Hypergeometric functions and parabolic partial differential equations*

Stéphane Malek
Abteilung Angewandte Analysis
Universität Ulm
89069 Ulm, Germany
malek@mathematik.uni-ulm.de
Fax +731 5023619

Abstract

We investigate the summability of formal power series solutions of parabolic linear partial differential equations with exponential coefficients.

Key words: Partial differential equations, power series solutions, summability. 2000 MSC: 35C10, 35C20.

1 Introduction

We consider the following parabolic partial differential equation

$$(P_m) \qquad (\partial_z + c_1 - 1)^{m_1} \cdots (\partial_z + c_{k_0} - 1)^{m_{k_0}} \partial_t^m y(t,z) = e^z (\partial_z + b_1) \cdots (\partial_z + b_k) y(t,z),$$

where $k_0, k, m, m_1, \ldots, m_{k_0} \geq 0$ are non negative integers and $c_1, \ldots, c_{k_0}$, $b_1, \ldots, b_k$ are complex numbers. We assume that

$$m + \sum_{j=1}^{k_0} m_j < k.$$

The initial condition $y(0,z)$ is supposed to be a Laplace transform

$$y(0,z) = \int_{\gamma_\epsilon} \lambda_0(w) e^{w(z-a)} dw,$$

along a path γ_ϵ described in the section 2, $a \in \mathbb{C}^*$ and z is in an open disc $D(0,\rho)$ centered at 0 of radius ρ. The function λ_0, is defined on a neighborhood of γ_ϵ in $\mathbb{C}$ and satisfies the following growth estimates. There exist $C > 0$, $Q > 0$, $M > 0$, such that, for all $y_w \in \mathbb{R}_+$, $y_w > M$ implies

$$|\lambda_0(\epsilon + i y_w)| \leq C e^{Q y_w}.$$

Notice that the integral $y(0,z)$ is absolutely convergent if a is such that

$$Q - \mathrm{Im}(z - a) < 0$$

*This research has been supported by a Marie Curie Fellowship of the European Community programme "Improving Human Research Potential" under contract number HPMF-CT-2002-01818.

for all $z \in D(0,\rho)$. We are looking for formal solutions of (P_m)

$$(\hat{y}) \qquad \hat{y}(t,z) = \sum_{j=0}^{\infty} \frac{y_j(z)}{j!} t^j,$$

where y_j, $j \geq 0$, are holomorphic functions on $D(0,\rho)$. We are interested in the properties of summability of these formal series with respect to t in a direction $d \in \mathbb{R}$. In this direction we will show the following result (sections 2 and 3).

Theorem 1 *For notational convenience, we put*

$$A = k - 1 - \sum_{j=1}^{k_0} m_j \quad , \quad \sigma_q = \sum_{j=1}^{q} m_j.$$

Let the following assumption hold.

i) *The complex number a is such that*

$$(A+1)\pi + Q - \mathrm{Im}(z-a) < 0 \quad , \quad z - a \neq 0,$$

for all $z \in \bar{D}(0,\rho)$.

Then, there exists a formal series $(\hat{y})$ solution of (P_m) which is $m/(A+1-m)$-summable in every directions $d \in \mathbb{R}$ such that

$$md + \mathrm{Im}(z) \neq 0,$$

for all $z \in D(0,\rho)$.

The parabolic partial differential equation (P_m) can be seen as a small perturbation of the following partial differential equation with constant coefficients

$$(C) \qquad (\partial_z + c_1 - 1)^{m_1} \cdots (\partial_z + c_{k_0} - 1)^{m_{k_0}} \partial_t^m y(t,z) = (\partial_z + b_1) \cdots (\partial_z + b_k) y(t,z),$$

whose summability property is a consequence of the general results of the paper [2]. The result of Theorem 1 shows that under some additional conditions, the summability properties of (P_m) and the latter equation (C) are essentially the same.

In the paper [13], G.P. Sobol' and I.M. Sobol' considered particular ordinary differential equations with exponential coefficients and showed that their solutions can be expressed in term of generalized hypergeometric functions. Likewise, in the proof of Theorem 1, we will use the fact that the formal series solutions of (P_m) and their Borel transforms can be expressed in term of generalized hypergeometric functions. On the other hand, for a physical interpretation of the equations (P_m) see [11].

Acknowledgement The author is very grateful to Werner Balser and to Boele Braaksma for usefull discussions about the subject of this paper.

2 Parabolic partial differential equations of first order

In this section, we consider the following parabolic partial differential equation

$$(P_1) \qquad (\partial_z + c_1 - 1)^{m_1} \cdots (\partial_z + c_{k_0} - 1)^{m_{k_0}} \partial_t u(t,z) = e^z (\partial_z + b_1) \cdots (\partial_z + b_k) u(t,z),$$

We assume that the condition **i**) of the theorem 1 is satisfied. We consider a formal solution of (P_1)

$$(\hat{u}) \qquad \hat{u}(t,z) = \sum_{j=0}^{\infty} \frac{u_j(z)}{j!} t^j,$$

where u_j, $j \geq 0$, are holomorphic functions on a open disc $D(0,\rho)$ centered at 0 of radius ρ and

$$\hat{u}(0,z) = \int_{\gamma_\epsilon} \lambda_0(w) e^{w(z-a)} dw.$$

Inserting the formal series $(\hat{u})$ in the equation (P_1), we get the following recursion for the functions u_j, $j \geq 0$,

$$(u_j) \qquad (\partial_z + c_1 - 1)^{m_1} \cdots (\partial_z + c_{k_0} - 1)^{m_{k_0}} u_{j+1}(z) = e^z (\partial_z + b_1) \cdots (\partial_z + b_k) u_j(z),$$

where

$$u_0(z) = \int_{\gamma_\epsilon} \lambda_0(w) e^{w(z-a)} dw.$$

In the following, we choose functions u_j written as Laplace transforms.

Lemma 1 *Consider a real number ϵ satisfying $\epsilon \notin -\mathrm{Re}(c_q) - \mathbb{N}_0$, $1 \leq q \leq k_0$. Let the symbol $(x)_j$ be the Pochhammer symbol $(x)_j = x \cdots (x+j-1)$.*
Consider a path $\gamma_\epsilon \subset \mathbb{C}$ described as follows,

$$\gamma_\epsilon = \{w \in \mathbb{C} : w = \epsilon + iy_w, y_w \in [0, +\infty)\}.$$

Then there exists a formal series $(\hat{u})$, solution of (P_1), whose coefficients $u_j(z)$ can be written in the form

$$u_j(z) = \int_{\gamma_\epsilon} e^{jz} \frac{(b_1+w)_j \cdots (b_k+w)_j}{(c_1+w)_j^{m_1} \cdots (c_{k_0}+w)_j^{m_{k_0}}} \lambda_0(w) e^{w(z-a)} dw.$$

Proof Assume that the function $u_j(z)$ can be written in the form

$$u_j(z) = \int_{\gamma_\epsilon} \lambda_j(z,w) e^{w(z-a)} dw.$$

where the latter integral is absolutely convergent and $\lambda_j(z,w)$ is holomorphic with respect to z and w. We get

$$(\beta_q + \partial_z) u_j(z) = \int_{\gamma_\epsilon} \{(\beta_q + w + \partial_z)\lambda_j(z,w)\} e^{w(z-a)} dw,$$

for $\beta_q \in \mathbb{C}$. From the latter formula and the relations (u_j), we get

$$(c_1 - 1 + w + \partial_z)^{m_1} \cdots (c_{k_0} - 1 + w + \partial_z)^{m_{k_0}} \lambda_{j+1}(z,w) = e^z (b_1 + w + \partial_z) \cdots (b_k + w + \partial_z) \lambda_j(z,w),$$

with $\lambda_0(z,w) = \lambda_0(w)$. If we put $\lambda_j(z,w) = e^{zj} p_j(w)$, we deduce the recursion

$$(c_1 + w + j)^{m_1} \cdots (c_{k_0} + w + j)^{m_{k_0}} p_{j+1}(w) = p_j(w)(b_1 + w + j) \cdots (b_k + w + j),$$

with $p_0(w) = \lambda_0(w)$, so that for $j \geq 1$,

$$p_j(w) = \frac{(b_1+w)_j \cdots (b_k+w)_j}{(c_1+w)_j^{m_1} \cdots (c_{k_0}+w)_j^{m_{k_0}}} \lambda_0(w).$$

The lemma follows from the latter formula. □

We consider now the following auxiliary function associated to the formal series $(\hat{u})$, constructed in Lemma 1,

$$v(t,z) = \sum_{j=0}^{\infty} \frac{u_j(z)}{(j!)(j!)^A} t^j.$$

In order to investigate analytic continuation properties of $v(t,z)$, we specify the path γ_ϵ, from now on and to the end of the paper.

Consider complex numbers $\tilde{b}_q = b_q - r_q$, $1 \le q \le k$, where $r_q \in \mathbb{N}_0$ and $\tilde{c}_q = c_q$, $1 \le q \le k_0$, satisfying the following conditions.

i) The intersection

$$R_b = \cap_{q=1}^{A+1}(-\mathrm{Re}(\tilde{b}_q), 1 - \mathrm{Re}(\tilde{b}_q)) \bigcap \cap_{q=A+2}^{k}(-\mathrm{Re}(\tilde{b}_q), +\infty)$$

is not empty.

ii) For all $1 \le q \le k_0$, for all $\tau \in [A+2+\sigma_{q-1}, A+1+\sigma_q]$,

$$\mathrm{Re}(\tilde{c}_q) > \mathrm{Re}(\tilde{b}_\tau).$$

The real number ϵ is now assumed to satisfy $\epsilon \in R_b$, $\epsilon \notin -\mathrm{Re}(c_q) - \mathbb{N}_0$, $1 \le q \le k_0$, and $\epsilon \notin -\mathrm{Re}(b_q) - \mathbb{Z}$, $1 \le q \le k$. The corresponding path γ_ϵ is described as above in Lemma 1.

In the next lemma we study the analytic continuation of $v(t,z)$ with respect to t.

Lemma 2 *There exists $r > 0$, such that the series $v(t,z)$ is convergent for $t \in D(0,r)$ and $z \in D(0,\rho)$. The function $v(t,z)$ has an analytic continuation with respect to $t \in S_\rho$, for all $z \in D(0,\rho)$, where S_ρ is any sector centered at 0 such that*

$$S_\rho \subset \{t \in \mathbb{C} : e^z t \notin [1,+\infty) \quad , \quad \forall z \in D(0,\rho)\}.$$

Moreover, there exist constants $\mathcal{A}, M > 0$ such that for all $t \in S_\rho$ and $|t| \ge M$, we have $|v(t,z)| \le \mathcal{A}$, for all $z \in D(0,\rho)$.

Proof We consider the hypergeometric series

$$F(\tilde{b}_1+w,\dots,\tilde{b}_k+w;\tilde{c}_1+w,\dots,\tilde{c}_{k_0}+w;1;u) = \sum_{j=0}^{\infty} \frac{(\tilde{b}_1+w)_j \cdots (\tilde{b}_k+w)_j}{j!(1)_j^A (\tilde{c}_1+w)_j^{m_1} \cdots (\tilde{c}_{k_0}+w)_j^{m_{k_0}}} u^j$$

which converge absolutely for $|u| < 1$ and $w \in \gamma_\epsilon$ (see for instance [7], [9]).

Under the upper mentioned conditions **i**) and **ii**), we get from [12], page 108, an integral representation for this generalized hypergeometric function. We define the functions

$$B(t_2,\dots,t_k,w) = \prod_{q=2}^{A+1} t_q^{\tilde{b}_q-1+w}(1-t_q)^{-(\tilde{b}_q+w)} \prod_{q=1}^{k_0}\prod_{\tau\in\mathcal{S}_q} t_\tau^{\tilde{b}_\tau+w-1}(1-t_\tau)^{\tilde{c}_q-\tilde{b}_\tau-1},$$

$$A(w) = \frac{\Pi_{q=2}^{A+1}\sin\pi(\tilde{b}_q+w)}{\pi^A}\frac{\Pi_{q=1}^{k_0}\{\Gamma(\tilde{c}_q+w)\}^{m_q}}{\Pi_{q=A+2}^{k}\Gamma(\tilde{b}_q+w)},$$

and the constant

$$\Gamma = 1/\prod_{q=1}^{k_0}\prod_{\tau\in\mathcal{S}_q}\Gamma(\tilde{c}_q - \tilde{b}_\tau),$$

where $\mathcal{S}_q = [A+2+\sigma_{q-1}, A+1+\sigma_q]$. We have the following integral representation which is an analytic continuation of the generalized hypergeometric function

$$\begin{aligned} &F(\tilde{b}_1+w,\ldots,\tilde{b}_k+w;\tilde{c}_1+w,\ldots,\tilde{c}_{k_0}+w;1;u) \\ &= \Gamma A(w)\int_{[0,1]^{k-1}} B(t_2,\ldots,t_k,w)\frac{1}{(1-ut_2\cdots t_k)^{(\tilde{b}_1+w)}}dt_2\cdots dt_k \end{aligned}$$

with $w\in\mathbb{C}$ such that $\mathrm{Re}(w)\in R_b$ and $u\notin[1,+\infty)$.

In the following, we assume that $w\in\gamma_\epsilon$ and is written in the form $w=\epsilon+iy_w$, $y_w\in[0,\infty)$. For y_w sufficiently large, we have

$$|\sin\pi(\tilde{b}_q+w)|\le e^{\pi(y_w+|\mathrm{Im}(\tilde{b}_q)|)}.$$

For $u\notin[1,\infty)$, we have

$$\begin{aligned} &|(1-t_2\cdots t_k u)^{-(\tilde{b}_1+w)}| \\ &= \exp(-(\mathrm{Re}(\tilde{b}_1)+\epsilon)\ln|1-t_2\cdots t_k u| + (y_w+\mathrm{Im}(\tilde{b}_1))\arg(1-t_2\cdots t_k u)) \end{aligned}$$

with $-\pi<\arg(1-t_2\cdots t_k u)<\pi$. Moreover,

$$|(1-t_q)^{-(\tilde{b}_q+w)}t_q^{(\tilde{b}_q-1+w)}| = (1-t_q)^{-(\mathrm{Re}(\tilde{b}_q)+\epsilon)}t_q^{(\mathrm{Re}(\tilde{b}_q)-1+\epsilon)},$$

and

$$|(1-t_\tau)^{(\tilde{c}_q-\tilde{b}_\tau-1)}t_\tau^{(\tilde{b}_\tau+w-1)}| = (1-t_\tau)^{(\mathrm{Re}(\tilde{c}_q)-\mathrm{Re}(\tilde{b}_\tau)-1)}t_\tau^{(\mathrm{Re}(\tilde{b}_\tau)-1+\epsilon)}.$$

Using the Stirling formula (see [2]),

$$\Gamma(z)\sim(2\pi)^{1/2}z^z z^{-1/2}e^{-z},$$

when $|z|\to\infty$ with $-\pi+\delta\le\arg(z)\le\pi-\delta$, $0<\delta<\pi$, and for all $f\in\mathbb{C}$,

$$(f+z)^{(f+z)}\sim z^{f+z}e^f,$$

when $|z|\to\infty$, we get the following estimates. There exist $M,\tilde{K}_1>0$, such that for all $|w|>M$,

$$\frac{\Pi_{q=1}^{k_0}\{\Gamma(\tilde{c}_q+w)\}^{m_q}}{\Pi_{q=A+2}^{k}\Gamma(\tilde{b}_q+w)}\le\tilde{K}_1|w|^{\mathrm{Re}(\sum_{q=1}^{k_0}\tilde{c}_q m_q-\sum_{q=A+2}^{k}\tilde{b}_q)}.$$

In the following estimates, we always assume that $w\in\gamma_\epsilon$, $|w|>M$, *and* $u=e^z t$ *where* $z\in D(0,\rho)$ *and* $t\in S_\rho\cup D(0,r)$, $r<1$ *being small enough.*

From the latter estimates, we deduce that there exist constants $\hat{K},\hat{C}\in\mathbb{R}_+$, independent from u, w, such that

$$(E_1)\qquad |F(\tilde{b}_1+w,\ldots,\tilde{b}_k+w;\tilde{c}_1+w,\ldots,\tilde{c}_{k_0}+w;1;u)|\le\hat{C}y_w^{\hat{K}}e^{y_w(A+1)\pi}.$$

We recall now one contiguity relation between generalized hypergeometric functions (see [9], p. 45 and [1]),

$$(Cr)\qquad \begin{aligned} &(ud/du+(\tilde{b}_j+w))F(\ldots,\tilde{b}_j+w,\ldots;\tilde{c}_1+w,\ldots,1;u) \\ &= (\tilde{b}_j+w)F(\ldots,(\tilde{b}_j+1)+w,\ldots;\tilde{c}_1+w,\ldots,1;u). \end{aligned}$$

We write

$$\frac{d}{du}F(\ldots,\tilde{b}_j+w,\ldots;\tilde{c}_1+w,\ldots,1;u)=\frac{1}{2i\pi}\int_{C_\delta}\frac{F(\ldots,\tilde{b}_j+w,\ldots;\tilde{c}_1+w,\ldots,1;\zeta)}{(\zeta-u)^2}d\zeta$$

where

$$C_\delta=\{\zeta\in\mathbb{C}:|\zeta-u|=|u|\delta\},$$

and δ is sufficiently small. From $(E1)$, we deduce that

$$|u\frac{d}{du}F(\tilde{b}_1+w,\ldots,\tilde{b}_k+w;\tilde{c}_1+w,\ldots,\tilde{c}_{k_0}+w;1;u)|\leq\hat{C}\frac{1}{\delta}y_w^{\hat{K}}e^{y_w(A+1)\pi}.$$

From the latter estimate and the relation (Cr), we deduce that there exist constants $\hat{K},\hat{C}\in\mathbb{R}_+$, independent from u, w, such that,

$$|F(\ldots,(\tilde{b}_j+1)+w,\ldots;\tilde{c}_1+w,\ldots;1;u)|\leq\hat{C}y_w^{\hat{K}}e^{y_w(A+1)\pi}.$$

In a finite number of steps, one gets that there exist constants $\hat{K},\hat{C}\in\mathbb{R}_+$, independent from u, w, such that,

$$(E_2)\qquad |F(b_1+w,\ldots,b_k+w;c_1+w,\ldots,c_{k_0}+w;1;u)|\leq\hat{C}y_w^{\hat{K}}e^{y_w(A+1)\pi}.$$

If we put $z-a=\alpha+i\beta$, we finally get the following estimates. There exists constants $\hat{K},\hat{C}\in\mathbb{R}_+$, independent from u, w, such that,

$$\begin{aligned}&|\lambda_0(w)e^{w(z-a)}F(b_1+w,\ldots,b_k+w;c_1+w,\ldots,c_{k_0}+w;1;u)|\\&\leq\hat{C}y_w^{\hat{K}}\exp(y_w((A+1)\pi+Q-\beta)+\epsilon\alpha)\end{aligned}$$

From the latter inequality, we deduce that under the condition **i)** of the theorem 1, the integral

$$v_\epsilon^1(t,z)=\int_{\gamma_\epsilon}\lambda_0(w)F(b_1+w,\ldots,b_k+w;c_1+w,\ldots,c_{k_0}+w;1;e^zt)e^{w(z-a)}dw,$$

exist for $t\in S_\rho\cup D(0,r)$, where S_ρ is any sector centered at 0 satisfying the hypothesis of the lemma 2, and $z\in D(0,\rho)$. Moreover, we see that the function $v_\epsilon^1(t,z)$ has an analytic continuation with respect to $t\in S_\rho\cup D(0,r)$, for all $z\in D(0,\rho)$, where S_ρ is any sector centered at 0 satisfying the hypothesis of the lemma 2 and there exist constants $\mathcal{A},M>0$ such that for all $t\in S_\rho$ and $|t|\geq M$, we have

$$|v_\epsilon^1(t,z)|\leq\mathcal{A},$$

for all $z\in D(0,\rho)$.

On the other hand, due to the integral formula for $u_j(z)$ and the Fubini theorem, we get that there exists $r>0$ such that $v(t,z)=v_\epsilon^1(t,z)$, for all $t\in D(0,r)$ and $z\in D(0,\rho)$. In conclusion, $v(t,z)$ has an analytic continuation which satisfies the conditions of the lemma 2. □

3 Summability properties

We recall now the definition of k-summability for a formal series $(\hat{u})$.

Definition 1 *A series*

$$\hat{u}(t,z)=\sum_{j=0}^{\infty}\frac{t^j}{j!}u_j(z)$$

where $u_j(z)$, $0\le j\le\infty$ *are holomorphic functions in a disc* $D(0,\rho)$, *is said to be* $k-$*summable in the direction* $d\in\mathbb{R}$ *if*

i) *There exist* $r>0$, $R>0$, *such that the formal Borel transform with respect to the variable* t *of order* k,

$$w(t,z)=\sum_{j=0}^{\infty}\frac{u_j(z)t^j}{j!\Gamma(1+j/k)},$$

is absolutely convergent for $|z|<r$ *and* $|t|<R$,

ii) *There exists* $\delta>0$ *such that for all* $z\in\bar{D}(0,r)$, *the series* $w(t,z)$ *can be analytically continued with respect to* t *in a sector* $S_{d,\delta}=\{t\in\mathbb{C}:|d-\arg(t)|<\delta\}$. *Moreover, there exist* $\delta_1<\delta$, $C>0$, *and* $K>0$ *such that*

$$\sup_{|z|\le r}|w(t,z)|\le Ce^{K|t|^k}$$

for all $t\in S_{d,\delta_1}$.

If this is so, the Laplace transform of order k of $w(t,z)$,

$$u(t,z)=t^{-k}\int_0^{\infty(\gamma)}v(\tau,z)e^{-(\tau/t)^k}\,d\tau^k,$$

where the integration is made along the ray $\arg(\tau)=\gamma$ with $|d-\gamma|<\delta$, is called the $k-$ sum of the formal series $\hat{u}(t,z)$.

We consider the following formal series

$$\hat{a}(t,z)=\sum_{j=0}^{\infty}\frac{u_j(z)(j!)}{(mj)!(j!)}t^j$$

where the u_j satisfy the recursion (u_j). We notice that

$$\partial_t^m\hat{a}(t^m,z)=\sum_{j=1}^{\infty}\frac{u_j(z)}{(m(j-1))!}t^{m(j-1)}=\sum_{j=0}^{\infty}\frac{u_{j+1}(z)}{(mj)!}t^{mj}$$

so that the formal series $\hat{y}(t,z)=\hat{a}(t^m,z)$ satisfy the equation (P_m).

Due to the general theory of moment summability (see [3] page 107-109), we know that the function

$$v_a(t,z)=\sum_{j=0}^{\infty}\frac{u_j(z)(j!)t^j}{(mj)!(j!)\Gamma(1+(A+1-m)j)}$$

satisfy the same properties of convergence around the origin, analytic continuation and growth estimates along sectors

$$S_\rho\subset\{t\in\mathbb{C}:e^zt\notin[1,+\infty)\quad,\quad\forall z\in D(0,\rho)\}.$$

as the function $v(t,z)$ in the lemma 2. From this, we deduce that the series $\hat{a}(t,z)$ is $1/(A+1-m)$-summable in every direction $d \in \mathbb{R}$ such that

$$d + \mathrm{Im}(z) \neq 0,$$

for all $z \in D(0,\rho)$. In conclusion, we deduce that the series $\hat{y}(t,z) = \hat{a}(t^m, z)$ is $m/(A+1-m)$-summable in every direction $d \in \mathbb{R}$ such that

$$md + \mathrm{Im}(z) \neq 0,$$

for all $z \in D(0,\rho)$. So that the theorem 1 is proved.

References

[1] A. Adolphson, B.M. Dwork, *Contiguity relations for generalized hypergeometric functions*, Trans. Amer. Math. Soc. 347 (2) : 615–625, 1995.

[2] W. Balser, *Summability of formal power series solutions of partial differential equations with constant coefficients*, preprint, submitted to Proceedings of the International Conference on Differential and Functional Differential Equations, Moscow (2002).

[3] W. Balser, *Formal power series and linear systems of meromorphic ordinary differential equations*, Springer-Verlag, New-York, 2000.

[4] W. Balser, *Power series solutions to the inhomogeneous heat equation*, preprint, 2003.

[5] J. Barros-Neto, F. Cardoso, *Hypergeometric functions and the Tricomi operator*, preprint arXiv:math.AP/0310480.

[6] O. Costin, S. Tanveer, *On the existence and uniqueness of solutions of nonlinear evolution systems of PDEs in $\mathbb{R}^+ \times \mathbb{C}^d$, their asymptotic and Borel summability properties.* Manuscript, 2003.

[7] A. Erdelyi, M. Wilhelm, F. Oberhettinger, F. G. Tricomi, *Higher transcendental functions*, Vol. I., Robert E. Krieger Publishing Co., Inc., Melbourne, Fla., 1981.

[8] K. Ichinobe, *The Borel sum of divergent Barnes hypergeometric series and its application to a partial differential equation*, Publ. Res. Inst. Math. Sci., 37 (2001), pp. 91-117.

[9] K. Iwasaki; H. Kimura and All, *From Gauss to Painlevé. A modern theory of special functions.* Aspects of Mathematics, E16. Braunschweig, 1991.

[10] M. Miyake, *Borel summability of divergent solutions of the Cauchy problem to non-Kovalevskian equations,* in Partial Differential Equations and Their Applications, C. Hua and L. Rodino, eds., Singapore, 1999, World Scientific, pp. 225-239.

[11] A.D. Polyanin, *Handbook of Linear Partial Differential Equations for Engineers and scientists*, CRC press, 1999.

[12] L.J. Slater, *Generalized Hypergeometric Functions*, Cambridge Press, 1966.

[13] G.P. Sobol' and I.M. Sobol', *Hypergeometric functions and ordinary linear differential equations with exponential coefficients*, Differential Equations **15** (1979), no. 7, 859-862 (1980).