s}$ is the space of all 2π-periodic in time functions with values in $H_0^1((0,\pi),\mathbf{R})$, which have a bounded analytic extension in the complex strip $|\mathrm{Im}\ t| < \sigma$ with trace function on $|\mathrm{Im}\ t| = \sigma$ belonging to $H^s(\mathbf{T}, H_0^1((0,\pi),\mathbf{C}))$.

Note that if $u \in X_{\sigma,s}$ is a solution of (3) in a weak sense then the map $x \mapsto u_{xx}(t,x) = \omega^2 u_{tt}(t,x) - f(x,u(t,x))$ belongs to $H_0^1(0,\pi)$ for all $t \in \mathbf{T}$; hence $u(t,.) \in H^3(0,\pi) \subset C^2([0,\pi])$ and it is easy to justify that u is a classical solution.

For $2s > 1$, $X_{\sigma,s}$ is a Banach algebra with respect to multiplication of functions, namely[3]

$$u_1, u_2 \in X_{\sigma,s} \implies u_1u_2 \in X_{\sigma,s} \qquad \text{and} \qquad ||u_1u_2||_{\sigma,s} \leq C||u_1||_{\sigma,s}||u_2||_{\sigma,s}.$$

The space of the solutions of the linear equation $v_{tt} - v_{xx} = 0$ that belong to $X_{\sigma,s}$ is[4]

$$V := \Big\{ v(t,x) = \sum_{l\in\mathbf{Z}} \exp(\mathrm{i}lt) u_l \sin(lx) \ \Big| \ u_l \in \mathbf{C},\ \overline{u_l} = -u_{-l},\ \sum_{l\in\mathbf{Z}} \exp(2\sigma|l|)(l^{2s}+1)l^2|u_l|^2 < +\infty \Big\}.$$

Let $\varepsilon := \dfrac{\omega^2 - 1}{2}$. Instead of looking for solutions of (3) in a shrinking neighborhood of 0 it is a convenient devise to perform the rescaling $u \to \delta u$ with $\delta := |\varepsilon|^{1/p-1}$ (in most cases, see however subsection 5.3), obtaining

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + \varepsilon g(\delta,x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{4}$$

where

$$g(\delta,x,u) := s^* \frac{f(x,\delta u)}{\delta^p} = s^* \Big(a_p(x)u^p + \delta a_{p+1}(x)u^{p+1} + \dots \Big)$$

and $s^* := \mathrm{sign}(\varepsilon)$, namely $s^* = 1$ if $\omega > 1$ and $s^* = -1$ if $\omega < 1$.

The main result of this paper is:

Theorem 1.1 *Consider the completely resonant nonlinear wave equation (1) where the nonlinearity $f(x,u) = a_p(x)u^p + O(u^{p+1})$, $p \geq 2$, satisfies assumption* **(H)**.

1) There exists an open set $\mathcal{A}_p$ in $H^1((0,\pi),\mathbf{R})$ such that, for all $a_p \in \mathcal{A}_p$, there is $\sigma > 0$ and a C^∞-curve $[0,\delta_0) \ni \delta \to u(\delta)(t,x) \in X_{\sigma,s}$ with the following properties:

- *(i) There exists $s^* \in \{-1,1\}$ and a Cantor set $\mathcal{C}_{a_p} \subset [0,\delta_0)$ satisfying*

$$\lim_{\eta\to 0^+} \frac{\mathrm{meas}(\mathcal{C}_{a_p} \cap (0,\eta))}{\eta} = 1 \tag{5}$$

such that, for all $\delta \in \mathcal{C}_{a_p}$, $u(\delta)$ is a $2\pi/\omega$-periodic in time classical solution of (1) with $\omega = \sqrt{2s^*\delta^{p-1}+1}$;

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[3]The proof is as in [23] recalling that $H_0^1((0,\pi),\mathbf{C})$ is a Banach algebra with respect to multiplication of functions.

[4]V can also be written as

$$V := \Big\{ v(t,x) = \eta(t+x) - \eta(t-x) \ \Big| \ \eta \in C^\omega_{\sigma,s+1}(\mathbf{T},\mathbf{R}) \text{ with } \int_{\mathbf{T}} \eta = 0 \Big\}$$

where $C^\omega_{\sigma,s+1}(\mathbf{T},\mathbf{R})$ denotes the space of all 2π-periodic functions with have a bounded analytic extension in the complex strip $|\mathrm{Im}\ t| < \sigma$ with trace function on $|\mathrm{Im}\ t| = \sigma$ belonging to $H^{s+1}(\mathbf{T},\mathbf{C})$.

- *(ii)* $||\widetilde{u}(\delta) - \delta u_0||_{\sigma,s} = O(\delta^2)$ *for some* $u_0 \in V\backslash\{0\}$*, where* $\widetilde{u}(\delta)(t,x) = u(\delta)(t/\omega, x)$*.*

All $a_3(x) \in H^1(0,\pi)$ *such that* $\langle a_3\rangle := (1/\pi)\int_0^\pi a_3(x)dx \neq 0$ *is in* $\mathcal{A}_3$*. Hence (i) and (ii) hold true for any nonlinearity like* $f(x,u) = a_3(x)u^3 + \sum_{k\geq 4} a_k(x)u^k$*,* $\langle a_3\rangle \neq 0$*, with* $s^* = \mathrm{sign}(\langle a_3\rangle)$*.*

2) In the case $f(x,u) = a_2u^2 + \sum_{k\geq 4} a_k(x)u^k$*,* $a_2 \neq 0$*, conclusions (i) and (ii) still hold true with* $\omega = \sqrt{-2\delta^2+1}$*.*

Remark 1.1 *(i) Since equation (1) is autonomous, any non-trivial time periodic solution* $u(t,x)$ *of (1) generates a circle of solutions, i.e. for any* $\theta \in \mathbf{R}$*,* $u_\theta := u(t-\theta,x)$ *is a solution too.*

(ii) Multiplicity of periodic solutions $u(t,x)$ *of (1) with increasing norm could be found, as in [5].*

Sketch of the proof. We require first, for simplicity of exposition (see note 5), that

$$\exists v \in V \quad \text{such that} \quad \int_\Omega a_p(x)v^{p+1}(t,x)\ dtdx \neq 0, \qquad \Omega := \mathbf{T} \times (0,\pi). \tag{6}$$

To fix the ideas, we assume there is $v \in V$ such that $\int_\Omega a_p(x)v^{p+1} > 0$ and we look for periodic solutions with frequency $\omega > 1$, so that $s^* = 1$ and $\omega = \sqrt{2\delta^{p-1}+1}$ (if $\int_\Omega a_p(x)v^{p+1} < 0$ for some $v \in V$, then we can look for solutions of frequency $\omega < 1$, so that $s^* = -1$ and $\omega = \sqrt{-2\delta^{p-1}+1}$).

Condition (6) is verified for p odd, iff $a(x)$ is not antisymmetric w.r.t. to $x = \pi/2$, and for p even, iff $a(x)$ is not symmetric w.r.t. to $x = \pi/2$, see Lemma 6.1 in the Appendix.

In order to find solutions of (4), we try to implement the usual Lyapunov-Schmidt reduction according to the decomposition $X_{\sigma,s} = V \oplus W$ where

$$W := \Big\{ w = \sum_{l\in\mathbf{Z}} \exp(\mathrm{i}lt)\ w_l(x) \in X_{\sigma,s} \mid w_{-l} = \overline{w_l} \ \text{ and } \int_0^\pi w_l(x)\sin(lx)\ dx = 0,\ \forall l \in \mathbf{Z} \Big\}. \tag{7}$$

Looking for solutions $u = v + w$ with $v \in V$, $w \in W$ and we are led to solve the bifurcation equation (called the (Q)-equation) and the range equation (called the (P)-equation)

$$\begin{cases} -\Delta v = \Pi_V g(\delta, x, v+w) & (Q) \\ L_\omega w = \varepsilon \Pi_W g(\delta, x, v+w) & (P) \end{cases} \tag{8}$$

where

$$\Delta v := v_{xx} + v_{tt}, \qquad L_\omega := -\omega^2\partial_{tt} + \partial_{xx}$$

and $\Pi_V : X_{\sigma,s} \to V$, $\Pi_W : X_{\sigma,s} \to W$ denote the projectors respectively on V and W.

Since V is infinite dimensional a difficulty arises in the application of the method of [11] in presence of small divisors: if $v \in V \cap X_{\sigma_0,s}$ then the solution $w(\delta,v)$ of the range equation, obtained with any Nash-Moser iteration scheme will have a lower regularity, e.g. $w(\delta,v) \in X_{\sigma_0/2,s}$. Therefore in solving next the bifurcation equation for $v \in V$, the best estimate we can obtain is $v \in V \cap X_{\sigma_0/2,s+2}$, which makes the scheme incoherent. Moreover we have to ensure that the 0^{th}-order bifurcation equation[5], i.e. the (Q)-equation for $\delta = 0$,

$$-\Delta v = \Pi_V\Big(a_p(x)v^p\Big) \tag{9}$$

has solutions $v \in V$ which are analytic, a necessary property to initiate an analytic Nash-Moser scheme (in [11]-[12] this problem does not arise since, dealing with non-resonant or partially resonant Hamiltonian PDEs like $u_{tt} - u_{xx} + a_1(x)u = f(x,u)$, the bifurcation equation is finite dimensional).

We overcome this difficulty thanks to a reduction to a *finite dimensional* bifurcation equation (on a subspace of V of dimension N independent of ω). This reduction can be implemented, in spite of the complete resonance of equation (1), thanks to the compactness of the operator $(-\Delta)^{-1}$.

[5]The right hand side $\Pi_V(a_p(x)v^p)$ is not identically equal to 0 in V iff condition (6) holds. If not verified, as for $f(x,u) = u^2$, the 0^{th}-order non-trivial bifurcation equation will involve higher order nonlinear terms, see [4] and subsection 5.3.

We introduce the decomposition $V = V_1 \oplus V_2$ where

$$\begin{cases} V_1 := \Big\{ v \in V \mid v(t,x) = \sum_{l=1}^{N} \Big(\exp{(\mathrm{i}lt)} u_l + \exp{(-\mathrm{i}lt)} \overline{u_l} \Big) \ \sin(lx), \ u_l \in \mathbf{C} \Big\} \\ V_2 := \Big\{ v \in V \mid v(t,x) = \sum_{l \geq N+1} \Big(\exp{(\mathrm{i}lt)} u_l + \exp{(-\mathrm{i}lt)} \overline{u_l} \Big) \sin(lx), \ u_l \in \mathbf{C} \Big\} \end{cases}$$

Setting $v := v_1 + v_2$, with $v_1 \in V_1, v_2 \in V_2$, (8) is equivalent to

$$\begin{cases} -\Delta v_1 = \Pi_{V_1} g(\delta, x, v_1 + v_2 + w) & (Q_1) \\ -\Delta v_2 = \Pi_{V_2} g(\delta, x, v_1 + v_2 + w) & (Q_2) \\ L_\omega w = \varepsilon \Pi_W g(\delta, x, v_1 + v_2 + w) & (P) \end{cases} \tag{10}$$

where $\Pi_{V_i} : X_{\sigma,s} \to V_i$ $(i = 1, 2)$, denote the orthogonal projectors on V_i $(i = 1, 2)$. We specify that all the norms $|| \ ||_{\sigma,s}$ are equivalent on V_1. In the sequel $B(\rho, V_1)$ will denote $B(\rho, V_1) := \{v_1 \in V_1 \mid ||v_1||_{0,s} < \rho\}$.

Our strategy to find solutions of system (10) - and hence to prove Theorem 1.1- is the following. We solve first the (Q_2)-equation obtaining $v_2 = v_2(\delta, v_1, w) \in V_2 \cap X_{\sigma,s}$ by a standard Implicit Function Theorem provided we have chosen N large enough and $0 < \sigma < \overline{\sigma}$ small enough - depending on the nonlinearity f but *independent of* δ, see section 2.

Next we solve the (P)-equation, obtaining $w = w(\delta, v_1) \in W \cap X_{\sigma/2,s}$ by means of a Nash-Moser Implicit Function Theorem for (δ, v_1) belonging to some Cantor-like set of parameters, see section 3.

A major role is played by the inversion of the *linearized operators* obtained at any stage of the Nash-Moser iteration. As usual, the main difficulty in controlling such inverse operators is due to the fact the diagonal elements may be arbitrarily small.

Our approach -presented in section 4- is different from the one in [11]-[7]-[8] which is based on the Frölich-Spencer technique [14]. It allows to deal with nonlinearities $f(x, u)$ with finite regularity in the spatial variable x and which are not the restriction to $(0, \pi) \times \mathbf{R}$ of a smooth odd function.

We first develop $u(t, \cdot) \in H^1_0((0, \pi), \mathbf{R})$ in time-Fourier expansion only and we distinguish the "diagonal part" $D = \mathrm{diag}\{D_k\}_{k \in \mathbf{Z}}$ of the operator that we want to invert. Next, using Sturm-Liouville theory (see Lemma 4.1), we diagonalize each D_k in a suitable basis of $H^1_0((0, \pi), \mathbf{R})$ (close, but different from $(\sin jx)_{j \geq 1}$). Assuming a "first order Melnikov non resonance condition" (Definition 3.3) we prove that its eigenvalues are polynomially bounded away from 0 and so we invert D with sufficiently good estimates (Corollary 4.2). The presence of the "off-diagonal" Toepliz operators requires to analyze the "small divisors": for our method it is sufficient to prove that the product of two "small divisors" is larger than a constant if the corresponding "singular sites" are close enough, see Lemma 4.6.

If the nonlinearity $f(x, u)$ can be extended to an analytic (in both variables) odd function then the Dirichlet problem on $[0, \pi]$ is actually equivalent to the 2π-periodic problem within the space of all odd functions and a natural configuration space to use is $Y := \{u(x) = \sum_{j \geq 1} u_j \sin(jx) \mid \sum_j \exp{(2aj)} j^{2\rho} |u_j|^2 < +\infty\}$. On the other hand, for (still analytic) non odd nonlinearities f, it is not possible in general to find a smooth periodic solution $u(t, .)$ that belongs to Y for all t. For example, if $f(x, u) = u^2$ then it is easy to see (deriving twice the equation w.r.t. x and using that $u_{xx}(t, 0) = 0$) that any smooth solution u must satisfy $-u_{xxxx}(t, 0) + 2u_x^2(t, 0) = 0$. Moreover, if u is not trivial then $u_x(., 0)$ is not identically 0. Hence $u_{xxxx}(., 0)$ does not vanish everywhere, which implies that $u(t, .)$ is not in Y for all t. For these reasons we shall consider as configuration space $H^1_0((0, \pi), \mathbf{R})$ (and the solutions that we shall find satisfy $u(t, \ .) \in H^1_0(0, \pi) \cap H^3(0, \pi)$).

Finally (section 5) we solve the finite dimensional (Q_1)-equation for a new set of nonlinearities : for these nonlinearities, we can define a smooth curve $(\delta \mapsto v_1(\delta) \in V_1)$ such that, if δ belongs to some "large" set, $(\delta, v_1(\delta))$ gives rise to an exact solution of equation (1).

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Notations: $B(R; X)$ denotes the open ball of radius R in the space X centered at 0. $f(z) = O(g(z))$ means that there is a universal constant C such that $|f(z)| \leq C|g(z)|$. $\overline{z}$ is the complex conjugated of z.

2 Solution of the (Q_2)-equation

The 0^{th}-order bifurcation equation (9) is the Euler-Lagrange equation of the functional $\Phi_0 : V \to \mathbf{R}$, defined by

$$\Phi_0(v) = \frac{||v||^2_{H^1}}{2} - \int_\Omega a_p(x) \frac{v^{p+1}}{p+1}\, dxdt, \qquad \Omega = \mathbf{T} \times (0,\pi), \tag{11}$$

where $||v||^2_{H^1} = \int_\Omega v_t^2 + v_x^2 \, dx\, dt$. By the Mountain-pass Theorem [1] and condition (6) (recall that we assume $\int_\Omega a_p(x) v^{p+1} > 0$ for some $v \in V$), the critical set

$$K_{0,c} := \Big\{ v \in V \mid \Phi_0'(v) = 0, \ \Phi_0(v) \le c \Big\} \tag{12}$$

is non-trivial (*i.e* $\neq \{0\}$) for $c > 0$ large enough and is compact for the H^1-topology, see [4].

In fact, by a direct bootstrap argument, $K_{0,c}$ is a compact subset of $V \cap H^k(\Omega)$, for any $k \ge 0$. Therefore $K_{0,c} \subset V \cap C^\infty(\Omega)$ (even if $a_p(x) \in H^1((0,\pi),\mathbf{R})$ only, because the projection Π_V has a regularizing effect in the variable x) and we have (*a priori estimate*)

$$\sup_{v \in K_{0,c}} ||v||_{0,s} < R < +\infty.$$

By the analyticity assumption $(\mathbf{H})$ on the nonlinearity f and the Banach algebra property of $X_{\sigma,s}$, the Nemitsky operator $X_{\sigma,s} \ni u \to g(\delta,x,u) \in X_{\sigma,s}$ is in $C^\infty(\mathcal{U}_\delta, X_{\sigma,s})$, where $\mathcal{U}_\delta = \{u \in X_{\sigma,s} \mid |\delta| \, ||u||_{\sigma,s} \le \widetilde{\delta}\}$ and $\widetilde{\delta}$ depends on the radius of convergence ρ of the power series that defines $f(x,u)$.

Lemma 2.1 (Solution of the (Q_2)-equation) *There exists $\overline{\sigma} > 0, N \in \mathbf{N}_+, \delta_0 > 0$ such that, $\forall 0 \le \sigma < \overline{\sigma}$, $\forall ||v_1||_{0,s} \le 2R$, $\forall ||w||_{\sigma,s} \le 1$, $\forall \delta \in [0,\delta_0)$, there exists a unique $v_2 = v_2(\delta, v_1, w) \in X_{\sigma,s}$ with $||v_2(\delta,v_1,w)||_{\sigma,s} \le 1$ which solves the (Q_2)-equation.*

Moreover, for any $v \in K_{0,c}$, $v_2(0, \Pi_{V_1} v, 0) = \Pi_{V_2} v$.

Furthermore $v_2(\delta,v_1,w) \in X_{\sigma,s+2}$, $v_2(\cdot,\cdot,\cdot) \in C^\infty\Big([0,\delta_0) \times B(2R;V_1) \times B(1; W \cap X_{\sigma,s}), V_2 \cap X_{\sigma,s+2}\Big)$ and Dv_2, $D^2 v_2$ are bounded on $[0,\delta_0) \times B(2R;V_1) \times B(1;W \cap X_{\sigma,s})$.

Finally, if in addition $||w||_{\sigma,s'} < +\infty$ for some $s' \ge s$, then (provided δ_0 has been chosen small enough) $||v_2(\delta,v_1,w)||_{\sigma,s'+2} \le K(s', ||w||_{\sigma,s'})$.

PROOF. Fixed points of the nonlinear operator $\mathcal{N}(\delta,v_1,w,\cdot) : V_2 \to V_2$ defined by

$$\mathcal{N}(\delta,v_1,w,v_2) := (-\Delta)^{-1} \Pi_{V_2} g(\delta, x, v_1 + w + v_2)$$

are solutions of equation (Q_2). For $w \in W \cap X_{\sigma,s}$, $v_2 \in V_2 \cap X_{\sigma,s}$ we conclude that $\mathcal{N}(\delta,v_1,w,v_2) \in V_2 \cap X_{\sigma,s+2}$ since $g(\delta,x,v_1+w+v_2) \in X_{\sigma,s}$ and because of the *regularizing property* of the operator $(-\Delta)^{-1}\Pi_{V_2} : X_{\sigma,s} \to V_2 \cap X_{\sigma,s+2}$.

Let $B := \{v_2 \in V_2 \cap X_{\sigma,s} \mid ||v_2||_{\sigma,s} \le 1\}$. We claim that there exists $N \in \mathbf{N}$, $\overline{\sigma} > 0$ and $\delta_0 > 0$, such that for any $0 \le \sigma < \overline{\sigma}$, $||v_1||_{0,s} \le 2R$, $||w||_{\sigma,s} \le 1$, $\delta \in [0,\delta_0)$ the operator $v_2 \to \mathcal{N}(\delta,v_1,w,v_2)$ is a contraction in B, more precisely

- *(i)* $||v_2||_{\sigma,s} \le 1 \Rightarrow ||\mathcal{N}(\delta,v_1,w,v_2)||_{\sigma,s} \le 1$;
- *(ii)* $\forall v_2, \widetilde{v}_2 \in B \Rightarrow ||\mathcal{N}(\delta,v_1,w,v_2) - \mathcal{N}(\delta,v_1,w,\widetilde{v}_2)||_{\sigma,s} \le (1/2)||v_2 - \widetilde{v}_2||_{\sigma,s}$.

Let us prove *(i)*. $\forall u \in X_{\sigma,s}$, $||(-\Delta)^{-1}\Pi_{V_2} u||_{\sigma,s} \le (C/(N+1)^2)||u||_{\sigma,s}$ and so $\forall ||w||_{\sigma,s} \le 1$, $||v_1||_{0,s} \le 2R$, $\delta \in [0,\delta_0)$,

$$\begin{aligned} ||\mathcal{N}(\delta,v_1,w,v_2)||_{\sigma,s} &\le \frac{C}{(N+1)^2}\Big|\Big| g(\delta,x,v_1+v_2+w)\Big|\Big|_{\sigma,s} \le \frac{C'}{(N+1)^2}\Big(||v_1||^p_{\sigma,s} + ||v_2||^p_{\sigma,s} + ||w||^p_{\sigma,s}\Big) \\ &\le \frac{C'}{(N+1)^2}\Big(\exp(\sigma p N) ||v_1||^p_{0,s} + ||v_2||^p_{\sigma,s} + 1\Big) \le \frac{C'}{(N+1)^2}\Big((4R)^p + ||v_2||^p_{\sigma,s} + 1\Big) \end{aligned}$$

for $\exp(\sigma N) \leq 2$, where we have used that $||v_1||_{\sigma,s} \leq \exp(\sigma N)||v_1||_{0,s} \leq 4R$. For N large enough (depending on R) and $\overline{\sigma} = \ln 2/N$, (i) follows. Property (ii) can be proved similarly and the existence of a unique solution $v_2(\delta, v_1, w) \in V_2 \cap X_{\sigma,s}$ follows.

Since $K_{0,c}$ is compact in $V \cap H^s(\Omega)$, we may assume that N has been chosen so large that for all $v \in K_{0,c}$, $||\Pi_{V_2} v||_{0,s} \leq 1/2$. Any $v \in K_{0,c}$ solves (9), hence $\Pi_{V_2} v$ solves the (Q_2)-equation associated with $(0, \Pi_{V_1} v, 0)$. Hence $\Pi_{V_2} v \in B$ and $\Pi_{V_2} v = \mathcal{N}(0, \Pi_{V_1} v, 0, \Pi_{V_2} v)$, which implies that $\Pi_{V_2} v = v_2(0, \Pi_{V_1} v, 0)$.

Because the map $(\delta, v_1, w, v_2) \mapsto \mathcal{N}(\delta, v_1, w, v_2)$ is C^∞, by the Implicit function Theorem $v_2 : \{(\delta, v_1, w) \mid \delta \in [0, \delta_0), ||v_1||_{0,s} \leq 2R, ||w||_{\sigma,s} \leq 1\} \to V_2 \cap X_{\sigma,s}$ is a C^∞ map. Finally, since $(-\Delta)^{-1}\Pi_{V_2}$ is a continuous linear operator from $X_{\sigma,s}$ to $V_2 \cap X_{\sigma,s+2}$ and

$$v_2(\delta, v_1, w) = (-\Delta)^{-1}\Pi_{V_2}\Big(g(\delta, x, v_1 + w + v_2(\delta, v_1, w)\Big),$$

by the regularity of the Nemitsky operator induced by g, $v_2(\cdot,\cdot,\cdot) \in C^\infty([0,\delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s}), V_2 \cap X_{\sigma,s+2})$. The estimates for the derivatives can be obtained similarly.

Finally, assume that $||w||_{\sigma,s'} < +\infty$ for some $s' \geq s$. We have

$$v_2(\delta, v_1, w) = (-\Delta)^{-1}\Pi_{V_2} g(\delta, x, v_1 + w + v_2(\delta, v_1, w)).$$

Using the regularizing properties of the operator $(-\Delta)^{-1}\Pi_{V_2}$ and the fact that $||v_1||_{\sigma,r} < +\infty$ for all $r \geq s$, we can derive by a simple bootstrap argument a bound on $||v_2(\delta, v_1, w)||_{\sigma,s'}$. It is useful here to observe that[6] $||a_l(x)(u^l)||_{\sigma,r} \leq C^l ||a_l||_{H^1} ||u||^{l-1}_{\sigma,s} ||u||_{\sigma,r}$ for $r \geq s$, $l \geq p$, so that if $u \in X_{\sigma,r}$ and $\delta||u||_{\sigma,s}$ is small enough, then $g(\delta, x, u) \in X_{\sigma,r}$. ∎

Remark 2.1 *Lemma 2.1 implies, in particular, that any solution $v \in K_{0,c}$ of the 0^{th}-order bifurcation equation (9) is not only in $V \cap C^\infty(\Omega)$ but actually belongs to $V \cap X_{\sigma,s+2}$ and therefore is analytic in t and hence in x.*

We conclude this section with a Lemma which is a standard consequence of the Lyapunov-Schmidt decomposition dealing with variational equations.

Let us define the "reduced" functional $\Psi_0 : B(2R, V_1) = \{v_1 \in V_1 \mid ||v_1||_{0,s} < 2R\} \to \mathbf{R}$ by

$$\Psi_0(v_1) := \Phi_0(v_1 + v_2(0, v_1, 0)). \tag{13}$$

Lemma 2.2 *If $\overline{v}_1$ is a critical point of Ψ_0, then $\overline{v} := \overline{v}_1 + v_2(0, \overline{v}_1, 0)$ is a critical point of $\Phi_0 : V \to \mathbf{R}$, i.e. $\overline{v} \in V$ is a solution of the 0^{th}-order bifurcation equation (9). Conversely, if $\overline{v}$ is a critical point of Φ_0 of critical value $\leq c$ then there is a critical point $\overline{v}_1 \in B(2R, V_1)$ of Ψ_0 such that $\overline{v} = \overline{v}_1 + v_2(0, \overline{v}_1, 0)$.*

PROOF. Since $v_2(0, v_1, 0)$ solves the (Q_2)-equation (for $\delta = 0$, $w = 0$), $d\Phi_0(v_1 + v_2(0, v_1, 0))[k] = 0$, $\forall k \in V_2$. Moreover, since $\forall v_1 \in V_1$ $v_2(0, v_1, 0) \in V_2$, $\partial_{v_1} v_2(0, v_1, 0)[h] \in V_2$, $\forall h \in V_1$. Therefore

$$\begin{aligned} d\Psi_0(v_1)[h] &= d\Phi_0(v_1 + v_2(0, v_1, 0))\Big[h + \partial_{v_1} v_2(0, v_1, 0)[h]\Big] = d\Phi_0(v_1 + v_2(0, v_1, 0))[h] \\ &= \int_\Omega \Big[-\Delta v_1 - \Pi_{V_1}\Big(a_p(x)(v_1 + v_2(0, v_1, 0))\Big)^p\Big] h \, dx \, dt. \end{aligned} \tag{14}$$

Therefore if $\overline{v}_1$ is a critical point of Ψ_0, then $\overline{v} := \overline{v}_1 + v_2(0, \overline{v}_1, 0)$ is a solution of equation (9).

Conversely, assume that $\overline{v}$ is a critical point of Φ_0, with $\Phi_0(v) \leq c$ and let $\overline{v}_1 = \Pi_{V_1}\overline{v}$. By Lemma 2.1, $\Pi_{V_2}\overline{v} = v_2(0, \overline{v}_1, 0)$ and it is clear that $\overline{v}_1$ is a critical point of Ψ_0. ∎

[6] This inequality can be obtained considering the extension of the maps to the complex strip of width σ, using that $H^1_0(0,\pi)$ is a Banach algebra, and the inequality $||u^l||_{H^r,\text{per}} \leq K^l ||u||^{l-1}_\infty ||u||_{H^r,\text{per}}$ for 2π-periodic functions. The last inequality follows from $||uv||_{H^r,\text{per}} \leq C_r(||u||_\infty ||v||_{H^r,\text{per}} + ||v||_\infty ||u||_{r,\text{per}})$ which is related to the Gagliardo-Nirenberg inequalities.

3 Solution of the (P)-equation

By the previous section we are reduced to solve the (P)-equation with $v_2 = v_2(\delta, v_1, w)$, namely

$$L_\omega w = \varepsilon \Pi_W \Gamma(\delta, v_1, w) \tag{15}$$

where

$$\Gamma(\delta, v_1, w)(t,x) := g\Big(\delta, x, v_1(t,x) + w(t,x) + v_2(\delta, v_1, w)(t,x)\Big).$$

The solution $w = w(\delta, v_1)$ of the (P)-equation (15) is obtained by means of a Nash-Moser Implicit Function Theorem for (δ, v_1) belonging to a Cantor-like set of parameters.

We consider the orthogonal splitting $W = W^{(n)} \oplus W^{(n)\perp}$ where

$$W^{(n)} = \Big\{ w \in W \ \Big| \ w = \sum_{l=-L_n}^{L_n} \exp(\mathrm{i}lt)\ w_l(x) \Big\} \quad \text{and} \quad W^{(n)\perp} = \Big\{ w \in W \ \Big| \ w = \sum_{|l|>L_n} \exp(\mathrm{i}lt)\ w_l(x) \Big\}, \tag{16}$$

and L_n are integer numbers (we will choose $L_n = L_0 2^n$ with $L_0 \in \mathbf{N}$ large enough). We denote by

$$P_n : W \to W^{(n)} \qquad \text{and} \qquad P_n^\perp : W \to W^{(n)\perp}$$

the orthogonal projectors onto $W^{(n)}$ and $W^{(n)\perp}$.

The convergence of the recursive scheme is based on properties $(P1)$-$(P2)$-$(P3)$ below.

- **(P1)** (**Regularity**) $\Gamma(\cdot,\cdot,\cdot) \in C^\infty\Big([0,\delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s}), X_{\sigma,s}\Big)$. Moreover, Γ, $D\Gamma$ and $D^2\Gamma$ are bounded on $[0,\delta_0) \times B(2R, V_1) \times B(1; W \cap X_{\sigma,s})$.

$(P1)$ is a consequence of the C^∞-regularity of the Nemistky operator induced by $g(\delta, x, u)$ on $X_{\sigma,s}$ and of the properties of the map v_2 stated in Lemma 2.1.

- **(P2)** (**Smoothing estimate**) $\forall\, w \in W^{(n)\perp} \cap X_{\sigma,s}$ and $\forall\, 0 \le \sigma' \le \sigma$, $||w||_{\sigma',s} \le \exp^{(-L_n(\sigma-\sigma'))} ||w||_{\sigma,s}$.

The next property $(P3)$ is an *invertibility property* of the linearized operator $\mathcal{L}_n(\delta, v_1, w) : W^{(n)} \to W^{(n)}$ defined by

$$\mathcal{L}_n(\delta, v_1, w)[h] := L_\omega h - \varepsilon P_n \Pi_W D_w \Gamma(\delta, v_1, w)[h] \tag{17}$$

where w is the approximate solution obtained at a given step of the Nash-Moser iteration.

The invertibility of $\mathcal{L}_n(\delta, v_1, w)$ is obtained excising the set of parameters (δ, v_1) for which 0 is an eigenvalue of $\mathcal{L}_n(\delta, v_1, w)$. Moreover, in order to have bounds for the norm of the inverse operator $\mathcal{L}_n^{-1}(\delta, v_1, w)$ which are sufficiently good for the recursive scheme, we also excise the parameters (δ, v_1) for which the eigenvalues of $\mathcal{L}_n(\delta, v_1, w)$ are too small. We prefix some definitions.

Definition 3.1 **(Mean value)** *For* $\Omega := \mathbf{T} \times (0,\pi)$ *we define*

$$M(\delta, v_1, w) := \frac{1}{|\Omega|} \int_\Omega \partial_u g\Big(\delta, x, v_1(t,x) + w(t,x) + v_2(\delta, v_1, w)(t,x)\Big)\ dxdt.$$

Definition 3.2 *We define for* $1 < \tau < 2$

$$[w]_{\sigma,s} := \inf \Big\{ \sum_{i=0}^q \frac{||h_i||_{\sigma_i,s}}{(\sigma_i - \sigma)^{\frac{2(\tau-1)}{\beta}}} \ ; \ q \ge 1,\ \overline{\sigma} \ge \sigma_i > \sigma, \ \ h_i \in W^{(i)},\ w = \sum_{i=0}^q h_i \Big\}$$

where $\beta := \frac{2-\tau}{2}$ *and we set* $[w]_{\sigma,s} := \infty$ *if the above set is empty.*

Definition 3.3 (First order Melnikov non-resonance condition) *Let $0 < \gamma < 1$ and $1 < \tau < 2$. We define (recall that $\omega = \sqrt{2\delta^{p-1}+1}$ and $\varepsilon = \delta^{p-1}$)*

$$\Delta_n^{\gamma,\tau}(v_1, w) \quad := \quad \Big\{ \delta \in [0,\delta_0) \ \Big| \ |\omega k - j| \geq \frac{\gamma}{(k+j)^\tau}, \ \Big|\omega k - j - \varepsilon \frac{M(\delta, v_1, w)}{2j}\Big| \geq \frac{\gamma}{(k+j)^\tau}$$
$$\forall k \in \mathbf{N}, \ j \geq 1, \ k \neq j, \ \frac{1}{3|\varepsilon|} < k, \ k \leq L_n, \ j \leq 2L_n \Big\}.$$

We claim that:

- **(P3) (Invertibility of $\mathcal{L}_n$)** There exist positive constants μ, δ_0 and C such that, if $[w]_{\sigma,s} \leq \mu$, $||v_1||_{0,s} \leq 2R$ and $\delta \in \Delta_n^{\gamma,\tau}(v_1, w) \cap [0, \delta_0)$ for some $0 < \gamma < 1$, $1 < \tau < 2$, then $\mathcal{L}_n(\delta, v_1, w)$ is invertible and the inverse operator $\mathcal{L}_n^{-1}(\delta, v_1, w) : W^{(n)} \to W^{(n)}$ satisfies

$$\Big|\Big|\mathcal{L}_n^{-1}(\delta, v_1, w)[h]\Big|\Big|_{\sigma,s} \leq \frac{C}{\gamma}(L_n)^{\tau-1} ||h||_{\sigma,s}. \tag{18}$$

Property $(P3)$ is the real core of the convergence proof and where the analysis of the small divisors enters into play. Property $(P3)$ is proved in section 4.

3.1 The Nash-Moser scheme

We are going to define recursively a sequence $\{w_n\}_{n\geq 0}$ with $w_n = w_n(\delta, v_1) \in W^{(n)}$, $w_0 = 0$, defined on smaller and smaller sets of "non-resonant" parameters (δ, v_1), $A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_1 \subseteq A_0 := \{(\delta, v_1) \mid \delta \in [0, \delta_0), \ ||v_1||_{0,s} \leq 2R\}$. The sequence $(w_n(\delta, v_1))$ will converge to a solution $w(\delta, v_1)$ of the (P)-equation (15) for $(\delta, v_1) \in A_\infty := \cap_{n\geq 1} A_n$. The main goal of the construction is to show that, at the end of the recurrence, the set of parameters $A_\infty := \cap_{n\geq 1} A_n$ for which we have the solution $w(\delta, v_1)$, remains sufficiently large.

We define inductively the sequence $\{w_n\}_{n\geq 0}$. Define the "loss of analyticity" γ_n by

$$\gamma_n := \frac{\gamma_0}{n^2+1}, \qquad \sigma_0 = \overline{\sigma}, \qquad \sigma_{n+1} = \sigma_n - \gamma_n, \qquad \forall \ n \geq 0,$$

where we choose $\gamma_0 > 0$ small such that the "total loss of analyticity" $\sum_{n\geq 0} \gamma_n = \gamma_0 \sum_{n \geq 0} 1/(n^2+1) \leq \overline{\sigma}/2$, i.e. $\lim_{n\to+\infty} \sigma_n \geq \overline{\sigma}/2 > 0$. We also assume

$$L_n := L_0 2^n, \qquad \forall \ n \geq 0,$$

for some large integer L_0 specified in the next proposition.

Proposition 3.1 (Induction) *Let $w_0 = 0$ and $A_0 := \{(\delta, v_1) \mid \delta \in [0, \delta_0), \ ||v_1||_{0,s} \leq 2R\}$. There exists $\varepsilon_0 := \varepsilon_0(\gamma, \tau)$, $L_0 := L_0(\gamma, \tau) > 0$ such that $\forall |\varepsilon|\gamma^{-1} < \varepsilon_0$, there exists a sequence $\{w_n\}_{n\geq 0}$, $w_n = w_n(\delta, v_1) \in W^{(n)}$, of solutions of the equation*

$$(P_n) \qquad\qquad L_\omega w_n - \varepsilon P_n \Pi_W \Gamma(\delta, v_1, w_n) = 0,$$

defined for $(\delta, v_1) \in A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_1 \subseteq A_0$, with $w_n(\delta, v_1) = \sum_{i=0}^n h_i(\delta, v_1)$, $h_0 := w_0 = 0$, $h_i = h_i(\delta, v_1) \in W^{(i)}$ satisfying $||h_i||_{\sigma_i,s} \leq |\varepsilon|\gamma^{-1} \exp(-\chi^i)$ for some $1 < \chi < 2$, $\forall i = 0, \ldots, n$.

Proof. Fix some $\chi \in (1,2)$. We assume $\varepsilon_0\gamma^{-1} > 0$ small enough such that

$$\frac{\varepsilon_0}{\gamma} \sum_{i\geq 0} \exp(-\chi^i) \Big(\frac{1+i^2}{\gamma_0}\Big)^{\frac{2(\tau-1)}{\beta}} < \mu. \tag{19}$$

The proof proceeds by induction. We recall that $\beta := (2-\tau)/2$ and that μ is defined in property $(P3)$.

Suppose we have already defined a solution $w_n = w_n(\delta, v_1) \in W^{(n)}$ of equation (P_n) satisfying the properties stated in the proposition. We want to define

$$w_{n+1} = w_{n+1}(\delta, v_1) := w_n(\delta, v_1) + h_{n+1}(\delta, v_1), \qquad h_{n+1}(\delta, v_1) \in W^{(n+1)}$$

as an *exact* solution of the equation

$$(P_{n+1}) \qquad\qquad L_\omega w_{n+1} - \varepsilon P_{n+1} \Pi_W \Gamma(\delta, v_1, w_{n+1}) = 0.$$

In order to find a solution $w_{n+1} = w_n + h_{n+1}$ of equation (P_{n+1}) we write, for $h \in W^{(n+1)}$,

$$\begin{aligned} L_\omega(w_n + h) - \varepsilon P_{n+1} \Pi_W \Gamma(\delta, v_1, w_n + h) &= L_\omega w_n - \varepsilon P_{n+1} \Pi_W \Gamma(\delta, v_1, w_n) & (20)\\ &+ \; L_\omega h - \varepsilon P_{n+1} \Pi_W D_w \Gamma(\delta, v_1, w_n)[h] + R(h) \\ &= r_n + \mathcal{L}_{n+1}(\delta, v_1, w_n)[h] + R(h), & (21) \end{aligned}$$

where, since w_n solves equation (P_n),

$$\begin{cases} r_n := L_\omega w_n - \varepsilon P_{n+1} \Pi_W \Gamma(\delta, v_1, w_n) = -\varepsilon P_n^\perp P_{n+1} \Pi_W \Gamma(\delta, v_1, w_n) \in W^{(n+1)} \\ R(h) := \; -\varepsilon P_{n+1} \Pi_W \Big(\Gamma(\delta, v_1, w_n + h) - \Gamma(\delta, v_1, w_n) - D_w \Gamma(\delta, v_1, w_n)[h] \Big) \end{cases}$$

satisfy, by properties $(P1)$ and $(P2)$,

$$||r_n||_{\sigma_{n+1},s} \le |\varepsilon| \; C \exp(-L_n \gamma_n) \Big|\Big| P_{n+1} \Pi_W \Gamma(\delta, v_1, w_n) \Big|\Big|_{\sigma_n,s} \le |\varepsilon| \; C' \exp(-L_n \gamma_n) \qquad (22)$$

and, by property $(P1)$, $\forall h, h' \in W^{(n+1)}$ with $||h||_{\sigma_{n+1},s}$, $||h'||_{\sigma_{n+1},s}$ small enough

$$\begin{cases} ||R(h)||_{\sigma_{n+1},s} \le C|\varepsilon| \; ||h||^2_{\sigma_{n+1},s} \\ ||R(h) - R(h')||_{\sigma_{n+1},s} \le C|\varepsilon| \; (||h||_{\sigma_{n+1},s} + ||h'||_{\sigma_{n+1},s}) \; ||h - h'||_{\sigma_{n+1},s}. \end{cases} \qquad (23)$$

Since $||h_i||_{\sigma_i,s} \le |\varepsilon|\gamma^{-1} \; \exp(-\chi^i)$, $\forall i = 0, \ldots, n$, by (19),

$$[w_n]_{\sigma_{n+1},s} \le \sum_{i=0}^n \frac{||h_i||_{\sigma_i,s}}{(\sigma_i - \sigma_{n+1})^{\frac{2(\tau-1)}{\beta}}} \le \frac{|\varepsilon|}{\gamma} \sum_{i=0}^n \frac{\exp(-\chi^i)}{\gamma_i^{2(\tau-1)/\beta}} \le \frac{|\varepsilon|}{\gamma} \sum_{i \ge 0} \exp(-\chi^i) \Big(\frac{1 + i^2}{\gamma_0} \Big)^{\frac{2(\tau-1)}{\beta}} < \mu. \qquad (24)$$

Hence by property $(P3)$, the linear operator $\mathcal{L}_{n+1}(\delta, v_1, w_n) : W^{(n+1)} \to W^{(n+1)}$ is invertible for (δ, v_1) restricted to the set of parameters

$$A_{n+1} := \Big\{ (\delta, v_1) \in A_n \mid \delta \in \Delta^{\gamma,\tau}_{n+1}(v_1, w_n) \Big\} \subseteq A_n. \qquad (25)$$

Moreover

$$\Big|\Big| \mathcal{L}_{n+1}(\delta, v_1, w_n)^{-1} \Big|\Big|_{\sigma_{n+1},s} \le \frac{C}{\gamma} (L_{n+1})^{\tau-1}, \qquad \forall (\delta, v_1) \in A_{n+1}. \qquad (26)$$

By (20) equation (P_n) is equivalent to the fixed point problem

$$w_{n+1} = w_n + h, \qquad \mathcal{G}(\delta, v_1, w_n, h) = h, \qquad (27)$$

for the nonlinear operator $\mathcal{G}(\delta, v_1, w_n, \cdot) : W^{(n+1)} \to W^{(n+1)}$, defined by

$$\mathcal{G}(\delta, v_1, w_n, h) := -\mathcal{L}_{n+1}(\delta, v_1, w_n)^{-1} \Big(r_n + R(h) \Big).$$

To complete the proof of the proposition we need the following Lemma.

Lemma 3.1 (Contraction) *$\mathcal{G}(\delta, v_1, w_n, \cdot)$ maps the ball $B(\rho_{n+1}; W^{(n+1)}) := \{ w \in W^{(n+1)} \mid ||w||_{\sigma_{n+1},s} \le \rho_{n+1} \}$ of radius $\rho_{n+1} := |\varepsilon|\gamma^{-1} \; \exp(-\chi^{n+1})$ into itself and is a contraction in this ball.*

Proof. By (26), (22) and (23),

$$\begin{aligned} ||\mathcal{G}(\delta, v_1, w_n, h)||_{\sigma_{n+1},s} &= \Big|\Big|\mathcal{L}_{n+1}(\delta, v_1, w_n)^{-1}\Big(r_n + R(h)\Big)\Big|\Big|_{\sigma_{n+1},s} \\ &\le \frac{C}{\gamma}(L_{n+1})^{\tau-1}\Big(||r_n||_{\sigma_{n+1},s} + ||R(h)||_{\sigma_{n+1},s}\Big) \\ &\le \frac{C'}{\gamma}(L_{n+1})^{\tau-1}\Big(|\varepsilon| \ \exp(-L_n\gamma_n) + |\varepsilon| \ ||h||^2_{\sigma_{n+1},s}\Big). \end{aligned} \tag{28}$$

By (28), if $||h||_{\sigma_{n+1},s} \le \rho_{n+1}$ then $||\mathcal{G}(\delta, v_1, w_n, h)||_{\sigma_{n+1},s} \le C'(L_{n+1})^{\tau-1}\gamma^{-1}|\varepsilon|(\exp(-L_n\gamma_n) + \rho^2_{n+1}) \le \rho_{n+1}$, provided that

$$\frac{C'}{\gamma}(L_{n+1})^{\tau-1}|\varepsilon| \ \exp(-L_n\gamma_n) \le \frac{\rho_{n+1}}{2} \qquad \text{and} \qquad \frac{C'}{\gamma}(L_{n+1})^{\tau-1}|\varepsilon|\rho_{n+1} \le \frac{1}{2}. \tag{29}$$

It is easy to check that for $L_{n+1} := L_0 2^{n+1}$ and $\rho_{n+1} := |\varepsilon|\gamma^{-1} \ \exp(-\chi^{n+1})$ both inequalities in (29) are satisfied $\forall n \ge 0$, choosing L_0 large enough and $|\varepsilon|\gamma^{-1} \le \varepsilon_0$ small enough. With similar estimates, using (23), we can prove that $\forall h, h' \in B(\rho_{n+1}; W^{(n+1)})$, $||\mathcal{G}(\delta, v_1, w_n, h') - \mathcal{G}(\delta, v_1, w_n, h)||_{\sigma_{n+1},s} \le (1/2)||h - h'||_{\sigma_{n+1},s}$ again for L_0 large enough and $|\varepsilon|\gamma^{-1} \le \varepsilon_0$ small enough, uniformly in n, and we conclude that $\mathcal{G}(\delta, v_1, w_n, \cdot)$ is a contraction on $B(\rho_{n+1}; W^{(n+1)})$. ∎

By the standard Contraction Mapping Theorem we deduce the existence, for L_0 large enough and ε small enough, of a unique $h_{n+1} \in W^{(n+1)}$ solving (27) and satisfying

$$||h_{n+1}||_{\sigma_{n+1},s} \le \rho_{n+1} = \frac{|\varepsilon|}{\gamma} \ \exp(-\chi^{n+1}).$$

Summarizing, $w_{n+1}(\delta, v_1) = w_n(\delta, v_1) + h_{n+1}(\delta, v_1)$ is a solution in $W^{(n+1)}$ of equation (P_{n+1}), defined for $(\delta, v_1) \in A_{n+1} \subseteq A_n \subseteq \ldots \subseteq A_1 \subseteq A_0$, and $w_{n+1}(\delta, v_1) = \sum_{i=0}^{n+1} h_i(\delta, v_1)$, $h_i = h_i(\delta, v_1) \in W^{(i)}$ satisfying $||h_i||_{\sigma_i,s} \le |\varepsilon|\gamma^{-1} \ \exp(-\chi^i)$ for some $\chi \in (1, 2)$, $\forall i = 0, \ldots, n+1$. ∎

Remark 3.1 *A difference with respect to the usual "quadratic" Nash-Moser scheme, is that $h_n(\delta, v_1)$ is found as an exact solution of equation (P_n). It appears to be more convenient to prove the regularity of $h_n(\delta, v_1)$ with respect to the parameters (δ, v_1), see Lemma 3.2.*

Corollary 3.1 (Solution of the (P)-equation) *For $(\delta, v_1) \in A_\infty := \cap_{n\ge0} A_n$, $\sum_{i\ge0} h_i(\delta, v_1)$ converges normally in $X_{\overline{\sigma}/2,s}$ to a solution $w(\delta, v_1) \in W \cap X_{\overline{\sigma}/2,s}$ of equation (15) with $||w(\delta, v_1)||_{\overline{\sigma}/2,s} \le C|\varepsilon|\gamma^{-1}$.*

Proof. By proposition 3.1, for $(\delta, v_1) \in A_\infty := \cap_{n\ge0} A_n$, $\sum_{i=0}^{\infty} ||h_i(\delta, v_1)||_{\overline{\sigma}/2,s} < \infty$. Hence $\sum_{i\ge0} h_i(\delta, v_1)$ converges normally in $X_{\overline{\sigma}/2,s}$ to some $w(\delta, v_1) \in W \cap X_{\overline{\sigma}/2,s}$, and we have

$$||w(\delta, v_1)||_{\overline{\sigma}/2,s} \le \sum_{i\ge0} ||h_i(\delta, v_1)||_{\overline{\sigma}/2,s} \le \sum_{i\ge0} ||h_i(\delta, v_1)||_{\sigma_i,s} \le \sum_{i\ge0} |\varepsilon|\gamma^{-1} \ \exp(-\chi^i) = O(|\varepsilon|\gamma^{-1}). \tag{30}$$

Let us justify that $L_\omega w = \varepsilon \Pi_W \Gamma(\delta, v_1, w)$. Since w_n solves equation (P_n) ,

$$L_\omega w_n = \varepsilon P_n \Pi_W \Gamma(\delta, v_1, w_n) = \varepsilon \Pi_W \Gamma(\delta, v_1, w_n) - \varepsilon P_n^\perp \Pi_W \Gamma(\delta, v_1, w_n). \tag{31}$$

We have

$$\Big|\Big|P_n^\perp \Pi_W \Gamma(\delta, v_1, w_n)\Big|\Big|_{\overline{\sigma}/2,s} \le C\exp(-L_n(\sigma_n - (\overline{\sigma}/2))) \le C\exp(-\gamma_0 L_0 2^n/(n^2+1)).$$

Hence, by $(P1)$, the right hand side in (31) converges in $X_{\overline{\sigma}/2,s}$ to $\Gamma(\delta, v_1, w)$. Moreover $(L_\omega w_n) \to L_\omega w$ in the sense of distributions. Hence $L_\omega w = \varepsilon \Pi_W \Gamma(\delta, v_1, w)$. ∎

Before proving the key property $(P3)$ on the linearized operator we prove a "Whitney-differentiability" property for $w(\delta, v_1)$ extending $w(\cdot, \cdot)$ in a smooth way on the whole A_0.

For this, some bound on the derivatives of $h_n = w_n - w_{n-1}$ is required.

Lemma 3.2 (Estimates for the derivatives of h_n and w_n) *For $|\varepsilon|\gamma^{-1}$ small enough and L_0 large enough, $\forall n \geq 0$, the function $(\delta, v_1) \to h_n(\delta, v_1)$ is in $C^\infty(A_n, W^{(n)})$ and the k^{th}-derivative $D^k h_n(\delta, v_1)$ satisfies*

$$\left|\left|D^k h_n(\delta, v_1)\right|\right|_{\sigma_n,s} \leq \frac{|\varepsilon|}{\gamma}\, K_1(k, \overline{\chi}) \exp(-\overline{\chi}^n), \tag{32}$$

for any $\overline{\chi} \in (0, \chi)$ and a suitable positive constant $K_1(k, \overline{\chi})$.

As a consequence, the function $(\delta, v_1) \to w_n(\delta, v_1) = \sum_{i=1}^n h_n(\delta, v_1)$ is in $C^\infty(A_n, W^{(n)})$ and the k^{th}-derivative $D^k w_n(\delta, v_1)$ satisfies

$$\left|\left|D^k w_n(\delta, v_1)\right|\right|_{\sigma_n,s} \leq \frac{|\varepsilon|}{\gamma}\, K_2(k), \tag{33}$$

for a suitable positive constant $K_2(k)$.

Proof. First, $||\partial_\lambda h_0||_{\sigma_0,s} = ||\partial_\lambda w_0||_{\sigma_0,s} = 0$ (we denote $\lambda := (\delta, v_1)$). Next, assume, by induction, that $h_n = h_n(\delta, v_1)$ is a C^∞ map defined in A_n. We shall prove that $h_{n+1} = h_{n+1}(\delta, v_1)$ is C^∞ too. First recall that $h_{n+1} = h_{n+1}(\delta, v_1)$ is defined, in Proposition 3.1, for $(\delta, v_1) \in A_{n+1}$ as a solution in $W^{(n+1)}$ of equation (P_{n+1}), namely

$$(P_{n+1}) \qquad U_{n+1}\Big(\delta, v_1, h_{n+1}(\delta, v_1)\Big) = 0,$$

where

$$U_{n+1}(\delta, v_1, h) := L_\omega(w_n + h) - \varepsilon P_{n+1}\Pi_W \Gamma(\delta, v_1, w_n + h).$$

The map $U_{n+1} : [0, \delta_0) \times V_1 \times W^{(n+1)} \to W^{(n+1)}$ is C^∞, and we claim that $D_h U_{n+1}(\delta, v_1, h_{n+1}) = \mathcal{L}_{n+1}(\delta, v_1, w_{n+1})$ is invertible and that

$$\left|\left|\mathcal{L}_{n+1}(\delta, v_1, w_{n+1})^{-1}\right|\right|_{\sigma_{n+1},s} \leq \frac{C'}{\gamma}(L_{n+1})^{\tau-1}. \tag{34}$$

As a consequence, by the Implicit Function Theorem, the map $(\delta, v_1) \mapsto h_{n+1}(\delta, v_1)$ is in $C^\infty(A_{n+1}, W^{(n+1)})$.

Let us prove (34). By Proposition 3.1, $||h_{n+1}||_{\sigma_{n+1},s} \leq C|\varepsilon|\gamma^{-1}\exp(-\chi^{n+1})$. Hence, by $(P1)$,

$$\left|\left|\mathcal{L}_{n+1}(\delta, v_1, w_{n+1}) - \mathcal{L}_{n+1}(\delta, v_1, w_n)\right|\right|_{\sigma_{n+1},s} \leq C|\varepsilon|\ ||h_{n+1}||_{\sigma_{n+1},s} \leq C\frac{\varepsilon^2}{\gamma}\exp(-\chi^{n+1}).$$

Recalling (26), $\mathcal{L}_{n+1}(\delta, v_1, w_n)$ is invertible and $||\mathcal{L}_{n+1}(\delta, v_1, w_n)^{-1}||_{\sigma_{n+1},s} \leq (C/\gamma)(L_{n+1})^{\tau-1}$. Hence, provided that $\varepsilon\gamma^{-1}$ is small enough (note that $(L_{n+1})^{\tau-1} = (L_0 2^{n+1})^{\tau-1} << \exp(\chi^{n+1})$ for n large), $\mathcal{L}_{n+1}(\delta, v_1, w_{n+1})$ is invertible and (34) holds.

We now prove in detail estimate (32) for $k = 1$. Deriving equation (P_{n+1}) with respect to some coordinate λ of $(\delta, v_1) \in A_{n+1}$, we obtain

$$(P'_{n+1}) \qquad \mathcal{L}_{n+1}(\delta, v_1, w_{n+1})\Big[\partial_\lambda h_{n+1}(\delta, v_1)\Big] = -(\partial_\lambda U_{n+1})\Big(\delta, v_1, h_{n+1}(\delta, v_1)\Big).$$

Hence, by (34), $\partial_\lambda h_{n+1}$ satisfies the estimate

$$\left|\left|\partial_\lambda h_{n+1}\right|\right|_{\sigma_{n+1},s} \leq \frac{C}{\gamma}(L_{n+1})^{\tau-1}\left|\left|(\partial_\lambda U_{n+1})(\delta, v_1, h_{n+1})\right|\right|_{\sigma_{n+1},s}. \tag{35}$$

There holds

$$(\partial_\lambda U_{n+1})(\delta, v_1, h) = L_\omega \partial_\lambda w_n - \varepsilon P_{n+1}\Pi_W\Big[(\partial_\lambda \Gamma)(\delta, v_1, w_n + h) + \partial_w \Gamma(\delta, v_1, w_n + h)[\partial_\lambda w_n]\Big] \tag{36}$$

that we can write as

$$(\partial_\lambda U_{n+1})(\delta, v_1, h) = (\partial_\lambda U_{n+1})(\delta, v_1, 0) + r(\delta, v_1, h), \tag{37}$$

where $r(\delta, v_1, h) := (\partial_\lambda U_{n+1})(\delta, v_1, h) - (\partial_\lambda U_{n+1})(\delta, v_1, 0)$ satisfies, by $(P1)$,

$$\Big|\Big| r(\delta, v_1, h)\Big|\Big|_{\sigma_{n+1},s} \leq C|\varepsilon| \; ||h||_{\sigma_{n+1},s}\Big(1 + ||\partial_\lambda w_n||_{\sigma_{n+1},s}\Big). \tag{38}$$

Now, since $w_n = w_n(\delta, v_1) \in W^{(n)}$ solves equation (P_n),

$$P_n U_{n+1}(\delta, v_1, 0) = L_\omega w_n - \varepsilon P_n \Pi_W \Gamma(\delta, v_1, w_n)) = 0, \quad \forall (\delta, v_1) \in A_n.$$

Hence differentiating w.r.t. λ we get $P_n(\partial_\lambda U_{n+1})(\delta, v_1, 0) = 0$, and so

$$\begin{aligned}
(\partial_\lambda U_{n+1})(\delta, v_1, 0) &= P_n^\perp (\partial_\lambda U_{n+1})(\delta, v_1, 0) \\
&= P_n^\perp L_\omega \partial_\lambda w_n - \varepsilon P_n^\perp P_{n+1} \Pi_W \widetilde{\Gamma}(\delta, v_1) \\
&= -\varepsilon P_n^\perp P_{n+1} \Pi_W \widetilde{\Gamma}(\delta, v_1),
\end{aligned} \tag{39}$$

where, by (36), $\widetilde{\Gamma}(\delta, v_1) := (\partial_\lambda \Gamma)(\delta, v_1, w_n) + \partial_w \Gamma(\delta, v_1, w_n)[\partial_\lambda w_n]$. By (39), $(P2)$, $(P1)$

$$\begin{aligned}
\Big|\Big|(\partial_\lambda U_{n+1})(\delta, v_1, 0)\Big|\Big|_{\sigma_{n+1},s} &\leq |\varepsilon| \; \exp(-L_n \gamma_n) \Big|\Big| \Pi_W \widetilde{\Gamma}(\delta, v_1)\Big|\Big|_{\sigma_n,s} \\
&\leq C|\varepsilon| \exp(-L_n \gamma_n)\Big(1 + ||\partial_\lambda w_n||_{\sigma_n,s}\Big).
\end{aligned} \tag{40}$$

Combining (35), (37), (38), (40) and the bound $||h_{n+1}||_{\sigma_{n+1},s} \leq |\varepsilon|\gamma^{-1} \exp(-\chi^{n+1})$, we get

$$\begin{aligned}
||\partial_\lambda h_{n+1}||_{\sigma_{n+1},s} &\leq \frac{C}{\gamma}(L_{n+1})^{\tau-1}|\varepsilon|\Big(\frac{|\varepsilon|}{\gamma}\exp(-\chi^{n+1}) + \exp(-L_n\gamma_n)\Big)\Big(1 + ||\partial_\lambda w_n||_{\sigma_n,s}\Big) \\
&\leq C(\overline{\chi})\frac{|\varepsilon|}{\gamma} \; \exp(-\overline{\chi}^{n+1})\Big(1 + ||\partial_\lambda w_n||_{\sigma_n,s}\Big) \leq C(\overline{\chi})\frac{|\varepsilon|}{\gamma} \; \exp(-\overline{\chi}^{n+1})\Big(1 + \sum_{i=0}^{n} ||\partial_\lambda h_i||_{\sigma_i,s}\Big)
\end{aligned}$$

for any $\overline{\chi} \in (1, \chi)$, $\varepsilon\gamma^{-1}$ small and L_0 large. Hence, setting $a_n := ||\partial_\lambda h_n||_{\sigma_n,s}$ we get

$$a_0 = 0 \quad \text{ and } \quad a_{n+1} \leq C(\overline{\chi})\frac{|\varepsilon|}{\gamma} \; \exp(-\overline{\chi}^{n+1})\Big(1 + a_0 + \ldots + a_n\Big)$$

which implies $||\partial_\lambda h_n||_{\sigma_n,s} = a_n \leq K(\overline{\chi})|\varepsilon|\gamma^{-1} \; \exp(-\overline{\chi}^n)$, $\forall n \geq 0$, for a suitable positive constant $K(\overline{\chi})$. We can prove in the same way the general estimate (32) from which (33) follows. ■

Since, by (32), $h_n(\delta, v_1) = O(\varepsilon\gamma^{-1}\exp(-\overline{\chi}^n))$ with all its derivatives, and the "non-resonant" set A_n is obtained at each step deleting strips of size $O(\gamma/L_n^\tau)$, we can define (by interpolation, say) a C^∞-extension $\widetilde{w}(\delta, v_1)$ of $w(\delta, v_1)$ for all $(\delta, v_1) \in A_0$. More precisely we can prove:

Lemma 3.3 (Smooth Extension $\widetilde{w}$ of w on A_0) *Given $\nu > 0$, there exists a function $\widetilde{w} \in C^\infty(A_0, W \cap X_{\overline{\sigma}/2,s})$ such that, if $(\delta, v_1) \in A_\infty := \cap_{n\geq 0} A_n$ and $\text{dist}((\delta, v_1), \partial A_n) \geq 2\nu/L_n^3$, $\forall n \geq 0$, then $\widetilde{w}(\delta, v_1)$ solves the (P)-equation (15).*

More precisely $\widetilde{w}(\delta, v_1)$ satisfies, $\forall k \in \mathbf{N}$,

$$\Big|\Big| D^k \widetilde{w}(\delta, v_1)\Big|\Big|_{\overline{\sigma}/2,s} \leq \frac{|\varepsilon|}{\gamma}\frac{C(k)}{\nu^k}, \qquad \forall (\delta, v_1) \in A_0, \tag{41}$$

for suitable constants $C(k) > 0$. Moreover $\widetilde{w}(\delta, v_1) := \lim_{n\to+\infty} \widetilde{w}_n(\delta, v_1)$ where $\widetilde{w}_n$ is in C^∞ $(A_0, W^{(n)})$, and the sequence $(\widetilde{w}_n)$ converges uniformly in A_0 to $\widetilde{w}$ for the norm $|| \; ||_{\overline{\sigma}/2,s}$, more precisely

$$\forall (\delta, v_1) \in A_0, \quad \Big|\Big| \widetilde{w}(\delta, v_1) - \widetilde{w}_n(\delta, v_1)\Big|\Big|_{\overline{\sigma}/2,s} \leq C\frac{|\varepsilon|}{\gamma} \; \exp(-\overline{\chi}^n). \tag{42}$$

PROOF. First we endow $\mathbf{R}\times V_1$ with the Borelian positive measure defined by $\mu(E) = m(L^{-1}(E))$, where L is some automorphism from $\mathbf{R}^{N+1}$ to $\mathbf{R}\times V_1$ and m is the Lebesgue measure in $\mathbf{R}^{N+1}$. Let $\varphi : \mathbf{R}\times V_1 \to \mathbf{R}_+$ be a C^∞-map supported in the open ball of radius 1 centered at 0 with $\int \varphi \, d\mu = 1$. Let

$$\widetilde{A}_n := \left\{ \lambda = (\delta, v_1) \in A_n \mid \mathrm{dist}(\lambda, \partial A_n) \geq \frac{\nu}{L_n^3} \right\} \subset A_n,$$

where ν is some small constant to be specified later. We define φ_n, $\psi_n : \mathbf{R}\times V_1 \to \mathbf{R}_+$ as

$$\varphi_n(\lambda) := \Big(\frac{L_n^3}{\nu}\Big)^{N+1} \varphi\Big(\frac{L_n^3}{\nu}\lambda\Big), \qquad \psi_n(\lambda) := \int_{\widetilde{A}_n} \varphi_n(\lambda - \eta)\, d\mu(\eta) = \Big(\varphi_n * \delta_{\widetilde{A}_n}\Big)(\lambda),$$

where $\delta_{\widetilde{A}_n}$ is the characteristic function of the set $\widetilde{A}_n$ namely, $\delta_{\widetilde{A}_n}(\lambda) := 1$ if $\lambda \in \widetilde{A}_n$ and $\delta_{\widetilde{A}_n}(\lambda) := 0$ if $\lambda \notin \widetilde{A}_n$. Clearly φ_n is a C^∞ map supported in the open ball of radius ν/L_n^3 centered at 0 and $\int \varphi_n \, d\mu = 1$. It follows that $0 \leq \psi_n(\lambda) \leq 1$, $\mathrm{supp}\psi_n \subset \mathrm{int}A_n$ and $\psi_n(\lambda) = 1$ if $\lambda \in \widetilde{A}_n$ satisfies $\mathrm{dist}(\lambda, \partial A_n) \geq 2\nu/L_n^3$. Moreover ψ_n is C^∞ and it is easy to check that $|D^k\psi_n(\lambda)| \leq C(k)(L_n^3/\nu)^k$, $\forall k \in \mathbf{N}$, $\forall \lambda \in \mathbf{R}\times V_1$ for a suitable positive constant $C(k)$.

Now we can define $\widetilde{w}_n : A_0 \to W^{(n)}$ by

$$\widetilde{w}_0(\lambda) := w_0(\lambda) = 0, \qquad \widetilde{w}_{n+1}(\lambda) := \widetilde{w}_n(\lambda) + \widetilde{h}_{n+1}(\lambda)$$

where $\widetilde{h}_{n+1}(\lambda) := \psi_{n+1}(\lambda) h_{n+1}(\lambda)$ if $\lambda \in A_{n+1}$ and $\widetilde{h}_{n+1}(\lambda) = 0$ if $\lambda \notin A_{n+1}$. (note that h_{n+1} is C^∞ because $\mathrm{supp}\psi_{n+1} \subset \mathrm{int}A_{n+1}$). We define $\widetilde{w}_n \in C^\infty(A_0, W^{(n)})$ by $\widetilde{w}_n(\lambda) := \sum_{i=1}^n \widetilde{h}_i(\lambda)$.

By the bound $|D^k\psi_n(\lambda)| \leq C(k)(L_n^3/\nu)^k$ given above and (32) we obtain

$$\Big|\Big| D^k \widetilde{h}_{n+1}(\lambda) \Big|\Big|_{\sigma_{n+1},s} \leq \frac{|\varepsilon|}{\gamma}\, C(k, \overline{\chi}) \Big(\frac{L_{n+1}^3}{\nu}\Big)^k \exp(-\overline{\chi}^n), \qquad \forall k \in N,\ \forall \lambda \in A_0,\ \forall n \geq 0.$$

As a consequence, the sequence $(\widetilde{w}_n)$ (and all its derivatives) converges uniformally in A_0 for the norm $|| \ ||_{\overline{\sigma}/2,s}$ on W, to some function $\widetilde{w}(\delta, v_1) \in C^\infty(A_0, W \cap X_{\overline{\sigma}/2,s})$ which satisfies (41) and (42).

Note that if $\lambda \notin A_\infty := \cap_{n\geq 0} A_n$ then the series $\widetilde{w}(\lambda) = \sum_{n\geq 1} \widetilde{h}_n(\lambda)$ is a finite sum. On the other hand, if $\lambda \in A_\infty$ and $\mathrm{dist}(\lambda, \partial A_n) \geq 2\nu/L_n^3$, $\forall n \geq 0$, then $\widetilde{w}(\lambda) = w(\lambda)$ solves the (P)-equation (15). ■

We complete this part with the following Lemma which will be used in section 5. Define

$$B_n := \Big\{ (\delta, v_1) \in A_0 \mid \delta \in \Delta_n^{2\gamma,\tau}(v_1, \widetilde{w}(\delta, v_1)) \Big\}$$

where we have replaced γ with 2γ in the definition of $\Delta_n^{\gamma,\tau}$, see Definition 3.3.

Lemma 3.4 *If $\nu\gamma^{-1} > 0$ and $\varepsilon\gamma^{-1}$ are small enough, then $B_n \subset \{(\delta, v_1) \in A_n \mid \mathrm{dist}((\delta, v_1), \partial A_n) \geq 2\nu/L_n^3\}$, $\forall n \geq 1$ and hence, if $(\delta, v_1) \in B_\infty := \cap_{n\geq 1} B_n$, then $\widetilde{w}(\delta, v_1)$ solves the (P)-equation (15).*

PROOF. This is a consequence of (41), (42) and the previous Lemma (we use that $L_n^\tau = o(\exp(\widetilde{\chi}^n))$ and $L_n^\tau = o(L_n^2)$ as $n \to \infty$. ■

Remark 3.2 Up to now, we have not justified that A_∞, B_∞ are not empty. It can be proved as in Proposition 5.1 that for any $v_1 \in B(2R, V_1)$ the set $\mathcal{B}_{v_1} := \{\delta \in [0, \delta_0) \mid (\delta, v_1) \in B_\infty\}$ satisfies $\lim_{\eta\to 0+} \mathrm{meas}((0,\eta) \cap \mathcal{B}_{v_1})/\eta = 0$. Hence $A_\infty \neq \emptyset$. ■

4 Analysis of the linearized problem: proof of (P3)

We prove in this section the key property $(P3)$ on the inversion of the linear operator $\mathcal{L}_n(\delta, v_1, w) : W^{(n)} \to W^{(n)}$ defined in (17). Let

$$a(t,x) := \partial_u g\Big(\delta, x, v_1(t,x) + w(t,x) + v_2(\delta, v_1, w)(t,x)\Big)$$

and define the linear operators D, M_1, $M_2 : W^{(n)} \to W^{(n)}$ by

$$\begin{cases} Dh := L_\omega h - \varepsilon P_n \Pi_W (a_0(x)\ h) \\ M_1 h := \varepsilon P_n \Pi_W (\overline{a}(t,x)\ h) \\ M_2 h := \varepsilon P_n \Pi_W (a(t,x)\ \partial_w v_2[h]) \end{cases} \tag{43}$$

where

$$\begin{cases} a_0(x) := (1/2\pi) \int_0^{2\pi} a(t,x)\ dt \\ \overline{a}(t,x) := a(t,x) - a_0(x). \end{cases}$$

$\mathcal{L}_n(\delta, v, w)$ can be written as

$$\begin{aligned} \mathcal{L}_n(\delta, v, w)[h] \quad &:= \quad L_\omega h - \varepsilon P_n \Pi_W D_w \Gamma(\delta, v_1, w)[h] \\ &= \quad L_\omega h - \varepsilon P_n \Pi_W \Big(\partial_u g(\delta, x, v_1 + w + v_2(\delta, v_1, w)) \Big(h + \partial_w v_2(\delta, v_1, w)[h] \Big) \Big) \\ &= \quad L_\omega h - \varepsilon P_n \Pi_W \Big(a(t,x)\ h \Big) - \varepsilon P_n \Pi_W \Big(a(t,x)\ \partial_w v_2(\delta, v_1, w)[h] \Big) \\ &= \quad Dh - M_1 h - M_2 h. \end{aligned}$$

First (Step 1) we prove that, assuming the "first order Melnikov non-resonance condition $\delta \in \Delta_n^{\gamma,\tau}(v_1, w)$ (see Definition 3.3) the linear operator D is invertible, see Corollary 4.2. Next (Step 2) we prove that M_1, M_2 are small enough with respect to D, yielding the invertibility of the whole $\mathcal{L}_n$.

Through this section we shall use the notations $F_k := \{ f \in H_0^1((0,\pi); \mathbf{C}) \mid \int_0^\pi f(x) \sin(kx)\ dx = 0 \}$ whence the space W, defined in (7), and its corresponding projector $\Pi_W : X_{\sigma,s} \to W$, are written, for any $h = \sum_{k \in \mathbf{Z}} \exp(\mathrm{i}kt) h_k$

$$W = \Big\{ h \in X_{\sigma,s} \mid h_k \in F_k\ \forall k \in \mathbf{Z} \Big\}, \qquad \Pi_W h(t,x) = \sum_{k \in \mathbf{Z}} \exp(\mathrm{i}kt) (\pi_k h_k)(x),$$

where $\pi_k : H_0^1((0,\pi); \mathbf{C}) \to F_k$ is the L^2-orthogonal projector onto F_k (note that $\pi_{-k} = \pi_k$ and $\pi_k \overline{u} = \overline{\pi_k u}$, hence $\pi_{-k} h_{-k} = \overline{\pi_k h_k}$).

Step 1: Inversion of $D : W^{(n)} \to W^{(n)}$.

In term of time-Fourier series, D is defined by $\forall\ h \in W^{(n)}$,

$$(Dh)_k = D_k h_k \qquad \forall\ |k| \le L_n,$$

where $D_k : \mathcal{D}(D_k) \subset F_k \to F_k$ is the linear operator

$$D_k u = \omega^2 k^2 u - S_k u \qquad \text{and} \qquad S_k u := -\partial_{xx} u + \varepsilon \pi_k (a_0(x)\ u).$$

Note that $S_k = S_{-k}$.

We now analyze the spectral properties of the Sturm-Liouville type operator S_k. We shall assume that $|\varepsilon| |a_0|_\infty < 1$, so that

$$\langle u, v \rangle_\varepsilon := \int_0^\pi u_x \overline{v}_x + \varepsilon a_0(x) u \overline{v}\ dx$$

defines a scalar product on $H_0^1((0,\pi); \mathbf{C})$, hence on F_k, and its associated norm is equivalent to the standard H^1-norm. More precisely [7],

$$||u||_\varepsilon = ||u||_{H^1} \Big(1 + O(\varepsilon |a_0|_\infty) \Big) \qquad \forall u \in F_k. \tag{44}$$

[7] $\forall u \in H_0^1(0,\pi)$, $\int_0^\pi u^2(x)\ dx \le \int_0^\pi u_x^2(x)\ dx$ since the least eigenvalue of $-\partial_{xx}$ with Dirichelet B.C. on $(0,\pi)$ is 1.

Lemma 4.1 (Sturm-Liouville) *The operator* $S_k : \mathcal{D}(S_k) \subset F_k \to F_k$ *possesses a* $\langle\ ,\ \rangle_\varepsilon$*-orthonormal basis* $(v_{k,j})_{j\ge 1, j\neq |k|}$ *of real eigenvectors with real eigenvalues* $(\lambda_{k,j})_{j\ge 1, j\neq |k|}$*, i.e.* $S_k v_{k,j} = \lambda_{k,j} v_{k,j}$*,* $\lambda_{k,j} \in \mathbf{R}$*, and* $\lambda_{k,j} = \lambda_{-k,j}$*,* $v_{-k,j} = v_{k,j}$*.*

Moreover, $(v_{k,j})_{j\ge 1, j\neq |k|}$ *is an orthogonal basis also for the* L^2*-scalar product in* F_k*. Defining* $\varphi_{k,j} = v_{k,j}/||v_{k,j}||_{L^2}$*,* $\lambda_{k,j}$ *and* $\varphi_{k,j}$ *have the asymptotic expansion as* $j \to +\infty$

$$\lambda_{k,j} = \lambda_{k,j}(\delta, v_1, w) = j^2 + \varepsilon M(\delta, v_1, w) + O\Big(\frac{\varepsilon||a_0||_{H^1}}{j}\Big), \qquad \Big|\varphi_{k,j} - \sqrt{\frac{2}{\pi}}\sin(jx)\Big|_{L^2} = O\Big(\frac{\varepsilon|a_0|_\infty}{j}\Big) \tag{45}$$

where $M(\delta, v_1, w)$*, introduced in Definition 3.1, is the mean value of* $a_0(x)$ *on* $(0, \pi)$*.*

Proof. In the Appendix. ■

By Lemma 4.1, the linear operator $D_k : \mathcal{D}(D_k) \subset F_k \to F_k$ possesses a $\langle\ ,\ \rangle_\varepsilon$-orthonormal basis $(v_{k,j})_{j\ge 1, j\neq|k|}$ of real eigenvectors with real eigenvalues $(\omega^2 k^2 - \lambda_{k,j})_{j\ge 1, j\neq |k|}$. As a consequence we derive

Corollary 4.1 (Diagonalization of D) *The operator* $D : W^{(n)} \to W^{(n)}$ *is the diagonal operator* $\mathrm{diag}\{\omega^2 k^2 - \lambda_{k,j}\}$ *in the basis* $\{\varphi_{0,j}\ ;\ j \ge 1\} \cup (\cup_{j,k\ge 1, j\neq k}\{\cos(kt)\varphi_{k,j}, \sin(kt)\varphi_{k,j}\}$ *of* $W^{(n)}$*.*

Noting that $\min_{|k|\le L_n} |\omega^2 k^2 - \lambda_{k,j}| \to \infty$ as $j \to +\infty$, we deduce from Corollary 4.1 that the linear operator $D : W^{(n)} \to W^{(n)}$ is invertible iff all its eigenvalues $\{\omega^2 k^2 - \lambda_{k,j}(\delta, v_1, w)\ \}_{|k|\le L_n, j\ge 1, j\neq|k|}$ are different from zero. If this holds, we can define D^{-1} as well as $|D|^{-1/2} : W^{(n)} \to W^{(n)}$ by

$$W^{(n)} \ni h = \sum_{k=-L_n}^{L_n} \exp(\mathrm{i}kt) h_k \longrightarrow |D|^{-1/2} h := \sum_{k=-L_n}^{L_n} \exp(\mathrm{i}kt) |D_k|^{-1/2} h_k \in W^{(n)}$$

where $|D_k|^{-1/2} : F_k \to F_k$ is the compact operator defined by

$$|D_k|^{-1/2}\varphi_{k,j} := \frac{\varphi_{k,j}}{\sqrt{|\omega^2 k^2 - \lambda_{k,j}|}}, \qquad \forall j \ge 1,\ j \neq |k|.$$

The "small divisor problem" (i) is that some of the eigenvalues of D, $\omega^2 k^2 - \lambda_{k,j}$, can become arbitrarily small for $(k, j) \in \mathbf{Z}^2$ sufficiently large and therefore the norm of $|D|^{-1/2}$ can become arbitrarily large as $n \to \infty$.

In order to quantify this phenomenon we define for all k

$$\alpha_k := \min_{j\neq|k|} |\omega^2 k^2 - \lambda_{k,j}|. \tag{46}$$

Note that $\alpha_{-k} = \alpha_k$. If $\forall |k| \le L_n,\ \alpha_k \neq 0$, then D is invertible and, since $(\varphi_{k,j})_{k\neq j}$ is an orthogonal basis for the $\langle\ ,\ \rangle_\varepsilon$ scalar product, $||\ |D_k|^{-1/2} u\ ||_\varepsilon \le \alpha_k^{-1/2} ||u||_\varepsilon$. Hence, by (44),

$$\Big|\Big||D_k|^{-1/2} u\Big|\Big|_{H^1} \le \frac{C}{\sqrt{\alpha_k}} ||u||_{H^1}, \qquad \forall k \in \mathbf{Z}. \tag{47}$$

The condition $\forall |k| \le L_n,\ \alpha_k \neq 0$, depends very sensitively on the parameters (δ, v_1). Assuming the "first order Melnikov non-resonance condition" $\delta \in \Delta_n^{\gamma,\tau}(v_1, w)$ (see Definition 3.3), we obtain, in Lemma 4.2, a lower bound of the form $c/|k|^{\tau-1}$ for the moduli of the eigenvalues of D_k and, therefore, in Corollary 4.2, sufficiently good estimates for the inverse of D.

Lemma 4.2 (Lower bound for the eigenvalues of D) *If* $\delta \in \Delta_n^{\gamma,\tau}(v_1, w) \cap [0, \delta_0)$ *and* δ_0 *is small enough (depending on* γ*), then (recall that* $1 < \tau < 2$ *)*

$$\alpha_k := \min_{j\ge 1, j\neq|k|} |\omega^2 k^2 - \lambda_{k,j}| \ge \frac{\gamma}{|k|^{\tau-1}} > 0, \qquad \forall\ |k| \le L_n. \tag{48}$$

PROOF. Since $\alpha_{-k} = \alpha_k$ it is sufficient to consider $k \geq 0$. By the asymptotic expansion (45) for the eigenvalues $\lambda_{k,j}$, using that $||a_0||_{H^1}$, $|M(\delta, v_1, w)| \leq C$,

$$\begin{aligned}
|\omega^2 k^2 - \lambda_{k,j}| &= \Big|\omega^2 k^2 - j^2 - \varepsilon M(\delta, v_1, w) + O\Big(\frac{\varepsilon ||a_0||_{H^1}}{j}\Big)\Big| \\
&= \Big|\Big(\omega k - \sqrt{j^2 + \varepsilon M(\delta, v_1, w)}\Big)\Big(\omega k + \sqrt{j^2 + \varepsilon M(\delta, v_1, w)}\Big) + O\Big(\frac{|\varepsilon|}{j}\Big)\Big| \\
&\geq \Big|\omega k - j - \varepsilon \frac{M(\delta, v_1, w)}{2j} + O\Big(\frac{\varepsilon^2}{j^3}\Big)\Big|\omega k - C\frac{|\varepsilon|}{j} \\
&\geq \Big|\omega k - j - \varepsilon \frac{M(\delta, v_1, w)}{2j}\Big| \frac{k}{2} - C'\Big(\frac{\varepsilon^2 k}{j^3} + \frac{|\varepsilon|}{j}\Big) \geq \frac{c\gamma}{k^{\tau-1}} - C\Big(\frac{\varepsilon^2 k}{j^3} + \frac{|\varepsilon|}{j}\Big), \qquad (49)
\end{aligned}$$

since $\delta \in \Delta_n^{\gamma,\tau}(v_1, w)$. If $\alpha_k := \min_{j \geq 1, j \neq k} |\omega^2 k^2 - \lambda_{k,j}|$ is attained at $j = j(k)$, i.e. $\alpha_k = |\omega^2 k^2 - \lambda_{k,j}|$ then, $j \leq 2k$ and therefore, by (49) and since $1 < \tau < 2$, we obtain (48). ■

Corollary 4.2 (Estimate of $|D|^{-1/2}$) *If $\delta \in \Delta_n^{\gamma,\tau}(v_1, w) \cap [0, \delta_0)$ and δ_0 is small enough, then $D : W^{(n)} \to W^{(n)}$ is invertible and $\forall s' \geq 0$*

$$\Big|\Big||D|^{-1/2} h\Big|\Big|_{\sigma,s'} \leq \frac{C}{\sqrt{\gamma}} ||h||_{\sigma, s' + \frac{\tau-1}{2}} \qquad \forall h \in W^{(n)}. \qquad (50)$$

PROOF. Use (47) and (48). ■

Step 2: Inversion of $\mathcal{L}_n : W^{(n)} \to W^{(n)}$.

In order to show the invertibility of $\mathcal{L}_n : W^{(n)} \to W^{(n)}$ it is a convenient devise to write

$$\mathcal{L}_n = D - M_1 - M_2 = |D|^{1/2}\Big(U - R_1 - R_2\Big)|D|^{1/2}$$

where

$$U := |D|^{-1/2} D |D|^{-1/2} = |D|^{-1} D \qquad \text{and} \qquad R_i := |D|^{-1/2} M_i |D|^{-1/2}, \quad i = 1, 2.$$

We shall prove the invertibility of $U - R_1 - R_2 : W^{(n)} \to W^{(n)}$ showing that, for ε small enough, R_1 and R_2 are small perturbations of U.

Lemma 4.3 (Estimate of $||U^{-1}||$) *$U : W^{(n)} \to W^{(n)}$ is an invertible operator and its inverse U^{-1} satisfies, $\forall s' \geq 0$,*

$$\Big|\Big|U^{-1} h\Big|\Big|_{\sigma,s'} = ||h||_{\sigma,s'}\Big(1 + O(\varepsilon ||a_0||_{H^1})\Big) \qquad \forall\ h \in W^{(n)}. \qquad (51)$$

PROOF. $U_k := |D_k|^{-1} D_k : F_k \to F_k$ being orthogonal for the $\langle\ ,\ \rangle_\varepsilon$ scalar product, it is invertible and $\forall u \in F_k, ||U_k^{-1} u||_\varepsilon = ||u||_e$. Hence, by (44), there is $C \geq 0$ such that

$$\forall u \in F_k, \quad ||U_k^{-1} u||_{H^1} \leq ||u||_e (1 + C\varepsilon ||a_0||_{H^1})$$

Therefore, $U = |D|^{-1} D$, being defined by $(Uh)_k = U_k h_k$, $\forall\ |k| \leq L_n$, U is invertible, $(U^{-1} h)_k = U_k^{-1} h_k$ and (51) holds. ■

For proving the smallness of R_1 and R_2 we need the following preliminary Lemma.

Lemma 4.4 *There are $\mu > 0$, $\delta_0 > 0$ and $C > 0$ with the following property : if $||v_1||_{0,s} \leq 2R$, $[w]_{\sigma,s} \leq \mu$ and $\delta \in [0, \delta_0)$, then $||a||_{\sigma, s + \frac{2(\tau-1)}{\beta}} \leq C$.*

PROOF. By the Definition 3.2 of $[w]_{\sigma,s}$ there are $h_0, h_1, \ldots, h_q \in W^{(i)}$ and a sequence $(\sigma_i)_{0\le i\le q}$ with $\sigma_i > \sigma$, such that $w = h_0 + h_1 + \ldots + h_q$ and

$$\sum_{i=0}^{q} \frac{||h_i||_{\sigma_i,s}}{(\sigma_i-\sigma)^{\frac{2(\tau-1)}{\beta}}} \le 2\mu. \tag{52}$$

An elementary calculus shows that[8]

$$||h_i||_{\sigma,s+\frac{2(\tau-1)}{\beta}} \le C(\tau)\frac{||h_i||_{\sigma_i,s}}{(\sigma_i-\sigma)^{\frac{2(\tau-1)}{\beta}}}. \tag{53}$$

Hence, by (52)-(53)

$$||w||_{\sigma,s+\frac{2(\tau-1)}{\beta}} \le \sum_{i=0}^{q} ||h_i||_{\sigma,s+\frac{2(\tau-1)}{\beta}} \le \sum_{i=0}^{q} C(\tau)\frac{||h_i||_{\sigma_i,s}}{(\sigma_i-\sigma)^{\frac{2(\tau-1)}{\beta}}} \le 2C(\tau)\mu$$

and by Lemma 2.1, provided μ is small enough, $v_2(\delta,v_1,w)$ is well defined and $||v_2(\delta,v_1,w)||_{\sigma,s+\frac{2(\tau-1)}{\beta}} \le C'$. Hence, by the algebra property of the norm $||\ ||_{\sigma,s+\frac{2(\tau-1)}{\beta}}$ and the analyticity of f, $||a||_{\sigma,s+\frac{2(\tau-1)}{\beta}}$ is bounded by some constant, provided δ_0 has been chosen small enough. ■

The "smallness" of $R_2 : W^{(n)} \to W^{(n)}$ is just a consequence of the regularizing property of $\partial_w v_2 : X_{\sigma,s} \to X_{\sigma,s+2}$ proved in Lemma 2.1 and Lemma 4.4.

Lemma 4.5 (Estimate of R_2) *Under the hypotheses of (P3), there exists a constant $C > 0$ depending on μ such that*

$$\Big|\Big|R_2 h\Big|\Big|_{\sigma,s+\frac{\tau-1}{2}} \le C\frac{|\varepsilon|}{\gamma}\ ||h||_{\sigma,s+\frac{\tau-1}{2}} \qquad \forall h \in W^{(n)}.$$

PROOF. Using (50) and the regularizing estimates $||\partial_w v_2[u]||_{\sigma,s+2} \le C||u||_{\sigma,s}$ of Lemma 2.1 we get

$$\begin{aligned}
\Big|\Big|R_2 h\Big|\Big|_{\sigma,s+\frac{\tau-1}{2}} &\le \frac{C}{\sqrt{\gamma}}\Big|\Big|M_2|D|^{-1/2}h\Big|\Big|_{\sigma,s+\tau-1} = C\frac{|\varepsilon|}{\sqrt{\gamma}}\ \Big|\Big|P_n\Pi_W\Big(a\ \partial_w v_2\Big[|D|^{-1/2}h\Big]\Big)\Big|\Big|_{\sigma,s+\tau-1} \\
&\le C\frac{|\varepsilon|}{\sqrt{\gamma}}\ ||a||_{\sigma,s+\tau-1}\Big|\Big|\partial_w v_2\Big[|D|^{-1/2}h\Big]\Big|\Big|_{\sigma,s+\tau-1} \\
&\le C'\frac{|\varepsilon|}{\sqrt{\gamma}}\ ||a||_{\sigma,s+\tau-1}\Big|\Big|\partial_w v_2\Big[|D|^{-1/2}h\Big]\Big|\Big|_{\sigma,s+2} \\
&\le C\frac{|\varepsilon|}{\sqrt{\gamma}}\ ||a||_{\sigma,s+\tau-1}\Big|\Big||D|^{-1/2}h\Big|\Big|_{\sigma,s} \le C'\frac{|\varepsilon|}{\gamma}\ ||h||_{\sigma,s+\frac{\tau-1}{2}}
\end{aligned}$$

since $1 < \tau < 2$ and by Lemma 4.4, $||a||_{\sigma,s+\tau-1} \le ||a||_{\sigma,s+\frac{2(\tau-1)}{\beta}} \le C$. ■

The estimate of the "off-diagonal" operator $R_1 : W^{(n)} \to W^{(n)}$ requires, on the contrary, a careful analysis of the "*small divisors*" and the use of the "first order Melnikov non-resonance condition" $\delta \in \Delta_n^{\gamma,\tau}(v_1,w)$, see Definition 3.3. For clarity, we enounce such property separately.

Lemma 4.6 (Analysis of the Small Divisors) *Let $\delta \in \Delta_n^{\gamma,\tau}(v_1,w) \cap [0,\delta_0)$, with δ_0 small. There exists $C > 0$ such that, $\forall l \ne k$,*

$$\frac{1}{\alpha_k\alpha_l} \le C\frac{|k-l|^{2\frac{\tau-1}{\beta}}}{\gamma^2|\varepsilon|^{\tau-1}} \qquad \text{where} \qquad \beta := \frac{2-\tau}{\tau}. \tag{54}$$

PROOF. To obtain (54) we distinguish different cases.

[8] Using that $\max_{k\ge1} k^\alpha \exp\{-(\sigma_i-\sigma)k\} \le C(\alpha)/(\sigma_i-\sigma)^\alpha$

- FIRST CASE: $|k-l| \geq [\max(|k|,|l|)]^{\beta}$. Then $(\alpha_k\alpha_l)^{-1} \leq C|k-l|^{2\frac{\tau-1}{\beta}}/\gamma^2$.

Indeed we estimate both α_k, α_l with the lower bound (48), $\alpha_k \geq C\gamma/|k|^{\tau-1}$, $\alpha_l \geq C\gamma/|l|^{\tau-1}$ and therefore

$$\frac{1}{\alpha_k\alpha_l} \leq C\frac{|k|^{\tau-1}|l|^{\tau-1}}{\gamma^2} \leq C\frac{[\max(|k|,|l|)]^{2(\tau-1)}}{\gamma^2} \leq C\frac{|k-l|^{2\frac{\tau-1}{\beta}}}{\gamma^2}.$$

- SECOND CASE: $|k-l| < [\max(|k|,|l|)]^{\beta}$ and ($|k| \leq 1/3|\varepsilon|$ or $|l| \leq 1/3|\varepsilon|$). Then $(\alpha_k\alpha_l)^{-1} \leq C/\gamma$.

Note that, in this case, $\mathrm{sign}(l) = \mathrm{sign}(k)$ and, to fix the ideas, we assume in the sequel that $l, k \geq 0$ (the estimate for $k, l < 0$ is the same, since $\alpha_k\alpha_l = \alpha_{-k}\alpha_{-l}$).

Suppose, for example, that $0 \leq k \leq 1/3|\varepsilon|$. We claim that if ε is small enough, then $\alpha_k \geq (k+1)/8$. Indeed, $\forall j \neq k$,

$$|\omega k - j| = \Big|\omega k - k + k - j\Big| \geq |k-j| - |\omega - 1|\ |k| \geq 1 - 2|\varepsilon|\ k \geq \frac{1}{3}.$$

Therefore $\forall 1 \leq k < 1/3|\varepsilon|$, $\forall j \neq k$, $j \geq 1$, $|\omega^2k^2 - j^2| = |\omega k - j|\ |\omega k + j| \geq (\omega k + 1)/3 \geq (k+1)/6$ and so

$$\alpha_k := \min_{j\geq 1, k\neq j}\Big|\omega^2k^2 - \lambda_{k,j}\Big| = \min_{j\geq 1, k\neq j}\Big|\omega^2k^2 - j^2 - \varepsilon M(\delta, v_1, w) + O\Big(\frac{\varepsilon||a_0||_{H^1}}{j}\Big)\Big| \geq \frac{k+1}{6} - |\varepsilon|\ C \geq \frac{k+1}{8}.$$

Next, we estimate α_l. If also $0 \leq l \leq 1/3|\varepsilon|$ then $\alpha_l \geq 1/8$ and therefore $(\alpha_k\alpha_l)^{-1} \leq 64$. Otherwise, if $l > 1/3|\varepsilon|$, we estimate α_l with the lower bound (48) and so, since[9] $l \leq 2k$ and $1 < \tau < 2$

$$\frac{1}{\alpha_k\alpha_l} \leq C\frac{l^{\tau-1}}{k\gamma} \leq \frac{C'}{k^{2-\tau}\gamma} \leq \frac{C'}{\gamma}.$$

In the remaining cases we consider $|k-l| < [\max(|k|,|l|)]^{\beta}$ and both $|k|, |l| > 1/3|\varepsilon|$. We have to distinguish two sub-cases. For this, $\forall k \in \mathbf{Z}$, let $j = j(k) \geq 1$ be the unique integer such that $\alpha_k := \min_{n\neq|k|}|\omega^2k^2 - \lambda_{k,n}| = |\omega^2k^2 - \lambda_{k,j}|$. Analogously let $i = i(k) \geq 1$ be the unique integer such that $\alpha_l = |\omega^2l^2 - \lambda_{l,i}|$.

- THIRD CASE: $0 < |k-l| < [\max(|k|,|l|)]^{\beta}$, $|k|, |l| > 1/3|\varepsilon|$ and $|k-l| = |j-i|$. Then $(\alpha_k\alpha_l)^{-1} \leq C/\gamma|\varepsilon|^{\tau-1}$.

Indeed $|(\omega k - j) - (\omega l - i)| = |\omega(k-l) - (j-i)| = |\omega - 1||k-l| \geq |\varepsilon|/2$ and therefore or $|\omega k - j| \geq |\varepsilon|/4$ or $|\omega l - i| \geq |\varepsilon|/4$. It follows that $|\omega^2k^2 - j^2| = |\omega k - j|\ |\omega k + j| \geq |\varepsilon|\omega k/2 \geq |\varepsilon|k/3$ and so, for ε small enough, $|\alpha_k| \geq |\varepsilon|k/4$, or $|\alpha_l| \geq |\varepsilon|l/4$. Hence, since $l \leq 2k$ and $k > 1/3$,

$$\frac{1}{\alpha_k\alpha_l} \leq C\frac{l^{\tau-1}}{\gamma|\varepsilon|k} \leq \frac{C}{\gamma k^{2-\tau}|\varepsilon|} \leq \frac{C}{\gamma|\varepsilon|^{\tau-1}}$$

- FOURTH CASE: $0 < |k-l| < [\max(k,l)]^{\beta}$, $k, l > 1/3|\varepsilon|$ and $|k-l| \neq |j-i|$. Then $(\alpha_k\alpha_l)^{-1} \leq C/\gamma^2$.

Using that ω is $\gamma-\tau$-Diophantine, that $|k-l| < [\max(k,l)]^{\beta}$ and so $l \geq 2k$,

$$\Big|(\omega k - j) - (\omega l - i)\Big| = \Big|\omega(k-l) - (j-i)\Big| \geq \frac{\gamma}{|k-l|^{\tau}} \geq \frac{\gamma}{[\max(k,l)]^{\beta\tau}} \geq c\frac{\gamma}{k^{\beta\tau}}$$

so that or $|\omega k - j| \geq c\gamma/2k^{\beta\tau}$ or $|\omega l - i| \geq c\gamma/2l^{\beta\tau}$. Therefore $|\omega^2k^2 - j^2| \geq C\gamma k^{1-\beta\tau}$ and, for ε small enough, $\alpha_k \geq C\gamma k^{1-\beta\tau}/2$. We estimate α_l with the worst possible lower bound and so, using also $l \leq 2k$,

$$\frac{1}{\alpha_k\alpha_l} \leq \frac{Cl^{\tau-1}}{\gamma^2k^{1-\beta\tau}} \leq C\frac{k^{\tau-2+\beta\tau}}{\gamma^2} \leq \frac{C}{\gamma^2},$$

since $\beta := (2-\tau)/\tau$. Collecting the estimates of all the previous cases, (54) follows. ■

[9] Indeed $|k-l| \leq (\max\{k,l\})^{\beta}$ and so or $l \leq k$ or $l \leq k + l^{\beta}$ and so, $l/2 < k$, since $l \geq 1/3|\varepsilon|$.

Lemma 4.7 (Bound of an off-diagonal operator) *Assume that $\delta \in \Delta_n^{\gamma,\tau}(v_1, w) \cap [0, \delta_0)$ and let, for some $s' \geq s$, $b(t,x) \in X_{\sigma,s'+\frac{\tau-1}{\beta}}$ satisfy $b_0(x) = 0$, i.e. $\int_0^{2\pi} b(t,x)\, dt \equiv 0$, $\forall x \in (0,\pi)$. Defining the operator $T_n : W^{(n)} \to W^{(n)}$ by*

$$T_n h := |D|^{-1/2} P_n \Pi_W \Big(b(t,x)\ |D|^{-1/2} h \Big),$$

there is a constant $\widetilde{C}$, independent of $b(x,t)$ and of p, such that

$$\Big|\Big| T_n h \Big|\Big|_{\sigma,s'} \leq \frac{\widetilde{C}}{|\varepsilon|^{\frac{\tau-1}{2}} \gamma} ||b||_{\sigma,s'+\frac{\tau-1}{\beta}} ||h||_{\sigma,s'} \qquad \forall h \in W^{(n)}.$$

Proof. For $h \in W^{(n)}$, we have $(T_n h)(t,x) = \sum_{|k| \leq L_n} (T_n h)_k(x) \exp(ikt)$, with

$$\begin{aligned} (T_n h)_k &= |D_k|^{-1/2} \pi_k \Big(b\ |D|^{-1/2} h \Big)_k \\ &= |D_k|^{-1/2} \pi_k \Big[\sum_{|l| \leq L_n} b_{k-l} |D_l|^{-1/2} h_l \Big] \end{aligned} \tag{55}$$

(since $\overline{b}_{k-l}(x) = b_{l-k}(x)$, the linear operator T_n is represented by a self-adjoint Toepliz matrix in $\mathrm{Mat}(2L_n \times 2L_n,\ H^1((0,\pi), \mathbf{C}))$ which is zero on the diagonal, as $b_0(x) = 0$). Abbreviating $B_m := ||b_m(x)||_{H^1}$, we get from (55) and (47), using that $B_0 := ||b_0(x)||_{H^1} = 0$,

$$\Big|\Big| (T_n h)_k \Big|\Big|_{H^1} \leq C \sum_{|l| \leq L_n, l \neq k} \frac{B_{k-l}}{\sqrt{\alpha_k}\sqrt{\alpha_l}} ||h_l||_{H^1}. \tag{56}$$

Hence, by (54),

$$\Big|\Big| (T_n h)_k \Big|\Big|_{H^1} \leq \frac{C}{\gamma |\varepsilon|^{\frac{\tau-1}{2}}} s_k \qquad \text{where} \qquad s_k := \sum_{|l| \leq L_n} B_{k-l} |k-l|^{\frac{\tau-1}{\beta}} ||h_l||_{H^1}. \tag{57}$$

By (57), setting $\widetilde{s}(t) := \sum_{|k| \leq L_n} s_k \exp(ikt)$,

$$\begin{aligned} \Big|\Big| T_n h \Big|\Big|_{\sigma,s'}^2 &= \sum_{|k| \leq L_n} \exp(2\sigma|k|)(k^{2s'}+1) \Big|\Big| (T_n h)_k \Big|\Big|_{H^1}^2 \\ &\leq \frac{C^2}{\gamma^2 |\varepsilon|^{\tau-1}} \sum_{|k| \leq L_n} \exp(2\sigma|k|)(k^{2s'}+1) s_k^2 = \frac{C^2}{\gamma^2 |\varepsilon|^{\tau-1}} ||\widetilde{s}||_{\sigma,s'}^2 \end{aligned} \tag{58}$$

It turns out that $\widetilde{s} = P_n(\widetilde{b}\widetilde{c})$ where $\widetilde{b}(t) := \sum_{l \in \mathbf{Z}} |l|^{\frac{\tau-1}{\beta}} B_l \exp(ilt)$ and $\widetilde{c}(t) := \sum_{|l| \leq L_n} ||h_l||_{H^1} \exp(ilt)$. Therefore, by (58),

$$\Big|\Big| T_n h \Big|\Big|_{\sigma,s'} \leq \frac{C}{\gamma |\varepsilon|^{\frac{\tau-1}{2}}} ||\widetilde{b}\widetilde{c}||_{\sigma,s'} \leq \frac{C}{\gamma |\varepsilon|^{\frac{\tau-1}{2}}} ||\widetilde{b}||_{\sigma,s'} ||\widetilde{c}||_{\sigma,s'} \leq \frac{C}{\gamma |\varepsilon|^{\frac{\tau-1}{2}}} ||b||_{\sigma,s'+\frac{\tau-1}{\beta}} ||h||_{\sigma,s'}$$

since $||\widetilde{b}||_{\sigma,s'} \leq ||b||_{\sigma,s'+\frac{\tau-1}{\beta}}$ and $||\widetilde{c}||_{\sigma,s'} = ||h||_{\sigma,s'}$. ■

Lemma 4.8 (Estimate of R_1) *Under the hypotheses of (P3), there exists a constant $C > 0$ depending on μ such that*

$$\Big|\Big| R_1 h \Big|\Big|_{\sigma,s+\frac{\tau-1}{2}} \leq |\varepsilon|^{\frac{3-\tau}{2}} \frac{C}{\gamma} ||h||_{\sigma,s+\frac{\tau-1}{2}} \qquad \forall h \in W^{(n)}.$$

PROOF. Recalling the definition of $R_1 := |D|^{-1/2} M_1 |D|^{-1/2}$ and M_1, and using Lemma 4.7 since $\overline{a}(t,x)$ has zero time-average,

$$\begin{aligned}
\left|\left|R_1 h\right|\right|_{\sigma, s+\frac{\tau-1}{2}} &= \left|\left||D|^{-1/2} M_1 |D|^{-1/2} h\right|\right|_{\sigma, s+\frac{\tau-1}{2}} = |\varepsilon| \left|\left||D|^{-1/2} P_n \Pi_W \Big(\overline{a}\ |D|^{-1/2} h\Big)\right|\right|_{\sigma, s+\frac{\tau-1}{2}} \\
&\leq |\varepsilon| \frac{\widetilde{C}}{|\varepsilon|^{\frac{\tau-1}{2}} \gamma} ||\overline{a}||_{\sigma, s+\frac{\tau-1}{2}+\frac{\tau-1}{\beta}} ||h||_{\sigma, s+\frac{\tau-1}{2}} \leq |\varepsilon|^{\frac{3-\tau}{2}} \frac{\widetilde{C}}{\gamma} ||\overline{a}||_{\sigma, s+\frac{2(\tau-1)}{\beta}} ||h||_{\sigma, s+\frac{\tau-1}{2}} \\
&\leq |\varepsilon|^{\frac{3-\tau}{2}} \frac{C}{\gamma} ||h||_{\sigma, s+\frac{\tau-1}{2}}
\end{aligned}$$

since $0 < \beta < 1$ and, by Lemma 4.4, $||\overline{a}||_{\sigma, s+\frac{2(\tau-1)}{\beta}} \leq ||a||_{\sigma, s+\frac{2(\tau-1)}{\beta}} \leq C$. ■

PROOF OF PROPERTY $(P3)$ COMPLETED. Under the hypothesis of $(P3)$, the linear operator $U : W^{(n)} \to W^{(n)}$ is invertible by Lemma 4.3 and, by Lemmas 4.5 and 4.8, provided δ is small enough, $||U^{-1} R_1||_{\sigma, s+\frac{\tau-1}{2}}$ and $||U^{-1} R_2||_{\sigma, s+\frac{\tau-1}{2}} < 1/4$. Therefore also the linear operator $U - R_1 - R_2 : W^{(n)} \to W^{(n)}$ is invertible and its inverse satisfies

$$\begin{aligned}
||(U - R_1 - R_2)^{-1} h||_{\sigma, s+\frac{\tau-1}{2}} &= ||(I - U^{-1} R_1 - U^{-1} R_2)^{-1} U^{-1} h||_{\sigma, s+\frac{\tau-1}{2}} && (59) \\
&\leq 2||U^{-1} h||_{\sigma, s+\frac{\tau-1}{2}} \leq C||h||_{\sigma, s+\frac{\tau-1}{2}} \qquad \forall h \in W^{(n)}. && (60)
\end{aligned}$$

Hence $\mathcal{L}_n$ is invertible, $\mathcal{L}_n^{-1} = |D|^{-1/2}(U - R_1 - R_2)^{-1} |D|^{-1/2} : W^{(n)} \to W^{(n)}$, and by (50), (59),

$$\begin{aligned}
||\mathcal{L}_n^{-1} h||_{\sigma, s} &= |||D|^{-1/2}(U - R_1 - R_2)^{-1} |D|^{-1/2} h||_{\sigma, s} \leq \frac{C}{\sqrt{\gamma}} ||(U - R_1 - R_2)^{-1} |D|^{-1/2} h||_{\sigma, s+\frac{\tau-1}{2}} \\
&\leq \frac{C'}{\sqrt{\gamma}} |||D|^{-1/2} h||_{\sigma, s+\frac{\tau-1}{2}} \leq \frac{C''}{\gamma} ||h||_{\sigma, s+\tau-1} \leq \frac{C''}{\gamma} (L_n)^{\tau-1} ||h||_{\sigma, s}
\end{aligned}$$

which completes the proof of property $(P3)$.

5 Solution of the (Q_1)-equation

Finally, we have to solve the finite dimensional (Q_1)-equation

$$-\Delta v_1 = \Pi_{V_1} \mathcal{G}(\delta, v_1) \tag{61}$$

where

$$\mathcal{G}(\delta, v_1)(t,x) := g\Big(\delta, x, v_1(t,x) + \widetilde{w}(\delta, v_1)(t,x) + v_2(\delta, v_1, \widetilde{w}(\delta, v_1))(t,x)\Big)$$

and we have to ensure that there are solutions $(\delta, v_1) \in B_\infty$ for a set[10] of δ's of positive.

5.1 The set $\mathcal{A}_p$

By Lemma 2.2 the 0^{th}-order (Q_1)-equation

$$-\Delta v_1 = \Pi_{V_1} \mathcal{G}(0, v_1) \tag{62}$$

is the Euler-Lagrange equation of the "reduced" functional $\Psi_0 : B(2R, V_1) \to \mathbf{R}$ defined in (13) (note that $\mathcal{G}(0, v_1) = a_p(x)(v_1 + v_2(0, v_1, 0))^p$ and see formula (14)).

[10] It is rather easy to see that for ω close to 1 and $\delta = [(\omega^2 - 1)/2]^{1/(p-1)}$, the set $E_\omega = \{v_1 \in V_1 \mid (\delta, v_1) \in B_\infty\}$ contains the whole ball $B(2R, V_1)$ only if ω is strongly nonresonant, i.e. belongs to the zero-measure set $\mathcal{W}_\gamma := \{\omega \in \mathbf{R} \mid |\omega l - j| \geq \gamma/l,\ \forall j \neq l, l \geq 0, j \geq 1\}$ for some $\gamma > 0$. If ω is strongly nonresonant, the existence of $2\pi/\omega$-periodic solutions can be proved ([4]-[5]) for ANY nonlinearity f. For more general frequencies, the set E_ω has gaps, which makes the analysis of the $(Q1)$ equation more delicate.

For any $a_p(x) \in H^1((0,\pi),\mathbf{R})$ such that condition (6) is verified, Ψ_0, or the functional $\widetilde{\Psi}_0$ obtained replacing $a_p(x)$ by $-a_p(x)$, possesses, by the Mountain-pass Theorem [1], a non-trivial critical point $\overline{v}_1 \in V_1$ with $||\overline{v}_1||_{0,s} \le R$ (R depending on a_p). More precisely, due to time translation invariance, we have a circle of critical points. In fact the functional Ψ_0 is invariant under the action of $\mathbf{R}/2\pi\mathbf{Z}$ on V_1 defined by

$$(\theta * v_1)(t,x) := v_1(t-\theta,x).$$

We shall say that a circle of critical points $[v_1] := \{\theta * v_1 \,;\theta \in \mathbf{R}/2\pi\mathbf{Z}\}$ is non degenerate when $\ker \Psi_0''(v_1)$ is spanned by $(\partial/\partial\theta)(\theta * v_1)_{|\theta=0} = \partial_t v_1$.

We recall that, by Lemma 6.1, condition (6) holds for any $a_p(x) \in \mathcal{O} \subset H^1((0,\pi),\mathbf{R})$ where

$$\mathcal{O} := \Big\{ a(x) \in H^1((0,\pi),\mathbf{R}) \text{ such that } \begin{cases} a(\pi-x) \neq -a(x), \text{ for some } x \in [0,\pi], \text{ if } p \text{ is odd} \\ a(\pi-x) \neq a(x), \text{ for some } x \in [0,\pi], \text{ if } p \text{ is even} \end{cases} \Big\}.$$

We can define similarly the nondegenerate critical circles of Φ_0 in V.

Remark 5.1 *$[\overline{v}_1]$ is a non-degenerate critical circle of $\Psi_0 : V_1 \to \mathbf{R}$ iff $[\overline{v}] = [\overline{v}_1 + v_2(0,\overline{v}_1,0)]$ is a non-degenerate critical circle of $\Phi_0 : V \to \mathbf{R}$, i.e. iff $\overline{v} \in V$ is, up to time translations, a non-degenerate solution of the 0^{th}-order bifurcation equation (9). Moreover, if $[\overline{v}]$ is a nondegenerate critical circle of Φ_0 of critical value $\le c$ then $[\Pi_{V_1}\overline{v}]$ is a nondegenerate critical circle of Ψ_0.*

In [3] it is proved that, for the nonlinearity $f(x,u) = u^3$, the critical points of Φ_0 are non-degenerate up to time translations.

Let us define

$$\mathcal{A}_p^1 := \Big\{ a_p(x) \in \mathcal{O} \subset H^1(0,\pi) \mid \text{there is a non-degenerate critical circle } [\overline{v}] \subset V\backslash\{0\} \text{ of } \Phi_0 \text{ or } \widetilde{\Phi}_0 \Big\},$$

where $\widetilde{\Phi}_0$ is obtained from Φ_0 replacing a_p with $-a_p$.

For any $a_p \in \mathcal{A}_p^1$ the 0^{th}-order bifurcation equation (9) (or the one obtained substituting $-a_p$ for a_p) possesses a non-trivial, non-degenerate solution (up to time translations) $\overline{v} \in V$. We can always assume that we have chosen c large enough in (12), so that, by Lemma 2.2, $\overline{v} = \overline{v}_1 + v_2(0,\overline{v}_1,0)$ for some non degenerate (up to time translations) solution $\overline{v}_1 \in B(2R,V_1)$ of equation (62). By the Implicit function Theorem, there exists a C^∞-curve of solutions of the (Q_1)-equation (61)

$$v_1(\cdot) : \ [0,\delta_0) \to V_1 \qquad \text{with} \qquad v_1(0) = \overline{v}_1.$$

Let us define the Cantor-like set

$$\mathcal{C}_{a_p,\overline{v}_1} := \Big\{ \delta \in [0,\delta_0) \mid (\delta, v_1(\delta)) \in B_\infty \Big\}.$$

The smoothness of $v_1(\cdot)$ implies that the Cantor set $\mathcal{C}_{a_p,\overline{v}_1}$ has full density at the origin, i.e. satisfies the measure estimate (5) of Theorem 1.1-(i).

Proposition 5.1 (Measure estimate of $\mathcal{C}_{a_p,\overline{v}_1}$) $\forall a_p(x) \in \mathcal{A}_p^1$, $\lim_{\eta\to 0^+}$ meas $(\mathcal{C}_{a_p,\overline{v}_1} \cap (0,\eta))/\eta = 1$.

Proof. Recall that

$$\begin{aligned} B_\infty := \cap_{n\ge 1} B_n \ = \ \Big\{ (\delta, v_1) \in A_0 \ : \ & \Big| \omega(\delta) l - j - \delta^{p-1} \frac{M(\delta, v_1, \widetilde{w}(\delta,v_1))}{2j} \Big| \ge \frac{2\gamma}{(l+j)^\tau}, \\ & \Big| \omega(\delta) l - j \Big| \ge \frac{2\gamma}{(l+j)^\tau}, \ \forall l,j \ge \frac{1}{3\delta^{p-1}}, \ l \neq j \ \Big\} \end{aligned}$$

where $\omega(\delta) = \sqrt{1+2\delta^{p-1}}$ (or $\omega(\delta) = \sqrt{1-2\delta^{p-1}}$), $M(\delta,v_1,w)$ is defined in Definition 3.1 and $\widetilde{w}(\delta,v_1)$ in Lemma 3.3.

Let $0 < \eta < \delta_0$. The complementary set of $\mathcal{C}_{a_p,\overline{v}_1}$ in $(0,\eta)$ is

$$\mathcal{C}^c_{a_p,\overline{v}_1} := \Big\{ \delta \in (0,\eta) \ \Big| \ \Big| \omega(\delta) l - j - \frac{\delta^{p-1} m(\delta)}{2j} \Big| < \frac{2\gamma}{(l+j)^\tau} \quad \text{or} \quad \Big| \omega(\delta) l - j \Big| < \frac{2\gamma}{(l+j)^\tau} \quad \text{for some } l, j > \frac{1}{3\delta^{p-1}}, \ l \neq j \ \Big\}$$

where $m(\delta) := M(\delta, v_1(\delta), \widetilde{w}(\delta, v_1(\delta)))$ is a function in $C^\infty([0,\delta_0), \mathbf{R})$ since $\delta \mapsto v_1(\delta)$ is C^∞ and $\widetilde{w}(\delta, v_1)$ is, by Lemma 3.3, in $C^\infty(A_0, W \cap X_{\overline{\sigma}/2,s}))$. This implies, in particular,

$$|m(\delta)| + |m'(\delta)| \leq C, \qquad \forall \delta \in [0, \delta_0/2] \tag{63}$$

for some positive constant C.

We claim that, for any interval $[\delta_1/2, \delta_1] \subset [0,\eta] \subset [0, \delta_0/2]$ the following measure estimate holds:

$$\mathrm{meas}\Big(\mathcal{C}^c_{a_p,\overline{v}_1} \cap \Big[\frac{\delta_1}{2}, \delta_1 \Big] \Big) \leq K_1(\tau) \gamma \eta^{(p-1)(\tau-1)} \mathrm{meas}\Big(\Big[\frac{\delta_1}{2}, \delta_1 \Big] \Big) \tag{64}$$

for some constant $K_1(\tau) > 0$.

Before proving (64) we show how to conclude the proof of the Lemma. Writing $(0,\eta] = \cup_{n\geq 0} \, [\eta/2^{n+1}, \eta/2^n]$ and applying the measure estimate (64) to any interval $[\delta_1/2, \delta_1] = [\eta/2^{n+1}, \eta/2^n]$, we get

$$\mathrm{meas}(\mathcal{C}^c_{a_p,\overline{v}_1} \cap [0,\eta]) \leq K_1(\tau) \gamma \eta^{(p-1)(\tau-1)} \eta,$$

whence $\lim_{\eta \to 0^+}$ meas $(\mathcal{C}_{a_p,\overline{v}_1} \cap (0,\eta))/\eta = 1$, proving the Lemma.

We now prove (64). We have

$$\mathcal{C}^c_{a_p,\overline{v}_1} \bigcap \Big[\frac{\delta_1}{2}, \delta_1 \Big] \subset \bigcup_{(l,j) \in \mathcal{I}_R} \mathcal{R}_{l,j}(\delta_1) \tag{65}$$

where

$$\mathcal{R}_{l,j}(\delta_1) := \Big\{ \delta \in \Big[\frac{\delta_1}{2}, \delta_1 \Big] \ | \ \Big| \omega(\delta) l - j - \frac{\delta^{p-1} m(\delta)}{2j} \Big| < \frac{2\gamma}{(l+j)^\tau} \quad \text{or} \quad \Big| \omega(\delta) l - j \Big| < \frac{2\gamma}{(l+j)^\tau} \Big\}$$

and

$$I_R := \Big\{ (l,j) \ | \ l, j > \frac{1}{3\delta_1^{p-1}}, \ l \neq j, \ \frac{j}{l} \in [1 - c_0 \delta_1^{p-1}, 1 + c_0 \delta_1^{p-1}] \Big\}$$

(note indeed that $\mathcal{R}_{j,l}(\delta_1) = \emptyset$ unless $j/l \in [1 - c_0 \delta_1^{p-1}, 1 + c_0 \delta_1^{p-1}]$ for some constant $c_0 > 0$ large enough).

Next, let us prove that

$$\mathrm{meas}(\mathcal{R}_{lj}(\delta_1)) = O\Big(\frac{\gamma}{l^{\tau+1} \delta_1^{p-2}} \Big). \tag{66}$$

Define $f_{lj}(\delta) := \omega(\delta) l - j - (\delta^{p-1} m(\delta)/2j)$ and $\mathcal{S}_{j,l}(\delta_1) := \{ \delta \in [\delta_1/2, \delta_1] \ : \ |f_{l,j}(\delta)| < 2\gamma/(l+j)^\tau \}$. Provided δ_0 has been chosen small enough (recall that $j, l \geq 1/3\delta_0^{p-1}$),

$$f'_{lj}(\delta) = \frac{l(p-1)\delta^{p-2}}{\sqrt{1+2\delta^{p-1}}} - \frac{(p-1)\delta^{p-2} m(\delta)}{2j} - \frac{\delta^{p-1} m'(\delta)}{2j} \geq \frac{(p-1)\delta^{p-2}}{2} \Big(l - \frac{C}{j} \Big) \geq \frac{(p-1)\delta^{p-2} l}{4}$$

and therefore $f'_{lj}(\delta) \geq (p-1) \delta_1^{p-2} l / 2^p$ for any $\delta \in [\delta_1/2, \delta_1]$. This implies

$$\mathrm{meas}(\mathcal{S}_{lj}(\delta_1)) \leq \frac{4\gamma}{(l+j)^\tau} \times \Big(\min_{\delta \in [\delta_1/2, \delta_1]} f'_{lj}(\delta) \Big)^{-1} \leq \frac{4\gamma}{(l+j)^\tau} \times \frac{2^p}{(p-1) l \delta_1^{p-2}} = O\Big(\frac{\gamma}{l^{\tau+1} \delta_1^{p-2}} \Big).$$

Similarly we can prove

$$\mathrm{meas}\Big(\Big\{ \delta \in \Big[\frac{\delta_1}{2}, \delta_1 \Big] \ : \ |\omega(\delta) l - j| < \frac{2\gamma}{(l+j)^\tau} \Big\} \Big) = O\Big(\frac{\gamma}{l^{\tau+1} \delta_1^{p-2}} \Big)$$

and the measure estimate (66) follows.

Now, by (65), (66) and since, for a given l, the number of j for which $(l,j) \in I_R$ is $O(\delta_1^{p-1} l)$,

$$\mathrm{meas}\Big(\mathcal{C}^c_{a_p,\overline{v}_1} \cap \Big[\frac{\delta_1}{2}, \delta_1\Big]\Big) \le \sum_{(l,j)\in I_R} \mathrm{meas}(\mathcal{R}_{j,l}(\delta_1)) \le C \sum_{l \ge 1/3\delta_1^{p-1}} \delta_1^{p-1} l \times \frac{\gamma}{l^{\tau+1}\delta_1^{p-2}} \le K_2(\tau)\gamma\delta_1^{1+(p-1)(\tau-1)}$$

whence we obtain (64) since $0 < \delta_1 < \eta$. ∎

Now for $a_p \in \mathcal{A}_p^1$, formula

$$\widetilde{u}(\delta) := \delta\Big[v_1(\delta) + \widetilde{w}(\delta, v_1(\delta)) + v_2\Big(\delta, v_1(\delta), \widetilde{w}(\delta, v_1(\delta))\Big)\Big]$$

defines a smooth path $\widetilde{u} : [0, \delta_0) \to X_{\overline{s}/2,s}$, and $\widetilde{u}(\delta) = \delta u_0 + O(\delta^2)$ with $u_0 = \overline{v}_1 + v_2(0, \overline{v}_1, 0) \in V$. By Lemma 3.4, $\widetilde{u}$ is a solution of (3) for $\delta \in \mathcal{C}_{a_p,\overline{v}_1}$ and conclusions (i) and (ii) of Theorem 1.1 hold by Proposition 5.1.

Now we can look for $2\pi/(n\omega)$ time-periodic solutions of (1) (*i.e.* $2\pi/n$ time-periodic solutions of (3)) as well. Let

$$X_{\sigma,s,n} := \Big\{u \in X_{\sigma,s} \mid u \text{ is } \frac{2\pi}{n} \text{ time} - \text{periodic}\Big\}.$$

Replacing $X_{\sigma,s}$ with $X_{\sigma,s,n}$, we can develop similarly the arguments of sections 2 and 3. Define the linear map $\mathcal{H}_n : V \to V$ by : for $v(t,x) = \eta(t+x) - \eta(t-x) \in V$,

$$(\mathcal{H}_n v)(t,x) := \eta(n(t+x)) - \eta(n(t-x))$$

and denote by $V_n := \mathcal{H}_n V$ (resp. W_n) the subspace of V (resp. W) formed by the functions $2\pi/n$-periodic in t.

Using the decomposition $X_{\sigma,s,n} = V_n \oplus W_n$ and introducing an appropriate finite dimensional subspace $V_{1,n}$ of V_n, we obtain associated $(Q1), (Q2), (P)$-equations (like in (10)), which can be solved exactly as in the case $n = 1$.

The 0^{th}-order bifurcation equation is the same (but in V_n) and the corresponding functional is just the restriction of Φ_0 to V_n. As Φ_0, $\Phi_{0|V_n}$ (or $\widetilde{\Phi}_{0|V_n}$) possesses "mountain pass" critical circles. Let, for $n \ge 2$,

$$\mathcal{A}_p^n := \Big\{a_p(x) \in \mathcal{O} \subset H^1(0,\pi) \mid \text{there is a non-degenerate critical circle } [\overline{v}] \subset V \text{ of } \Phi_{0|V_n} \text{ or } \widetilde{\Phi}_{0|V_n}\Big\},$$

and $\mathcal{A}_p = \cup_{n=1}^{\infty} \mathcal{A}_p^n$. By the implicit function theorem, $\mathcal{A}_p$ is an open subset of $H^1(0,\pi)$.

By the arguments of Proposition 5.1, we obtain that if $a_p \in \mathcal{A}_p$, then conclusions (i) and (ii) of Theorem 1.1 hold (since $2\pi/(n\omega)$ time-periodic solutions are just peculiar $2\pi/\omega$ time-periodic solutions).

This proves the first part of Theorem 1.1.

5.2 Case $f(x,u) = a_3(x)u^3 + O(u^4)$

We assume here that $f(x,u) = a_3(x)u^3 + O(u^4)$, where

$$\frac{1}{\pi}\int_0^\pi a_3(x)dx := \langle a_3 \rangle \neq 0 \ . \tag{67}$$

To fix the ideas, we deal with the case $\langle a_3 \rangle > 0$.

Note that assumption (67) also implies condition (6), i.e. that there is $v \in V$ such that $\int_\Omega a_3(x)v^4 \neq 0$. Indeed we know that $\int_\Omega a_3(x)v^4 = 0$, $\forall v \in V$ iff $a_3(\pi - x) = -a_3(x)$, $\forall x \in [0,\pi]$. But in this case $\langle a_3 \rangle = 0$.

Therefore the 0^{th}-order bifurcation equation is (9) (with $p = 3$), *i. e.* the Euler-Lagrange equation of

$$\Phi_0(v) = \frac{||v||^2_{H^1}}{2} - \int_\Omega a_3(x)\frac{v^4}{4} \tag{68}$$

The functional $\Phi_n(v) = \Phi_0(\mathcal{H}_n v)$ has the following development: for $v(t,x) = \eta(t+x) - \eta(t-x) \in V$ we obtain, using that $\int_\Omega v^4 = \int_\Omega (\mathcal{H}_n v)^4$,

$$\Phi_n(v) = 2\pi n^2 \int_{\mathbf{T}} \dot\eta^2(t)dt - \langle a_3 \rangle \int_\Omega \frac{v^4}{4} \; - \int_\Omega \Big(a_3(x) - \langle a_3 \rangle \Big) \frac{(\mathcal{H}_n v)^4}{4}.$$

Hence

$$\begin{aligned} \Phi_n\Big(\frac{\sqrt{2}n}{\sqrt{\langle a_3\rangle}} v\Big) &= \frac{8\pi n^4}{\langle a_3 \rangle}\Big[\frac{1}{2}\int_{\mathbf{T}} \dot\eta^2(s)ds - \frac{1}{8\pi}\int_\Omega v^4 \; + \frac{1}{8\pi}\int_\Omega \Big(\frac{a_3(x)}{\langle a_3\rangle} - 1\Big)(\mathcal{H}_n v)^4 \; dt \; dx\Big] \\ &= \frac{8\pi n^4}{\langle a_3 \rangle}[\Psi(\eta) + R_n(v)] \end{aligned}$$

where

$$\begin{aligned} \Psi(\eta) &:= \frac{1}{2}\int_{\mathbf{T}} \dot\eta^2(s)ds - \frac{1}{4}\int_{\mathbf{T}} \eta^4(s)ds - \frac{3}{8\pi}\Big(\int_{\mathbf{T}} \eta^2(s)ds\Big)^2, \\ R_n(v) &:= \frac{1}{8\pi}\int_\Omega b(x)(\mathcal{H}_n v)^4 \; dt \; dx, \quad b(x) := \frac{a_3(x)}{\langle a_3 \rangle} - 1. \end{aligned}$$

To get that $a_3 \in \mathcal{A}_3$, it is enough to prove that Ψ has a non-degenerate critical circle and that R_n is a small perturbation for large n, more precisely that $D^2 R_n \to 0$, $DR_n \to 0$ uniformly on bounded sets as $n \to +\infty$. Then, by the implicit function theorem, for n large enough, Φ_n too (hence $\Phi_{0|V_n}$) has a non-degenerate critical circle, which implies that $a_3 \in \mathcal{A}_3$.

The critical points of Ψ in $E := \{ \ \eta \in H^1(\mathbf{T}) \ | \ \int_{\mathbf{T}} \eta = 0 \ \}$ are the 2π-periodic solutions with zero mean value of

$$\ddot\eta + \eta^3 + 3\langle \eta^2 \rangle \eta = C, \quad C \in \mathbf{R} \tag{69}$$

By [BP] it is known that there exists a solution to this problem (with $C = 0$) which is a non-degenerate (up to time translations) critical point of Ψ in E. Finally

Lemma 5.1 *There holds*

$$||DR_n(v)||, \ ||D^2 R_n(v)|| \to 0 \qquad \text{as} \qquad n \to +\infty \tag{70}$$

uniformly for v in bounded sets.

Proof. We shall prove the estimate only for $D^2 R_n$. We have

$$\begin{aligned} |D^2 R_n(v)[h,k]| &= \frac{3}{2\pi}\int_\Omega b(x)(\mathcal{H}_n v)^2(\mathcal{H}_n h) \ (\mathcal{H}_n k) \\ &= \frac{3}{2\pi}\int_0^\pi b(x) g(nx)\, dx \end{aligned}$$

where $g(y)$ is the π-periodic function defined by

$$g(y) := \int_{\mathbf{T}} (\eta(t+y) - \eta(t-y))^2 (\beta(t+y) - \beta(t-y))(\gamma(t+y) - \gamma(t-y))dt \ ,$$

β and γ being associated with h and k as η is with v. Developing in Fourier series $g(y) = \sum_{l\in\mathbf{Z}} g_l \exp(\mathrm{i}2ly)$ we have $g(nx) = \sum_{l\in\mathbf{Z}} g_l \exp(\mathrm{i}2lnx)$. Extending $b(x)$ to a π-periodic function, we also write $b(x) = \sum_{l\in\mathbf{Z}} b_l \exp(\mathrm{i}2lx)$, with $b_0 = \langle b \rangle = 0$. Therefore

$$\begin{aligned} |D^2 R_n(v_n)[h,k]| &= \frac{3}{2}\Big|\sum_{l\neq 0} g_l b_{-ln}\Big| \leq \frac{3}{2}\Big(\sum_{l\neq 0} g_l^2\Big)^{1/2}\Big(\sum_{l\neq 0} b_{ln}^2\Big)^{1/2} \\ &\leq \frac{3}{2}||g||_{L^2(0,\pi)}\Big(\sum_{l\neq 0} b_{ln}^2\Big)^{1/2} \\ &\leq C||v_0||_\infty^2 ||\beta||_\infty ||\gamma||_\infty \ \Big(\sum_{l\neq 0} b_{ln}^2\Big)^{1/2}. \end{aligned}$$

Since $(\sum_{l\neq 0} b_{ln}^2)^{1/2} \to 0$ as $n \to \infty$ it proves (70). With a similar calculus we can prove that $DR_n(v) \to 0$ as $n \to +\infty$. ■

This completes the proof of part 1) of Theorem 1.1.

5.3 Case $f(x,u) = a_2 u^2 + O(u^4)$

We now prove part 2) of Theorem 1.1.

In this case condition (6) is violated. It turns out (see [5]) that we must take $\omega < 1$ (*i.e.* $\varepsilon < 0$), and $\delta = |\varepsilon|^{1/2}$ in the rescaling. By the computations of [5], the 0^{th}-order bifurcation equation is the Euler Lagrange equation of the functional $\Phi_0 : V \to \mathbf{R}$ defined by

$$\Phi_0(v) = \frac{||v||^2_{H^1}}{2} + \frac{a_2^2}{2}\int_\Omega v^2 L^{-1} v^2,$$

where $L^{-1} : W \to W$ is the inverse operator of $-\partial_{tt} + \partial_{xx}$.

We can still use the same arguments to solve the $(Q2)$ and (P) equations. As explained in subsection 5.1, if the functional $\Phi_n : V \to \mathbf{R}$, defined by $\Phi_n(v) = \Phi_0(\mathcal{H}_n v)$ possesses a nondegenerate critical circle, then (i) and (ii) hold in Theorem 1.1 (with $\omega = \sqrt{1-2\delta^2}$).

Φ_n admits the following development (Lemmae 3.7 and 3.8 in [5]): for $v(t,x) = \eta(t+x) - \eta(t-x)$

$$\Phi_n(v) = 2\pi n^2 \int_{\mathbf{T}} \dot\eta^2(t)dt - \frac{\pi^2 a_2^2}{12}\Big(\int_{\mathbf{T}} \eta^2(t)\ dt\Big)^2 + \frac{a_2^2}{2n^2}\Big(\int_\Omega v^2 L^{-1} v^2 + \frac{\pi^2}{6}\Big(\int_{\mathbf{T}} \eta^2(t)\ dt\Big)^2\Big). \tag{71}$$

Hence we can write

$$\Phi_n\Big(\frac{\sqrt{12}n}{\sqrt{\pi}a_2}v\Big) = \frac{48n^4}{a_2^2}\Big[\frac{1}{2}\int_{\mathbf{T}} \dot\eta^2(s)\ ds - \frac{1}{4}\Big(\int_{\mathbf{T}} \eta^2(s)\ ds\Big)^2 + \frac{1}{n^2}R(\eta)\Big] = \frac{48n^4}{a_2^2}\Big[\Psi(\eta) + \frac{1}{n^2}R(\eta)\Big] \tag{72}$$

where

$$\Psi(\eta) = \frac{1}{2}\int_{\mathbf{T}} \dot\eta^2(s)\ ds - \frac{1}{4}\Big(\int_{\mathbf{T}} \eta^2(s)\ ds\Big)^2 \tag{73}$$

and $R : E \to \mathbf{R}$ is a smooth map defined on $E := \{\eta \in H^1(\mathbf{T}) \mid \int_{\mathbf{T}} \eta = 0\}$. In order to prove that Φ_n has a non-degenerate critical circle for n large enough, it is enough to prove that:

Lemma 5.2 $\Psi : E \to \mathbf{R}$ *possesses a critical point which is non degenerate, up to time translations.*

Proof. The critical points of Ψ in E are the 2π-periodic solutions of zero mean value of the equation

$$\ddot\eta + \Big(\int_{\mathbf{T}} \eta^2(t)\ dt\Big)\eta = C, \quad C \in \mathbf{R}. \tag{74}$$

Since $\ddot\eta$ and η have both zero mean value, any solution of (74) must satisfy $C = 0$. Equation (74) (with $C = 0$) has a 2π-periodic solution of the form $\bar\eta(t) = (1/\sqrt{\pi})\sin t$.

We claim that $\bar\eta$ is non-degenerate, up to time-translations, i.e. Ker $(D^2\Psi)(\bar\eta) = \langle \dot{\bar\eta} \rangle$.

The linearized equation of (74) at $\bar\eta$ is

$$\ddot h + \Big(\int_{\mathbf{T}} \bar\eta^2(t)\ dt\Big)h + 2\Big(\int_{\mathbf{T}} \bar\eta(t)h(t)\ dt\Big)\bar\eta = c,$$

and again c must be equal to 0. We get

$$\ddot h + h + \frac{2}{\pi}\Big(\int_{\mathbf{T}} \sin t\ h(t)\ dt\Big)\sin t = 0. \tag{75}$$

Developing in time-Fourier series

$$h(t) = \sum_{k\geq 1}\Big(a_k \sin kt + b_k \cos kt\Big)$$

we find out that any solution of the linearized equation (75) satisfies

$$-k^2 b_k + b_k = 0, \ \ \forall k \geq 1, \qquad -k^2 a_k + a_k = 0, \ \ \forall k \geq 2, \qquad a_1 = 0$$

and therefore $h \in \langle \cos t \rangle = \langle \dot{\bar{\eta}} \rangle$. ■

By the previous Lemma, for n large enough there is a non-degenerate critical circle of Φ_n in V. This completes the proof of Theorem 1.1.

6 Appendix

Lemma 6.1 *If q is an even integer, then*

$$\int_\Omega a(x) v^q(t,x) \ dt \ dx = 0, \ \forall v \in V \iff \Big\{ a(\pi - x) = -a(x), \ \forall x \in [0,\pi] \Big\}.$$

If $q \geq 3$ is an odd integer, then

$$\int_\Omega a(x) v^q(t,x) \ dt \ dx = 0, \ \forall v \in V \iff \Big\{ a(\pi - x) = a(x), \ \forall x \in [0,\pi] \Big\}.$$

PROOF. We first assume that $q = 2s$ is even. If $a(\pi - x) = -a(x) \ \forall x \in (0,\pi)$, then, for all $v \in V$,

$$\begin{aligned} \int_\Omega a(x) v^{2s}(t,x) \ dt \ dx &= \int_\Omega a(\pi - x) v^{2s}(t, \pi - x) \ dt \ dx \\ &= \int_\Omega -a(x)(-v(t+\pi, x))^{2s} \ dt \ dx \\ &= -\int_\Omega a(x) v^{2s}(t,x) \ dt \ dx \end{aligned}$$

and so $\int_\Omega a(x) v^{2s}(t,x) \ dt \ dx = 0$.

Now assume that $\Sigma(v) := \int_\Omega a(x) v^{2s}(t,x) \ dt \ dx = 0 \ \forall v \in V$. Writing that $D^{2s}\Sigma = 0$, we get

$$\int_\Omega a(x) v_1(t,x) \ldots v_{2s}(t,x) \ dt \ dx = 0, \qquad \forall (v_1, \ldots, v_{2s}) \in V^{2s}.$$

Choosing $v_{2s}(t,x) = v_{2s-1}(t,x) = \cos(lt)\sin(lx)$, we obtain

$$\frac{1}{4} \int_\Omega a(x) v_1(t,x) \ldots v_{2(s-1)}(t,x)(\cos(2lt) + 1)(1 - \cos(2lx)) \ dt \ dx = 0$$

Taking limits as $l \to \infty$, there results $\int_\Omega a(x) v_1(t,x) \ldots v_{2(s-1)}(t,x) \, dt \, dx = 0 \ \forall (v_1, \ldots, v_{2(s-1)}) \in V^{2(s-1)}$. Iterating this operation, we finally get

$$\forall (v_1, v_2) \in V^2 \ \ \int_\Omega a(x) v_1(t,x) v_2(t,x) \ dt \ dx = 0, \quad \text{and} \quad \int_0^\pi a(x) \ dx = 0.$$

Choosing $v_1(t,x) = v_2(t,x) = \cos(lt)\sin(lx)$ in the first equality, we derive that $\int_0^\pi a(x)\sin^2(lx) \ dx = 0$. Hence

$$\forall l \in \mathbf{N} \ \ \int_0^\pi a(x)\cos(2lx) \ dx = 0.$$

This implies that a is orthogonal in $L^2(0,\pi)$ to

$$F = \Big\{ b \in L^2(0,\pi) \mid b(\pi - x) = b(x) \text{ a.e.} \Big\}.$$

Hence $a(\pi - x) = -a(x)$ a.e., and, since a is continuous, the identity holds everywhere.

We next assume that $q = 2s+1$ is odd, $q \geq 3$. The first implication is derived in a similar way. Now assume that $\int_\Omega a(x)v^q(t,x)\ dt\ dx = 0\ \ \forall v \in V$. We can prove exactly as in the first part that

$$\forall (v_1, v_2, v_3) \in V^3 \quad \int_\Omega a(x)v_1(t,x)v_2(t,x)v_3(t,x)\ dt\ dx = 0.$$

Choosing $v_1(t,x) = \cos(l_1 t)\sin(l_1 x)$, $v_2(t,x) = \cos(l_2 t)\sin(l_2 x)$, $v_3(t,x) = \cos((l_1+l_2)t)\sin((l_1+l_2)x)$ and using the fact that $\int_0^{2\pi} \cos(l_1 t)\cos(l_2 t)\cos((l_1+l_2)t)\ dt \neq 0$, we obtain

$$\begin{aligned} &\int_0^\pi a(x)\Big[\sin^2(l_1 x)\sin(l_2 x)\cos(l_2 x) + \sin^2(l_2 x)\sin(l_1 x)\cos(l_1 x)\Big]\ dx = \\ &\int_0^\pi a(x)\sin(l_1 x)\sin(l_2 x)\sin\Big((l_1+l_2)x\Big)\ dx = 0. \end{aligned} \tag{76}$$

Letting l_2 go to infinity and taking limits, (76) yields $\int_0^\pi (1/2)a(x)\sin(l_1 x)\cos(l_1 x)\ dx = 0$. Hence

$$\forall l > 0 \quad \int_0^\pi a(x)\sin(2lx) = 0.$$

This implies that, in $L^2(0,\pi)$, a is orthogonal to $G = \{b \in L^2(0,\pi) \mid b(\pi - x) = -b(x) \text{ a.e.}\}$. Hence $a(\pi - x) = a(x)\ \forall x \in (0,\pi)$. ■

Proof of Lemma 4.1. Let $K_k(\varepsilon) = S_k^{-1}(\varepsilon)$ be the selfadjoint compact operator of F_k defined by

$$\langle K_k(\varepsilon)u, v\rangle_\varepsilon = (u,v)_{L^2}, \qquad \forall u, v \in F_k$$

(in other words $K_k(\varepsilon)u$ is the unique weak solution $z \in F_k$ of $S_k z := u$).

Note that $K_k(\varepsilon)$ is a positive operator, i.e. $\langle K_k(\varepsilon)u, u\rangle_\varepsilon > 0$, $\forall u \neq 0$, and that $K_k(\varepsilon)$ is also selfadjoint for the L^2-scalar product.

Let $F_{k,r} = \{u \in F_k \mid u((0,\pi)) \subset \mathbf{R}\}$. We have $S_k(F_{k,r} \subset F_{k,r}$, and by the spectral theory of compact selfadjoint operators in Hilbert spaces, there is a $\langle\ ,\ \rangle_\varepsilon$-orthonormal basis $(v_{k,j})_{j\geq 1, j\neq k}$ of F_k, such that $v_{k,j} \in F_{k,r}$ is an eigenvector of $K_k(\varepsilon)$ associated to a positive eigenvalue $\nu_{k,j}(\varepsilon)$, the sequence $(\nu_{k,j}(\varepsilon))_j$ is non-increasing and tends to 0 as $j \to +\infty$. Each $v_{k,j}(\varepsilon)$ belongs to $D(S_k)$ and is an eigenvector of S_k with associated eigenvalue $\lambda_{k,j}(\varepsilon) = 1/\nu_{k,j}(\varepsilon)$. $(\lambda_{k,j}(\varepsilon))_{j\geq 1}$ is a sequence non decreasing and tending to $+\infty$ as $j \to +\infty$.

The map $\varepsilon \mapsto K_k(\varepsilon) \in \mathcal{L}(F_k, F_k)$ is differentiable and $K_k'(\varepsilon) = -K_k(\varepsilon)MK_k(\varepsilon)$, where $Mu := \pi_k(a_0 u)$.

For $u = \sum_{j\neq k} \alpha_j v_{k,j}(\varepsilon) \in F_k$,

$$\langle u, u\rangle_\varepsilon = \sum_{j\neq k} |\alpha_j|^2 \quad \text{and} \quad (u,u)_{L^2} = \sum_{j\neq k} \frac{|\alpha_j|^2}{\lambda_{k,j}(\varepsilon)}.$$

As a consequence,

$$\lambda_{k,j}(\varepsilon) = \min\Big\{ \max_{u\in F, ||u||_{L^2}=1} \langle u,u\rangle_\varepsilon\ ;\ F \text{ subspace of } F_k \text{ of dimension } j \text{ (if } j<k)\ ,\ j-1 \text{ (if } j>k)\Big\}. \tag{77}$$

It is clear by inspection that $\lambda_{k,j}(0) = j^2$ and that we can choose $v_{k,j}(0) = \sqrt{2/\pi}\sin(jx)/j$. Hence, by (77), $|\lambda_{k,j}(\varepsilon) - j^2| \leq |\varepsilon|\ ||a_0||_\infty < 1$, from which we derive

$$\forall l \neq j \quad |\lambda_{k,l}(\varepsilon) - \lambda_{k,j}(\varepsilon)| \geq (l+j) - 2 \geq 2\min(l,j) - 1 \quad (\geq 1). \tag{78}$$

In particular, the eigenvalues $\lambda_{k,j}(\varepsilon)$ ($\nu_{k,j}(\varepsilon)$) are simple. By the variational characterization (77) we also see that $\lambda_{k,j}(\varepsilon)$ depends continuously on ε, and we can assume without loss of generality that $\varepsilon \mapsto v_{k,j}(\varepsilon)$ is a continuous map to F_k.

Let $\varphi_{k,j}(\varepsilon) := \sqrt{\lambda_{k,j}(\varepsilon)} v_{k,j}(\varepsilon)$. $(\varphi_{k,j}(\varepsilon))_{j\neq k}$ is a L^2-orthogonal family in F_k and

$$\forall \varepsilon \quad \begin{cases} K_k(\varepsilon)\varphi_{k,j}(\varepsilon) = \nu_{k,j}(\varepsilon)\varphi_{k,j}(\varepsilon) \\ (\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon))_{L^2} = 1 \end{cases}$$

We observe that the L^2-orthogonality w.r.t. $\varphi_{k,j}(\varepsilon)$ is equivalent to the $\langle\ ,\ \rangle_\varepsilon$-orthogonality w.r.t. $\varphi_{k,j}(\varepsilon)$, and that $E_{k,j}(\varepsilon) := [\varphi_{k,j}(\varepsilon)]^\perp$ is invariant under $K_k(\varepsilon)$. Using that $L_{k,j} := (K_k(\varepsilon) - \nu_{k,j}(\varepsilon)I)_{|E_{k,j}(\varepsilon)}$ is invertible, it is easy to derive from the Implicit Function Theorem that the maps $(\varepsilon \mapsto \nu_{k,j}(\varepsilon))$ and $(\varepsilon \mapsto \varphi_{k,j}(\varepsilon))$ are differentiable.

Denoting by P the orthogonal projector onto $E_{k,j}(\varepsilon)$, we have

$$\varphi'_{k,j}(\varepsilon) = L^{-1}(-PK'_k(\varepsilon)\varphi_{k,j}(\varepsilon)) = L^{-1}(PK_kMK_k\varphi_{k,j}(\varepsilon)) = \nu_{k,j}(\varepsilon)L^{-1}K_kPM\varphi_{k,j}(\varepsilon),$$

$$\begin{aligned} \nu'_{k,j}(\varepsilon) &= \Big(K'_k(\varepsilon)\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon)\Big)_{L^2} = -\Big(K_kMK_k\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon)\Big)_{L^2} \\ &= -\Big(MK_k\varphi_{k,j}(\varepsilon), K_k\varphi_{k,j}(\varepsilon)\Big)_{L^2} = -\nu^2_{k,j}(\varepsilon)\Big(M\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon)\Big)_{L^2}. \end{aligned} \tag{79}$$

We have

$$\nu_{k,j}L^{-1}K_k\Big(\sum_{l\neq j}\alpha_l v_{k,l}\Big) = \sum_{l\neq j}\frac{\nu_{k,j}\nu_{k,l}}{\nu_{k,l}-\nu_{k,j}}\alpha_l v_{k,l} = \sum_{l\neq j}\frac{\alpha_l}{\lambda_{k,j}-\lambda_{k,l}}v_{k,l}.$$

Hence, by (78), $|\nu_{k,j}L^{-1}K_kPu|_{L^2} \leq |u|_{L^2}/j$. We obtain $|\varphi'_{k,j}(\varepsilon)|_{L_2} = O(|a_0|_\infty/j)$. Hence

$$\Big|\varphi_{k,j}(\varepsilon) - \sqrt{\frac{2}{\pi}}\sin(jx)\Big|_{L^2} = O\Big(\frac{\varepsilon|a_0|_\infty}{j}\Big).$$

Hence, by (79),

$$\begin{aligned} \lambda'_{k,j}(\varepsilon) &= \Big(M\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon)\Big)_{L^2} = \int_0^\pi a_0(x)(\varphi_{k,j})^2\ dx \\ &= \frac{2}{\pi}\int_0^\pi a_0(x)(\sin(jx))^2\ dx + O\Big(\frac{\varepsilon|a_0|^2_\infty}{j}\Big) \end{aligned}$$

Writing $\sin^2(jx) = (1-\cos(2jx))/2$, and since $\int_0^\pi a_0(x)\cos(2jx)\ dx = -\int_0^\pi (a_0)_x(x)\sin(2jx)/2j\ dx$, we get

$$\lambda'_{k,j}(\varepsilon) = \frac{1}{\pi}\int_0^\pi a_0(x)\ dx + O\Big(\frac{||a_0||_{H^1}}{j}\Big) = M(\delta, v_1, w) + O\Big(\frac{||a_0||_{H^1}}{j}\Big).$$

Hence $\lambda_{k,j}(\varepsilon) = j^2 + \varepsilon M(\delta, v_1, w) + O(\varepsilon||a_0||_{H^1}/j)$, which is the first estimate in (45).

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