Generalising Group Algebras

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Abstract

We generalise group algebras to algebraic objects with bounded Hilbert space representation theory - the generalised group algebras are called "host" algebras. The main property of a host algebra, is that its representation theory should be isomorphic (in the sense of the Gelfand–Raikov theorem) to a specified subset of representations of the algebraic object. Here we obtain both existence and uniqueness theorems for host algebras as well as general structure theorems for host algebras. Abstractly, this solves the question of when a set of Hilbert space representations is isomorphic to the representation theory of a C*-algebra. To make contact with harmonic analysis, we consider general convolution algebras associated to representation sets, and consider conditions for a convolution algebra to be a host algebra.

1 Introduction

In [10] we developed a general theory of group algebras which is applicable to topological groups which are not necessary locally compact. This theory has an easy extension to other algebraic objects, and here we want to develop this extension as well as to analyze the difficult question of when such a generalised group algebra exists.

The Gelfand–Raikov theorem [9] proved that the continuous (unitary) representation theory of any locally compact group is isomorphic in a natural sense to the (nondegenerate Hilbert space) representation theory of a C*–algebra. The proof is constructive, in that it explicitly constructs the group algebra as the enveloping C*-algebra of the convolution algebra $L^1(G)$ and faithfully embeds the group as unitaries in the multiplier algebra of the group algebra. Subsequently group algebras for locally compact groups have been generalised in many directions (e.g. twisted group algebras, groupoid algebras, some semigroup algebras and cross–products of a C*-algebra by a group action), and has been a central component of harmonic analysis. Local compactness and continuous behaviour of measures on compact sets are important requirements for all these generalisations, and this is a severe limitation. We want to generalise outside of this class, i.e. consider for algebraic objects other than locally compact groups, whether their continuous (Hilbert) space representation theory is isomorphic to the representation theory of a C*–algebra. In fact, we will analyze the general question of when a set of (Hilbert space) representations of an algebraic object is isomorphic to the representation theory of a C*–algebra (in the sense of Gelfand–Raikov).

Counterexamples are easy to find, e.g. it is not true for all topological groups that their continuous representation theory is isomorphic (in the sense of Gelfand–Raikov) to the representation theory of a C*–algebra. For instance, there are Abelian groups with NO nontrivial continuous unitary representations, cf. Banaszczyk [4], and there are Abelian groups with continuous representations, but no irreducible ones cf. Example 5.2 in Pestov [14]. However, we can still ask the question of which subsets of representations of an algebraic object are isomorphic to the representation theory of a C*–algebra.

To make the discussion more precise, we define:

1.1 Definition *For a set* X, *a (nondegenerate)* **representation theory**, $\mathrm{Rep}\,X$ *is a set of maps* π *from* X *to bounded operators on some Hilbert space* $\mathcal{H}_\pi$, *i.e.* $\pi : X \to \mathcal{B}(\mathcal{H}_\pi)$ *such that*

(i) Each $\pi \in \mathrm{Rep}\,X$ *is nondegenerate, i.e.* $\mathcal{A}^*(\pi(X))\mathcal{H}_\pi$ *is dense in* $\mathcal{H}_\pi$, *where* $\mathcal{A}^*(\cdot)$ *denotes the *-algebra generated by its argument.*

(ii) $(\alpha \circ \pi){\restriction}_{\mathrm{ess}} \in \mathrm{Rep}\,X$ *for all* $\pi \in \mathrm{Rep}\,X$ *and *-homomorphisms* $\alpha : \mathcal{B}(\mathcal{H}_\pi) \to \mathcal{B}(\mathcal{K}_\alpha)$ *where* $\mathcal{K}_\alpha$ *is a Hilbert space. (The notation* $\pi{\restriction}_{\mathrm{ess}}$ *for a map* $\pi : X \to \mathcal{B}(\mathcal{H}_\pi)$ *denotes* π *followed by restriction to its essential subspace* $\overline{\mathcal{A}^*(\pi(X))\mathcal{H}_\pi}$.*)*

(iii) The direct sum of any family of cyclic representations in $\mathrm{Rep}\,X$ *is again in* $\mathrm{Rep}\,X$ *(repetitions are allowed).*

1.2 Remark (1) As usual, a *cyclic* representation $\pi \in \mathrm{Rep}\,X$ is a map $\pi : X \to \mathcal{B}(\mathcal{H}_\pi)$ such that there is a cyclic vector $\psi \in \mathcal{H}_\pi$, i.e. $\mathcal{A}^*(\pi(X))\psi$ is dense in $\mathcal{H}_\pi$. Note that if $\mathrm{Rep}\,X$ is nonempty, then it contains the cyclic components of all $\pi \in \mathrm{Rep}\,X$, because the projection of a representation $\pi \in \mathrm{Rep}\,X$ to one of its cyclic components is a *-homomorphism so by (ii), that cyclic component is in $\mathrm{Rep}\,X$. Thus by (iii) it is clear that $\mathrm{Rep}\,X$ is closed under finite direct sums.

(2) Usually X has some algebraic structure, i.e. operations and relations, and $\mathrm{Rep}\,X$ is specified by requiring the maps π to respect some specified operation(s) or relation(s). This usually implies that the requirements in (ii) and (iii) are automatic.

(3) Unitary equivalence is an equivalence relation for $\mathrm{Rep}\,X$ and we will use this to identify representations on different spaces.

In particular, (iii) allows us to form the "universal representation" $\pi_u : X \to \mathcal{B}(\mathcal{H}_u)$ by

$$\pi_u = \oplus\{\pi \in \mathrm{Rep}\,X \mid \pi \quad \text{is cyclic}\,\} \in \mathrm{Rep}\,X\,.$$

Define the C*-algebra $\mathcal{A}_d(X) := C^*\left(\pi_u(X)\right) \subset \mathcal{B}(\mathcal{H}_u)$ where $C^*(\cdot)$ denotes the C*-algebra generated by its argument. We claim that there is a bijection between $\mathrm{Rep}\,X$ and $\mathrm{Rep}\,\mathcal{A}_d(X)(=$ the C*-representation set of $\mathcal{A}_d(X))$. It is obtained as follows;- any $\pi \in \mathrm{Rep}\,\mathcal{A}_d(X)$ defines a representation of X by restriction: $\pi(x) := \pi(\pi_u(x))$ (making use of (ii)), producing a map $\mathrm{Rep}\,\mathcal{A}_d(X) \to \mathrm{Rep}\,X$. That the map is injective, follows from the fact that $\pi_u(X)$ is a generating set for $\mathcal{A}_d(X)$. To see that it is surjective, note that any cyclic representation of X can be obtained from $\mathcal{A}_d(X) \subset \mathcal{B}(\mathcal{H}_u)$ by restricting to the subspace in the direct sum $\mathcal{H}_u$ corresponding to it (this restiction is a representation of $\mathcal{A}_d(X)$). Since $\mathrm{Rep}\,\mathcal{A}_d(X)$ is closed under direct sums and the map respects direct sums, it follows that the map is a bijection. We will henceforth take the map as an identification, e.g. use the notation $\pi\left(\mathcal{A}_d(X)\right)$ for a $\pi \in \mathrm{Rep}\,X$.

From the bijection between $\mathrm{Rep}\,X$ and $\mathrm{Rep}\,\mathcal{A}_d(X)$ we see that $\mathrm{Rep}\,X$ contains irreducible representations, and that any set of representations which separates $\mathcal{A}_d(X)$ will generate all of $\mathrm{Rep}\,X$ by direct sums as in (iii) and composition with concrete *-homomorphisms α as in (ii). For instance, by forming the direct sum of cyclic components of the separating set, one obtains a faithful representation of $\mathcal{A}_d(X)$ hence the set of its α becomes just $\mathrm{Rep}\,\mathcal{A}_d(X)$. In particular, the set of irreducible representations in $\mathrm{Rep}\,X$ is such a generating set for $\mathrm{Rep}\,X$.

Usually one is not interested in the full set $\mathrm{Rep}\,X$, but in some subset $\mathcal{R} \subset \mathrm{Rep}\,X$. For instance, X may have a topology, and $\mathcal{R}$ may be the set of those representations which are continuous w.r.t. the strong operator topology (we will have examples below). One is then interested in whether $\mathcal{R}$ is isomorphic to the representation theory of a C*-algebra in the following sense:

1.3 Definition *Let* X *and* $\mathrm{Rep}\,X$ *be as above, and let* $\mathcal{R}\subset\mathrm{Rep}\,X$ *be a given subset of representations of* X. *Then a* **host algebra** *for* $\mathcal{R}$ *is a* C^**–algebra* $\mathcal{L}$ *and a *-homomorphism* $\varphi:\mathcal{A}_d(X)\to M(\mathcal{L})$ *(= multiplier algebra of* $\mathcal{L}$*) such that the unique extension map* $\theta:\mathrm{Rep}\mathcal{L}\to\mathrm{Rep}\,X$ *is injective, and with image* $\theta(\mathrm{Rep}\,\mathcal{L})=\mathcal{R}$. *In this case we say that* $\mathcal{R}$ *is* **isomorphic** *to* $\mathrm{Rep}\,\mathcal{L}$. *Two host algebras* $\mathcal{L}_1,\,\mathcal{L}_2$ *for* $\mathcal{R}$ *are* **isomorphic** *if there is a *-isomorphism* $\Phi:\mathcal{L}_1\to\mathcal{L}_2$ *such that* $\Phi(\varphi_1(x)A)=\varphi_2(x)\Phi(A)$ *for all* $x\in X,\ A\in\mathcal{L}_1$.

1.4 Remark (1) Any nondegenerate representation of $\mathcal{L}$ has a unique extension (on the same space) to its multiplier algebra $M(\mathcal{L})$, and this defines the map $\theta:\mathrm{Rep}\mathcal{L}\to\mathrm{Rep}\,X$ by $\theta(\pi)(x):=\operatorname*{s\text{–}lim}_{\alpha\to\infty}\pi(\varphi(\pi_u(x))E_\alpha)$ where $\{E_\alpha\}\subset\mathcal{L}$ is any approximate identity of $\mathcal{L}$.

(2) Note that the map θ preserves direct sums, unitary conjugation, subrepresentations, and (as we will see) irreducibility, so that this notion of isomorphism between $\mathcal{R}$ and $\mathrm{Rep}\,\mathcal{L}$ involves strong structural correspondences, and restricts the class of sets $\mathcal{R}$ for which host algebras exist. However, this isomorphism is obviously not an equivalence relation, since it relates objects in two distinct sets. In the case that $\theta:\mathrm{Rep}\mathcal{L}\to\mathcal{R}$ is surjective but not injective, it is natural to say that $\mathrm{Rep}\,\mathcal{L}$ is *homomorphic* to $\mathcal{R}$, since θ still transfers some structure to $\mathcal{R}$ (but e.g. irreducibility of representations is lost). We will not examine this concept here.

(3) An isomorphism $\Phi:\mathcal{L}_1\to\mathcal{L}_2$ of host algebras extend canonically to an isomorphism of the multiplier algebras $\Phi:M(\mathcal{L}_1)\to M(\mathcal{L}_2)$ such that $\Phi(\varphi_1(x))=\varphi_2(x)$ for all $x\in X$.

The terminology of a host algebra was adopted from [11] (where it was the concept of an ideal host), and generalises group algebras and crossed products. Of course host algebras need not exist, and if they do, it is not clear that they are unique. Below we want to analyze these questions. First, we present a set of examples to motivate the preceding definitions.

1.5 Example (1) Let X be a topological group G, and let $\mathrm{Rep}\,X$ be the set of σ–representations of G, where σ is a fixed $\mathbb{T}$–valued two-cocycle. That is, $\mathrm{Rep}\,X$ consists of maps $U:G\to\mathcal{B}(\mathcal{H})$ such that each $U(g)$ is unitary, and $U(g)U(h)=\sigma(g,h)U(gh)$. Then $\mathcal{A}_d(G)=C^*_\sigma(G_d)$, i.e. the discrete σ–group algebra of G. The representation set in which one is interested is

$$\mathcal{R}=\{\,U\in\mathrm{Rep}\,X\ \mid\ g\to U(g)\quad\text{is strong operator continuous}\,\}$$

in which case a host algebra for $\mathcal{R}$ is a σ–group algebra in the sense of [10], which is isomorphic to the usual σ–group algebra when the latter is defined, i.e. if G is locally compact and σ suitably regular (by the uniqueness theorem of [10]). There are other possible subsets $\mathcal{R}$ for which one would like to have a host algebra, e.g. the set of separable representations, or those representations where some distinguished one–parameter subgroup is strong operator continuous, and has a positive generator (such examples occur in physics). Another interesting variant is to let $\mathrm{Rep}\, X$ consist of nonunitary representations.

(2) Let X be a topological semigroup S, and let $\mathrm{Rep}\, X$ be the set of bounded Hilbert space representations, i.e. maps $\pi : S \to \mathcal{B}(\mathcal{H})$ such that $\pi(s)\pi(t) = \pi(st)$. Then the representation set in which one is interested is

$$\mathcal{R} = \{\, \pi \in \mathrm{Rep}\, X \;\mid\; s \to \pi(s) \quad \text{is strong operator continuous} \,\}\,.$$

Then a host algebra for $\mathcal{R}$ is a semigroup algebra, and even if S is locally compact, its existence is a nontrivial problem due to the absence of Haar measure.

(3) Let X be the disjoint union $X = G \cup \mathcal{A}$ where G is a topological group, and $\mathcal{A}$ is a C*-algebra. To specify $\mathrm{Rep}\, X$, fix an action $\alpha : G \to \mathrm{Aut}\, \mathcal{A}$ (pointwise norm-continuous), and a $\mathbb{T}$–valued two-cocycle σ. Then $\mathrm{Rep}\, X$ is defined as all maps $\rho : X \to \mathcal{B}(\mathcal{H})$ such that $(\rho\restriction G,\, \rho\restriction \mathcal{A})$ is a σ–covariant pair, i.e. i.e. $\rho\restriction G =: U : G \to \mathcal{U}(\mathcal{H}) =$ unitaries on $\mathcal{H}$ such that $U(g)U(h) = \sigma(g,h)U(gh)$; and $\rho\restriction \mathcal{A} =: \pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a *-representation of $\mathcal{A}$ such that

$$\pi\left(\alpha_g(A)\right) = U_g \pi(A) U_g^* \qquad \forall\, g \in G,\; A \in \mathcal{A}.$$

Then $\mathcal{A}_d(X) = \mathcal{A} \underset{\alpha,\sigma}{\times} G_d =$ the discrete σ–crossed product of $\mathcal{A}$ by G, cf. [13]. The representation set in which one is interested is

$$\mathcal{R} = \{\, \rho \in \mathrm{Rep}\, X \;\mid\; g \in G \to \rho(g) = U(g) \quad \text{is strong operator continuous} \,\}\,.$$

In the case that G is locally compact, host algebras for $\mathcal{R}$ of course exist for suitably regular σ, and are the usual σ–crossed products of $\mathcal{A}$ by G, denoted $\mathcal{A} \underset{\alpha,\sigma}{\times} G$. In the case that G is not locally compact, one will *define* $\mathcal{A} \underset{\alpha,\sigma}{\times} G$ to be a host algebra for $\mathcal{R}$. Of course then there are serious existence and uniqueness questions to analyze.

(4) Let X be a C*-algebra $\mathcal{A}$, and let $\mathrm{Rep}\, X$ be the full set of C*-representations of $\mathcal{A}$ (then clearly $\mathcal{A}_d(X) = \mathcal{A}$). A possible choice for $\mathcal{R}$ consists of all the representations which are normal w.r.t. some fixed set of representations (this example arises in physics). There are also plenty of other selection conditions for subsets of representations within which one may want to restrict the analysis.

It is possible to extend the analysis to objects with unbounded Hilbert space representations if one can associate in a consistent way bounded families of operators with a given set of unbounded operators. This problem has been analyzed by Woronowicz [19].

In the rest of this paper we will analyze basic structures associated with host algebras, prove uniqueness and existence theorems, and study the connection with convolution algebras.

2 Basic properties of host algebras

Let $\mathcal{L}$ be a C*-algebra, and recall that the **strict topology** of its multiplier algebra $M(\mathcal{L})$ is given by the family of seminorms on $M(\mathcal{L})$:

$$B \to \|BA\| + \|AB\|, \quad A \in \mathcal{L},\ B \in M(\mathcal{L})\ .$$

Then $\mathcal{L}$ is strictly dense in $M(\mathcal{L})$, cf. Prop. 3.5 and 3.6 in [6].

2.1 Proposition *Let X be a set with $\mathcal{R} \subset \mathrm{Rep}\, X$ given, and let $\mathcal{L}$ be a host algebra for $\mathcal{R}$. Then*

(1) *$\mathcal{B}(X)$ (hence $\varphi(\mathcal{A}_d(X))$) is strictly dense in $M(\mathcal{L})$, where $\mathcal{B}(X)$ denotes the *-algebra generated by $\varphi(\pi_u(X))$.*

(2) *Each $\pi \in \mathrm{Rep}\, \mathcal{L}$ is strict–strong operator continuous, and $\theta(\pi)$ is the strict extension of π to $\varphi(\pi_u(X))$.*

(3i) *Each $\pi \in \mathcal{R}$ extends uniquely to $\varphi(\mathcal{A}_d(X))$ as a *-representation which is strict–strict continuous (the second strict topology referred to is that of $M(\theta^{-1}(\pi)(\mathcal{L}))$). These unique extensions are also the unique extensions as strict-strong operator continuous representations.*

(3ii) *Conversely, the restrictions to $\varphi(\pi_u(X))$ of the strict–strong operator continuous representations of $\varphi(\mathcal{A}_d(X))$ are in $\mathcal{R}$ (and by (3i) these are automatically strict-strict continuous). Thus we can identify $\mathcal{R}$ with the set of strict–strong operator continuous representations of $\varphi(\mathcal{A}_d(X))$.*

(4) *The inverse map of the bijection θ, is the map $\theta^{-1} : \mathcal{R} \to \mathrm{Rep}\, \mathcal{L}$ obtained by $\theta^{-1}(\pi)(A) := \text{s–}\lim_{\alpha} \widetilde{\pi}(B_\alpha)$ where $\widetilde{\pi}$ is the unique strict–strong operator continuous extension in (3i), and $\{B_\alpha\} \subset \varphi(\mathcal{A}_d(X))$ is a net strictly converging to $A \in \mathcal{L}$.*

(5) *If $\varphi(\pi_u(X))$ is commutative then $\mathcal{L}$ is commutative.*

Proof: (1) Let $\mathcal{Q}$ be the strict closure of $\mathcal{B}(X)$. This is a *-algebra, so since $\varphi(\pi_u(X))$ separates $\mathcal{R} = \theta(\mathrm{Rep}\, \mathcal{L})$, it follows that $\mathcal{Q}$ separates $\mathrm{Rep}\, \mathcal{L}$. Thus by Prop. 2.2 in [19], we have that $\mathcal{Q} = M(\mathcal{L})$.

(2) Let $\pi \in \mathrm{Rep}\,\mathcal{L}$, which is a *–homomorphism $\pi : \mathcal{L} \to \pi(\mathcal{L}) =: \mathcal{C} \subset \mathcal{B}(\mathcal{H}_\pi)$, and by Prop. 3.8 and 3.9 in [6], this extends uniquely to a *–homomorphism $\pi : M(\mathcal{L}) \to M(\mathcal{C}) \subseteq \mathcal{B}(\mathcal{H}_\pi)$ which is strict–strict continuous (using nondegeneracy of π). Since on $M(\mathcal{C}) \subseteq \mathcal{B}(\mathcal{H}_\pi)$ the strong operator topology is coarser than the strict topology, it follows that $\pi : M(\mathcal{L}) \to \mathcal{B}(\mathcal{H}_\pi)$ is strict–strong operator continuous. If $\{E_\alpha\} \subset \mathcal{L}$ is an approximate identity of $\mathcal{L}$, then for each $B \in M(\mathcal{L})$ the net $\{BE_\alpha\}$ strictly converges to B, hence $\pi(BE_\alpha)$ converges in strong operator topology to $\pi(B)$, and by definition this is $\theta(\pi)(x)$ when $B = \varphi(\pi_u(x))$.

(3i) By the bijection $\theta : \mathrm{Rep}\,\mathcal{L} \to \mathcal{R}$, for each $\pi \in \mathcal{R}$ there is a $\rho \in \mathrm{Rep}\,\mathcal{L}$ such that its strict extension to $M(\mathcal{L})$ produces $\pi \in \mathcal{R}$ by (2). Hence each $\pi \in \mathcal{R}$ has a strictly continuous extension $\widetilde{\pi} = \rho\restriction\varphi\left(\mathcal{A}_d(X)\right)$ to $\varphi\left(\mathcal{A}_d(X)\right)$. If $\widehat{\pi}$ is another strictly continuous extension of π to $\varphi\left(\mathcal{A}_d(X)\right)$, then since $\varphi\left(\mathcal{A}_d(X)\right)$ is strictly dense in $M(\mathcal{L})$, it extends uniquely to $\mathcal{L}$, so by definition we get $\theta(\widetilde{\pi}) = \pi = \theta(\widehat{\pi})$. Since θ is injective, we have that $\widetilde{\pi}\restriction\mathcal{L} = \widehat{\pi}\restriction\mathcal{L}$ and as $\mathcal{L}$ is strictly dense we have that $\widehat{\pi} = \widetilde{\pi}$. Since $\rho = \theta^{-1}(\pi)$ is strict–strict continuous, hence strict–strong operator continuous, the proof is now clear.

(3ii) Conversely, if π is a *-representation of $\varphi\left(\mathcal{A}_d(X)\right)$ which is strict–strong operator continuous, then we will show that it extends uniquely to $M(\mathcal{L})$ as a *-representation, in which case $\theta(\pi\restriction\mathcal{L}) = \pi\restriction\varphi(\pi_u(X)) \in \mathcal{R}$. First, π extends by strict continuity to a well-defined continuous linear map on $M(\mathcal{L})$ because addition in $M(\mathcal{L})$ (resp. $\mathcal{B}(\mathcal{H}_\pi)$) is strictly (resp. strong operator) continuous. By linearity, the extension π is uniquely determined by its values on $M(\mathcal{L})^1_{\rm sa} =$ selfadjoint part of $M(\mathcal{L})$. From the strict density of $\varphi\left(\mathcal{A}_d(X)\right)$ in $M(\mathcal{L})$ we have a Kaplansky–type density theorem, that $\varphi\left(\mathcal{A}_d(X)\right)^1_{\rm sa}$ is strictly dense in $M(\mathcal{L})^1_{\rm sa}$ (cf. p50 in [18]). Let $A,\, B \in M(\mathcal{L})^1_{\rm sa}$ then this implies there are nets $\{A_\mu\} \subset \varphi\left(\mathcal{A}_d(X)\right)^1_{\rm sa} \supset \{B_\nu\}$ strictly converging: $A_\mu \to A$, $B_\nu \to B$. Since multiplication in $M(\mathcal{L})$ is jointly continuous w.r.t. the strict topology on *bounded* subsets, it follows that $A_\mu B_\nu \to AB$ strictly. Now $\|\pi(A_\mu)\| \leq 1 \geq \|\pi(B_\nu)\|$, so since multiplication in $\mathcal{B}(\mathcal{H}_\pi)$ is jointly continuous in the strong operator topology on *bounded* subsets, we have that

$$\pi(AB) = \operatorname*{s\text{--}lim}_{\mu,\nu\to\infty} \pi(A_\mu B_\nu) = \operatorname*{s\text{--}lim}_{\mu,\nu\to\infty} \pi(A_\mu)\pi(B_\nu) = \pi(A)\pi(B)\,.$$

Thus π is a homomorphism. Finally, note that involution on $M(\mathcal{L})$ is strictly continuous, whereas involution in $\mathcal{B}(\mathcal{H}_\pi)$ is only strong operator continuous for normal operators. Thus, by selfadjointness:

$$\pi(A) = \operatorname*{s\text{--}lim}_{\mu\to\infty} \pi(A_\mu) = \operatorname*{s\text{--}lim}_{\mu\to\infty} \pi(A_\mu)^* = \pi(A)^*\,.$$

Thus π is a *-homomorphism on $M(\mathcal{L})$, which completes the proof.

(4) This is clear from the previous parts.

(5) The strict topology on $M(\mathcal{L}) \subset \mathcal{L}''$ is finer than the weak operator topology of $\mathcal{L}''$ on $M(\mathcal{L})$. Thus the strict closure of $\mathcal{B}(X)$ (i.e. $M(\mathcal{L})$) is contained in its weak operator closure, and this is in the double commutant $\mathcal{B}(X)''$. Now if $\mathcal{B}(X)$ is commutative we have that $\mathcal{B}(X)'' \supset \mathcal{L}$ is commutative. ■

Since $\varphi(\mathcal{A}_d(X))$ and $\mathcal{L}$ are both strictly dense in $M(\mathcal{L})$, and the strictly continuous representations on $M(\mathcal{L})$ are the extensions of representations in $\operatorname{Rep}\mathcal{L}$, it follows that all the properties of these representations are determined by their restrictions to either $\varphi(\pi_u(X))$ or $\mathcal{L}$. In particular, we find the following structural properties for $\mathcal{R}$.

2.2 Corollary *Let $\mathcal{L}$ be a host algebra for $\mathcal{R} \subset \operatorname{Rep} X$. Then*

(1) $\pi \in \operatorname{Rep}\mathcal{L}$ is cyclic (resp. irreducible) iff $\theta(\pi) \in \mathcal{R}$ is cyclic (resp. irreducible).

(2) If $\mathcal{C} \subset \operatorname{Rep}\mathcal{L}$ is a set of cyclic representations, then $\theta(\underset{\pi\in\mathcal{C}}{\oplus}\pi) = \underset{\pi\in\mathcal{C}}{\oplus}\theta(\pi)$ and conversely if $\mathcal{D} \subset \mathcal{R}$ is a set of cyclic representations, then $\theta^{-1}(\underset{\pi\in\mathcal{C}}{\oplus}\pi) = \underset{\pi\in\mathcal{C}}{\oplus}\theta^{-1}(\pi)$.

(3) $\mathcal{R}$ is closed with respect to:

(i) formation of direct sums of sets of cyclic representations in $\mathcal{R}$,

*(ii) composition with strict–strong operator continuous concrete *-homomorphisms, i.e. $(\alpha\circ\pi)\restriction_{\rm ess} \in \mathcal{R}$ if $\pi \in \mathcal{R}$ and $\alpha : \pi(\varphi(\mathcal{A}_d(X))) \to \mathcal{B}(\mathcal{K})$ is a strict–strong operator continuous *-homomorphism. The strict topology referred to here is that of $M(\theta^{-1}(\pi)(\mathcal{L}))$.*

(4) $\mathcal{R}$ is generated from its subset of irreducible representations $\mathcal{R}_{\rm irr}$, by the two operations in part(3).

(5) Define $\pi_{\mathcal{R}} := \oplus\{\pi \in \mathcal{R} \mid \pi \text{ cyclic}\}$. Then $\mathcal{L}'' = \pi_{\mathcal{R}}(X)'' = \pi_{\mathcal{R}}(\mathcal{A}_d(X))''$.

Proof: (1) By strict continuity, the closures of $\pi(\mathcal{L})\psi$ and $\theta(\pi)(\mathcal{A}_d(X))\psi$ are equal for each $\psi \in \mathcal{H}_\pi$. Thus ψ is a cyclic vector for $\pi(\mathcal{L})$ iff it is a cyclic vector for $\theta(\pi)(\mathcal{A}_d(X))$. Since a representation is irreducible iff each nonzero vector is cyclic, it follows that π is irreducible iff $\theta(\pi)$ is irreducible.

(2) For a given set $\mathcal{C} \subset \operatorname{Rep}\mathcal{L}$ of cyclic representations, $\underset{\pi\in\mathcal{C}}{\oplus}\pi \in \operatorname{Rep}\mathcal{L}$ hence both $\underset{\pi\in\mathcal{C}}{\oplus}\pi$ and $\theta(\underset{\pi\in\mathcal{C}}{\oplus}\pi)$ are strict–strong operator continuous. Thus the closures of $\underset{\pi\in\mathcal{C}}{\oplus}\pi(\mathcal{L})\mathcal{K}$ and $\theta(\underset{\pi\in\mathcal{C}}{\oplus}\pi)(\mathcal{A}_d(X))\mathcal{K}$ are the same for any subspace $\mathcal{K} \subset \underset{\pi\in\mathcal{C}}{\oplus}\mathcal{H}_\pi$. In particular, let $\mathcal{K}$ be any of the invariant subspaces $\mathcal{H}_\pi$ or $\mathcal{H}_\pi^\perp$ for $\underset{\pi\in\mathcal{C}}{\oplus}\pi(\mathcal{L})$, then it is clear that these are also invariant subspaces for $\theta(\underset{\pi\in\mathcal{C}}{\oplus}\pi)(\mathcal{A}_d(X))$. So since $\theta(\pi)$ is just the strict extension of π, it follows that $\theta(\underset{\pi\in\mathcal{C}}{\oplus}\pi)$ restricted to $\mathcal{H}_\pi$ is just $\theta(\pi)$. Since a similar statement holds on the complementary

subspace $\mathcal{H}_\pi^\perp$, it follows that $\theta(\pi)$ is a direct summand of $\theta(\underset{\pi\in\mathcal{C}}{\oplus}\pi)$. Since the full representation space is just the direct sum of all $\mathcal{H}_\pi$, $\pi\in\mathcal{C}$, it follows that $\theta(\underset{\pi\in\mathcal{C}}{\oplus}\pi)$ is the direct sum of all $\theta(\pi)$, $\pi\in\mathcal{C}$.

(3i) That $\mathcal{R}$ is closed w.r.t. direct sums of cyclic sets, follows from the fact that this is true for $\mathrm{Rep}\,\mathcal{L}$, and from (1) and (2) above.

(3ii) Let $\pi\in\mathcal{R}$, then by construction it extends uniquely to $\mathcal{A}_d(X)$, and this extension is strict–strict continuous by Theorem 2.1(3i). If $\alpha:\pi\left(\mathcal{A}_d(X)\right)\to\mathcal{B}(\mathcal{H})$ is a strict–strong operator continuous *-homomorphism, then the composition $\alpha\circ\pi:\mathcal{A}_d(X)\to\mathcal{B}(\mathcal{H})$ is a strict–strong operator continuous representation, hence by Theorem 2.1(3ii) $\big(\alpha\circ\pi\restriction\varphi(\pi_u(X))\big)\restriction_{\rm ess}\in\mathcal{R}$.

(4) That $\mathcal{R}$ has irreducible representations, follows from part (1) above. Consider the atomic representation $\pi_a:=\oplus\{\,\pi\in\mathrm{Rep}\,\mathcal{L}\;\mid\;\pi\ \text{irreducible}\,\}$ which is faithful, hence its extension to $M(\mathcal{L})$ is faithful, cf. Prop. 2.4 [3], and so π_a is faithful on $\varphi(\mathcal{A}_d(X))$. Since π_a is faithful on $\mathcal{L}$, any $\pi\in\mathrm{Rep}\,\mathcal{L}$ is of the form $\alpha\circ\pi_a$ where $\alpha:\pi_a(\mathcal{L})\to\mathcal{B}(\mathcal{H})$ is a concrete *-homomorphism. Now α extends to $M(\pi_a(\mathcal{L}))\supset\theta(\pi_a)(X)$ to a strict–strong operator continuous *-homomorphism $\widetilde{\alpha}$, and it satisfies $\theta(\pi)=\theta(\alpha\circ\pi_a)=\widetilde{\alpha}\circ\theta(\pi_a)\in\mathcal{R}$. Since $\mathcal{L}$ is a host, all of $\mathcal{R}$ is therefore of the form $\widetilde{\alpha}\circ\theta(\pi_a)$. However by parts (1) and (2) $\theta(\pi_a)$ is the direct sum of the irreducible representations in $\mathcal{R}$, so the claim is clear.

(5) Observe from parts (1) and (2) that $\theta^{-1}(\pi_{\mathcal{R}})$ is the universal representation of $\mathcal{L}$, so it follows from the strict–strong operator continuity of its extension to $M(\mathcal{L})$ and the strict denseness of $\varphi(\mathcal{A}_d(X))$ and $\mathcal{L}$ in $M(\mathcal{L})$ that $\mathcal{L}''=\theta^{-1}(\pi_{\mathcal{R}})\left(\varphi(\mathcal{A}_d(X))\right)''$. Since $\theta^{-1}(\pi_{\mathcal{R}})$ restricts on $\varphi(X)$ to $\pi_{\mathcal{R}}$, and $\mathcal{A}_d(X)$ is generated by X, it follows that $\mathcal{L}''=\theta^{-1}(\pi_{\mathcal{R}})\left(\varphi(\mathcal{A}_d(X))\right)''=\pi_{\mathcal{R}}(\mathcal{A}_d(X))''=\pi_{\mathcal{R}}(X)''$. ■

Host algebras do not behave naturally w.r.t. containment, i.e. if $\mathcal{L}_i$ is a host algebra for $\mathcal{R}_i$, $i=1,\,2$ where $\mathcal{R}_1\subset\mathcal{R}_2$, then it does not always follow that $\mathcal{L}_1\subset\mathcal{L}_2$ with $\varphi_2(\pi_u(x))\restriction\mathcal{L}_1=\varphi_1(\pi_u(x))$, $x\in X$. This is because:

2.3 Proposition *Let $\mathcal{L}_i$ be a host algebra for $\mathcal{R}_i$, $i=1,\,2$ such that $\mathcal{L}_1\subset\mathcal{L}_2$, and such that $\varphi_1(\pi_u(x))A=\varphi_2(\pi_u(x))A$ for all $x\in X$, $A\in\mathcal{L}_1$. Then $\mathcal{L}_1$ is a closed two-sided ideal of $\mathcal{L}_2$, and hence $\mathrm{Rep}\,\mathcal{L}_2=\mathrm{Rep}\,\mathcal{L}_1\oplus\mathrm{Rep}\,(\mathcal{L}_2/\mathcal{L}_1)$ where $\mathrm{Rep}\,\mathcal{L}_1$ is identified in $\mathrm{Rep}\,\mathcal{L}_2$ by unique extensions, and $\mathrm{Rep}\,(\mathcal{L}_2/\mathcal{L}_1)$ corresponds to those representations which vanish on $\mathcal{L}_1$.*

Proof: Recall that $M(\mathcal{L}_1)\subset\mathcal{L}_1''\subset\mathcal{L}_2''\supset M(\mathcal{L}_2)$. Recall that $\mathrm{Span}\varphi_i(X)$ is $\mathcal{L}_i$–strictly dense in $M(\mathcal{L}_i)$. Since the actions of both $\varphi_i(X)$, $i=1,\,2$ coincide on $\mathcal{L}_1$, it follows that $\mathcal{B}(X):=\mathcal{A}^*(\varphi_2(\pi_u(X)))$ is $\mathcal{L}_i$–strictly dense in $M(\mathcal{L}_i)$, $i=1,\,2$ by Proposition 2.1. Since $\mathcal{L}_1\subset\mathcal{L}_2$ it now follows from the definition of strict topologies that the $\mathcal{L}_1$–strict closure of $\mathcal{B}(X)$ contains the $\mathcal{L}_2$–strict closure of $\mathcal{B}(X)$. Thus $M(\mathcal{L}_1) \supseteq M(\mathcal{L}_2) \supset \mathcal{L}_2$, and hence $\mathcal{L}_1$ is an ideal of $\mathcal{L}_2$. The direct sum decomposition of $\text{Rep}\,\mathcal{L}_2$ follows from the ideal property, cf. [7] ∎

Thus we can have natural containment of host algebras only for direct summands.

3 Existence of host algebras.

Above we saw examples of pairs $\{X,\,\mathcal{R}\}$ for which host algebras do exist, as well as examples for which they do not exist. Here we want to develop an existence theorem, i.e. to find a property of $\{X,\,\mathcal{R}\}$ which characterises the existence of a host algebra $\mathcal{L}$ exactly. We will examine the Von Neumann algebra $\pi_{\mathcal{R}}(X)''$ and the C*-algebra $\pi_{\mathcal{R}}(\mathcal{A}_d(X))$ contained in it, and try to characterise when there is a strongly dense C*-algebra $\mathcal{L} \subset \pi_{\mathcal{R}}(X)''$ such that the $\pi_{\mathcal{R}}\restriction\mathcal{L}$ is the universal representation for $\mathcal{L}$, and such that $\mathcal{L}$ contains $\pi_{\mathcal{R}}(\mathcal{A}_d(X))$ in its relative multiplier algebra. For this, we need to generalise slightly Pedersen's concept of an open projection to arbitrary Von Neumann algebras (cf. Prop. 3.11.9 [15]), as well as Akemann's concept of q-continuous operators cf. [1].

Let $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be a given Von Neumann algebra, and recall that for any positive functional $\omega \in \mathcal{N}^*_+$, its left kernel is $N_\omega := \{\, A \in \mathcal{N} \mid \omega(A^*A) = 0 \,\}$. These are closed left ideals. Define

$$\mathcal{S}(\mathcal{N}) := \left\{ \bigcap_{\omega\in S} N_\omega \;\mid\; S \subseteq (\mathcal{N}_*)_+ \text{ arbitrary subsets} \right\}$$

i.e. all possible intersections of the left kernels of the normal positive functionals. Observe that each $L \in \mathcal{S}(\mathcal{N})$ is ultraweakly, i.e. $\sigma(\mathcal{N},\,\mathcal{N}_*)$–closed. To see this it suffices to show that the left kernels $N_\omega = \{\, A \in \mathcal{N} \mid \omega(BA) = 0 \;\; \forall B \in \mathcal{N} \,\}$ for $\omega \in (\mathcal{N}_*)_+$ are ultraweakly closed. This follows directly from the fact that all normal functionals are ultraweakly continuous, hence have ultraweakly closed kernels, and that the map $A \to BA$ is ultraweakly continuous for fixed B (cf. Theorem 1.7.8 [16]).

3.1 Definition *The fact that each $L \in \mathcal{S}(\mathcal{N})$ is ultraweakly closed, implies that there is for each a unique projection $P \in \mathcal{N}$ such that $L = \mathcal{N}P$ (cf. Prop. 1.10.1 [16]). Define these as the* **open** *projections of $\mathcal{N}$.*

By the next lemma, this agrees with the usual definitions of open projections (cf. [2] and Prop. 3.11.9 [15]) when $\mathcal{N}$ is a universal enveloping von Neumann algebra $\mathcal{A}''$, which is the only circumstance where they were defined before.

3.2 Lemma *In the case that $\mathcal{N} = \mathcal{A}''$ for some C*-algebra $\mathcal{A}$, then a projection $P \in \mathcal{N}$ is open iff $L = \mathcal{N}P \cap \mathcal{A}$ is a closed left ideal of $\mathcal{A}$.*

Proof: We already know from Theorem 3.10.7, Proposition 3.11.9 and Remark 3.11.10 in Pedersen [15] that when $\mathcal{N} = \mathcal{A}''$ then the usual open projections are in bijection with

(i) hereditary C*-subalgebras of $\mathcal{A}$ by $P \to P\mathcal{N}P \cap \mathcal{A}$,

(ii) closed left ideals of $\mathcal{A}$ by $P \to \mathcal{N}P \cap \mathcal{A}$ and

(iii) weak *-closed faces containing 0 of the quasi-state space $Q(\mathcal{A})$ by

$$P \to \{\, \omega \in Q(\mathcal{A}) \mid \omega(P) = 0 \,\}\,.$$

Since $\mathcal{N}_* = \mathcal{A}^*$ (after extension by weak operator continuity) it follows from (iii) that $\mathcal{S}(\mathcal{N})$ is also in bijection with these objects, hence in this case our definition of open projections coincides with the usual one. ∎

Following Akemann [1] we define:

3.3 Definition *If $\mathcal{N}$ is a general Von Neumann algebra, then an $A \in \mathcal{N}_{\rm sa}$ is* **q-continuous** *if for each open set in its spectrum $T \subset \sigma(A) \subset \mathbb{R}$ the corresponding spectral projection $E(T) = \chi_T(A)$ is an open projection. Denote the set of q-continuous elements of $\mathcal{N}$ by $\mathcal{N}_q$.*

The real usefulness of the q-continuous elements, lies in the result of Akemann, Pedersen and Tomiyama that when $\mathcal{N} = \mathcal{A}''$, then $\mathcal{N}_q = M(\mathcal{A})_{\rm sa}$ and so it provides a method of constructing $M(\mathcal{A})$ if only the Von Neumann algebra $\mathcal{A}''$ is given (cf. Theorem 2.2 [3]).

Finally, let us recall that for a general Von Neumann algebra $\mathcal{N}$ there is a unique projection $P_* \in Z(\mathcal{N}'')$ (where $\mathcal{N}''$ is the universal von Neumann algebra of $\mathcal{N}$) such that for any functional $\varphi \in \mathcal{N}^*$ we have $\varphi \in \mathcal{N}_*$ iff $\varphi(P_*A) = \varphi(A)$ for all $A \in \mathcal{N}$ (cf. Prop. 10.1.14 and Prop. 10.1.18 in [12]). We will call P_* the **normal projection** of $\mathcal{N}$.

3.4 Theorem *Let a set of representations $\mathcal{R} \subset \operatorname{Rep} X$ be given, and define $\mathcal{N} := \pi_{\mathcal{R}}(X)''$. Then $\mathcal{R}$ has a host algebra iff*

(1) $\mathcal{R}$ is the set of normal representations of $\mathcal{N}$,

(2) The finite span (without closure) of q-continuous elements, $\operatorname{span}(\mathcal{N}_q)$, is a C-subalgebra of $\mathcal{N}$,*

(3) $\pi_{\mathcal{R}}(\mathcal{A}_d(X))_{\rm sa} \subset \mathcal{N}_q$,

(4) $P_*\operatorname{span}(\mathcal{N}_q)$ is a strong operator dense subalgebra of $\mathcal{N}$, where P_* is the normal projection of $\mathcal{N}$.

In this case, the host algebra is $\mathcal{L} = P_* \mathrm{span}(\mathcal{N}_q)$*. Hence host algebras are unique up to isomorphism.*

Proof: Let $\mathcal{R}$ have a host algebra $\mathcal{L}$, so $\varphi : \mathcal{A}_d(X) \to M(\mathcal{L})$ and $\theta(\mathrm{Rep}\,\mathcal{L}) = \mathcal{R}$. Then by Corollary 2.2(5), $\mathcal{N} = \mathcal{L}''$, so (1) is satisfied and by Theorem 2.2 in [3] we have $\mathrm{span}(\mathcal{N}_q) = M(\mathcal{L}) \supset \varphi\left(\mathcal{A}_d(X)\right)$, so condition (2) is satisfied. For the universal representation $\pi_{\mathcal{L}}$ of $\mathcal{L}$, we get $\theta(\pi_{\mathcal{L}}) = \pi_{\mathcal{R}}$ by $\theta(\mathrm{Rep}\,\mathcal{L}) = \mathcal{R}$, hence on $\mathcal{H}_{\mathcal{L}}$, $\varphi\left(\mathcal{A}_d(X)\right) \subset M(\mathcal{L}) \subset \mathcal{L}''$ is precisely $\pi_{\mathcal{R}}\left(\mathcal{A}_d(X)\right)$, so we get that $\pi_{\mathcal{R}}\left(\mathcal{A}_d(X)\right)_{\mathrm{sa}} \subset M(\mathcal{L})_{\mathrm{sa}} = \mathcal{N}_q$, and condition (3) is satisfied. For condition (4), it suffices to show that $\mathcal{L} = P_* \mathrm{span}(\mathcal{N}_q)$. Recall that corresponding to the ideal $\mathcal{L}$ in $M(\mathcal{L})$ there is a central open projection $Q \in Z(M(\mathcal{L})'')$ given by $Q = \operatorname*{s\text{–}lim}_{\alpha} E_\alpha$ where $\{E_\alpha\} \subset \mathcal{L}$ is any approximate identity of $\mathcal{L}$, and the limit is taken in the strong operator topology of $M(\mathcal{L})''$. It satisfies $M(\mathcal{L}) \cap Q \cdot M(\mathcal{L})'' = \mathcal{L}$, and hence $Q \cdot M(\mathcal{L}) = \mathcal{L} = Q\mathcal{L}$. So it will suffice to show that $Q = P_*$. By Kadison and Ringrose Prop. 10.1.14 and Prop. 10.1.18 [12], P_* is the projection onto the space of vectors whose vector states are in $\mathcal{N}_* = \mathcal{L}^*$. Then we have that $P_* \mathcal{N}'' = \pi_{\mathcal{L}}(\mathcal{N}'') = \pi_{\mathcal{L}}(\mathcal{N})'' = \pi_{\mathcal{L}}(\mathcal{L})'' = \mathcal{L}'' \subset \mathcal{N}''$ using the fact that universal representation $\pi_{\mathcal{L}}$ of $\mathcal{L}$ is normal, $\mathcal{L}$ is strong operator dense in $\mathcal{N}$, and that $\mathcal{N}$ is just the strong closure of $\mathcal{L}$ in $\mathcal{N}'' \supset M(\mathcal{L})''$ cf. Corollary 3,7.9 in [15]. So $P_* = \operatorname*{s\text{–}lim}_{\alpha} E_\alpha = Q$ for any approximate identity $\{E_\alpha\}$ of $\mathcal{L}$.

Conversely, assume that for $\mathcal{N}$ the conditions (1), (2), (3) and (4) are satisfied. Put $\mathcal{L} := P_* \mathrm{span}(\mathcal{N}_q)$, then this is a C*-algebra by condition (2) and the fact that P_* is central. By (3), the algebra $\pi_{\mathcal{R}}(\mathcal{A}_d(X))$ is then in the relative multiplier of $\mathcal{L}$, and hence there is a *-homomorphism $\varphi : \mathcal{A}_d(X) \to M(\mathcal{L})$. By definition of $\mathcal{N}$, we have that $\pi_{\mathcal{R}}(\mathcal{A}_d(X))$ is strong operator dense in $\mathcal{N}$, and by (4) the algebra $\mathcal{L}$ is strong operator dense in $\mathcal{N}$. Thus any normal representation $\pi \in \mathcal{R}$ is uniquely determined by its restrictions to $\pi_{\mathcal{R}}(\mathcal{A}_d(X))$ or to $\mathcal{L}$. The normal representations are precisely $\mathcal{R}$ by (1), so we only need to show that $\mathcal{R} = \mathrm{Rep}\,\mathcal{L}$ to establish that $\mathcal{L}$ is a host algebra for $\mathcal{R}$. Since $\mathcal{L}$ is a sub-C*-algebra of $\mathcal{N}$, any representation of $\mathcal{L}$ is a restriction of a $\pi \in \mathrm{Rep}\,\mathcal{N}$ to $\mathcal{L}$ on the subspace $\overline{\pi(\mathcal{L})\mathcal{H}_\pi}$. By construction of $\mathcal{L} = P_* \mathrm{span}(\mathcal{N}_q)$, this can only produce normal representations. Since $\mathcal{L}$ is strong operator dense in $\mathcal{N}$ we get all the normal representations, hence $\mathrm{Rep}\,\mathcal{L} = \mathcal{R}$ by (1). Hence $\mathcal{L}$ is a host algebra for $\mathcal{R}$. The isomorphism claim follows from the fact that $P_* \mathrm{span}(\mathcal{N}_q)$ is constructed only from the given set $\mathcal{R}$. ■

The conditions in theorem 3.4 are not easy to verify in interesting examples, e.g. we do not know whether they are satisfied when $\mathcal{R}$ is the set of strong operator continuous representations of the gauge group X (e.g. the group of smooth maps from the 4-sphere to $SU(n)$

with pointwise multiplication and the natural topology coming from the differential seminorms). However, from the structures above, it is easy to generate many examples of pairs $(X, \mathcal{R})$ with host algebras. For instance, let $\mathcal{L}$ be any nonunital C*-algebra, and in $M(\mathcal{L}) \subset \mathcal{L}''$ choose any sub-C*-algebra $\mathcal{A}$ which is strongly dense in $\mathcal{L}''$. Let $\mathcal{R}$ be the restriction of $\mathrm{Rep}\,\mathcal{L}$ to $\mathcal{A}$, then $\mathcal{L}$ is a host algebra for the pair $(\mathcal{A}, \mathcal{R})$.

4 Uniqueness of host algebras.

Above, we obtained uniqueness of host algebras for free in Theorem 3.4. There is however an independent proof of uniqueness, which is interesting because it gives a different construction of host algebras (as operator valued sections), and here we want to give this proof. The proof draws on Takesaki–Bichteler duality theory cf. [5, 17].

Assume we have a host algebra $\mathcal{L}$ for a fixed set $\mathcal{R} \subset \mathrm{Rep}\, X$. Let $\mathcal{H}$ be a Hilbert space which is large enough so that each cyclic representation $\pi \in \mathrm{Rep}\,\mathcal{L}$ is unitarily equivalent to a (possibly degenerate) representation of $\mathcal{L}$ on $\mathcal{H}$ on its essential subspace. Denote the set of all representations of $\mathcal{L}$ on $\mathcal{H}$ (possibly degenerate) by $\mathrm{Rep}\,(\mathcal{L}, \mathcal{H})$. For instance we can choose for $\mathcal{H}$ the space $\mathcal{H}_\mathcal{R} = \mathcal{H}_\mathcal{L} =$ space of the universal representation of $\mathcal{L}$ (cf. Corollary 2.2(5)), and henceforth we will maintain this choice. Of course restricting representations to their essential subspaces defines a map $\mathrm{Rep}\,(\mathcal{L}, \mathcal{H}) \to \mathrm{Rep}\,\mathcal{L}$. We equip $\mathrm{Rep}\,(\mathcal{L}, \mathcal{H})$ with the pointwise strong topology, i.e. a net $\{\pi_\nu\} \subset \mathrm{Rep}\,(\mathcal{L}, \mathcal{H})$ converges to $\pi \in \mathrm{Rep}\,(\mathcal{L}, \mathcal{H})$ iff $\pi_\nu(A) \to \pi(A)$ in the strong operator topology of $\mathcal{B}(\mathcal{H})$ for all $A \in \mathcal{L}$.

We also have $\mathrm{Rep}\,(X, \mathcal{H}) \equiv$ maps $\pi : X \to \mathcal{B}(\mathcal{H})$ such that π restricted to its essential subspace $\overline{\mathcal{A}^*(\pi(X))\mathcal{H}}$ is unitarily equivalent to a element of $\mathrm{Rep}\, X$. Restricting representations to their essential subspaces defines a map $\mathrm{Rep}\,(X, \mathcal{H}) \to \mathrm{Rep}\, X$, and by definition of $\mathcal{H} = \mathcal{H}_\mathcal{R}$ all cyclic representations in $\mathcal{R}$ are obtained this way. We topologise $\mathrm{Rep}\,(X, \mathcal{H})$ also with the pointwise (on X) strong operator topology i.e. $\{\pi_\nu\} \subset \mathrm{Rep}\,(X, \mathcal{H})$ converges to $\pi \in \mathrm{Rep}\,(X, \mathcal{H})$ iff $\pi_\nu(x) \to \pi(x)$ in the strong operator topology of $\mathcal{B}(\mathcal{H})$ for all $x \in X$.

Now the unique extension map $\theta : \mathrm{Rep}\,\mathcal{L} \to \mathrm{Rep}\, X$ is of course also well defined for degenerate representations, i.e. we have a map $\theta : \mathrm{Rep}\,(\mathcal{L}, \mathcal{H}) \to \mathrm{Rep}\,(X, \mathcal{H})$ given by $\theta(\pi)(x) := \operatorname*{s\text{–}lim}_{\alpha\to\infty} \pi(\varphi(\pi_u(x))E_\alpha)$ where $\{E_\alpha\}$ is any approximate identity of $\mathcal{L}$. Since $\mathcal{L}$ is a host algebra for $\mathcal{R}$, its image $\widetilde{\mathcal{R}} := \theta\,(\mathrm{Rep}\,(\mathcal{L}, \mathcal{H}))$ consists of all $\pi \in \mathrm{Rep}\,(X, \mathcal{H})$ such that π restricted to its essential subspace $\overline{\mathcal{A}^*(\pi(X))\mathcal{H}}$ is unitarily equivalent to an element of $\mathcal{R}$. We endow $\widetilde{\mathcal{R}}$ with the relative topology of $\mathrm{Rep}\,(X, \mathcal{H})$.

From Prop. 2.1 and Corr. 2.2 we also get:

4.1 Corollary *Let* $\mathcal{L}$ *be a host algebra for* $\mathcal{R}$, *and* $\mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$ *as above. Denote the essential subspace of* $\pi \in \mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$ *by* $\mathcal{H}_\pi$, *with essential projection* $P_\pi : \mathcal{H} \to \mathcal{H}_\pi$.
(1) *If* $\pi,\,\pi' \in \mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$ *satisfy* $\mathcal{H}_\pi \perp \mathcal{H}_{\pi'}$, *then* $\theta(\pi \oplus \pi') = \theta(\pi) \oplus \theta(\pi')$. *Conversely, if* $\pi,\,\pi' \in \widetilde{\mathcal{R}} \subset \mathrm{Rep}\,(X,\,\mathcal{H})$ *with* $\mathcal{H}_\pi \perp \mathcal{H}_{\pi'}$, *then* $\theta^{-1}(\pi \oplus \pi') = \theta^{-1}(\pi) \oplus \theta^{-1}(\pi')$.
(2) *The essential projections of* $\pi \in \mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$ *and* $\theta(\pi) \in \widetilde{\mathcal{R}} \subset \mathrm{Rep}\,(X,\,\mathcal{H})$ *are the same*
(3) *for* $\pi \in \mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$ *let* $U \in \mathcal{B}(\mathcal{H})$ *be any partial isometry with initial projection* $U^*U \geq P_\pi$, *and define* $\pi^U(A) := U\pi(A)U^*$, $A \in \mathcal{L}$. *Then* $\theta(\pi^U) = U\theta(\pi)U^* =: \theta(\pi)^U$, *and conversely* $\theta^{-1}(\pi^U) = \theta^{-1}(\pi)^U$ *for all* $\pi \in \widetilde{\mathcal{R}}$.

Thus by this corollary, θ preserves much of the structure of $\mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$. In fact, it also preserves the topology:

4.2 Proposition *Let* $\mathcal{L}$ *be a host algebra for* $\mathcal{R}$, *then* $\theta : \mathrm{Rep}\,(\mathcal{L},\mathcal{H}) \to \widetilde{\mathcal{R}}$ *is a homeomorphism.*

Proof: From Prop. 2.1, it suffices to show that for the strict extensions of $\mathrm{Rep}\,(\mathcal{L},\,\mathcal{H})$ to $M(\mathcal{L})$, for a net $\{\pi_\nu\}$ we have the convergence $\pi_\nu(A) \to \pi(A)$ for all $A \in \mathcal{L}$ in strong operator topology iff $\pi_\nu(B) \to \pi(B)$ for all $B \in \varphi(\pi_u(X))$ in strong operator topology.
Assume that π and $\{\pi_\nu\}$ are strict–strong operator continuous representations in $\mathrm{Rep}\,(M(\mathcal{L}),\mathcal{H})$ such that $\pi_\nu(A) \to \pi(A)$ for all $A \in \mathcal{L}$ in strong operator topology. For any $B \in \varphi(\pi_u(X))$, let $\{A_\alpha\} \subset \mathcal{L}$ be a net strictly converging to B. Then for all $\psi \in \mathcal{H}$ we have:

$$\begin{aligned}\|(\pi_\nu(B) - \pi(B))\psi\| &\leq \|\pi_\nu(B - A_\alpha)\psi\| + \|(\pi_\nu(A_\alpha) - \pi(A_\alpha))\psi\| \\ &\quad + \|\pi(A_\alpha - B)\psi\|\,. \qquad -(1)\end{aligned}$$

Since $A_\alpha \to B$ strictly, we have for the extension of the universal representation $\pi_{\mathcal{L}}$ to $M(\mathcal{L})$ that $\|\pi_{\mathcal{L}}(B - A_\alpha)\psi\| \to 0$ for all $\psi \in \mathcal{H}$. Since π and π_ν are subrepresentations of $\pi_{\mathcal{L}}$, we also have that

$$\|\pi_\nu(B - A_\alpha)\psi\| \leq \|\pi_{\mathcal{L}}(B - A_\alpha)\psi\| \geq \|\pi(B - A_\alpha)\psi\|\,.$$

Thus for each $\varepsilon > 0$ there is an α_1 such that for all ν

$$\|\pi_\nu(B - A_\alpha)\psi\| + \|\pi(B - A_\alpha)\psi\| \leq \varepsilon \quad \forall\, \alpha > \alpha_1\,.$$

Thus from (1) we get for all $\alpha > \alpha_1$ that

$$\begin{aligned}\lim_{\nu\to\infty} \|(\pi_\nu(B) - \pi(B))\psi\| &\leq \varepsilon + \lim_{\nu\to\infty} \|(\pi_\nu(A_\alpha) - \pi(A_\alpha))\psi\| \\ &= \varepsilon\,.\end{aligned}$$

So, since $\varepsilon > 0$ is arbitrary, we have for all $\psi \in \mathcal{H}$ that $\lim\limits_{\nu\to\infty} \|(\pi_\nu(B) - \pi(B))\psi\| = 0$.
Conversely, assume that for π and $\{\pi_\nu\}$ strict–strong operator continuous representations in $\mathrm{Rep}\,(M(\mathcal{L}), \mathcal{H})$, that $\pi_\nu(B) \to \pi(B)$ for all $B \in \varphi(\pi_u(X))$ in strong operator topology. By triangle inequalities, if $\pi_\nu(B_i) \to \pi(B_i)$ in strong operator topology, then $\pi_\nu(B_1 + B_2) \to \pi(B_1 + B_2)$ in strong operator topology, and moreover

$$\begin{aligned}
\|\left(\pi_\nu(B_1B_2) - \pi(B_1B_2)\right)\psi\| &= \|\left(\pi_\nu(B_1)\pi_\nu(B_2) - \pi_\nu(B_1)\pi(B_2) + \pi_\nu(B_1)\pi(B_2) - \pi(B_1)\pi(B_2)\right)\psi\| \\
&\leq \|B_1\|\|\left(\pi_\nu(B_2) - \pi(B_2)\right)\psi\| + \|\left(\pi_\nu(B_1) - \pi(B_1)\right)\pi(B_2)\psi\| \\
&\to 0 \qquad \text{as } \nu \to \infty.
\end{aligned}$$

Thus $\pi_\nu(B) \to \pi(B)$ in strong operator topology for all $B \in \mathcal{B}(X)$ = the *-algebra $\mathcal{B}(X)$ generated by $\varphi(\pi_u(X))$. Now $\mathcal{B}(X)$ is strictly dense in $M(\mathcal{L})$, so for any $A \in \mathcal{L}$, let $\{B_\alpha\} \subset \mathcal{B}(X)$ be a strictly convergent net to $A \in \mathcal{L}$. As above, we get that for any $\varepsilon > 0$ and $\psi \in \mathcal{H}$ there is an α_1 such that for all ν

$$\begin{aligned}
\|\pi_\nu(A - B_\alpha)\psi\| \;&+\; \|\pi(A - B_\alpha)\psi\| \leq \varepsilon \quad \forall\, \alpha > \alpha_1\,. \\
\text{Thus:} \qquad \|(\pi_\nu(A) - \pi(A))\psi\| &\leq \|\pi_\nu(A - B_\alpha)\psi\| + \|(\pi_\nu(B_\alpha) - \pi(B_\alpha))\psi\| \\
&\qquad + \|\pi(B_\alpha - A)\psi\| \\
&\leq \varepsilon + \|(\pi_\nu(B_\alpha) - \pi(B_\alpha))\psi\|
\end{aligned}$$

for all $\alpha > \alpha_1$. Take the limit $\nu \to \infty$ on both sides, and use the fact that $\varepsilon > 0$ is arbitrary to find that $\lim\limits_{\nu\to\infty} \|(\pi_\nu(A) - \pi(A))\psi\| = 0$. ■

Following Takesaki and Bichteler [5, 17], we define:

Def. An **admissible operator field** on $\mathrm{Rep}\,(\mathcal{L}, \mathcal{H})$ (resp. $\widetilde{\mathcal{R}}$) is a map $T : \mathrm{Rep}\,(\mathcal{L}, \mathcal{H}) \to \mathcal{B}(\mathcal{H})$ (resp. $T : \widetilde{\mathcal{R}} \to \mathcal{B}(\mathcal{H})$) such that:

(i) $\|T\| = \sup\{\, \|T(\pi)\| \;\mid\; \pi \in \mathrm{Rep}\,(\mathcal{L}, \mathcal{H})\,\} < \infty$
(resp. $\|T\| = \sup\left\{\, \|T(\pi)\| \;\mid\; \pi \in \widetilde{\mathcal{R}}\,\right\} < \infty$),

(ii) $T(\pi) = P_\pi T(\pi) = T(\pi)P_\pi$ for all $\pi \in \mathrm{Rep}\,(\mathcal{L}, \mathcal{H})$ (resp. $\pi \in \widetilde{\mathcal{R}}$) where P_π denotes the essential projection of π,

(iii) $T(\pi \oplus \pi') = T(\pi) \oplus T(\pi')$ whenever $\mathcal{H}_\pi \perp \mathcal{H}_{\pi'}$ in $\mathcal{H}$,

(iv) $T(\pi^U) = UT(\pi)U^*$ for all $\pi \in \mathrm{Rep}\,(\mathcal{L}, \mathcal{H})$ (resp. $\pi \in \widetilde{\mathcal{R}}$) where $U \in \mathcal{B}(\mathcal{H})$ is a partial isometry with $U^*U > P_\pi$.

The set of admissible operator fields form a C*–algebra under pointwise operations and the sup–norm, and we denote the two resultant C*-algebras by $\mathcal{A}(\mathcal{L},\mathcal{H})$ and $\mathcal{A}(\mathcal{R},\mathcal{H})$ respectively. We have a *-homomorphism $\Phi : M(\mathcal{L}) \to \mathcal{A}(\mathcal{L},\mathcal{H})$ by $\Phi(A) \equiv T_A : \mathrm{Rep}\,(\mathcal{L},\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ where $T_A(\pi) = \pi(A) = \text{s–}\lim_\alpha \pi(AE_\alpha)$ and $\{E_\alpha\}$ is any approximate identity of $\mathcal{L}$. We used the canonical extension of $\pi \in \mathrm{Rep}\,(\mathcal{L},\mathcal{H})$ from $\mathcal{L}$ to $M(\mathcal{L})$. Since the C*-algebra $\Phi(\mathcal{L})$ is obviously isomorphic to $\mathcal{L}$, it follows that Φ is a *-isomorphism on $M(\mathcal{L})$. We have the Takesaki–Bichteler duality theorem:

$$\Phi(\mathcal{L}) \cong \{\, T \in \mathcal{A}(\mathcal{L},\mathcal{H}) \;\mid\; T \quad \text{is strong–operator continuous}\,\}$$

where $\mathrm{Rep}\,(\mathcal{L},\mathcal{H})$ has the defined topology. That is, $\mathcal{L}$ is isomorphic to the algebra of continuous admissible operator fields on $\mathrm{Rep}\,(\mathcal{L},\mathcal{H})$. Using this, it is now easy to prove:

4.3 Theorem *Let $\mathcal{R} \subset \mathrm{Rep}\,X$ be given. If $\mathcal{R}$ has a host algebra $\mathcal{L}$, then up to isomorphism it is unique.*

Proof: Define a map $\widetilde{\theta} : \mathcal{A}(\mathcal{L},\mathcal{H}) \to \mathcal{A}(\mathcal{R},\mathcal{H})$ by $\widetilde{\theta}(T) := T \circ \theta^{-1}$. That $\widetilde{\theta}$ takes admissible operator fields to admissible operator fields follows from Corollary 4.1. Since θ is bijective, $\widetilde{\theta}$ is a *-isomorphism of C*-algebras. Thus $\widetilde{\theta} \circ \Phi : M(\mathcal{L}) \to \mathcal{A}(\mathcal{R},\mathcal{H})$ is a *-monomorphism. Note that

$$\widetilde{\theta} \circ \Phi(\varphi(x)) = \theta^{-1}(\pi)(\varphi(x)) = \text{s–}\lim_\alpha \theta^{-1}(\pi)\,(\varphi(x)E_\alpha) = \theta(\theta^{-1}(\pi))(x) = \pi(x)$$

so the host algebra $\mathcal{L}$ is isomorphic to the host algebra $\widetilde{\theta}(\Phi(\mathcal{L}))$ with embedding map $X \to M\left(\widetilde{\theta} \circ \Phi(\mathcal{L})\right)$ given by the operator fields $T_x(\pi) := \pi(x)$. Since θ is a homeomorphism, it maps the strong operator continuous fields to the strong operator continuous fields on $\mathcal{R}$, i.e.

$$\widetilde{\theta}(\Phi(\mathcal{L})) = \left\{ T \in \mathcal{A}(\mathcal{R},\mathcal{H}) \;\mid\; T : \widetilde{\mathcal{R}} \to \mathcal{B}(\mathcal{H}) \quad \text{is strong–operator continuous} \right\}.$$

But now since we have defined $\mathcal{L} \cong \widetilde{\theta}(\Phi(\mathcal{L}))$ intrinsically on X, i.e. involving only X and $\mathcal{R}$, it follows that all host algebras $\mathcal{L}$ are isomorphic. ■

4.4 Remark Note that the proof above provides a method for constructing a host algebra, i.e. *if we know* that $\mathcal{R}$ has a host algebra, then we can construct it as the set of (strong operator) continuous admissible operator fields on $\widetilde{\mathcal{R}}$.

5 Convolution Algebras

Historically, group algebras and their generalisations were constructed from convolution algebras, and here we want to build a bridge to that point of view. In [10] we developed a "universal" convolution algebra for a topological group G, in which a group algebra $\mathcal{L}$ for $\mathcal{R} \subset \operatorname{Rep} G$ is guaranteed to be, if it exists. In essence, we started from the observation that when $\mathcal{L}$ exists, then $\mathcal{L} \subset \mathcal{L}'' = \mathcal{L}^{**} = (J(\mathcal{R}))^*$ where $J(\mathcal{R})$ denotes the space of coefficient functions (on G) of representations in $\mathcal{R}$ equipped with the norm of $C^*(G_d)^*$, using the fact that $J(\mathcal{R})$ is identified with $\mathcal{L}^* \subset C^*(G_d)^*$ in the natural way. In the case that $\mathcal{R}$ is the set of strong operator continuous representations of a locally compact group G, then $J(\mathcal{R})$ is of course the Fourier–Stieltjes algebra of G. When we do not know that $\mathcal{L}$ exists, the space $(J(\mathcal{R}))^*$ still makes sense (if $\mathcal{R}$ is closed w.r.t. direct sums), and working backwards, we endow it with a natural multiplication which coincides with the multiplication in $\mathcal{L}''$ when $\mathcal{L}$ exists, and which agrees with convolution of functionals. We will see below that this is in fact precisely $\pi_{\mathcal{R}}(G)''$. Since it has some interesting subalgebras, we will consider the structures of the C*-algebra $(J(\mathcal{R}))^*$ in this context.

We assume a representation theory $\operatorname{Rep} X$ is given for a set X as in the previous sections. For a set $\mathcal{R} \subset \operatorname{Rep} X$ which is closed w.r.t. finite direct sums, define its set of coefficient functions:

$$B(\mathcal{R}) := \{\, f : X \to \mathbb{C} \;\mid\; f(x) = (\psi,\, \pi(x)\varphi),\ \pi \in \mathcal{R};\ \psi,\, \varphi \in \mathcal{H}_\pi \,\}$$

which is clearly a linear space. Now $B(\mathcal{R})$ is the image of the restriction map of the vector functionals of $\pi_{\mathcal{R}}(\mathcal{A}_d(X))$ to $\pi_{\mathcal{R}}(X)$. We will assume that the restriction map is injective, i.e. a vector functional $f(A) = (\psi,\, \pi(A)\varphi)$, $A \in \mathcal{A}_d(X)$, $\pi \in \mathcal{R}$ is uniquely determined by its restriction to $\pi_u(X) \subset \mathcal{A}_d(X)$. (This is automatic in the group situation of [10] because $\operatorname{span}\pi_u(G)$ is dense in $\mathcal{A}_d(G)$.) There are two natural norms on $B(\mathcal{R})$, the uniform norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$ (when the coefficient functions are bounded), and the norm on the dual space $\mathcal{A}_d(X)^*$, where we identify $B(\mathcal{R})$ with a subspace of $\mathcal{A}_d(X)^*$ as above (i.e. with the vector functionals of $\pi_{\mathcal{R}}$). The latter norm is the more useful of the two, and we denote it by $\|\cdot\|_*$. Let $J(\mathcal{R})$ denote the completion of $B(\mathcal{R})$ in the norm $\|\cdot\|_*$. Below we will find it convenient to use the notation $\omega_x(f(x)) := \omega(f)$ for $\omega \in J(\mathcal{R})^*$, $f \in J(\mathcal{R})$, i.e. we explicitly indicate the argument of the function which a functional is evaluated on.

5.1 Proposition *Let $\mathcal{R} \subset \operatorname{Rep} X$ be closed w.r.t. finite direct sums.*

(1) *If $\mathcal{R}$ has a host algebra, then $B(\mathcal{R}) = J(\mathcal{R})$, i.e. $B(\mathcal{R})$ is closed w.r.t. the norm $\|\cdot\|_*$.*

(2) *For each $\omega \in J(\mathcal{R})^*$ and $\pi \in \mathcal{R} \cup \{\pi_{\mathcal{R}}\}$, there is a unique operator $\pi(\omega) \in \mathcal{B}(\mathcal{H}_\pi)$ such that $\|\pi(\omega)\| \leq \|\omega\|$ and $\omega_x((\psi,\, \pi(x)\,\varphi)) = (\psi,\, \pi(\omega)\,\varphi)$ for all $\psi,\ \varphi \in \mathcal{H}_\pi$. Moreover $\pi(\omega) \in \pi(X)''$.*

(3) *The map* $\omega \to \pi_{\mathcal{R}}(\omega)$ *is a continuous linear bijection from* $J(\mathcal{R})^*$ *to the Von Neumann algebra* $\pi_{\mathcal{R}}(X)''$.

Proof: (1) Since $\mathcal{R}$ has a host algebra $\mathcal{L}$, then by Proposition 2.1 each $f \in B(\mathcal{R})$ has a unique strictly continuous extension $\widehat{f}$ from the *-algebra generated by $\varphi(\pi_u(X))$ to $M(\mathcal{L})$. Now

$$\|f\|_* = \left\|\widehat{f}\restriction\varphi(\mathcal{A}_d(X))\right\| = \|\widehat{f}\| = \|\widehat{f}\restriction\mathcal{L}\|$$

because $\widehat{f}$ is strictly continuous, both $\varphi(\mathcal{A}_d(X))$ and $\mathcal{L}$ are strictly dense in $M(\mathcal{L})$ and the unit ball of any strictly dense C*-algebra in $M(\mathcal{L})$ is strictly dense in the unit ball of $M(\mathcal{L})$ (the last fact is Exercise 2.N in [18]). But $\mathcal{L}$ is a host algebra for $\mathcal{R}$, hence

$$\left\{\widehat{f}\restriction\mathcal{L} \;\mid\; f \in B(\mathcal{R})\right\} = \mathcal{L}^*$$

and this is complete in norm. Thus $B(\mathcal{R})$ is complete in the $\|\cdot\|_*$–norm and hence $B(\mathcal{R}) = J(\mathcal{R})$.

(2) Let $\pi \in \mathcal{R} \cup \{\pi_{\mathcal{R}}\}$ and $\omega \in J(\mathcal{R})^*$, then the function $f : x \to (\psi,\, \pi(x)\,\varphi)$ is in $B(\mathcal{R})$, hence $\omega(f) = \omega_x((\psi,\, \pi(x)\,\varphi))$ is defined. Now the map $\psi \to \omega_x((\psi,\, \pi(x)\,\varphi))$ is conjugate linear, and bounded as

$$\begin{aligned} |\omega_x((\psi,\, \pi(x)\,\varphi))| &\leq \|\omega\| \cdot \|f\|_* \\ &= \|\omega\| \cdot \sup\{\,|(\psi,\, \pi(A)\,\varphi)| \;\mid\; A \in \mathcal{A}_d(X)\,,\; \|A\| \leq 1\,\} \\ &\leq \|\omega\| \cdot \|\psi\| \cdot \|\varphi\| \qquad\qquad (*) \end{aligned}$$

hence it is a conjugate linear functional on $\mathcal{H}_\pi$. Thus by the Riesz representation theorem, there is a vector $\varphi_\omega \in \mathcal{H}_\pi$ such that

$$\omega_x((\psi,\, \pi(x)\,\varphi)) = (\psi,\, \varphi_\omega) \quad \forall\, \psi \in \mathcal{H}_\pi \qquad\qquad (+)$$

Denote φ_ω by $\pi(\omega)\varphi$, then by $(*)$ we see $\|\pi(\omega)\varphi\| \leq \|\omega\| \cdot \|\varphi\|$, hence by linearity of $\varphi \to \pi(\omega)\varphi$ (clear from $(+)$), we have defined a bounded operator $\pi(\omega) : \mathcal{H}_\pi \to \mathcal{H}_\pi$. Uniqueness comes from the fact that $\pi(\omega)$ is fully determined by the coefficients $(\psi,\, \pi(\omega)\varphi)$ as ψ and φ ranges over $\mathcal{H}_\pi$. Next observe that if $B \in \mathcal{B}(\mathcal{H}_\pi)$ commutes with $\pi(X)$, then

$$\begin{aligned} (\psi,\, \pi(\omega)B\,\varphi) &= \omega_x((\psi,\, \pi(x)B\,\varphi)) = \omega_x((\psi,\, B\pi(x)\,\varphi)) \\ &= (B^*\psi,\, \pi(\omega)\,\varphi) = (\psi,\, B\pi(\omega)\,\varphi) \end{aligned}$$

for all $\psi,\, \varphi \in \mathcal{H}_\pi$, hence B commutes with $\pi(\omega)$. Thus $\pi(\omega) \in \pi(X)''$.

(3) From part (2) we get a continuous linear map $\omega \to \pi_{\mathcal{R}}(\omega)$ from $J(\mathcal{R})^*$ to the Von Neumann

algebra $\pi_{\mathcal{R}}(X)''$ (linearity is obvious from the defining relation). To see that it is a surjection, observe that for any $A \in \pi_{\mathcal{R}}(X)''$ we get a functional $\omega \in J(\mathcal{R})^*$ by $\omega(f) := (\psi,\, A\varphi)$ when $f(x) = (\psi,\, \pi_{\mathcal{R}}(x)\varphi)$. By the uniqueness in part (2) we get that $\pi(\omega) = A$. Injectivity of the map $\omega \to \pi_{\mathcal{R}}(\omega)$ follows from the fact that the vector functionals of the representation $\pi_{\mathcal{R}}$ comprises all of $B(\mathcal{R})$, hence a functional $\omega \in J(\mathcal{R})^*$ is uniquely specified by the set of values $\omega_x((\psi,\, \pi_{\mathcal{R}}(x)\,\varphi)) = (\psi,\, \pi_{\mathcal{R}}(\omega)\,\varphi)$. ■

By the linear bijection $\pi_{\mathcal{R}} : J(\mathcal{R})^* \to \pi_{\mathcal{R}}(X)''$, we now make $J(\mathcal{R})^*$ into a *-algebra by defining a product and involution by

$$\begin{aligned} (\omega * \beta)_x((\psi,\, \pi_{\mathcal{R}}(x)\,\varphi)) &:= (\psi,\, \pi_{\mathcal{R}}(\omega)\pi_{\mathcal{R}}(\beta)\,\varphi)\,, \\ (\omega^*)_x((\psi,\, \pi_{\mathcal{R}}(x)\,\varphi)) &:= (\psi,\, \pi_{\mathcal{R}}(\omega)^*\,\varphi) \end{aligned}$$

for all $\psi, \varphi \in \mathcal{H}_{\mathcal{R}}$. In the situation where $X = G$ is a group and $\operatorname{Rep} X$ is its unitary representation theory, these definitions just produce the usual convolution and involution of functionals, cf. [10].

5.2 Proposition *Let $\mathcal{R} \subset \operatorname{Rep} X$ be closed w.r.t. finite direct sums, then the norm $\|\cdot\|$ of the dual space $J(\mathcal{R})^*$ is a C*–norm w.r.t. the *-algebra structure, and hence we have the C*-algebra isomorphism $J(\mathcal{R})^* \cong \pi_{\mathcal{R}}(X)''$.*

Proof: For any $\omega \in J(\mathcal{R})^*$ we have

$$\begin{aligned} \|\omega\| &= \sup\{\,|\omega(f)| \;\mid\; f \in J(\mathcal{R}),\ \|f\|_* \leq 1\,\} \\ &= \sup\{\,|\omega(f)| \;\mid\; f \in B(\mathcal{R}),\ \|f\|_* \leq 1\,\} \qquad \text{as } B(\mathcal{R}) \text{ is dense in } J(\mathcal{R}) \\ &= \sup\{\,|(\psi,\, \pi(\omega)\xi)| \;\mid\; \pi \in \mathcal{R};\ \psi,\xi \in \mathcal{H}_\pi,\ \|\psi\| \leq 1 \geq \|\xi\|\,\} \\ &= \sup\{\,\|\pi(\omega)\| \;\mid\; \pi \in \mathcal{R}\,\}\,. \end{aligned}$$

Since the operator norms $\|\pi(\omega)\|$ are C*-norms, it follows that $\|\cdot\|$ on $J(\mathcal{R})^*$ is a C*-norm. ■

By definition, for any $\pi \in \mathcal{R}$, the map $\pi : J(\mathcal{R})^* \to \mathcal{B}(\mathcal{H}_\pi)$ obtained via Proposition 5.1 is a C*-representation.

Note that by the inclusion $\pi_{\mathcal{R}}(X) \subset \pi_{\mathcal{R}}(X)'' \cong J(\mathcal{R})^*$ we have a map $\delta : X \to J(\mathcal{R})^*$ by evaluation $\delta_x(f) := f(x)$, $f \in J(\mathcal{R})$, hence $(\delta_x)_y((\psi,\, \pi(y)\xi)) = (\psi,\, \pi(x)\xi)$ on the coefficient functions. If a host algebra $\mathcal{L} \subset \pi_{\mathcal{R}}(X)'' \cong J(\mathcal{R})^*$ exists, then it is stable under multiplication by δ_X.

5.3 Definition *A d–ideal* $\mathcal{A}$ *of* $J(\mathcal{R})^*$ *is a nonzero norm–closed *–subalgebra such that* $\delta_x * \mathcal{A} \subseteq \mathcal{A} \supseteq \mathcal{A} * \delta_x$ *for all* $x \in X$, *(i.e.* δ_X *is in the relative multiplier algebra of* $\mathcal{A}$.)

Obviously d-ideals are C*-algebras. From any nonzero $A \in J(\mathcal{R})^*$ we can generate a d-ideal by just taking the closed *-algebra generated by the set $\delta_X * A$. For any d-ideal we have the usual map $\theta : \mathrm{Rep}\,\mathcal{A} \to \mathrm{Rep}\,X$ by $\theta(\pi)(x) = \text{s–}\lim_{\alpha} \pi(\delta_x * E_\alpha)$ where $\{E_\alpha\} \subset \mathcal{A}$ is any approximate identity of $\mathcal{A}$. (Equivalently, $\theta(\pi)$ is uniquely determined by the equation $\theta(\pi)(x) \cdot \pi(A)\psi = \pi(\delta_x * A)\psi$ for all $A \in \mathcal{A}$, $\psi \in \mathcal{H}_\pi$.)

A d-ideal must satisfy $\theta(\mathrm{Rep}\,\mathcal{A}) \subseteq \mathcal{R}$ if it is to be a host algebra for $\mathcal{R}$. So we denote

$$\mathcal{I}(\mathcal{R}) := \{\, \mathcal{A} \subset J(\mathcal{R})^* \;\mid\; \mathcal{A} \;\text{ is a d-ideal and }\; \theta(\mathrm{Rep}\,\mathcal{A}) \subseteq \mathcal{R}\,\}\ .$$

The natural map which we will want to be inverse to θ, is the map $\pi \in \mathrm{Rep}\,X \to \pi_{\mathcal{A}} \in \mathrm{Rep}\,\mathcal{A}$ defined by

$$\omega_x((\psi,\,\pi(x)\varphi)) = (\psi,\,\pi_{\mathcal{A}}(\omega)\varphi) \qquad \forall\,\psi,\,\varphi \in \mathcal{H}_\pi,\ \omega \in \mathcal{A}\,,$$

via Proposition 5.1.

5.4 Theorem *If a d-ideal* $\mathcal{A} \in \mathcal{I}(\mathcal{R})$ *separates* $B(\mathcal{R})$, *then* $\theta : \mathrm{Rep}\,\mathcal{A} \to \mathcal{R}$ *is surjective.*

Proof: Let $\mathcal{A}$ separate $B(\mathcal{R})$. We first show that $\pi_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$ is nondegenerate for any $\pi \in \mathcal{R}$. If $\pi_{\mathcal{A}}$ were degenerate, there would be a nonzero $\varphi \in \mathcal{H}_\pi$ such that $\pi_{\mathcal{A}}(\mathcal{A})\varphi = 0$, i.e. $\omega_x((\psi,\,\pi(x)\varphi)) = 0$ for all $\psi \in \mathcal{H}_\pi$, $\omega \in \mathcal{A}$. Now $\pi \in \mathcal{R}$ is nondegenerate, hence there is a vector $\psi \in \mathcal{H}_\pi$ such that the function $x \to (\psi,\,\pi(x)\varphi)$ is nonzero, and by the previous sentence this is in $\mathrm{Ker}\,\omega$ for all $\omega \in \mathcal{A}$. This contradicts the hypothesis that $\mathcal{A}$ separates B_σ, and thus $\pi_{\mathcal{A}}$ is nondegenerate. We will now show that $\pi = \theta(\pi_{\mathcal{A}})$, which establishes surjectivity of θ. For all $\psi,\,\varphi \in \mathcal{H}_\pi$, $\omega \in \mathcal{A}$ we have:

$$\begin{aligned}(\varphi,\,\theta(\pi_{\mathcal{A}})(x)\,\pi_{\mathcal{A}}(\omega)\,\psi) &= (\varphi,\,\pi_{\mathcal{A}}(\delta_x * \omega)\psi)\\ = (\varphi,\,\pi_{\mathcal{A}}(\delta_x)\pi_{\mathcal{A}}(\omega)\psi) &= (\varphi,\,\pi(x)\,\pi_{\mathcal{A}}(\omega)\psi)\ ,\end{aligned}$$

i.e. $\theta(\pi_{\mathcal{A}})(x) \cdot \pi_{\mathcal{A}}(\omega)\psi = \pi(x) \cdot \pi_{\mathcal{A}}(\omega)\psi$ for all $\psi \in \mathcal{H}_\pi$, $\omega \in \mathcal{A}$. Since $\pi_{\mathcal{A}}$ is nondegenerate, $\pi_{\mathcal{A}}(\mathcal{A})\mathcal{H}_\pi$ is dense, hence $\theta(\pi_{\mathcal{A}})(x) = \pi(x)$ for all $x \in X$, which proves that θ is surjective. ∎

Recall that we have the canonical isometry $\iota : J(\mathcal{R}) \to J(\mathcal{R})^{**}$ by $\iota(f)(\omega) := \omega(f)$ for $\omega \in J(\mathcal{R})^*$, $f \in J(\mathcal{R})$, and that $J(\mathcal{R})$ is reflexive if $\iota(J(\mathcal{R})) = J(\mathcal{R})^{**}$. If $\mathcal{A} \subset J(\mathcal{R})^*$ is a d-ideal, we denote the restriction of ι by $j : J(\mathcal{R}) \to \mathcal{A}^*$ where $j(f)(\beta) := \beta(f)$, $\beta \in \mathcal{A}$, $f \in J(\mathcal{R})$. Note that j is injective if $\mathcal{A}$ separates $J(\mathcal{R})$. Now even if $J(\mathcal{R})$ is not reflexive, there may still be d-ideals $\mathcal{A}$ such that $j(J(\mathcal{R})) = \mathcal{A}^*$, and we need these because:

5.5 Theorem *For a d-ideal* $\mathcal{A} \in \mathcal{I}(\mathcal{R})$, *the map* $\theta : \mathrm{Rep}\,\mathcal{A} \to \mathcal{R}$ *is injective with inverse map* $\pi \in \mathcal{R} \to \pi_{\mathcal{A}} \in \mathrm{Rep}\,\mathcal{A}$ *iff* $j(J(\mathcal{R})) = \mathcal{A}^*$. *In this case,* $\mathcal{A}$ *is a host algebra for* $\theta(\mathrm{Rep}\,\mathcal{A})$.

Proof: We need to prove that $\theta(\pi)_{\mathcal{A}}(\omega) = \pi(\omega)$ for all $\pi \in \mathrm{Rep}\,\mathcal{A}$, $\omega \in \mathcal{A}$ iff $j(J(\mathcal{R})) = \mathcal{A}^*$. Assume that $\theta(\pi)_{\mathcal{A}} = \pi$. Let $f(x) := (\varphi,\, \theta(\pi)(x)\,\psi)$, then

$$j(f)(\omega) = \omega(f) = (\varphi,\, \theta(\pi)_{\mathcal{A}}(\omega)\psi) = (\varphi, \pi(\omega)\psi)$$

for all $\varphi,\, \psi \in \mathcal{H}_\pi$, $\pi \in \mathrm{Rep}\,\mathcal{A}$, $\omega \in \mathcal{A}$. By varying the rhs over $\pi \in \mathrm{Rep}\,\mathcal{A}$, $\varphi = \psi \in \mathcal{H}_\pi$, we obtain all states of $\mathcal{A}$, and since these span $\mathcal{A}^*$ and j is linear, it means any functional of $\mathcal{A}$ can be expressed as an element of $j(J(\mathcal{R}))$, i.e. $j(J(\mathcal{R})) = \mathcal{A}^*$.
Conversely, let $j(J(\mathcal{R})) = \mathcal{A}^*$. For a $f \in B(\mathcal{R})$ say $f(x) = (\varphi,\, \pi(x)\,\psi)$, we have:

$$\begin{aligned} j(f)(\omega * \beta) &= (\omega * \beta)(f) = (\varphi,\, \pi(\omega)\pi(\beta)\psi) \\ &= \omega_x((\varphi,\, \pi(x)\pi(\beta)\psi)) = \omega_x((\varphi,\, \pi(\delta_x * \beta)\psi)) \\ &= \omega_x((\delta_x * \beta)_y(\varphi,\, \pi(y)\,\psi)) \\ &= \omega_x(j(f)(\delta_x * \beta)) \qquad \forall\, \omega,\, \beta \in \mathcal{A},\ f \in B(\mathcal{R})\,. \end{aligned}$$

But j is an isometry and $B(\mathcal{R})$ is dense in $J(\mathcal{R})$, hence

$$j(f)(\omega * \beta) = \omega_x(j(f)(\delta_x * \beta)) \qquad \forall\, \omega,\, \beta \in \mathcal{A},\ f \in J(\mathcal{R})\,.$$

Thus, since $j(J(\mathcal{R})) = \mathcal{A}^*$, we have:

$$\xi(\omega * \beta) = \omega_x(\xi(\delta_x * \beta)) \qquad \forall\, \xi \in \mathcal{A}^*,\ \omega,\, \beta \in \mathcal{A}\,.$$

In particular, choose $\xi(\omega) = (\varphi,\, \pi(\omega)\psi)$, $\pi \in \mathrm{Rep}\,\mathcal{A}$, $\varphi,\, \psi \in \mathcal{H}_\pi$, then

$$\begin{aligned} (\varphi,\, \pi(\omega * \beta)\psi) &= \omega_x((\varphi,\, \pi(\delta_x * \beta)\psi)) \\ &= \omega_x((\varphi,\, \theta(\pi)(x)\,\pi(\beta)\psi) \\ &= (\varphi,\, \theta(\pi)_{\mathcal{A}}(\omega)\,\pi(\beta)\psi) \end{aligned}$$

for all $\pi \in \mathrm{Rep}\,\mathcal{A}$, $\varphi,\, \psi \in \mathcal{H}_\pi$, $\omega,\, \beta \in \mathcal{A}$. Thus

$$\pi(\omega) \cdot \pi(\beta)\psi = \theta(\pi)_{\mathcal{A}}(\omega) \cdot \pi(\beta)\psi\,.$$

By nondegeneracy of $\pi \in \mathrm{Rep}\,\mathcal{A}$ we get $\pi(\omega) = \theta(\pi)_{\mathcal{A}}(\omega)$ for all $\omega \in \mathcal{A}$. ■

The condition $j(J(\mathcal{R})) = \mathcal{A}^*$ is quite natural, if we keep in mind that if $\mathcal{A}$ is a host algebra, then its dual is the coefficient space of its representation space $\mathcal{R}$, and the latter is $B(\mathcal{R})\,(= J(\mathcal{R})$ in this case by Proposition 5.1 (1).)

5.6 Corollary *(i) Any d-ideal* $\mathcal{A} \in \mathcal{I}(\mathcal{R})$ *which separates* $B(\mathcal{R})$ *and satisfies* $j(J(\mathcal{R})) = \mathcal{A}^*$ *is a host algebra for* $\mathcal{R}$*.*
(ii) Conversely let $\mathcal{A} \subset J(\mathcal{R})^*$ *be a d-ideal which is a host algebra for* $\mathcal{R}$ *where the map* $\varphi : X \to M(\mathcal{A})$ *is obtained from the embedding of* δ_X *in the relative multiplier algebra of* $\mathcal{A}$*. Then* $\mathcal{A} \in \mathcal{I}(\mathcal{R})$*,* $\mathcal{A}$ *separates* $B(\mathcal{R})$ *and satisfies* $j(J(\mathcal{R})) = \mathcal{A}^*$*.*

Proof: (i) By Theorems 5.4 and 5.5, $\theta : \mathrm{Rep}\,\mathcal{A} \to \mathcal{R}$ is bijective.
(ii) If $\mathcal{A}$ is a host algebra as stated above, then by definition $\theta : \mathrm{Rep}\,\mathcal{A} \to \mathcal{R}$ so $\mathcal{A} \in \mathcal{I}(\mathcal{R})$. Moreover, by Proposition 2.1 each $f \in B(\mathcal{R})$ is strictly continuous, extends uniquely by strict continuity to $M(\mathcal{A})$ and is uniquely determined by its values on $\mathcal{A}$ (which is strictly dense in $M(\mathcal{A})$). Thus $\mathcal{A}$ separates $B(\mathcal{R})$. Finally, since θ is bijective it has inverse map $\pi \in \mathcal{R} \to \pi_{\mathcal{A}} \in \mathrm{Rep}\,\mathcal{A}$ by:

$$\begin{aligned}
(\phi,\, \theta(\pi_{\mathcal{A}})(x)\, \pi_{\mathcal{A}}(\omega)\psi) &= (\phi,\, \pi_{\mathcal{A}}(\delta_x * \omega)\psi) = (\delta_x * \omega)_y((\phi,\, \pi(y)\psi)) \\
&= (\delta_x)_y(\omega_z((\phi,\, \pi(y)\pi(z)\psi))) = \omega_z((\phi,\, \pi(x)\pi(z)\psi)) \\
&= ((\phi,\, \pi(x)\pi_{\mathcal{A}}(\omega)\psi)) \qquad \text{for all } \phi,\, \psi \in \mathcal{H}_\pi \text{ and } \omega \in \mathcal{A}.
\end{aligned}$$

Thus by nondegeneracy of $\pi_{\mathcal{A}}$ it follows that $\theta(\pi_{\mathcal{A}})(x) = \pi(x)$. Now it follows from Theorem 5.5, by the injectivity of θ that $j(J_\sigma) = \mathcal{A}^*$. ∎

6 Measure Algebras.

A natural class of functionals in $J(\mathcal{R})^*$ to consider, are those associated with finite (σ-additive, complex valued) measures on X according to $\omega_\mu(f) = \int_X f(x)\, d\mu(x)$, with f any bounded measurable function. Historically, these (with additional regularity properties) were the building blocks for group algebras and their generalisations.

To provide an adequate setting for the analysis, assume we have a measure space $(X,\, \mathcal{S},\, \nu)$ where $\mathcal{S}$ is a σ–algebra and ν is a positive (σ–additive) measure. Assume that there is a $c > 0$ such that all representations $\pi \in \mathrm{Rep}\,X$ satisfy $\|\pi(x)\| \leq c$ for all $x \in X$ (e.g. for unitary group representations we have $c = 1$). Let $\mathrm{Rep}_{\mathcal{S}}X$ denote those representations for which all their coefficient functions are $\mathcal{S}$–measurable. In this section we will only consider subsets $\mathcal{R} \subset \mathrm{Rep}_{\mathcal{S}}X$ closed w.r.t. direct sums, and measures defined w.r.t. the σ–algebra $\mathcal{S}$. We still assume that each vector functional $f(A) = (\psi,\, \pi(A)\varphi)$, $A \in \mathcal{A}_d(X)$, $\pi \in \mathcal{R}$ is uniquely determined by its restriction to $\pi_u(X) \subset \mathcal{A}_d(X)$.

For such measures μ which are finite, note that the functionals ω_μ are continous w.r.t. the supremum norm, i.e. $|\omega_\mu(f)| \leq \|\omega_\mu\| \cdot \|f\|_\infty$ for f bounded and measurable. It is obvious that for $f \in B(\mathcal{R})$ we have $\|f\|_\infty \leq c \cdot \|f\|_* = c \cdot \sup\{\, |f(A)| \;\mid\; A \in \mathcal{A}_d(X),\ \|A\| \leq 1\,\}$ because all $\pi_u(x)/c$ are in the unit ball of $\mathcal{A}_d(X)$. Since for any $\mathcal{R} \subset \mathrm{Rep}_{\mathcal{S}} X$ as above, the coefficient functions are bounded and measurable, we can restrict the functionals ω_μ to $B(\mathcal{R})$ and find $\omega_\mu \restriction B(\mathcal{R}) \in J(\mathcal{R})^*$. Denote the set of these functionals by $\mathcal{M}(X) \subset J(\mathcal{R})^*$. Then Proposition 5.1(2) has a well-known extension: given ω_μ as above, and a $\pi \in \mathrm{Rep}_{\mathcal{S}} X$, then there is a unique operator $\pi(\omega_\mu) \in \mathcal{B}(\mathcal{H}_\pi)$ such that $\|\pi(\omega_\mu)\| \leq \|\omega_\mu\|$ and

$$\int (\psi,\, \pi(x)\,\varphi) d\mu(x) = (\psi,\, \pi(\omega_\mu)\,\varphi)$$

for all $\psi,\, \varphi \in \mathcal{H}_\pi$. We will denote by $\mathcal{M}_\nu(X)$ the space of those functionals $\omega_\mu \in \mathcal{M}(X)$ for which μ is absolutely continuous w.r.t. ν.

We will also need to integrate the map $x \to \delta_x * \beta =: h_\beta(x) \in J(\mathcal{R})^*$, so recall the two conditions of measurability for a Banach space–valued function w.r.t. a measure μ, cf. Lemma 9, Sect III.6.7 of Dunford and Schwartz [8]: (i) inverse images of Borel sets are measurable, (ii) on the complement of a μ–null set, the range of the function must be separable. This last concept is called μ–essentially separably valued. So define for a given $\mathcal{R} \subset \mathrm{Rep}_{\mathcal{S}} X$:

$$\begin{aligned} D_\nu(\mathcal{R}) \;\; := \;\; \Big\{ \beta \in J(\mathcal{R})^* \;\Big|\; & h_\beta^{-1}(S) \in \mathcal{S} \text{ when } S \subset J(\mathcal{R})^* \text{ is Borel,} \\ & h_\beta \text{ is } \nu\text{–essentially separably valued} \Big\}\,. \end{aligned}$$

6.1 Theorem *Let* $\mathcal{R} \subset \mathrm{Rep}_{\mathcal{S}} X$ *as above, then*
(i) $D_\nu(\mathcal{R})$ *is a closed right ideal of* $J(\mathcal{R})^*$*,*
(ii) let $\mathcal{A} \subseteq D_\nu(\mathcal{R})$ *be a d-ideal, and let* $\omega \in \mathcal{M}_\nu(X) \cap \mathcal{A}$*. Then* $\pi(\omega) = \theta(\pi)_{\mathcal{A}}(\omega)$ *for all* $\pi \in \mathrm{Rep}\,\mathcal{A}$*, and hence* θ *is injective on* $(\mathrm{Rep}\,\mathcal{A})\restriction(\mathcal{M}_\nu(X) \cap \mathcal{A})$*. In particular, if* $\mathcal{A} \subset \overline{\mathcal{M}_\nu(X)} \cap D_\nu(\mathcal{R})$*, then* $\theta : \mathrm{Rep}\,\mathcal{A} \to \mathrm{Rep}\,X$ *is injective.*

Proof: (i) The two conditions in the definition of $D_\nu(\mathcal{R})$ are exactly the conditions which characterize a ν–measurable function. By Theorem 11, Sect III.6 of Dunford and Schwartz [8] such functions form a linear space and hence if $k := h_\beta + h_\alpha = h_{\beta+\alpha}$ with $\alpha,\, \beta \in D_\nu(\mathcal{R})$, then k is measurable, hence $\beta + \alpha \in D_\nu(\mathcal{R})$, so $D_\nu(\mathcal{R})$ is a linear space. The map $x \to h_\beta(x) * \alpha = h_{\beta*\alpha}(x)$ is the composition of the measurable map h_β with convolution by α. Since convolution by α is continuous, it is a Borel map on $J(\mathcal{R})^*$, and takes separable sets to separable sets. Thus the map $x \to h_\beta(x) * \alpha = h_{\beta*\alpha}(x)$ is measurable for all $\beta \in D_\nu(\mathcal{R})$ and $\alpha \in J(\mathcal{R})^*$, so $\beta * \alpha \in D_\nu(\mathcal{R})$, i.e. $D_\nu(\mathcal{R})$ is a right ideal in $J(\mathcal{R})^*$, hence an algebra.

We check norm closure. Let $\{\beta_n\} \subset D_\nu(\mathcal{R})$ be a sequence converging to $\beta \in J(\mathcal{R})^*$. Then $\|h_{\beta_n}(x) - h_\beta(x)\| \to 0$, so we obtain pointwise convergence. For pointwise limits we still have the measurability property that $h_\beta^{-1}(S) \in \mathcal{S}$ when $S \subset J(\mathcal{R})^*$ is Borel, so we only need to check that h_β is ν–essentially separably valued. For each h_{β_n} let $N_n \subset X$ be a ν–null set such that $h_{\beta_n}(N_n^c)$ is separable. Then $K := \bigcup_{n=1}^\infty N_n$ is a ν–null set such that

$$h_\beta(K^c) \subseteq \overline{\bigcup_{n=1}^\infty h_{\beta_n}(N_n^c)}$$

is separable. So h_β is ν–essentially separably valued, hence ν–measurable, i.e. $\beta \in D_\nu(\mathcal{R})$ and hence $D_\nu(\mathcal{R})$ is a Banach algebra and a right ideal of $J(\mathcal{R})^*$.

(ii) Let $\omega \in \mathcal{M}_\nu(X) \cap \mathcal{A}$ with associated Borel measure μ. Now for any $\beta \in \mathcal{A}$, the function $x \to \delta_x * \beta \in \mathcal{A}$ is μ–measurable by definition of $D_\nu(\mathcal{R})$ (using $\mu \ll \nu$), and bounded by $c \cdot \|\beta\|$. Thus, the Bochner integral $B := \int_X \delta_x * \beta \, d\mu(x)$ is well–defined (cf. Chapter III [8]), and $B \in \mathcal{A}$. Then

$$\xi(B) = \int_X \xi(\delta_x * \beta) \, d\mu(x) \qquad \forall\, \xi \in \mathcal{A}^* \tag{1}$$

and in particular for $\xi = j(f)$, $f \in J(\mathcal{R})$, we have

$$\begin{aligned} j(f)(B) = B(f) &= \int_X (\delta_x * \beta)(f) \, d\mu(x) = \omega_x((\delta_x * \beta)(f)) \\ &= (\omega * \beta)(f) \qquad \forall\, f \in J(\mathcal{R}) \,. \end{aligned}$$

Thus $B = \omega * \beta = \int_X \delta_x * \beta \, d\mu(x)$, and so, using Eq.(1) again:

$$\xi(\omega * \beta) = \int_X \xi(\delta_x * \beta) \, d\mu(x) = \omega_x(\xi(\delta_x * \beta)) \tag{2}$$

for all $\xi \in \mathcal{A}^*$, $\beta \in \mathcal{A}$, $\omega \in \mathcal{M}_\nu(X) \cap \mathcal{A}$. Now choose $\xi(\omega) = (\varphi,\, \pi(\omega)\psi)$, $\pi \in \operatorname{Rep}\mathcal{A}$, $\varphi,\ \psi \in \mathcal{H}_\pi$, then by Eq.(2) we find:

$$\begin{aligned} \xi(\omega * \beta) = (\varphi,\, \pi(\omega * \beta)\psi) &= \omega_x((\varphi,\, \pi(\delta_x * \beta)\psi)) \\ &= \omega_x((\varphi,\, \theta(\pi)(x)\, \pi(\beta)\psi) \\ &= (\varphi,\, \theta(\pi)_{\mathcal{A}}(\omega)\, \pi(\beta)\psi) \end{aligned}$$

for all $\pi \in \operatorname{Rep}\mathcal{A}$, $\varphi,\ \psi \in \mathcal{H}_\pi$, $\beta \in \mathcal{A}$, $\omega \in \mathcal{M}_\nu(X) \cap \mathcal{A}$. By nondegeneracy of $\pi \in \operatorname{Rep}\mathcal{A}$ we get $\pi(\omega) = \theta(\pi)_{\mathcal{A}}(\omega)$ for all $\omega \in \mathcal{M}_\nu(X) \cap \mathcal{A}$. Hence if $\mathcal{A} \subseteq \mathcal{M}_\nu(X)$, then $\pi = \theta(\pi)_{\mathcal{A}}$ for all $\pi \in \operatorname{Rep}\mathcal{A}$. ∎

In the case that X is a Borel semigroup and $\operatorname{Rep} X$ homomorphisms from X to $\mathcal{B}(\mathcal{H})$, we can show that $D_\nu(\mathcal{R}) \cap D_\nu(\mathcal{R})^*$ is a d-ideal (cf. [10]). Note that we did not require that $\mathcal{A} \in \mathcal{I}(\mathcal{R})$, and so $\theta(\operatorname{Rep}\mathcal{A})$ need not have anything to do with $\mathcal{R}$. In fact, it seems the proper context for studying convolution algebras of measures is where $\mathcal{R} = \operatorname{Rep}_{\mathcal{S}} X$.

7 Representations of X which are strong operator continuous

As the examples 1.5 indicate, the most common context for this analysis is where X has a topology, and the set of representations $\mathcal{R}$ for which one seeks a host algebra is

$$\mathcal{R}_c := \{\, \pi \in \operatorname{Rep} X \ \mid\ x \to \pi(x) \in \mathcal{B}(\mathcal{H}_\pi) \quad \text{is strong operator continuous} \,\}\ .$$

Below we study this situation, and maintain the notation and assumption of a topology on X for the rest of this section. We still assume that $\|\pi(x)\| \leq c$ for all $x \in X$, $\pi \in \operatorname{Rep} X$, hence that $\|\delta_x\| \leq c$ for all x, and that vector functionals are determined by their restrictions to $\pi_{\mathcal{R}}(X)$. Observe that $\mathcal{R}_c$ is closed w.r.t. finite direct sums. Our first task is to characterize $\mathcal{I}(\mathcal{R}_c)$ more explicitly. For each positive functional $\xi \in (J(\mathcal{R}_c)^*)^*_+$ define a seminorm $|\cdot|_\xi$ on $J(\mathcal{R}_c)^*$ by $|A|_\xi := [\xi(A^*A)]^{1/2}$. If $\xi(A) = (\psi,\, \pi(A)\psi)$ for $\pi \in \operatorname{Rep} J(\mathcal{R}_c)^*$, $\psi \in \mathcal{H}_\pi$, then $|A|_\xi = \|\pi(A)\psi\|$. By the Cauchy-Schwartz inequality we have $|A|_\xi \leq \|\xi\| \cdot \|A\|$ for all A. Define

$$\begin{aligned}
\mathcal{Q}_0(X) &:= \{A \in J(\mathcal{R}_c)^* \mid |(\delta_x - \delta_a) * D * A|_\xi \to 0 \quad \text{as} \quad x \to a \quad \forall\, a \in X, \\
&\qquad\qquad D \in \mathcal{A}_d(X),\ \xi \in (J(\mathcal{R}_c)^*)^*_+ \,\} \\
\mathcal{Q}(X) &:= \mathcal{Q}_0(X) \cap \mathcal{Q}_0(X)^* \quad \text{(adjoint is meant here, not dual)} \\
\mathcal{L}_0(X) &:= \{A \in J(\mathcal{R}_c)^* \mid \|(\delta_x - \delta_a) * D * A\| \to 0 \quad \text{as} \quad x \to a \ \ \forall\, a \in X, \\
&\qquad\qquad D \in \mathcal{A}_d(X)\} \\
\mathcal{L}(X) &:= \mathcal{L}_0(X) \cap \mathcal{L}_0(X)^* \quad \text{(adjoint here, not dual)}
\end{aligned}$$

Clearly if X has the discrete topology, then $\mathcal{Q}_0(X) = J(\mathcal{R}_c)^*$ so henceforth we assume that X is nondiscrete. Note that $\mathcal{Q}_0(X) \supseteq \mathcal{L}_0(X)$ and that we always have pointwise continuity $\lim\limits_{x\to a}(A^* * (\delta_x - \delta_a)^*(\delta_x - \delta_a) * A)(f) = 0$ for all $A \in J(\mathcal{R}_c)^*$, $f \in J(\mathcal{R}_c)$ since for $f(x) = (\psi,\, \pi(x)\varphi)$, $\pi \in \mathcal{R}_c$ we have $(A^* * (\delta_x - \delta_a)^*(\delta_x - \delta_a) * A)(f) = ((\pi(x) - \pi(a))\pi(A)\psi,\, (\pi(x) - \pi(a))\pi(A)\varphi)$ which goes to zero as $x \to a$ by the strong operator continuity of π. Thus we can only have $\mathcal{Q}_0(X) \neq J(\mathcal{R}_c)^*$ when $\iota(J(\mathcal{R}_c)) \neq J(\mathcal{R}_c)^{**}$, i.e. if $J(\mathcal{R}_c)$ is not reflexive. We will mainly be concerned with $\mathcal{Q}(X)$, but $\mathcal{L}(X)$ is more natural for measure algebras.

7.1 Theorem *(i) The spaces $\mathcal{Q}_0(X)$ and $\mathcal{L}_0(X)$ are norm-closed right ideals in $J(\mathcal{R}_c)^*$, hence Banach algebras. Thus $\mathcal{Q}(X)$ and $\mathcal{L}(X)$ are C^*-algebras.*

(ii) If $\mathcal{A}_d(X)$ has a state ω for which $x \to \omega(\delta_x)$ is discontinuous, then $\mathbb{1} \notin \mathcal{Q}_0(X) \supseteq \mathcal{L}_0(X)$ and hence $\mathcal{Q}_0(X) \neq J(\mathcal{R}_c)^*$.

(iii) Both $\mathcal{Q}(X)$ and $\mathcal{L}(X)$ are d-ideals, i.e. δ_X is in their relative multiplier algebras.

Proof: (i) We first prove norm closure. Consider a sequence $\{A_n\} \subset \mathcal{Q}_0(X)$ which converges in norm to $A \in J(\mathcal{R}_c)^*$. Then for all $\xi \in (J(\mathcal{R}_c)^*)^*_+$, $D \in \mathcal{A}_d(X)$ we have:

$$\begin{aligned} |(\delta_x - \delta_a) * D * A|_\xi &\leq |\delta_x * D * (A - A_n)|_\xi + |(\delta_x - \delta_a) * D * A_n|_\xi \\ &\qquad + |\delta_a * D * (A_n - A)|_\xi \\ &\leq \|\xi\| \cdot \|\delta_x * D * (A - A_n)\| + \|\xi\| \cdot \|\delta_a * D * (A_n - A)\| \\ &\qquad + |(\delta_x - \delta_a) * D * A_n|_\xi \\ &\leq 2c\|\xi\| \cdot \|D\| \cdot \|A - A_n\| + |(\delta_x - \delta_a) * D * A_n|_\xi \\ &\xrightarrow[a]{x} 2c\|\xi\| \cdot \|D\| \cdot \|A - A_n\| \xrightarrow[\infty]{n} 0 \end{aligned}$$

and thus $A \in \mathcal{Q}_0(X)$ i.e. $\mathcal{Q}_0(X)$ is norm closed. A similar calculation establishes that $\mathcal{L}_0(X)$ is also norm closed.
Next we show that $\mathcal{Q}_0(X)$ is a right ideal. Let $A \in \mathcal{Q}_0(X)$ and $B \in J(\mathcal{R}_c)^*$, then for all $\xi \in (J(\mathcal{R}_c)^*)^*_+$ we have $|(\delta_x - \delta_a) * D * (A * B)|_\xi = |(\delta_x - \delta_a) * D * A|_{\xi_B}$ where $\xi_B(A) := \xi(B^*AB)$. Obviously $\xi_B \in (J(\mathcal{R}_c)^*)^*_+$ hence by $A \in \mathcal{Q}_0(X)$ we get that $|(\delta_x - \delta_a) * D * A|_{\xi_B} \to 0$ as $x \to a$ for all $D \in \mathcal{A}_d(X)$, and hence $A * B \in \mathcal{Q}_0(X)$. Thus $\mathcal{Q}_0(X)$ is a closed right ideal in $J(\mathcal{R}_c)^*$. Next let $A \in \mathcal{L}_0(X)$ and $B \in J(\mathcal{R}_c)^*$, then

$$\|(\delta_x - \delta_a) * D * (A * B)\| \leq \|(\delta_x - \delta_a) * D * A\| \cdot \|B\| \xrightarrow[a]{x} 0$$

for all $D \in \mathcal{A}_d(X)$. Thus $A * B \in \mathcal{L}_0(X)$. To show that $\mathcal{L}(X)$ is a C*-subalgebra of $J(\mathcal{R}_c)^*$, note that we already have norm–closure, and that it is closed under involution, so it only remains to check that it is an algebra. Let $A,\ B \in \mathcal{L}(X)$, hence $A,\ A^* \in \mathcal{L}_0(X) \ni B,\ B^*$. Since $\mathcal{L}_0(X)$ is a right ideal, it contains $A * B$, as well as $B^* * A^* = (A * B)^*$. Thus $A * B \in \mathcal{L}(X)$. By a similar argument we find that $\mathcal{Q}(X)$ is a C*-algebra.
(ii) If $\mathbb{1} \in \mathcal{Q}_0(X)$ then by definition $|(\delta_x - \delta_a) * D|_\xi \to 0$ as as $y \to a$ for all $D \in \mathcal{A}_d(X)$ and $\xi \in (J(\mathcal{R}_c)^*)^*_+$. In particular, let $D = \mathbb{1}$ then by the Cauchy-Schwartz inequality $|\xi(\delta_y - \delta_a)| \leq |(\delta_x - \delta_a)|_\xi \to 0$ as $y \to a$ hence $\xi(\delta_y - \delta_a) \to 0$ as $y \to a$ for all $\xi \in (J(\mathcal{R}_c)^*)^*_+$. However, since $\mathcal{A}_d(X)$ is in $J(\mathcal{R}_c)^*$, by the Hahn-Banach theorem the restriction of $J(\mathcal{R}_c)^{**}$ to $\mathcal{A}_d(X)$ is exactly the dual of $\mathcal{A}_d(X)$, and by assumption this contains a state ω for which $\omega(\delta_y - \delta_a) \not\to 0$ as $y \to a$. Thus $\mathbb{1} \notin \mathcal{Q}_0(X)$.
(iii) By (i) we already know that $\mathcal{L}_0(X) * \delta_x \subseteq \mathcal{L}_0(X)$. Let $A \in \mathcal{L}_0(X)$, $z \in X$, then

$$\|(\delta_x - \delta_a) * D * (\delta_z * A)\| = \|(\delta_x - \delta_a) * (D * \delta_z) * A\| \xrightarrow[a]{x} 0$$

for all $D \in \mathcal{A}_d(X)$ because $D * \delta_z \in \mathcal{A}_d(X)$. Thus $\delta_z * A \in \mathcal{L}_0(X)$, i.e. $\delta_x * \mathcal{L}_0(X) \subseteq \mathcal{L}_0(X)$ for all $x \in X$. Now let $A \in \mathcal{L}(X) \subset \mathcal{L}_0(X)$, hence $\delta_x * A \in \mathcal{L}_0(X)$, and also $(\delta_x * A)^* = A^* * \delta_x^* \in \mathcal{L}_0(X)$ because $A^* \in \mathcal{L}_0(X)$ and this is a right ideal. Thus $\delta_x * A \in \mathcal{L}(X)$, and likewise $A * \delta_x \in \mathcal{L}(X)$, hence $\delta_x * \mathcal{L}(X) \subseteq \mathcal{L}(X) \supseteq \mathcal{L}(X) * \delta_x$. By replacing the norms $\|\cdot\|$ in the equation above by $|\cdot|_\xi$ we can transcribe this argument to prove also that $\mathcal{Q}(X)$ is a d-ideal. ■

Since $\mathcal{Q}(X)$ and $\mathcal{L}(X)$ are d-ideals, they contain the d-ideals generated by their subsets, and the reason why we are interested in them is due to the next theorem:

7.2 Theorem *Let X be nondiscrete and $\|\delta_x\| \leq c$ for all $x \in X$. Then for a d-ideal $\mathcal{A}$ we have that $\mathcal{A} \in \mathcal{I}(\mathcal{R}_c)$ iff $\mathcal{A} \subseteq \mathcal{Q}(X)$.*

Proof: Let $\mathcal{A} \in \mathcal{I}(\mathcal{R}_c)$ i.e. $\theta(\pi) \in \mathcal{R}_c$. Now any positive functional $\xi \in (J(\mathcal{R}_c)^*)^*_+$ is of the form $\xi(B) := (\psi, \pi(B)\psi)$, for $\pi \in \operatorname{Rep} J(\mathcal{R}_c)^*$, $\psi \in \mathcal{H}_\pi$ so for such a ξ we have for all $A \in \mathcal{A}$:

$$\begin{aligned} |(\delta_x - \delta_a) * \delta_y * A|_\xi &= \|\pi((\delta_x - \delta_a) * \delta_y * A)\psi\| \\ &= \|[\theta(\widetilde{\pi})(x) - \theta(\widetilde{\pi})(a)]\pi(\delta_y * A)\psi\| \\ &\xrightarrow[a]{x} 0 \end{aligned}$$

where $\widetilde{\pi}$ is the restriction of π to $\mathcal{A}$ on its essential subspace, where the latter is the closure of $\pi(\mathcal{A})\mathcal{H}_\pi \ni \pi(\delta_y * A)\psi$. In the last step we used $\mathcal{A} \in \mathcal{I}(\mathcal{R}_c)$. Thus $|(\delta_x - \delta_a) * \delta_y * A|_\xi \xrightarrow[a]{x} 0$ for all $y \in X$ and $\xi \in (J(\mathcal{R}_c)^*)^*_+$ i.e. $\mathcal{A} \subseteq \mathcal{Q}(X)$.
Conversely, let $\mathcal{A} \subseteq \mathcal{Q}(G)$ and recall from the Hahn-Banach theorem that the dual of $\mathcal{A}$ consists of the restriction of $J(\mathcal{R}_c)^{**}$ to $\mathcal{A}$. Thus $|(\delta_x - \delta_a) * A|_\xi \to 0$ as $x \to a$ for all $A \in \mathcal{A}$ and $\xi \in \mathcal{A}^*_+$. By choosing coefficient functions $\xi(A) = (\psi, \pi(A)\psi)$, for $\pi \in \operatorname{Rep}\mathcal{A}$ we find as above that

$$\|[\theta(\pi)(x) - \theta(\pi)(a)]\pi(A)\psi\| \xrightarrow[a]{x} 0$$

for all ψ and so $\theta(\pi)(x)$ is strong operator continuous. Hence $\mathcal{A} \in \mathcal{I}(\mathcal{R}_c)$. ■

Thus, if $\mathcal{R} \subseteq \mathcal{R}_c$ and $\mathcal{R}$ has a host algebra $\mathcal{L}$, then $\Psi(\mathcal{L}) \subseteq \mathcal{Q}(X)$ where Ψ is the isomorphism of Proposition 5.2.

Now in the light of Theorem 7.2, a d-ideal $\mathcal{A} \subset \mathcal{Q}(X)$ will be an adequate host algebra for $\mathcal{R}_c$ if we can show that θ is bijective. We can now use Corollary 5.6 and Theorem 6.1 to find sharp conditions for this. It is immediate that:

7.3 Corollary *Let* X *be nondiscrete and* $\|\delta_x\| \leq c$ *for all* $x \in X$. *Then* $\mathcal{A}$ *is a host algebra for* $\mathcal{R}_c$ *with embedding map* $\varphi : X \to M(\mathcal{A})$ *given by* $\varphi(x) = \delta_x \in J(\mathcal{R}_c)^*$ *iff* $\mathcal{A} \subset \mathcal{Q}(X)$, $\mathcal{A}$ *separates* $B(\mathcal{R}_c)$ *and satisfies* $j(J(\mathcal{R}_c)) = \mathcal{A}^*$.

8 Conclusions.

We have introduced a very general framework to study generalisations of group algebras, and through our existence theorem solved the question of when a set of representations is isomorphic to the representation theory of a C*-algebra. Since there are many arenas of mathematics as well as physics where one studies a particular subset of representations which has some desired property, the current framework has a wide field of potential application. To be useful however, one would need to find more concrete versions of the conditions in Theorem 3.4 in each such a setting. In particular an important future direction would be to find a more useful version of these conditions in the case of $\mathcal{R}_c$, i.e. strong operator continuous representations. This would then have immediate application to the construction of group algebras for groups which are not locally compact.

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