UNIQUENESS OF LIMIT CYCLES FOR A CLASS OF PLANAR VECTOR FIELDS.

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Abstract. In this paper we give sufficient conditions to ensure uniqueness of limit cycles for a class of planar vector fields. We also exhibit a class of examples with exactly one limit cycle.

1. Introduction

In this paper we consider the problem of determine the number of limit cycles, i.e. isolated closed trajectories, for planar vector fields. This is a classical problem, included as part of the XVI Hilbert's problem. The literature is huge and still growing, for a review see [CS1964] and the more recent [Zh1992, Ye1986].

An important subproblem is to study systems with a unique limit cycle, in fact in this case the dynamics of the cycle can "dominate" the global dynamics of the whole system.

Let us consider planar vector fields of the form:

$$\begin{cases} \dot{x} &= \beta(x)\left[\phi(y) - F(x,y)\right] \\ \dot{y} &= -\alpha(y)g(x)\,, \end{cases} \tag{1.1}$$

under the regularity assumptions (to ensure the existence and uniqueness of the Cauchy initial problem) there exists: $-\infty \leq a < 0 < b \leq +\infty$, such that:

A1) $\beta \in Lip\,(a,b)$ and $\alpha \in Lip\,(\mathbb{R})$;

A2) $\phi \in Lip\,(\mathbb{R})$, $g \in Lip\,(a,b)$ and $F \in Lip\,((a,b)\times\mathbb{R})$.

Without loss of generality we can assume α and β to be positive in their respective domains of definition, in fact the existence of x_0 such that $\beta(x_0) = 0$ (or y_0 s.t. $\alpha(y_0) = 0$), gives rise to *invariant lines*, which cannot intersect a limit cycle. Hence we can reparametrize time, by dividing the vector field by: $\alpha(y)\beta(x)$. The transformed system is:

$$\begin{cases} \dot{x} &= \tilde{\phi}(y) - \tilde{F}(x,y) \\ \dot{y} &= -\tilde{g}(x)\,, \end{cases} \tag{1.2}$$

where $\tilde{\phi}(y) = \phi(y)/\alpha(y)$, $\tilde{F}(x,y) = F(x,y)/\alpha(y)$ and $\tilde{g}(x) = g(x)/\beta(x)$. In the following we will drop out the $\tilde{}$–mark and consider the general system of previous type.

These systems can be though as "non–Hamiltonian perturbations" of Hamiltonian ones, with Hamilton function: $H(x,y) = \Phi(y) + G(x)$, where $\Phi(y) = \int_0^y \phi(s)\,ds$ and $G(x) = \int_0^x g(s)\,ds$, being $F(x,y)$ the "perturbation".

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One can also consider (1.2) as "generalized" Liénard equations:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0\,,$$

which in the Liénard plane can be rewritten as:

$$\begin{cases} \dot{x} &= y - F(x) \\ \dot{y} &= -g(x)\,, \end{cases} \tag{1.3}$$

where $F'(x) = f(x)$, hence our systems generalize (1.3) by allowing a dependence of F also on y.

Let $\lambda > 0$ and let us consider the energy level $\mathcal{H}_\lambda = \{(x,y) \in \mathbb{R}^2 : \Phi(y) + G(x) = \lambda\}$, the knowledge of the flow through $\mathcal{H}_\lambda$ can give informations about the existence of limit cycles. Because

$$< \nabla \mathcal{H}_\lambda, X(x,y) > \Big|_{\mathcal{H}_\lambda} = -F(x,y)g(x)\,,$$

where $X(x,y) = (\phi(y) - F(x,y), -g(x))$, no limit cycles can be completely contained in a region where gF doesn't change sign. We will see in a while that the set of zeros of F will play a fundamental role in our construction.

For Liénard systems the set of zeros of F is given by vertical lines $x = x_k$ s.t. $F(x_k) = 0$. In a recent paper [SV2004] authors, using ideas taken from Liénard systems [CV2003], proved a uniqueness result for systems (1.2) assuming that $F(x,y)$ vanishes only at three vertical lines $x = x_- < 0$, $x = 0$ and $x = x_+ > 0$. We generalize this condition by assuming that zeros of $F(x,y)$ lie on (quite) general curves. More precisely let us assume there exist $\psi_j : \mathbb{R} \to \mathbb{R}$, $j \in \{1,2\}$, such that [1]:

B0) $F(0,y) = 0$ for all real y;

B1) $y \mapsto \psi_1(y)$, is *positive* for all $y \in \mathbb{R}$, increasing for negative y, decreasing for positive y, $\psi_1(0) < b$;

B2) $y \mapsto \psi_2(y)$, is *negative* for all $y \in \mathbb{R}$, decreasing for negative y, increasing for positive y, $\psi_2(0) > a$;

B3) for all $y \in \mathbb{R}$, $j \in \{1,2\}$, we have:

$$F(\psi_j(y), y) \equiv 0\,,$$

these curves will be called "non–trivial zeros" of $F(x,y)$ (in opposition with the trivial zeros given by $x = 0$).

Let us divide the strip $(a,b) \times \mathbb{R}$ into four distinct domains:

- $D_1^> := \{(x,y) \in (a,b) \times \mathbb{R} : x > \psi_1(y)\}$;
- $D_1^< := \{(x,y) \in (a,b) \times \mathbb{R} : 0 < x < \psi_1(y)\}$;
- $D_2^> := \{(x,y) \in (a,b) \times \mathbb{R} : \psi_2(y) < x < 0\}$;
- $D_2^< := \{(x,y) \in (a,b) \times \mathbb{R} : x < \psi_2(y)\}$.

The following assumptions generalize "standard sign ones":

C1) $y\phi(y) > 0$ for all $y \neq 0$ and $xg(x) > 0$ for all $x \in (a,b) \setminus \{0\}$;

C2) $g(x)F(x,y) < 0$ for all $(x,y) \in D_1^< \cup D_2^>$.

We remark that hypothesis C2) can be weakened into:

C2') $g(x)F(x,y) \leq 0$ for all $(x,y) \in D_1^< \cup D_2^>$ except at some (x_0, y_0) where strictly inequality holds.

[1] We remark that our main result still holds, even if one assume there exist $\alpha_j < 0 < \beta_j$, $j \in \{1,2\}$, and the functions ψ_j to be defined in $[\alpha_j, \beta_j]$ and verify hypothesis B) on their new domain of definition.

With these hypotheses we ensures that $(0,0)$ is the only singular point of system (1.2) and trajectories wind clockwise around it.

We are now able to state our main result

Theorem 1.1. *Let us consider system* (1.2) *and let us assume Hypotheses A), B) and C) to hold. Then there is at most one limit cycles which intersect both curves $x=\psi_1(y)$ and $x=\psi_2(y)$ contained in $(a,b)\times\mathbb{R}$, provided:*

D1) *the function $y\mapsto F(x,y)/\phi(y)$ is strictly increasing for $(x,y)\in D_1^<$ and $y\neq 0$;*

D2) *the function $y\mapsto F(x,y)/\phi(y)$ is strictly decreasing for $(x,y)\in D_2^>$ and $y\neq 0$;*

E) *the function $x\mapsto F(x,y)$ is positive in $D_1^>$, negative in $D_2^<$ and increasing in $D_1^>\cup D_2^<$;*

F) *let $A_j(y)=[\phi(y)\partial_x F(x,y)-g(x)\partial_y F(x,y)]\Big|_{x=\psi_j(y)}$, then $A_j(y)y>0$ for $y\neq 0$, $j\in\{1,2\}$.*

Hypotheses D) and E) naturally generalize hypotheses used in the Liénard case [CV2003] or in the more general situation studied in [SV2004]. Also hypothesis F) is very natural: each closed trajectory intersects the non–trivial zeros of $F(x,y)$ at most once in any quadrant. We remark that this condition is trivially verified if the functions ψ_j are indentically constant, namely in the case considered in [SV2004].

The proof of this result will be given in the next section. In the last section (§ 3) we will provide a family of systems with exactly one limit cycle.

2. Proof of Theorem 1.1

The aim of this section is to prove our main result Theorem 1.1. The proof is based on the following remark,

Remark 2.1. *Along any closed curve $\gamma:[0,T]\to(a,b)\times\mathbb{R}$ one has:*

$$\int_0^T \frac{d}{dt}H\Big|_{flow}\circ\gamma(s)\,ds = H\circ\gamma(T)-H\circ\gamma(0)=0\,,$$

moreover if γ is an integral curve of system (1.2) *we can evaluate the integrand function to obtain:*

$$I_\gamma := \int_0^T g(x_\gamma(s))F(x_\gamma(s),y_\gamma(s))\,ds = 0\,, \tag{2.1}$$

where $\gamma(s)=(x_\gamma(s),y_\gamma(s))$.

The uniqueness result will be proved by showing that the existence of two limit cycles, γ_1 contained [2] in γ_2, both intersecting $x=\psi_1(y)$ and $x=\psi_2(y)$, will imply: $I_{\gamma_1}<I_{\gamma_2}$, which contradicts (2.1).

From now on we will assume the existence of two limit cycles, γ_1 contained in γ_2, which intersect both non–trivial zeros of F.

Let us now consider the set of zeros of the equation: $\phi(y)-F(x,y)=0$ inside $D_1^<\cup D_2^>$; we claim that this set is the graph of some function $x\mapsto\zeta(x)$. Moreover this function vanishes for $x\in\{\psi_2(0),0,\psi_1(0)\}$, it is positive for $x\in(\psi_2(0),0)$ and negative for $x\in(0,\psi_1(0))$.

[2] By this we mean γ_1 is properly contained in the compact set whose boundary is γ_2.

The first statement follows from hypotheses D). In fact, let assume the claim to be false, then there exist $(\bar{x}, y_k) \in D_2^>$, such that:

$$y_1 < y_2 \quad \text{and} \quad \phi(y_k) - F(\bar{x}, y_k) = 0 \quad k \in \{1,2\}\,.$$

Then using the decreasing property (hypothesis D)) we obtain a contradiction:

$$0 = \left[F(\bar{x}, y_1) - \phi(y_1)\right]/\phi(y_1) > \left[F(\bar{x}, y_2) - \phi(y_2)\right]/\phi(y_2) = 0\,.$$

We can do the same construction for $D_1^<$, and the statement is proved. The sign properties of ζ can be proved as follows. Let $\psi_2(0) < x < 0$, then by C2) $0 < F(x, \zeta(x)) = \phi(\zeta(x))$, using now C1) we conclude that $\zeta(x) > 0$. The case $0 < x < \psi_1(0)$ can be handle similarly and we omit. By continuity we get the result about the zeros of $\zeta(x)$.

Hypothesis F) guarantees that a closed trajectory can intersect the non–trivial zeros of $F(x,y)$ only once in each quadrant, in fact $A_j(y)$ gives a measure of the angle between the vector field and the normal to $\mathcal{F}_0 = \{(x,y) : x = \psi_1(y)\} \cup \{(x,y) : x = \psi_2(y)\}$ at $(\psi_j(y), y)$:

$$< \nabla\mathcal{F}_0, X(x,y)\Big|_{\mathcal{F}_0} >= \left[\phi(y)\partial_x F(x,y) - g(x)\partial_y F(x,y)\right]\Big|_{\mathcal{F}_0} = A_j(y) \quad j \in \{1,2\}\,.$$

For instance, because the angle between the vector field and $\{(x,y) : x = \psi_1(y)\} \cap \{y > 0\}$ is in absolute value smaller than $\pi/2$, a trajectory starting at $(0, \bar{y})$, for some $\bar{y} > 0$, which will intersect $\{(x,y) : x = \psi_1(y)\} \cap \{y > 0\}$, could not meet anew $\{(x,y) : x = \psi_1(y)\} \cap \{y > 0\}$.

From hypotheses D) and the sign of F on $D_2^< \cup D_1^>$, it follows easily that for all $(x,y) \in D_2^< \cup D_1^>$ one has: $(y - \zeta(x))\,(\phi(y) - F(x,y)) > 0$. Hence a cycle intersects $(D_2^< \cup D_1^>) \cap \{y > 0\}$ in a region where $\phi(y) - F(x,y) > 0$, whereas the intersection with $(D_2^< \cup D_1^>) \cap \{y < 0\}$ holds where $\phi(y) - F(x,y) < 0$. This remark allows us to divide the path of integration needed to evaluate I_{γ_j}, $j \in \{1,2\}$, in two parts: an "horizontal" one where $\dot{x} > 0$ and a vertical one, where $\dot{x}$ vanishes.

Let us define (see Figure 1), for $j \in \{1,2\}$, A_j (respectively B_j) the intersection point of γ_j with $x = \psi_1(y)$ for $y > 0$ (respectively $y < 0$), and C_j (respectively D_j) the intersection point of γ_j with $x = \psi_2(y)$ for $y < 0$ (respectively $y > 0$). Let also introduce, A_* being the intersection point of γ_1 and the line $x = x_{A_2}$ contained in the first quadrant, and A_{**} being the intersection point of γ_2 and the line $y = y_{A_1}$ contained in the first quadrant. Similarly we introduce points: B_*, B_{**}, C_*, C_{**} and D_*, D_{**} (see Figure 1).

According to this subdivision of the arcs of limit cycles, we evaluate I_{γ_j} as follows:

$$\text{(2.2)}\quad I_{\gamma_1} = \int_{D_*A_*} + \int_{A_*A_1} + \int_{A_1B_1} + \int_{B_1B_*} + \int_{A_*C_*} + \int_{C_*C_1} + \int_{C_1D_1} + \int_{D_1D_*}$$
$$I_{\gamma_2} = \int_{D_2A_2} + \int_{A_2A_{**}} + \int_{A_{**}B_{**}} + \int_{B_{**}B_2} + \int_{B_2C_2} + \int_{C_2C_{**}} + \int_{C_{**}D_{**}} + \int_{D_{**}D_2}\,.$$

Let now show that $I_{\gamma_1} < I_{\gamma_2}$, which prove the contradiction and conclude the proof.

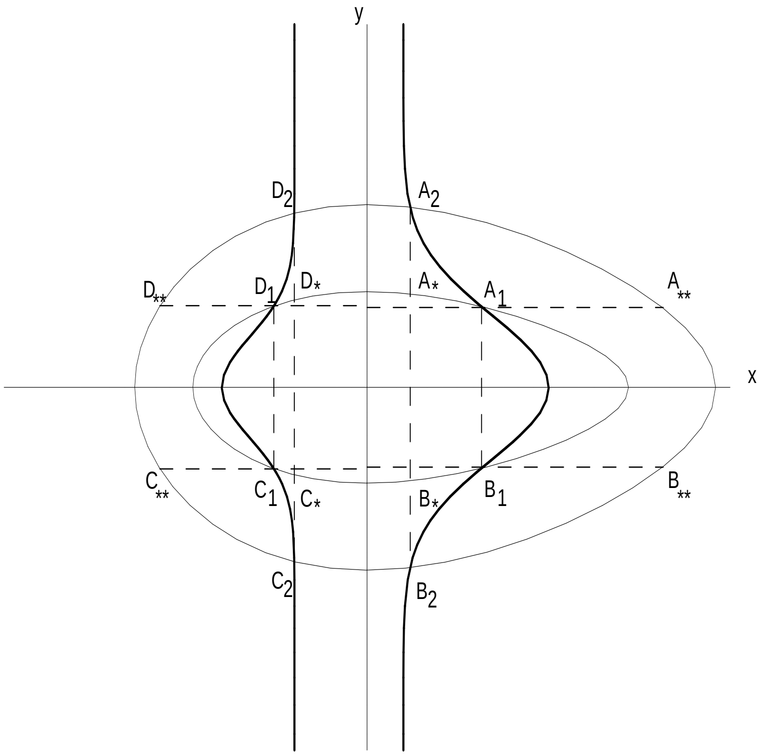

FIGURE 1. The non–trivial zeros of F (thick), the limit cycles γ_1 and γ_2 (thin) intersecting both non–trivial zeros of F and their subdivision into arcs.

2.1. **Integration along "horizontal arcs".** Because along horizontal arcs we have $\dot{x} \neq 0$, we can change integration variable from t to x, hence for example:

$$\int_{D_j A_j} g(x)F(x,y)\,dt = \int_{x_{D_j}}^{x_{A_j}} \frac{g(x)F(x,y_j(x))}{\phi(y_j(x)) - F(x,y_j(x))}\,dx\,,$$

where $y_j(x)$, $j \in \{1,2\}$, is the parametrization of γ_j as graph over x for $x \in (x_{D_j}, x_{A_j})$.

Because $y_2(x) > y_1(x)$ for all $x \in (x_{D_*}, x_{A_*})$, using hypotheses D) and the sign assumptions C) we get:

$$\frac{g(x)F(x,y_1(x))}{\phi(y_1(x)) - F(x,y_1(x))} < \frac{g(x)F(x,y_2(x))}{\phi(y_2(x)) - F(x,y_2(x))}\,,$$

hence:

$$\int_{x_{D_1}}^{x_{A_1}} \frac{g(x)F(x,y_1(x))}{\phi(y_1(x))-F(x,y_1(x))}\,dx < \int_{x_{D_1}}^{x_{D_*}} \frac{g(x)F(x,y_1(x))}{\phi(y_1(x))-F(x,y_1(x))}\,dx + \int_{x_{A_*}}^{x_{A_1}} \frac{g(x)F(x,y_1(x))}{\phi(y_1(x))-F(x,y_1(x))}\,dx$$
$$+ \int_{x_{D_2}}^{x_{A_2}} \frac{g(x)F(x,y_2(x))}{\phi(y_2(x))-F(x,y_2(x))}\,dx \leq \int_{x_{D_2}}^{x_{A_2}} \frac{g(x)F(x,y_2(x))}{\phi(y_2(x))-F(x,y_2(x))}\,dx \,, \tag{2.3}$$

the last step follows because the integrand function is negative by hypothesis C2) and from the previous discussion on the sign of $\phi(y)-F(x,y)$.

In a very similar way we can prove that:

$$\int_{x_{B_1}}^{x_{C_1}} \frac{g(x)F(x,y_1(x))}{\phi(y_1(x))-F(x,y_1(x))}\,dx < \int_{x_{B_2}}^{x_{C_2}} \frac{g(x)F(x,y_2(x))}{\phi(y_2(x))-F(x,y_2(x))}\,dx \,. \tag{2.4}$$

2.2. **Integration along "vertical arcs".** Along vertical arcs $\dot{y}$ never vanishes, hence we can perform the integration w.r.t. the y variable and getting for example:

$$\int_{A_jB_j} g(x)F(x,y)\,dt = \int_{y_{B_j}}^{y_{A_j}} F(x_j(y),y)\,dy \,,$$

where $x_j(y)$, $j\in\{1,2\}$, is the parametrization of γ_j as graph over y for $y\in(y_{B_j},y_{A_j})$.

Because $x_2(y)>x_1(y)$ for all $y\in(y_{A_{**}},y_{B_{**}})$, from hypotheses E) and the definition of A_{**} and B_{**}, we get:

$$\int_{y_{B_1}}^{y_{A_1}} F(x_1(y),y)\,dy < \int_{y_{B_{**}}}^{y_{A_{**}}} F(x_2(y),y)\,dy \,.$$

Again from the sign assumption on F in $D_1^>$, we get:

$$\int_{y_{A_{**}}}^{y_{A_2}} F(x_2(y),y)\,dy > 0 \quad \text{and} \quad \int_{y_{B_2}}^{y_{B_{**}}} F(x_2(y),y)\,dy > 0 \,,$$

hence we obtain:

$$\int_{y_{B_1}}^{y_{A_1}} F(x_1(y),y)\,dy < \int_{y_{B_2}}^{y_{A_2}} F(x_2(y),y)\,dy \,. \tag{2.5}$$

Analogously we can prove that:

$$\int_{y_{C_1}}^{y_{D_1}} F(x_1(y),y)\,dy < \int_{y_{C_2}}^{y_{D_2}} F(x_2(y),y)\,dy \,. \tag{2.6}$$

2.3. **Conclusion of the proof.** We are now able to complete our proof. In fact from (2.3) and (2.4) of § 2.1, from (2.5) and (2.6) of § 2.2 and the subdivision (2.2) we get:

$$I_{\gamma_1} < I_{\gamma_2} \,,$$

which contradicts (2.1), and so the Theorem is proved.

3. A system with exactly one limit cycle

In this last part we present an important class of examples exhibiting exactly one limit cycle. Let us assume that F has the following "special form":

$$F(x,y) = x\,[x-\psi_1(y)]\,[x-\psi_2(y)] \,,$$

where ψ_j verify hypotheses B). Then hypothesis F) is equivalent to:

F') the function $y\mapsto\Phi(y)+G(\psi_j(y))$ is strictly increasing for positive y and strictly decreasing for negative ones, $j\in\{1,2\}$.

In fact we have:

$$\begin{aligned} A_1(y) &= \psi_1(y)\,[\psi_1(y)-\psi_2(y)]\,[\phi(y)-g(\psi_1(y))\psi_1'(y)] \\ &= \psi_1(y)\,[\psi_1(y)-\psi_2(y)]\,\tfrac{d}{dy}\,[\Phi(y)-G(\psi_1(y))]\ , \end{aligned}$$

and the claim follows from the sign properties of ψ_j and the definitions of Φ and G. Similarly for A_2.

A concrete example is given by choosing:

$$\phi(y)=y\,,\ g(x)=x\,,\ \psi_1(y)=c_1e^{-d_1y^2}+e_1 \quad \text{and} \quad \psi_2(y)=-c_2e^{-d_2y^2}-e_2\,,$$

with c_j, d_j, and e_j positive real numbers such that:

(1) $c_1+e_1=c_2+e_2$,
(2) let $r=c_1+e_1$, then $c_jd_j\max\{r,r^2\}<1/2$ for $j\in\{1,2\}$.

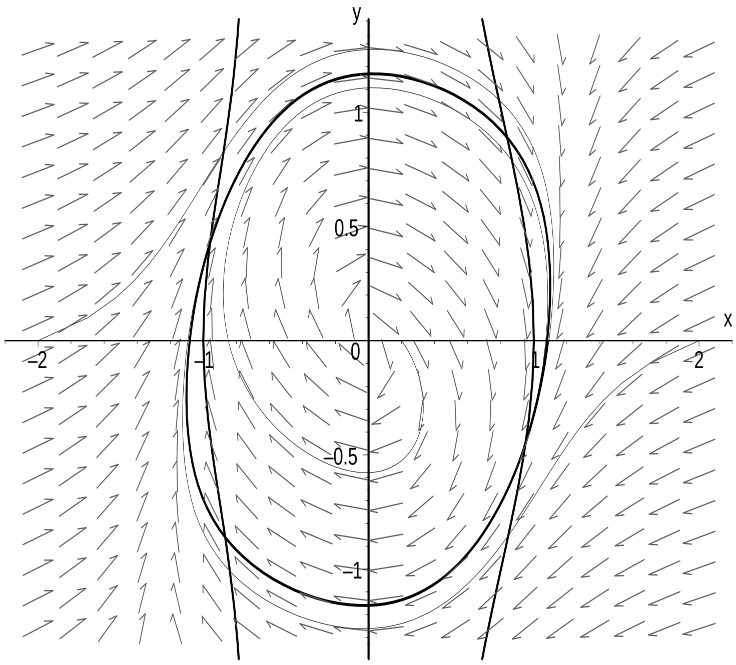

FIGURE 2. An example with $c_1=d_1=e_1=1/2$, $c_2=1/4$, $d_2=1$, and $e_2=3/4$. We numerically compute the attracting limit cycle (thick) and three attracted trajectories (thin), we also plot the non–trivial zeros of F (thick) and the direction of the vector field (arrows).

The example of Figure 2 verifies all the hypotheses of Theorem 1.1. Hypotheses A), B), C) and E) are readily verified, whereas hypotheses D) and F') (hence F)) follow [3] from (1) and (2) plus some easy calculations that we omit.

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[3] Using $c_j d_j r < 1/2$ hypothesis F') holds. Whereas (1) and $c_j d_j r^2 < 1/2$ ensure that the circle $x^2 + y^2 = r^2$ is contained inside $D_1^{<} \cup D_2^{>}$ and it is tangent to $x = \psi_j(y)$ at $y = 0$, for $j \in \{1, 2\}$, thus any limit cycle must intersect both curves $x = \psi_j(y)$, $j \in \{1, 2\}$.