Parametric resonance of ground states in the nonlinear Schrödinger equation

S. Cuccagna*, E. Kirr†, and D. Pelinovsky‡

September 16, 2004

Abstract

We study the global existence and long-time behavior of solutions of the initial-value problem for the cubic nonlinear Schrödinger equation with a linear attractive localized potential and a time-dependent nonlinearity coefficient. For small initial data, we show under some non-degeneracy assumptions that the solution approaches the profile of the ground state and decays in time like $t^{-1/4}$. The decay is due to resonant coupling between the ground state and the radiation field induced by the time-dependent nonlinearity coefficient.

1 Introduction

We consider the initial-value problem for the nonlinear Schrödinger (NLS) equation in three spatial dimensions:

$$\begin{aligned} i\partial_t u(t,x) &= (-\Delta + V(x))u(t,x) + \gamma(t)|u(t,x)|^2 u(t,x), \\ u(0,x) &= u_0(x), \qquad (t,x) \in [0,\infty) \times \mathbb{R}^3, \end{aligned} \tag{1.1}$$

where $u(t,x)$ is a complex-valued function, Δ is the Laplacian in $x \in \mathbb{R}^3$, $V(x)$ is a real-valued potential, and $\gamma(t)$ is the nonlinearity coefficient in the form:

$$\gamma(t) = \gamma_0 + \gamma_1 \cos(\omega t), \qquad \gamma_0,\ \gamma_1 \in \mathbb{R}. \tag{1.2}$$

The NLS equation (1.1) is a mean field model for Bose-Einstein condensates, see [10]. The external potential $V(x)$ models the trapping mechanism of the condensate, while the periodic coefficient $\gamma(t)$ models the Feshbach resonance induced by the harmonic magnetic field [4, 6]. The homogenization (averaging) of the NLS equation (1.1) with $\gamma_1 \neq 0$ was studied recently in the limit $\omega \gg 1$ [13, 14], when the solution $u(t,x)$ was shown to approach to a periodically modulated ground state at the intermediate time scale $0 < t < T$ for some $T > 0$. Parametric resonance between the ground state and the continuous spectrum is removed from the leading order of the averaging method in the limit $\omega \gg 1$. It is expected however that when the

*DISMI, University of Modena and Reggio Emilia, Italy
†Department of Mathematics, University of Chicago, USA
‡Department of Mathematics, McMaster University, Canada

radiation field escapes the linear attractive localized potential $V(x)$, it leads to the radiative decay of the periodically modulated ground state in the long-time behavior of the solution $u(t,x)$ as $t \to \infty$. We study here this problem in the case of finite ω.

In what follows, we use the following notations. Let $L^p(\mathbb{R}^3)$, $p \geq 1$ be the usual Banach spaces of complex valued measurable functions whose p^{th} power is integrable. Let $H^n(\mathbb{R}^3)$, $n \geq 0$ be the Sobolev spaces of functions which have n generalized derivatives in $L^2(\mathbb{R}^3)$. Let $H^n(\mathbb{R}^3)$, $n < 0$ be the duals of $H^{-n}(\mathbb{R}^3)$ and $L^2_\sigma(\mathbb{R}^3)$ be the weighted L^2 space with the norm:

$$\|f\|_{L^2_\sigma} = \|(1+|x|^2)^{\sigma/2} f\|_{L^2}. \tag{1.3}$$

Let $\langle \cdot, \cdot \rangle$ denote the scalar product in $L^2(\mathbb{R}^3)$. We assume that the localized potential $V(x)$ satisfies two conditions:

$$|V(x)| \leq \frac{C}{(1+|x|)^\beta}, \qquad C > 0, \quad \beta > 3, \quad \forall x \in \mathbb{R}^3, \tag{1.4}$$

and

$$\nabla V \in L^p(\mathbb{R}^3), \ \ \forall p > \frac{3}{2}. \tag{1.5}$$

It follows from the decay condition (1.4) on $V(x)$ that the operator $\mathcal{H} = -\Delta + V(x)$ is self-adjoint on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$, see [12, Theorem 7.5.5]. As a result, the spectrum of $\mathcal{H}$ consists of finite-dimensional discrete spectrum for negative eigenvalues, absolutely continuous spectrum for non-negative eigenvalues, and a possible resonance at zero. We assume that the zero-eigenvalue resonance does not occur in the spectrum of $\mathcal{H}$ and denote the projection operator on the absolutely continuous spectrum of $\mathcal{H}$ by $\mathcal{P}_c$. Adding a technical assumption on the discrete spectrum of $\mathcal{H}$, we formulate our main result as Theorem 1.

Assumption 1. *The self-adjoint operator $\mathcal{H} = -\Delta + V(x)$ on $L^2(\mathbb{R}^3)$ has a unique simple eigenvalue λ of the discrete spectrum, such that*

$$\lambda < 0, \qquad \lambda + \omega > 0, \tag{1.6}$$

with the corresponding real-valued eigenvector $\psi(x) \in L^2(\mathbb{R}^3)$.

Remark 1.1. *Based on the decay of the potential (1.4), it follows from [11] that*

$$|\psi(x)| \leq C e^{-\sqrt{-\lambda}|x|}, \qquad \forall |x| > x_0 \gg 1, \tag{1.7}$$

such that

$$\psi(x) \in L^2_\sigma(\mathbb{R}^3) \cap L^p(\mathbb{R}^3), \ \forall \sigma \geq 0, \ \forall p \geq 2. \tag{1.8}$$

Theorem 1. *Let Assumption 1 hold and assume that*

$$\Gamma \stackrel{def}{=} \frac{\gamma_1^2}{4} \mathrm{Im} \langle \psi^3, (\mathcal{H} - \lambda - \omega - i0)^{-1} \mathcal{P}_c \psi^3 \rangle \neq 0, \tag{1.9}$$

Then there exists $\varepsilon_0 > 0$ such that for all initial conditions $u_0(x)$ satisfying

$$\varepsilon \stackrel{def}{=} \max\{\|u_0\|_{H^1}, \|u_0\|_{L^2_\sigma}\} < \varepsilon_0, \qquad \text{for some } \sigma > \frac{5}{2}, \tag{1.10}$$

the initial-value problem (1.1) is globally well-posed and satisfies the decay estimates for all $t \geq 0$*:*

$$u(t,x) = e^{-i\lambda t} A(t)\psi(x) + u_d(t,x), \tag{1.11}$$

where

$$|A(t)| \leq \frac{C_0 \varepsilon}{(1 + 4\Gamma\varepsilon^4 t)^{1/4}}, \tag{1.12}$$

and

$$\|u_d(t)\|_{L^2_{-\sigma}} \leq \frac{C_1 \varepsilon}{(1+t)^{3/2}} + \frac{C_2 \varepsilon^3}{(1 + 4\Gamma\varepsilon^4 t)^{3/4}} \tag{1.13}$$

$$\|u_d(t)\|_{L^4} \leq \frac{C_3 \varepsilon}{(1+t)^{3/4}} + \frac{C_4 \varepsilon^2}{(1 + 4\Gamma\varepsilon^4 t)^{1/2}}, \tag{1.14}$$

with some positive ϵ*-independent constants* C_0*–*C_4*.*

Remark 1.2. *Since* $\psi^3 \in L^2_\sigma(\mathbb{R}^3)$ *due to Remark 1.1, the parameter* Γ *is well-defined. Moreover:*

$$\mathrm{Im}\langle \psi^3, (\mathcal{H} - \mu - i0)^{-1} \mathcal{P}_c \psi^3 \rangle = \pi \langle \psi^3, E'(\mu)\psi^3 \rangle \geq 0,$$

where $E'(\mu)$*,* $\mu \in \mathbb{R}$ *is the spectral measure induced by* $\mathcal{H} = -\Delta + V(x)$*, see [5]. For* $\mu = \lambda + \omega > 0$ *the above expression is generically positive.*

Remark 1.3. *Although we believe that the decay estimate (1.14) can be sharpened to:*

$$\|u_d(t)\|_{L^4} \leq \frac{\tilde{C}_3 \varepsilon}{(1+t)^{3/4}} + \frac{\tilde{C}_4 \varepsilon^3}{(1 + 4\Gamma\varepsilon^4 t)^{3/4}}, \tag{1.15}$$

we have no proof of the improved estimate (1.15).

The existence of an asymptotically stable center manifold (nonlinear ground state) that bifurcates from the linear ground state $\psi(x)$ was proved in the case $\gamma(t) = \gamma_0$ in [17], see also [19, 20]. As a consequence, any solution of the initial-value problem (1.1) in the case $\gamma(t) = \gamma_0$ with sufficiently small initial data $u_0(x)$ in (1.10) converges to the nonlinear ground state. Theorem 1 shows that the central manifold is destroyed by the time periodic perturbation (1.2) with $\gamma_1 \neq 0$. The argument can be easily adapted to handle a general periodic and almost periodic functions $\gamma(t)$.

For the proof of Theorem 1, we decompose the solution $u(t,x)$ onto projections on the ground state $\psi(x)$ and the continuum spectrum of the linear dispersive operator $\mathcal{H}$, according to the representation (1.11). The non-degeneracy condition (1.9) states that there is a nontrivial Fermi Golden Rule for the parametric resonance between $\psi^3(x)$ and the continuous spectrum of $\mathcal{H}$, which is induced by the time frequency ω of the nonlinearity coefficient $\gamma(t)$. We employ dispersive estimates, contraction principles in a hierarchy of Banach spaces and comparison results for ODE's to prove that the resonance leads to a slow destruction of any small localized initial value $u_0(x)$, according to the decay rates (1.12)–(1.14).

Theorem 1 can be viewed as a nonlinear generalization of the linear quantum resonance results in [2, 7, 8, 9, 21, 26]. Nonlinear Fermi Golden rule and nonlinear resonances for the NLS equation (1.1) with $\gamma(t) = \gamma_0$ were introduced in [18, 22]. Relaxation of excited bound states to the ground state due to nonlinear resonances was recently studied in [23, 25]. Our analysis of global existence of solutions is different from [23, 25], since no conservation of energy exists in the time evolution of the NLS equation (1.1) with the time-periodic nonlinearity coefficient $\gamma(t)$. In addition, we use methods based on contraction principles, which provide not only a more systematical treatment of the resonance but also sharper information on the evolution of solution, when they are compared to previously employed bootstrapping arguments. Consequently, we can extend our results to the case, when $\lambda+\omega < 0$ but $\lambda+n\omega > 0$ for some integer n. This case corresponds to a higher-order resonance and it leads to a slower decay rate of the solution $u(t,x)$ like $t^{-1/(4n)}$.

The paper is structured as follows. Section 2 gives local existence results for the problem (1.1). The proof of Theorem 1 is given in Section 3, where we decompose the problem (1.1) in the form of a coupled system of two differential equations. The proof relies on analysis of each of the two equations which is reported in Sections 4 and 5, respectively. The H^1 estimates are needed to prevent the finite-time blow up and they are given in Section 6. Section 7 concludes the paper with generalizations and open problems. Appendix A presents applications of the contraction principle in a hierarchy of Banach spaces and it is used in Section 4.

2 Local Existence

We show here that the initial-value problem (1.1) is locally well-posed. The local well-posedness in $H^2(\mathbb{R}^3)$ follows from the decay condition (1.4). If the potential $V(x)$ is also sufficiently smooth, such that the condition (1.5) is satisfied, the initial-value problem (1.1) is also well-posed in $H^1(\mathbb{R}^3)$.

Theorem 2. *Assume that the decay condition (1.4) be satisfied. Then, for every $u_0 \in H^2(\mathbb{R}^3)$ there exists a unique solution $u(t)$ of the initial value problem (1.1) defined on a maximal interval $t \in [0, T_{\max})$ such that*

$$u \in C^1\left([0,T_{\max}), L^2\right) \cap C\left([0,T_{\max}), H^2\right), \tag{2.1}$$

where

$$\text{either } T_{\max} = +\infty \text{ or } \lim_{t\to T_{\max}} \|u(t)\|_{H^2} = \infty. \tag{2.2}$$

Moreover, $\|u(t)\|_{L^2} \equiv \|u(0)\|_{L^2}$, $\forall t \in [0, T_{\max})$, and $u(t)$ depends continuously on the initial data, i.e. if $\lim_{n\to\infty} u_0^n = u_0$ in $H^2(\mathbb{R}^3)$ then for any closed interval $I \subset [0, T_{\max})$ the solution $u^n(t)$ of the problem (1.1) with initial data u_0^n is defined on I for sufficiently large n, and $\lim_{n\to\infty} u^n(t) = u(t)$ in $C(I, H^2)$.

Proof. Let $\mathcal{T}(t)$, $t \in \mathbb{R}$ be the group of unitary operators on $L^2(\mathbb{R}^3)$ generated by $-i\mathcal{H}$, where $\mathcal{H} = -\Delta + V(x)$, see proof of Theorem 7.5.5 in [12]. In addition, $\mathcal{T}(t)$ is a group of unitary operators on $H^2(\mathbb{R}^3)$ endowed with the graph norm

$$\|f\|_{\mathcal{H}} = \|\mathcal{H}f\|_{L^2} + \|f\|_{L^2}. \tag{2.3}$$

Due to the decay condition (1.4) the above norm is equivalent with the standard one in $H^2(\mathbb{R}^3)$. The initial-value problem (1.1) can be reformulated in the integral form:

$$u(t) = \mathcal{T}(t)u_0 - i\int_0^t \mathcal{T}(t-s)\gamma(s)|u(s)|^2 u(s)ds. \tag{2.4}$$

Since $\gamma(t)$ is uniformly bounded, the map $F = \gamma(t)|u|^2 u : [0,\infty) \times H^2 \to H^2$ is locally Lipschitz, by the generalization of Lemma 8.1.2 in [12] to $x \in \mathbb{R}^3$. The contraction principle for the integral equation (2.4) finishes the proof, see Theorem 6.1.7 in [12]. $\square$

Theorem 3. *Assume that both conditions (1.4) and (1.5) be satisfied. Then, for every $u_0 \in H^1(\mathbb{R}^3)$ there exists a unique solution $u(t)$ of the initial-value problem (1.1) defined on a maximal interval $t \in [0, T_{\max})$ such that*

$$u \in C^1\left([0,T_{\max}), H^{-1}\right) \cap C\left([0,T_{\max}), H^1\right), \tag{2.5}$$

where

$$\text{either } T_{\max} = +\infty \text{ or } \lim_{t\to T_{\max}} \|u(t)\|_{H^1} = \infty. \tag{2.6}$$

Moreover, $\|u(t)\|_{L^2} \equiv \|u(0)\|_{L^2}$, $\forall t \in [0,T_{\max})$, and $u(t)$ depends continuously on the initial data, i.e. if $\lim_{n\to\infty} u_0^n = u_0$ in $H^1(\mathbb{R}^3)$ then for any closed interval $I \subset [0,T_{\max})$ the solution $u^n(t)$ of the problem (1.1) with initial data u_0^n is defined on I for sufficiently large n, and $\lim_{n\to\infty} u^n(t) = u(t)$ in $C(I,H^1)$.

Proof. Since $\gamma(t)$ is uniformly bounded on $[0,\infty)$, the argument of Theorem 4.4.6 and Remark 4.4.8 in [1] is applied to the proof of theorem. $\square$

3 Global existence and long time behavior

We decompose here the solution $u(t)$ into the slow and rapidly varying parts. Using this setting, we prove our main Theorem 1. By Theorems 2 or 3, the solution of the initial-value problem (1.1) satisfies $u(t) \in L^2(\mathbb{R}^3)$ for any $t \in [0,T_{\max})$. By the Spectral Theorem and Assumption 1, any L^2 function can be written as a sum of its projections onto the eigenvector ψ and the continuous spectrum of $\mathcal{H} = -\Delta + V$. Hence we can write:

$$u(t) = A(t)e^{-i\lambda t}\psi + u_d(t), \tag{3.1}$$

where $A(t) \in \mathbb{C}$ and $u_d(t) = \mathcal{P}_c u(t)$. Using the orthogonality between ψ and the range of $\mathcal{P}_c$, the NLS equation (1.1) becomes the coupled system of equations in $A(t)$ and $u_d(t)$:

$$\dot{A} = -i\gamma(t)e^{i\lambda t}\langle\psi, |u|^2 u\rangle \tag{3.2}$$

$$\dot{u}_d = -i\mathcal{H}u_d - i\gamma(t)\mathcal{P}_c|u|^2 u \tag{3.3}$$

where

$$\begin{aligned} |u|^2 u &= |A|^2 A e^{-i\lambda t}\psi^3 + 2|A|^2\psi^2 u_d + A^2 e^{-2i\lambda t}\psi^2\overline{u}_d \\ &+ 2Ae^{-i\lambda t}\psi|u_d|^2 + \overline{A}e^{i\lambda t}\psi u_d^2 + |u_d|^2 u_d. \end{aligned} \tag{3.4}$$

In the linear case, when $\gamma(t) \equiv 0$, we have

$$A(t) = A_0 = \langle\psi, u_0\rangle, \qquad u_d(t) = \mathcal{T}\mathcal{P}_c u_0, \tag{3.5}$$

where $\mathcal{T}(t) = e^{-i\mathcal{H}t}$ is the group of unitary operators generated by $-i\mathcal{H}$ on $L^2(\mathbb{R}^3)$. Hence, the solution $u(t)$ is a superposition of the bound state $A_0 e^{-i\lambda t}\psi(x)$ and the dispersive part

$u_d(t) = e^{-i\mathcal{H}t}\mathcal{P}_c u_0$. Under the decay condition (1.4), the dispersive part satisfies the decay estimates for all $t \geq 0$:

$$\|e^{-i\mathcal{H}t}\mathcal{P}_c f\|_{L^2_{-\sigma}} \leq \frac{D_1}{(1+t)^{3/2}}\|f\|_{L^2_\sigma}, \quad \sigma > \frac{5}{2} \tag{3.6}$$

$$\|e^{-i\mathcal{H}t}\mathcal{P}_c f\|_{L^4} \leq \frac{D_2}{(1+t)^{3/4}}\|f\|_{L^{4/3}}, \tag{3.7}$$

$$\|e^{-i\mathcal{H}t}\mathcal{P}_c f\|_{L^\infty} \leq \frac{D_3}{(1+t)^{3/2}}\|f\|_{L^1}, \tag{3.8}$$

where D_1, D_2, and D_3 are some positive constants. Estimates (3.6) and (3.8) were proved in [5] and [3], respectively. Estimate (3.7) follows from (3.8) by the Riesz-Thorin interpolation argument, since $e^{-i\mathcal{H}t}$ is unitary on $L^2(\mathbb{R}^3)$. The estimates (3.6)–(3.8) imply that $u_d(t)$ decays in the same way as the solution of the free Schrödinger equation in weighted L^2 and L^p norms. We note that slightly stronger assumptions on $V(x)$ would imply that $u_d(t)$ converges as $t \to \infty$ to the solution of the free Schrödinger equation in the above spaces, see [27].

We consider the resolvent $(\mathcal{H} - \mu - i\delta)^{-1}\mathcal{P}_c, \delta > 0$ as a map from $L^2_\sigma(\mathbb{R}^3)$ to $L^2_{-\sigma}(\mathbb{R}^3)$ for some $\sigma > \frac{5}{2}$. By Sections 3 and 8 in [7], the resolvent converges strongly as $\varepsilon \searrow 0$ even when μ is in the spectrum of $\mathcal{H}$ and it satisfies the following decay estimates for all $t \geq 0$:

$$\|e^{-i\mathcal{H}t}(\mathcal{H} - \mu - i0)^{-1}\mathcal{P}_c f\|_{L^2_{-\sigma}} \leq \frac{D_4}{(1+t)^{3/2}}\|f\|_{L^2_\sigma}, \quad \mu \neq 0, \ \sigma > \frac{5}{2} \tag{3.9}$$

$$\|e^{-i\mathcal{H}t}(\mathcal{H} - i0)^{-1}\mathcal{P}_c f\|_{L^2_{-\sigma}} \leq \frac{D_5}{(1+t)^{1/2}}\|f\|_{L^2_\sigma}, \quad \sigma > \frac{5}{2}, \tag{3.10}$$

where D_4,D_5 are some positive constants and $(\mathcal{H} - \mu - i0)^{-1}$ denotes the limit as $\varepsilon \searrow 0$.

In the nonlinear case, when $\gamma(t) \not\equiv 0$, the dispersive equation (3.3) can be analyzed with the Duhamel principle in the equivalent integral form:

$$u_d(t) = e^{-i\mathcal{H}t}u_d(0) - i\int_0^t \gamma(s)|A|^2 A(s) e^{-i\lambda s} e^{-i\mathcal{H}(t-s)}\mathcal{P}_c\psi^3 ds + K_A[u_d](t), \tag{3.11}$$

where

$$\begin{aligned} K_A[u_d](t) &= -2i\int_0^t \gamma(s)|A|^2(s) e^{-i\mathcal{H}(t-s)}\mathcal{P}_c\psi^2 u_d(s) ds \\ &- i\int_0^t \gamma(s)A^2(s)e^{-2i\lambda s}e^{-i\mathcal{H}(t-s)}\mathcal{P}_c\psi^2\overline{u}_d(s)ds \\ &- 2i\int_0^t \gamma(s)A(s)e^{-i\lambda s}e^{-i\mathcal{H}(t-s)}\mathcal{P}_c\psi|u_d(s)|^2 ds \\ &- i\int_0^t \gamma(s)\overline{A}(s)e^{i\lambda t}e^{-i\mathcal{H}(t-s)}\mathcal{P}_c\psi u_d^2(s)ds \\ &- i\int_0^t \gamma(s)e^{-i\mathcal{H}(t-s)}\mathcal{P}_c|u_d|^2 u_d(s)ds. \end{aligned} \tag{3.12}$$

We show in Section 4 that $K_A[u_d]$ is locally contractive in certain Banach spaces and that the solution $u_d(t)$ can be approximated by:

$$u_d(t) \approx e^{-i\mathcal{H}t}u_d(0) - i\int_0^t \gamma(s)e^{-i\lambda s}|A|^2 A(s)e^{-i\mathcal{H}(t-s)}\mathcal{P}_c\psi^3 ds, \tag{3.13}$$

see Lemma 4.2. When the representation (3.13) is substituted into the amplitude equation (3.2) via the expression (3.4), we obtain a closed integro-differential equation in $A(t)$. We show in Section 5 that the main contribution in the evolution of $|A(t)|$ comes from the terms that are linear in $u_d(t)$. Their leading effect can be computed with the Fermi Golden rule, introduced in [21]. Comparison results for ODE's together with linearization and contraction principles allow us to control the non-dominant terms and obtain the decay estimate (1.12), which implies the decay estimates (1.13)-(1.14) by using the representation (3.13).

We note that the proof of Theorem 1 relies heavily on the tight control of the error in the representation (3.13). In this analysis, the smallness of the initial condition, defined by the small parameter ε in (1.10), is crucial. The nonlinearity of operator $K_A[u_d]$ raises two important issues that we need to address before we can treat it perturbatively.

The first issue is related to the presence of two time scales in the integral equation (3.11). The first term on the right hand side is initially $\mathcal{O}(\varepsilon)$ and decays fast, according to the decay estimates (3.6)-(3.8). The second term is initially $\mathcal{O}(\varepsilon^3)$ but decays slowly, as shown in Section 4. Because $K_A[u_d]$ is a nonlinear operator in $u_d(t)$, it couples the two time scale and leads to a slowly varying error of order $\mathcal{O}(\varepsilon^2)$. We keep the time scale separated and obtain a slowly varying error of order $\mathcal{O}(\varepsilon^4)$ by using the contraction principle in a hierarchy of Banach spaces, described in Appendix A.

The second issue is raised by the nonlinear dependence of $K_A[u_d]$ on $A(t)$. In order to treat $K_A[u_d]$ perturbatively, we need to obtain some information on the decay rate of $|A(t)|$. However, such information would come from analysis of the amplitude equation (3.2) which can be performed only after we have obtained some estimates on the decay of $u_d(t)$. We show how to get around this vicious circle in the following proof of Theorem 1.

Proof of Theorem 1. We fix $\sigma > 5/2$ and $\varepsilon > 0$, where ε is defined by the initial condition u_0 in (1.10). By Theorem 3, there exists $T_{\max} > 0$ such that the solution $u(t)$ is continuous and belongs to $H^1(\mathbb{R}^3)$ for all $t \in [0, T_{\max})$. We decompose the solution $u(t)$ according to the representation (3.1). Let the set τ be defined by

$$\tau = \left\{ t \in [0, T_{\max}) : \ |A(s)| \leq \frac{\mathcal{C}\varepsilon}{(1 + 4\Gamma\varepsilon^4 s)^{1/4}} \text{ for all } 0 \leq s \leq t \right\}, \tag{3.14}$$

where the constant Γ is defined by (1.9) and the constant $\mathcal{C}$ satisfies the inequality:

$$\mathcal{C} > \mathcal{C}_0 + \mathcal{D}, \tag{3.15}$$

where $\mathcal{C}_0$ is introduced from the condition $|A(0)| \leq \mathcal{C}_0 \varepsilon$ and the positive constant $\mathcal{D}$ is defined in Lemma 5.1. We will set $\mathcal{C}_0 = 1$ without loss of generality. The interval $[0, T_{\max})$ is a connected topological space with the topology inherited from the standard one on real numbers. Since $\mathcal{C} > \mathcal{C}_0$, we have $0 \in \tau$, such that the set τ is nonempty. The set τ is closed in $[0, T_{\max})$ because $A(t)$ is continuous. The continuity of $A(t)$ is a consequence of the continuity of $u(t)$ in $H^1(\mathbb{R}^3)$ and the continuity of the projection operator onto ψ in $L^2(\mathbb{R}^3)$. In addition, we prove in Section 5 that the set τ is open in $[0, T_{\max})$, see Lemma 5.3. Since the set τ is both open and closed nonempty subset of the connected space $[0, T_{\max})$ we have

$$\tau = [0, T_{\max}). \tag{3.16}$$

Finally, we prove in Section 6 that there exists a positive constant $\mathcal{E}$, such that

$$\|u(t)\|_{H^1} \leq \mathcal{E} \log(1 + 4\Gamma\varepsilon^4 t), \qquad \text{for } t \in \tau, \tag{3.17}$$

see Lemma 6.1. The blow up alternative (2.6) and the estimate (3.17) combined with the closure relation (3.16) imply that $T_{\max} = \infty$. Consequently, the solution $u(t)$ is globally defined and the decay estimate (1.12) holds. The decay estimates (1.13)–(1.14) follow from Lemma 4.2 by taking T be any number in $\tau = [0, \infty)$. Hence, the decay estimates (1.13)–(1.14) hold true for any $t \geq 0$. ∎

Some remarks follow from the proof of Theorem 1. The parameter ε measures the size of the initial condition. On the time scale $0 < t < (4\Gamma\varepsilon^4)^{-1}$, the estimates (1.12)–(1.14) show that the amplitude $|A(t)|$ remains nearly constant, order of $\mathrm{O}(\varepsilon)$, while the dispersive part $u_d(t)$ shrinks to the size $u_d(t) \sim \varepsilon^3$. As a result, the initial stage of the dynamics is a fast relaxation of the initial value u_0 to the slowly varying bound state $e^{-i\lambda t}A(t)\psi$. Beyond this time scale, for $t > (4\Gamma\varepsilon^4)^{-1}$, the decay of the bound state occurs, according to decay rate (1.12). As a result, the solution $u(t)$ decays to zero as $t \to \infty$. Since

$$\Gamma = \frac{\gamma_1^2}{4} \mathrm{Im}\langle \psi^3, (\mathcal{H} - \lambda - \omega - i0)^{-1} \mathcal{P}_c \psi^3 \rangle = \frac{\pi\gamma_1^2}{4} \langle \psi^3, E'(\lambda + \omega)\psi^3 \rangle \searrow 0 \quad \text{as } \omega \to \infty,$$

the decay time scale diverges in the homogeneous limit $\omega \gg 1$. As a result, the solution $u(t)$ approaches a non-decaying bound state on intermediate time scales in the limit $\omega \gg 1$, as shown recently in [13, 14].

4 Reduction to the integro-differential equation

We fix here an arbitrary time instance $T \in \tau$ and focus on the dispersive equation (3.3) in the integral form (3.11)–(3.12). Main results of this section are given by Lemmas 4.1 and 4.2. In order to define the domain and range of the nonlinear operator K_A, we consider two Banach spaces X and Y, where

$$X = C([0, T], L^2_{-\sigma} \cap L^4 \cap L^2), \tag{4.1}$$

such that

$$\|v\|_X = \max \left\{ \sup_{t \in [0,T]} (1+t)^{3/2} \|v(t)\|_{L^2_{-\sigma}}, \sup_{t \in [0,T]} (1+t)^{3/4} \|v(t)\|_{L^4}, \sup_{t \in [0,T]} \|v(t)\|_{L^2} \right\},$$

and

$$Y = C([0, T], L^2_{-\sigma} \cap L^4), \tag{4.2}$$

such that

$$\|v\|_Y = \max \left\{ \sup_{t \in [0,T]} (1 + 4\Gamma\varepsilon^4 t)^{3/4} \|v(t)\|_{L^2_{-\sigma}}, \sup_{t \in [0,T]} (1 + 4\Gamma\varepsilon^4 t)^{1/2} \|v(t)\|_{L^4} \right\}.$$

We will be using the convolution estimates:

$$\int_0^t (1+t-s)^{-b}(1+\epsilon s)^{-a} ds \leq \left\{ \begin{array}{ll} D(1+\epsilon t)^{-\min(a,b)} & \text{if } b>1 \text{ or } a>1 \\ D\epsilon^{b-1}(1+\epsilon t)^{1-a-b} & \text{if } 0<b<1 \text{ and } 0<a<1 \end{array} \right. \tag{4.3}$$

and

$$\int_0^t (t-s)^{-b}(1+\epsilon s)^{-a} ds \leq \left\{ \begin{array}{ll} D\epsilon^{b-1}(1+\epsilon t)^{-\min(a,b)} & \text{if } 0<b<1 \text{ and } a>1 \\ D\epsilon^{b-1}(1+\epsilon t)^{1-a-b} & \text{if } 0<b<1 \text{ and } 0<a<1 \end{array} \right. \tag{4.4}$$

which are valid for $0 < \epsilon \leq 1$ and $t > 0$. Using the decay estimates (3.6)–(3.7), the unitarity of $e^{-i\mathcal{H}t}$, $t \in \mathbb{R}$ on $L^2(\mathbb{R}^3)$ and the convolution estimates (4.3)–(4.4), it is straightforward to show that the forcing terms in the integral equation (3.11) satisfy:

$$\begin{aligned} e^{-i\mathcal{H}t} u_d(0) &\in X && (4.5) \\ -i\int_0^t \gamma(s)|A|^2 A(s) e^{-i\lambda s} e^{-i\mathcal{H}(t-s)} \mathcal{P}_c \psi^3 ds &\in Y. && (4.6) \end{aligned}$$

Moreover X is continuously embedded in Y. We assume without loss of generality that $\varepsilon_0 \leq (4\Gamma)^{-1/4}$. Due to the bound (1.10), we have

$$4\Gamma\varepsilon^4 \leq 1 \quad \text{and} \quad \|\cdot\|_X \leq \|\cdot\|_Y.$$

In order to solve the integral equation (3.11) by using a local contraction principle, we obtain the Lipschitz properties of the nonlinear operator K_A.

Lemma 4.1. *The operator K_A is a locally Lipschitz operator on both X and Y. More precisely, there exist T-independent positive constants $\mathcal{D}_1$,$\mathcal{D}_2$, which depend only on Γ, $\sup_{t\in\mathbb{R}}|\gamma(t)|$ and the dispersion constants D_1–D_3 in (3.6)–(3.8), such that the following estimates are met:* $\forall \mathcal{R}_1 > 0$, $\forall u, v \in X: \|u\|_X, \|v\|_X \leq \mathcal{R}_1\varepsilon$,

$$\|K_A[u] - K_A[v]\|_X \leq \mathcal{D}_1(\mathcal{C}+\mathcal{R}_1)^2\varepsilon^2\|u-v\|_X, \tag{4.7}$$

$\forall \mathcal{R}_2 > 0$, $\forall u, v \in Y: \|u\|_Y, \|v\|_Y \leq \mathcal{R}_2\varepsilon$,

$$\|K_A[u] - K_A[v]\|_Y \leq \mathcal{D}_2(\mathcal{C}+\mathcal{R}_2)^2\varepsilon\|u-v\|_Y, \tag{4.8}$$

where $\mathcal{C}$ is defined in (3.14).

Proof. Since $K_A[0] = 0$, the invariance of X and Y under K_A will follow from the Lipschitz estimates (4.7)–(4.8). Let u, $v \in X$. For any fixed small $\varepsilon > 0$, there exists large $\mathcal{R}_1 > 0$ such that $\|u\|_X, \|v\|_X \leq \mathcal{R}_1\varepsilon$. Using the definition (3.12), we derive the following estimate in $L^2_{-\sigma}(\mathbb{R}^3)$:

$$\begin{aligned} & \|K_A[u](t) - K_A[v](t)\|_{L^2_{-\sigma}} && (4.9) \\ \leq\ & 3\sup_{0\leq s\leq t}|\gamma(s)|\int_0^t |A|^2(s)\|e^{-i\mathcal{H}(t-s)}\mathcal{P}_c\psi^2|u(s)-v(s)|\,\|_{L^2_{-\sigma}} ds \\ +\ & 3\sup_{0\leq s\leq t}|\gamma(s)|\int_0^t |A(s)|\ \|e^{-i\mathcal{H}(t-s)}\mathcal{P}_c\psi(|u(s)|+|v(s)|)\ |u(s)-v(s)|\,\|_{L^2_{-\sigma}} ds \\ +\ & \sup_{0\leq s\leq t}|\gamma(s)|\int_0^t \|e^{-i\mathcal{H}(t-s)}\mathcal{P}_c(|u(s)|^2+|u(s)v(s)|+|v(s)|^2)\ |u(s)-v(s)|\,\|_{L^2_{-\sigma}} ds. \end{aligned}$$

By the dispersive estimate (3.6) and Hölder inequalities we have for any $0 \leq s \leq t \leq T$:

$$\begin{aligned}
\| e^{-i\mathcal{H}(t-s)} \mathcal{P}_c \psi^2 |u(s) - v(s)| \, \|_{L^2_{-\sigma}} &\leq \frac{D_1}{(1+t-s)^{3/2}} \|(1+|x|^2)^{\sigma/2} \psi^2 |u(s)-v(s)| \, \|_{L^2} \\
&\leq \frac{D_1 \|(1+|x|^2)^{\sigma} \psi^2\|_{L^\infty}}{(1+t-s)^{3/2}} \|u(s)-v(s)\|_{L^2_{-\sigma}} \\
&\leq \frac{D_1 \|(1+|x|^2)^{\sigma} \psi^2\|_{L^\infty}}{(1+t-s)^{3/2}(1+s)^{3/2}} \|u-v\|_X,
\end{aligned}$$

and, similarly:

$$\begin{aligned}
& \| e^{-i\mathcal{H}(t-s)} \mathcal{P}_c \psi (|u(s)| + |v(s)|) \; |u(s) - v(s)| \, \|_{L^2_{-\sigma}} \\
\leq \; & \frac{D_1}{(1+t-s)^{3/2}} \|(1+|x|^2)^{\sigma/2} \psi (|u(s)| + |v(s)|) |u(s)-v(s)| \, \|_{L^2} \\
\leq \; & \frac{D_1 \|(1+|x|^2)^{\sigma/2} \psi\|_{L^\infty}}{(1+t-s)^{3/2}} \|u+v\|_{L^4} \|u-v\|_{L^4} \\
\leq \; & \frac{2 D_1 \mathcal{R}_1 \varepsilon \|(1+|x|^2)^{\sigma/2} \psi\|_{L^\infty}}{(1+t-s)^{3/2}(1+s)^{3/2}} \|u-v\|_X.
\end{aligned}$$

For the third term in the estimate (4.9), we avoid putting a weight like $(1+|x|^2)^{\sigma/2}$, $\sigma > 0$ on u or v and switch to L^p type estimates. This technical modification is needed due to the fact that if $u(t) \in H^1(\mathbb{R}^3)$ is the solution of the NLS equation (1.1), then $\|u(t)\|_{L^2_\sigma}$ grows in time, according to Theorem 1. Using the dispersive estimate (3.8), we have for any $0 \leq s \leq t \leq T$:

$$\begin{aligned}
& \| e^{-i\mathcal{H}(t-s)} \mathcal{P}_c (|u(s)|^2 + |u(s) v(s)| + |v(s)|^2) \; |u(s) - v(s)| \, \|_{L^2_{-\sigma}} \\
\leq \; & \|(1+|x|^2)^{-\sigma}\|_{L^2} \| e^{-i\mathcal{H}(t-s)} \mathcal{P}_c (|u(s)|^2 + |u(s) v(s)| + |v(s)|^2) |u(s) - v(s)| \, \|_{L^\infty} \\
\leq \; & \frac{3 D_3}{2(t-s)^{3/2}} \| |u|^2 + |v|^2 \|_{L^1} \|u-v\|_{L^2} \\
\leq \; & \frac{3 D_3 \mathcal{R}_1^2 \varepsilon^2}{(t-s)^{3/2}(1+s)^{3/2}} \|u-v\|_X.
\end{aligned}$$

It follows from (3.14) that $|A(s)| \leq \mathcal{C}\varepsilon$ for $0 \leq s \leq T$. As a result, the above estimates can be combined as follows:

$$\begin{aligned}
& \|K_A[u](t) - K_A[v](t)\|_{L^2_{-\sigma}} \qquad (4.10) \\
\leq \; & 3 \sup_{0 \leq s \leq t} |\gamma(s)| D_1 \mathcal{C}^2 \varepsilon^2 \int_0^t (1+t-s)^{-3/2} (1+s)^{-3/2} ds \|u-v\|_X \\
+ \; & 6 \sup_{0 \leq s \leq t} |\gamma(s)| D_1 \mathcal{C} \mathcal{R}_1 \varepsilon^2 \int_0^t (1+t-s)^{-3/2} (1+s)^{-3/2} ds \|u-v\|_X \\
+ \; & 3 \sup_{0 \leq s \leq t} |\gamma(s)| D_3 \mathcal{R}_1^2 \varepsilon^2 \int_0^{t_1} (t-s)^{-3/2} (1+s)^{-3/2} ds \|u-v\|_X \\
+ \; & \sup_{0 \leq s \leq t} |\gamma(s)| \int_{t_1}^t \| e^{-i\mathcal{H}(t-s)} \mathcal{P}_c (|u(s)|^2 + |u(s) v(s)| + |v(s)|^2) \; |u(s) - v(s)| \, \|_{L^2_{-\sigma}} ds,
\end{aligned}$$

where $t_1 = \max(0, t-1)$. We split the last integral term in order to avoid the non-integrable singularity of $(t-s)^{-3/2}$ at $s=t$. The first three terms on the right-hand-side of (4.10) are estimated with the convolution inequality (4.3). The fourth term is estimated with the dispersive estimate (3.7) as follows:

$$\begin{aligned}
& \int_{t_1}^{t} \|e^{-i\mathcal{H}(t-s)}\mathcal{P}_c(|u(s)|^2 + |u(s)v(s)| + |v(s)|^2)\ |u(s)-v(s)|\ \|_{L^2_{-\sigma}} ds \\
\leq\ & \|(1+|x|^2)^{-\sigma/2}\|_{L^4} \int_{t_1}^{t} \|e^{-i\mathcal{H}(t-s)}\mathcal{P}_c(|u(s)|^2 + |u(s)v(s)| + |v(s)|^2)\ |u(s)-v(s)|\ \|_{L^4} ds \\
\leq\ & \frac{3D_2}{2} \int_{t_1}^{t} (t-s)^{-3/4} \|(|u(s)|^2 + |v(s)|^2)\ |u(s)-v(s)|\ \|_{L^{4/3}} ds \\
\leq\ & 3D_2 \mathcal{R}_1^2 \varepsilon^2 \int_{t_1}^{t} (t-s)^{-3/4}(1+s)^{-9/4} ds \|u-v\|_X \\
\leq\ & 3D_2 \mathcal{R}_1^2 \varepsilon^2 (1+t)^{-9/4} \|u-v\|_X.
\end{aligned}$$

Combining all the estimates, we have:

$$\|K_A[u](t) - K_A[v](t)\|_{L^2_{-\sigma}} \leq (1+t)^{-3/2} \tilde{D}_1 (\mathcal{C} + \mathcal{R}_1)^2 \varepsilon^2 \|u-v\|_X, \tag{4.11}$$

where the positive constant $\tilde{D}_1$ depends only on $\sup_{t\in\mathbb{R}} |\gamma(t)|$ and the dispersion constant D_1–D_3 in the decay estimates (3.6)–(3.8). For the L^4 estimates in space X, we proceed similarly to the previous computations:

$$\begin{aligned}
& \|K_A[u](t) - K_A[v](t)\|_{L^4} \qquad\qquad (4.12)\\
\leq\ & 3 \sup_{0\leq s\leq t} |\gamma(s)| D_2 \int_0^t |A|^2(s)(t-s)^{-3/4} \|\psi^2 |u(s)-v(s)|\ \|_{L^{4/3}} ds \\
+\ & 3 \sup_{0\leq s\leq t} |\gamma(s)| D_2 \int_0^t |A(s)|(t-s)^{-3/4} \|\psi(|u(s)|+|v(s)|)\ |u(s)-v(s)|\ \|_{L^{4/3}} ds \\
+\ & \sup_{0\leq s\leq t} |\gamma(s)| D_2 \int_0^t (t-s)^{-3/4} \|(|u(s)|^2 + |u(s)v(s)| + |v(s)|^2)\ |u(s)-v(s)|\ \|_{L^{4/3}} ds.
\end{aligned}$$

By Hölder inequalities:

$$\begin{aligned}
\|\psi^2 |u(s)-v(s)|\ \|_{L^{4/3}} &= \|(1+|x|^2)^{\sigma/2}\psi^2 (1+|x|^2)^{-\sigma/2} |u(s)-v(s)|\ \|_{L^{4/3}} \\
&\leq \|(1+|x|^2)^{\sigma/2}\psi^2\|_{L^4} \|u(s)-v(s)\|_{L^2_{-\sigma}} \\
&\leq \frac{\|(1+|x|^2)^{\sigma/2}\psi^2\|_{L^4}}{(1+s)^{3/2}} \|u-v\|_X,
\end{aligned}$$

$$\begin{aligned}
\|\psi(|u(s)|+|v(s)|)|u(s)-v(s)|\|_{L^{4/3}} &\leq \|(1+|x|^2)^{\sigma/2}\psi\|_{L^\infty} \||u(s)|+|v(s)|\|_{L^4} \|u(s)-v(s)\|_{L^2_{-\sigma}} \\
&\leq \frac{2\mathcal{R}_1\varepsilon}{(1+s)^{9/4}} \|(1+|x|^2)^{\sigma/2}\psi\|_{L^\infty} \|u-v\|_X,
\end{aligned}$$

and

$$\begin{aligned}
\|(|u(s)|^2+|u(s)v(s)|+|v(s)|^2)|u(s)-v(s)|\|_{L^{4/3}} &\leq \frac{3}{2}\||u(s)|^2+|v(s)|^2\|_{L^2}\|u(s)-v(s)\|_{L^4} \\
&\leq \frac{3\max\left(\|u(s)\|_{L^4}^2,\|v(s)\|_{L^4}^2\right)}{(1+s)^{3/4}}\|u-v\|_X \\
&\leq \frac{3\mathcal{R}_1^2\varepsilon^2}{(1+s)^{9/4}}\|u-v\|_X.
\end{aligned}$$

By plugging the above estimates in (4.12) and using the convolution inequality (4.4) together with $|A(s)|\leq \mathcal{C}\varepsilon$ for $0\leq s\leq T$, we obtain:

$$\|K_A[u](t)-K_A[v](t)\|_{L^4}\leq (1+t)^{-3/4}\tilde{D}_2(\mathcal{C}+\mathcal{R}_1)^2\varepsilon^2\|u-v\|_X \tag{4.13}$$

where the positive constant $\tilde{D}_2$ depends only on $\sup_{t\in\mathbb{R}}|\gamma(t)|$ and the dispersion constants D_1–D_3 in the decay estimates (3.6)–(3.8). For the L^2 estimate we proceed as in (4.9) and (4.12) but, for the first two terms, we use the unitarity of $e^{-i\mathcal{H}t}$, $t\in\mathbb{R}$ on $L^2(\mathbb{R}^3)$:

$$\begin{aligned}
&\|K_A[u](t)-K_A[v](t)\|_{L^2} \qquad\qquad (4.14)\\
\leq\;& 3\sup_{0\leq s\leq t}|\gamma(s)|\int_0^t |A|^2(s)\|\psi^2|u(s)-v(s)|\;\|_{L^2}ds \\
+\;& 3\sup_{0\leq s\leq t}|\gamma(s)|\int_0^t |A(s)|\|\psi(|u(s)|+|v(s)|)\;|u(s)-v(s)|\;\|_{L^2}ds \\
+\;& \left\|\int_0^t \gamma(s)e^{-i\mathcal{H}(t-s)}\mathcal{P}_c(|u(s)|^2u(s)-|v(s)|^2v(s))ds\right\|_{L^2}.
\end{aligned}$$

By the Hölder inequalities:

$$\begin{aligned}
\|\psi^2|u(s)-v(s)|\;\|_{L^2} &\leq \|(1+|x|^2)^{\sigma/2}\psi^2\|_{L^\infty}\|u(s)-v(s)\|_{L^2_{-\sigma}} \\
&\leq \frac{\|(1+|x|^2)^{\sigma/2}\psi^2\|_{L^\infty}}{(1+s)^{3/2}}\|u-v\|_X
\end{aligned}$$

and

$$\begin{aligned}
\|\psi(|u(s)|+|v(s)|)\;|u(s)-v(s)|\;\|_{L^2} &\leq \|\psi\|_{L^\infty}\|\;|u(s)|+|v(s)|\;\|_{L^4}\|u(s)-v(s)|\|_{L^4} \\
&\leq \frac{2\mathcal{R}_1\varepsilon\|\psi\|_{L^\infty}}{(1+s)^{3/2}}\|u-v\|_X,
\end{aligned}$$

the first two terms in (4.14) are bounded. For the last term, we denote

$$w(s)=|u(s)|^2u(s)-|v(s)|^2v(s)$$

and use the Strichartz estimates, see formula (2.3.5) in [1],

$$\left\|\int_0^t \gamma(s)e^{-i\mathcal{H}(t-s)}\mathcal{P}_c w(s)ds\right\|_{L^2}^2 \leq C\|w(s,x)\|_{L^{8/5}([0,t],L^{4/3})},$$

where C is a constant, such that

$$\begin{aligned}\|w(s)\|_{L^{4/3}} &\leq \frac{3}{2}\|(|u(s)|^2+|v(s)|^2)(|u(s)-|v(s)|)\|_{L^{4/3}} \\ &\leq \frac{3}{2}\|\ |u(s)|^2+|v(s)|^2\ \|_{L^2}\|u(s)-v(s)\|_{L^4} \\ &\leq \frac{3\mathcal{R}_1^2\varepsilon^2}{(1+s)^{9/4}}\|u-v\|_X\end{aligned}$$

and

$$\left\|\int_0^t \gamma(s)e^{-i\mathcal{H}(t-s)}\mathcal{P}_c w(s)ds\right\|_{L^2} \leq 3D_1\mathcal{R}_1^2\varepsilon^2 \sup_{0\leq s\leq t}|\gamma(s)|\,\|u-v\|_X\|(1+s)^{-9/4}\|_{L^{8/5}[0,t]}.$$

Therefore,

$$\left\|\int_0^t \gamma(s)e^{-i\mathcal{H}(t-s)}\mathcal{P}_c w(s)ds\right\|_{L^2} \leq 3D_1 \sup_{0\leq s\leq t}|\gamma(s)|\mathcal{R}_1^2\varepsilon^2\ \|u-v\|_X.$$

Using the above estimates in the bound (4.14) and taking into account that $|A(s)|\leq \mathcal{C}\varepsilon$ for $0\leq s\leq T$, we obtain:

$$\|K_A[u](t)-K_A[v](t)\|_{L^2} \leq \tilde{D}_3(\mathcal{C}+\mathcal{R}_1)^2\varepsilon^2\|u-v\|_X, \tag{4.15}$$

where the positive constant $\tilde{D}_3$ depends only on $\sup_{t\in\mathbb{R}}|\gamma(t)|$ and the dispersion constants D_1–D_3 in the decay estimates (3.6)–(3.8). Combining the estimates (4.11), (4.13), and (4.15), we obtain the estimate (4.7), where $\mathcal{D}_1\geq\max\{\tilde{D}_1,\tilde{D}_2,\tilde{D}_3\}$.

Let u, $v\in Y$. For any fixed small $\varepsilon>0$, there exists large $\mathcal{R}_2>0$, such that $\|u\|_Y$, $\|v\|_Y\leq\mathcal{R}_2\varepsilon$. The proof of the estimate (4.8) follows closely that of the estimate (4.7) with the simplification that the space Y requires smaller decay in time. On the other hand, it also requires that $|A(t)|$ decays on $[0,T]$, which is not necessary in the proof of the bound (4.7). Similarly to the estimate (4.9), we use the dispersive estimate (3.6), the definition (3.12), and the decay estimate (3.14) and obtain the following estimate in $L^2_{-\sigma}(\mathbb{R}^3)$:

$$\begin{aligned}&\|K_A[u](t)-K_A[v](t)\|_{L^2_{-\sigma}} \\ \leq\ & 3\sup_{0\leq s\leq t}|\gamma(s)|\int_0^t \frac{\mathcal{C}^2\varepsilon^2}{(1+4\Gamma\varepsilon^4 s)^{1/2}}\frac{D_1}{(1+t-s)^{3/2}}\|\psi^2(u(s)-v(s))\ \|_{L^2_\sigma}ds \\ +\ & 3\sup_{0\leq s\leq t}|\gamma(s)|\int_0^t \frac{\mathcal{C}\varepsilon}{(1+4\Gamma\varepsilon^4 s)^{1/4}}\frac{D_1}{(1+t-s)^{3/2}}\|\psi(u^2(s)-v^2(s))\|_{L^2_\sigma}ds \\ +\ & \sup_{0\leq s\leq t}|\gamma(s)|\int_0^t \|e^{-i\mathcal{H}(t-s)}\mathcal{P}_c(|u(s)|^2+|u(s)v(s)|+|v(s)|^2)\ |u(s)-v(s)|\ \|_{L^2_{-\sigma}}ds.\end{aligned} \tag{4.16}$$

The first two integrals are bounded with the Hölder estimates and the properties of functions in Y:

$$\begin{aligned}\|\psi^2(u(s)-v(s))\|_{L^2_\sigma} &\leq \|(1+|x|^2)^\sigma\psi^2\|_{L^\infty}\|u(s)-v(s)\|_{L^2_{-\sigma}} \\ &\leq \frac{\|(1+|x|^2)^\sigma\psi^2\|_{L^\infty}}{(1+4\Gamma\varepsilon^4 s)^{3/4}}\|u-v\|_Y\end{aligned}$$

and

$$\begin{aligned}\|\psi(u^2(s)-v^2(s))\|_{L^2_\sigma} &\leq \|(1+|x|^2)^{\sigma/2}\psi\|_{L^\infty}\| \ |u(s)|+|v(s)| \ \|_{L^4}\| \ |u(s)-v(s)| \ \|_{L^4} \\ &\leq \frac{2\mathcal{R}_2\varepsilon\|(1+|x|^2)^{\sigma/2}\psi\|_{L^\infty}}{1+4\Gamma\varepsilon^4 s}\|u-v\|_Y.\end{aligned}$$

The last integral term is bounded with the estimates in $L^4(\mathbb{R}^3)$:

$$\begin{aligned}&\|e^{-i\mathcal{H}(t-s)}\mathcal{P}_c(|u(s)|^2+|u(s)v(s)|+|v(s)|^2) \ |u(s)-v(s)| \ \|_{L^2_{-\sigma}} \\ \leq \ & \|(1+|x|^2)^{-\sigma}\|_{L^4}\|e^{-i\mathcal{H}(t-s)}\mathcal{P}_c(|u(s)|^2+|u(s)v(s)|+|v(s)|^2)\|u(s)-v(s)| \ \|_{L^{4/3}} \\ \leq \ & \frac{3D_2}{2(t-s)^{3/4}}\||u|^2+|v|^2\|_{L^2}\|u-v\|_{L^4} \\ \leq \ & \frac{3D_2\mathcal{R}_2^2\varepsilon^2}{(t-s)^{3/4}(1+4\Gamma\varepsilon^4 s)^{3/2}}\|u-v\|_Y.\end{aligned}$$

Plugging the above estimates into (4.16) and using the convolution inequalities (4.3)–(4.4) we obtain:

$$\|K_A[u](t)-K_A[v](t)\|_{L^2_{-\sigma}} \leq \tilde{D}_4(\mathcal{C}+\mathcal{R}_2)^2\varepsilon(1+4\Gamma\varepsilon^4 t)^{-3/4}\|u-v\|_Y, \tag{4.17}$$

where the positive constant $\tilde{D}_4$ depends only on Γ, $\sup_{t\in\mathbb{R}}|\gamma(t)|$ and the dispersion constants D_1–D_3 in the decay estimates (3.6)–(3.8).

The estimates in X, given in (4.12)–(4.13), extend to similar estimates in Y. Although the functions u, v decay slowly in time but the slow decay is compensated by the decay in $|A(s)|$, given by (3.14). As a result, we have:

$$\|K_A[u](t)-K_A[v](t)\|_{L^4} \leq \tilde{D}_5(\mathcal{C}+\mathcal{R}_2)^2\varepsilon(1+t)^{-1/2}\|u-v\|_Y, \tag{4.18}$$

where the positive constant $\tilde{D}_5$ depends on Γ, $\sup_{t\in\mathbb{R}}|\gamma(t)|$, and the dispersion constant D_1–D_3 in the decay estimates (3.6)–(3.8). Combining (4.17) and (4.18), we find the estimate (4.8), where $\mathcal{D}_2 \geq \max\{\tilde{D}_4, \tilde{D}_5\}$. ☐

In Appendix A we extend the classical principle of superposition from forced linear equations to the fixed points of nonlinear, locally contractive, nonhomogeneous operators in a hierarchy of Banach spaces. In the particular case of the integral equation (3.11), this principle implies that the solution $u_d(t)$ can be written as a sum of a function in X and a function in Y. The former is the response to the first forcing term and, although it is relatively large at the beginning as $O(\varepsilon)$, it decays fast in time. The latter can be interpreted as the response to the second forcing term and, although it is relatively small at the beginning as $O(\varepsilon^3)$, it decays very slowly in time and eventually dominates. Because the two forcing terms have different initial sizes and evolve on different time scales, separation of each term is essential in the proof of our main result. We note that modulation equations were used in [22, 23, 25] to separate the time scales between these two terms. We completely avoid using the modulation equations by relying on the contraction principle of Appendix A.

Lemma 4.2. *There exists a T-independent positive constant $\mathcal{D}$, which depends only on Γ, $\sup_{t\in\mathbb{R}}|\gamma(t)|$, and the dispersion constants D_1–D_3 in the decay estimates (3.6)–(3.8), such* *that the solution $u_d(t)$ of the integral equation (3.11) belongs to Y for*

$$\varepsilon < \varepsilon_0 \leq \frac{1}{2}\mathcal{D}^{-1}\mathcal{C}^{-3}. \tag{4.19}$$

Moreover, the solution $u_d(t)$ can be written as:

$$u_d(t) = f(t) + v_d(t) + E_d(t) = e^{-i\mathcal{H}t}u_d(0) + f(t) + K_A[f + v_d](t) + F_d(t), \tag{4.20}$$

where

$$f(t) = -i\int_0^t \gamma(s)|A|^2 A(s) e^{-i\lambda s} e^{-i\mathcal{H}(t-s)}\mathcal{P}_c\psi^3 ds \tag{4.21}$$

and

$$\|v_d\|_X \leq \mathcal{D}\varepsilon, \qquad \|E_d\|_Y \leq \mathcal{D}\mathcal{C}^5\varepsilon^3, \qquad \|F_d\|_Y \leq \mathcal{D}\mathcal{C}^7\varepsilon^4. \tag{4.22}$$

Proof. We apply Proposition A.2 from Appendix A to the integral equations (3.11) and

$$v_d(t) = e^{-i\mathcal{H}t}u_d(0) + K_A[v_d](t). \tag{4.23}$$

Hypothesis (A.1), (A.2) and (A.11) hold because of Lemma 4.1. To verify (A.12)–(A.16), we fix $L_1 = L_2 = 1/2$. Due to inclusions (4.5)–(4.6), there exist constants $\mathcal{D}_3$ and $\mathcal{D}_4$, such that

$$\begin{aligned}\|e^{-i\mathcal{H}t}u_d(0)\|_X &\leq \mathcal{D}_3\varepsilon \\ \|e^{-i\mathcal{H}t}u_d(0) + f(t)\|_Y &\leq \|e^{-i\mathcal{H}t}u_d(0)\|_Y + \|f(t)\|_Y \leq \mathcal{D}_4\varepsilon(1 + \mathcal{C}^3\max\{\varepsilon\Gamma^{-1/4}, \varepsilon^2\}).\end{aligned}$$

Let $R_1 = 2\mathcal{D}_3\varepsilon$ and $R_2 = 3\mathcal{D}_4\varepsilon$. Then, the equations (A.13), (A.15) and (A.16) hold with the above choices of $L_{1,2}$ and $R_{1,2}$, provided that $\mathcal{D} \geq \max\{\Gamma^{-1/4}, 2^{-1/2}\}$. To simplify the expression of the Lipschitz constants in the bounds (4.7)–(4.8) we assume that $\mathcal{D} \geq \max\{2\mathcal{D}_3, 3\mathcal{D}_4\}$ and $\mathcal{C} > \mathcal{D}$, according to the inequality (3.15). The Lipschitz constants for $K_A[u]$ in the balls of radius R_1 for X and R_2 for Y are dominated by:

$$L_X(R_1) \leq 4\mathcal{D}_3\mathcal{C}^2\varepsilon^2, \qquad L_Y(R_2) \leq 4\mathcal{D}_4\mathcal{C}^2\varepsilon. \tag{4.24}$$

Conditions (A.12) and (A.14) with $L_1 = L_2 = 1/2$ hold provided that the bound (4.19) holds. With the above constraints, Proposition A.2 is now applied to our case, when

$$\|v_d\|_X \leq 2\mathcal{D}_3\varepsilon, \qquad \|f(t)\|_Y \leq \mathcal{D}_4\Gamma^{-1/4}\mathcal{C}^3\varepsilon^2. \tag{4.25}$$

For the Lipschitz constants, we can use (4.24) which are both less than 1/2. As a result, the estimates (4.22) follow directly from (A.19) and (A.21) provided that

$$\mathcal{D} \geq \max\{2\mathcal{D}_3, 8\mathcal{D}_1\mathcal{D}_4\Gamma^{-1/4}, 32\mathcal{D}_2^2\mathcal{D}_4\Gamma^{-1/4}\}, \tag{4.26}$$

where the constants $\mathcal{D}_{1,2}$ are defined in the bounds (4.7)–(4.8). □

Corollary 4.1. *The solution $u_d(t)$ of the integral equation (3.11) satisfies the decay estimates for $0 \leq t \leq T$:*

$$\|u_d(t) - f(t)\|_{L^2_{-\sigma}} \leq \frac{\mathcal{D}\varepsilon}{(1+t)^{3/2}} + \frac{\mathcal{D}\mathcal{C}^7\varepsilon^4}{(1+4\Gamma\varepsilon^4 t)^{3/4}} \tag{4.27}$$

$$\|u_d(t)\|_{L^2_{-\sigma}} \leq \frac{\mathcal{D}\varepsilon}{(1+t)^{3/2}} + \frac{\mathcal{D}\mathcal{C}^5\varepsilon^3}{(1+4\Gamma\varepsilon^4 t)^{3/4}} \tag{4.28}$$

$$\|u_d(t)\|_{L^4} \leq \frac{\mathcal{D}\varepsilon}{(1+t)^{3/4}} + \frac{\mathcal{D}\mathcal{C}^3\varepsilon^2}{(1+4\Gamma\varepsilon^4 t)^{1/2}}. \tag{4.29}$$

Proof. The bounds (4.28) and (4.29) follow from (4.20), (4.22), and (4.25). The bound (4.27) is obtained from (4.20) and (4.23) by direct calculation of $\|K_A[v_d + f]\|_{L^2_{-\sigma}}$ that uses the estimates (4.25) and the argument for the $L^2_{-\sigma}$ norms in Lemma 4.1. $\square$

5 Reduction to the amplitude equation

We analyze here the amplitude equation (3.2), which shows explicitly the decay of $A(t)$ due to the coupling with the dispersive part $u_d(t)$. In what follows $\mathcal{D}$, $\mathcal{D}_1$, $\mathcal{D}_2$, etc. denote T-independent positive constants that depend only on Γ, $\sup_{t\in\mathbb{R}}|\gamma(t)|$, and the dispersion constants D_1–D_4 in the decay estimates (3.6)–(3.9). In general, these constants are slightly bigger than the constant $\mathcal{D}$, defined in Lemma 4.2. Main results of this section are given by Lemmas 5.1, 5.2, and 5.3.

We eliminate the fast oscillations in the phase of $A(t)$ in the amplitude equation (3.2) with the cubic term (3.4):

$$\begin{aligned} \frac{d|A|^2}{dt} &= 2\gamma(t)|A|^2(t)\ \mathrm{Im}\left[\overline{A}(t)e^{i\lambda t}\langle\psi^3, u_d(t)\rangle\right] \\ &+ 2\gamma(t)\ \mathrm{Im}\left[\overline{A}^2(t)e^{2i\lambda t}\langle\psi^2, u_d^2(t)\rangle\right] \\ &+ 2\gamma(t)\ \mathrm{Im}\left[\overline{A}(t)e^{i\lambda t}\langle\psi, |u_d|^2 u_d(t)\rangle\right]. \end{aligned} \tag{5.1}$$

The leading-order contribution of $u_d(t)$ appears in the first term on the right hand side of (5.1). Let

$$g(t) = \frac{\gamma(t)}{|A(t)|}\ \mathrm{Im}\left[\overline{A}^2(t)e^{2i\lambda t}\langle\psi^2, u_d^2(t)\rangle\right] + \frac{\gamma(t)}{|A(t)|}\ \mathrm{Im}\left[\overline{A}(t)e^{i\lambda t}\langle\psi, |u_d|^2 u_d(t)\rangle\right], \tag{5.2}$$

and rewrite the amplitude equation (5.1) in the form:

$$\frac{d|A|}{dt} = \gamma(t)|A|(t)\ \mathrm{Im}\left[\overline{A}(t)e^{i\lambda t}\langle\psi^3, u_d(t)\rangle\right] + g(t). \tag{5.3}$$

By the Hölder estimates, we have for $t \in [0,T]$:

$$\begin{aligned} |\langle\psi^2, u_d^2(t)\rangle| &= |\langle(1+|x|)^{2\sigma}\psi^2, (1+|x|)^{-2\sigma}u_d^2(t)\rangle| \\ &\leq \|(1+|x|)^{\sigma}\psi\|^2_{L^\infty}\|u_d(t)\|^2_{L^2_{-\sigma}} \\ &\leq \frac{\mathcal{D}_1\varepsilon^2}{(1+t)^3} + \frac{\mathcal{D}_1\mathcal{C}^{10}\varepsilon^6}{(1+4\Gamma\varepsilon^4 t)^{3/2}}, \end{aligned}$$

where, for the last inequality, we have used the estimate (4.28), as well as the inequality:

$$\forall a, b > 0: \quad (a+b)^2 \leq 2(a^2+b^2). \tag{5.4}$$

Similarly,

$$\begin{aligned} |\langle\psi, |u_d|^2 u_d(t)\rangle| &= |\langle(1+|x|)^{\sigma}\psi, (1+|x|)^{-\sigma}u_d(t)|u_d|^2(t)\rangle| \\ &\leq \|(1+|x|)^{\sigma}\psi\|_{L^\infty}\|u_d(t)\|_{L^2_{-\sigma}}\|u_d(t)\|^2_{L^4} \\ &\leq \frac{\mathcal{D}_2\varepsilon^2}{(1+t)^{12/5}} + \frac{\mathcal{D}_2\mathcal{C}^{12}\varepsilon^7}{(1+4\Gamma\varepsilon^4 t)^{7/4}}, \end{aligned}$$

where, for the last inequality, we have used the estimates (4.28)–(4.29), as well as:

$$\forall a,b,c,d>0: \quad (a+b)(c+d)^2 \le 2^{4/3}\frac{a^{7/3}+b^{7/3}}{7/3} + 2^{5/2}\frac{c^{7/2}+d^{7/2}}{7/2}. \tag{5.5}$$

The inequality (5.5) can be obtained from Young and Hölder inequalities. Taking into account the decay of $|A(t)|$ in time, we obtain the estimate for $t \in [0,T]$:

$$|g(t)| \le \frac{\mathcal{D}_3\varepsilon^2}{(1+t)^{12/5}} + \frac{\mathcal{D}_3\mathcal{C}^{12}\varepsilon^7}{(1+4\Gamma\varepsilon^4 t)^{7/4}}. \tag{5.6}$$

Let $u_d(t) = f(t) + (u_d(t) - f(t))$, where $f(t)$ is the forcing term from the bound state, given in (4.21). We rewrite the amplitude equation (5.3) in the form:

$$\frac{d|A|}{dt} = \gamma(t)|A|(t)\ \mathrm{Im}\left[\overline{A}(t)e^{i\lambda t}\langle\psi^3, f(t)\rangle\right] + h(t), \tag{5.7}$$

where, due to the estimates (4.27) and (5.6), the correction term $h(t)$ satisfies the estimate for $t \in [0,T]$:

$$|h(t)| \le \frac{\mathcal{D}_4\mathcal{C}^2\varepsilon^3}{(1+t)^{3/2}(1+4\Gamma\varepsilon^4 t)^{1/2}} + \frac{\mathcal{D}_4\mathcal{C}^9\varepsilon^6}{(1+4\Gamma\varepsilon^4 t)^{5/4}} + \frac{\mathcal{D}_4\varepsilon^2}{(1+t)^{12/5}} + \frac{\mathcal{D}_4\mathcal{C}^{12}\varepsilon^7}{(1+4\Gamma\varepsilon^4 t)^{7/4}}. \tag{5.8}$$

The effect of $f(t)$ in the amplitude equation (5.7) becomes evident if we integrate by parts once. For rigorous computations, we use weighted L^2 spaces with weights $w_\pm = (1+|x|)^{\pm\sigma}$. Using notations:

$$\gamma(t) = \gamma_0 + \gamma_1\cos(\omega t) = \sum_{j=-1}^{1}\gamma^j e^{-i\omega_j t},$$

where $\gamma^0 = \gamma_0$, $\gamma^{\pm 1} = \gamma_1/2$, $\omega_0 = 0$, and $\omega_{\pm 1} = \pm|\omega|$, we compute the inner product in (5.7) as follows:

$$\begin{aligned}
\langle\psi^3, f(t)\rangle &= -i\langle\psi^3, \int_0^t \gamma(s)|A|^2A(s)e^{-i\lambda s}e^{-i\mathcal{H}(t-s)}\mathcal{P}_c\psi^3 ds\rangle \\
&= -i\sum_{j=-1}^{1}\gamma^j\langle w_+\psi^3, \int_0^t |A|^2A(s)w_-e^{-i(\lambda+\omega_j)s}e^{-i\mathcal{H}(t-s)}\mathcal{P}_c\psi^3 ds\rangle \\
&= -|A|^2A(s)\sum_{j=-1}^{1}\gamma^j\langle w_+\psi^3, w_-e^{-i\mathcal{H}t}e^{i(\mathcal{H}-\lambda-\omega_j)s}(\mathcal{H}-\lambda-\omega_j-i0)^{-1}\mathcal{P}_c\psi^3\rangle\Big|_{s=0}^{s=t} \\
&+ \sum_{j=-1}^{1}\gamma^j\int_0^t \partial_s(|A|^2A(s))e^{-i(\lambda+\omega_j)s}\langle w_+\psi^3, w_-e^{-i\mathcal{H}(t-s)}(\mathcal{H}-\lambda-\omega_j-i0)^{-1}\mathcal{P}_c\psi^3\rangle ds \\
&= -|A|^2A(t)\sum_{j=-1}^{1}\gamma^j e^{-i(\lambda+\omega_j)t}\langle w_+\psi^3, w_-(\mathcal{H}-\lambda-\omega_j-i0)^{-1}\mathcal{P}_c\psi^3\rangle + \tilde{h}(t),
\end{aligned} \tag{5.9}$$

where

$$\tilde{h}(t) = |A|^2A(0)\sum_{j=-1}^{1}\gamma^j\langle w_+\psi^3, w_-e^{-i\mathcal{H}t}(\mathcal{H}-\lambda-\omega_j-i0)^{-1}\mathcal{P}_c\psi^3\rangle$$

$$+ \sum_{j=-1}^{1} \gamma^j \int_0^t \partial_s(|A|^2 A(s)) e^{-i(\lambda+\omega_j)s} \langle w_+\psi^3, w_- e^{-i\mathcal{H}(t-s)}(\mathcal{H}-\lambda-\omega_j - i0)^{-1}\mathcal{P}_c\psi^3\rangle ds.$$

In order to estimate $\tilde{h}(t)$, we note that:

$$|\partial_s(|A|^2 A(s))| \leq 3|A|^2(s)|\partial_s A(s)|$$

and, by virtue of the amplitude equation (3.2),

$$|\partial_s A(s)| = |\gamma(s)\langle \psi, \mathcal{P}_c|u|^2 u(s)\rangle| \leq \sup \gamma \|w_+\psi\|_{L^\infty}\|u\|_{L^2_{-\sigma}}\|u\|^2_{L^4} \tag{5.10}$$

Using the decomposition (3.1) and the estimates (3.14) and (4.28)–(4.29) for $A(s)$ and $u_d(s)$ on $s \in [0,T]$, we obtain:

$$|\partial_s(|A|^2 A(s))| \leq \frac{\mathcal{D}_5 \mathcal{C}^5 \varepsilon^5}{(1+4\Gamma\varepsilon^4 t)^{5/4}}. \tag{5.11}$$

Using the estimates (3.9) and (5.11), we obtain for $t \in [0,T]$:

$$|\tilde{h}(t)| \leq \frac{\mathcal{D}_6 \varepsilon^3}{(1+t)^{3/2}} + \frac{\mathcal{D}_6 \mathcal{C}^5 \varepsilon^5}{(1+4\Gamma\varepsilon^4 t)^{5/4}}. \tag{5.12}$$

After straightforward calculations, the amplitude equation (5.7) simplifies to the form:

$$\frac{d|A|}{dt} = (-\Gamma + \rho(t))|A(t)|^5 + \hat{h}(t) \tag{5.13}$$

where the positive parameter Γ is given by (1.9), the function $\rho(t)$ is periodic with mean zero:

$$\begin{aligned}
\rho(t) &= -\cos(2\omega t)\frac{\gamma_1^2}{4}\langle \psi^3, \mathrm{Im}(\mathcal{H}-\lambda-\omega-i0)^{-1}\mathcal{P}_c\psi^3\rangle \\
&+ \sin(2\omega t)\frac{\gamma_1^2}{4}\left[\langle \psi^3, \mathrm{Re}(\mathcal{H}-\lambda-\omega-i0)^{-1}\mathcal{P}_c\psi^3\rangle - \langle \psi^3, (\mathcal{H}-\lambda+\omega)^{-1}\mathcal{P}_c\psi^3\rangle\right] \\
&- \cos(\omega t)\frac{\gamma_0\gamma_1}{2}\langle \psi^3, \mathrm{Im}(\mathcal{H}-\lambda-\omega-i0)^{-1}\mathcal{P}_c\psi^3\rangle \\
&+ \sin(\omega t)\frac{\gamma_0\gamma_1}{2}\left[\langle \psi^3, \mathrm{Re}(\mathcal{H}-\lambda-\omega-i0)^{-1}\mathcal{P}_c\psi^3\rangle - \langle \psi^3, (\mathcal{H}-\lambda+\omega)^{-1}\mathcal{P}_c\psi^3\rangle\right]
\end{aligned} \tag{5.14}$$

and the function $\hat{h}(t)$ is given by:

$$\hat{h}(t) = h(t) + \gamma(t)|A|(t)\ \mathrm{Im}\left[\overline{A}(t)e^{i\lambda t}\tilde{h}(t)\right]. \tag{5.15}$$

Due to the estimates (5.8) and (5.12), the function $\hat{h}(t)$ is bounded for $t \in [0,T]$ by:

$$\begin{aligned}
|\hat{h}(t)| \leq R(t) &= \frac{\mathcal{D}_7\mathcal{C}^2\varepsilon^3}{(1+t)^{3/2}(1+4\Gamma\varepsilon^4 t)^{1/2}} + \frac{\mathcal{D}_7\mathcal{C}^9\varepsilon^6}{(1+4\Gamma\varepsilon^4 t)^{5/4}} \\
&+ \frac{\mathcal{D}_7\varepsilon^2}{(1+t)^{12/5}} + \frac{\mathcal{D}_7\mathcal{C}^{12}\varepsilon^7}{(1+4\Gamma\varepsilon^4 t)^{7/4}}.
\end{aligned} \tag{5.16}$$

We note that an equivalent way of writing $\Gamma - \rho(t)$ is

$$\Gamma - \rho(t) = \sum_{j=-1}^{1} |\gamma_j|^2 \langle w_+\psi^3, w_-(\mathcal{H}-\lambda-\omega_j-i0)^{-1}\mathcal{P}_c\psi^3\rangle,$$

such that

$$|\Gamma - \rho(t)| \leq D_4 \sup_{t\in\mathbb{R}} |\gamma(t)|^2 \|\psi^3\|_{L^2_\sigma}, \tag{5.17}$$

where D_4 is the constant in the estimate (3.9). In the rest of this section, we analyze the decay estimate on the amplitude $|A(t)|$ from the amplitude equation (5.13). By taking the upper bound in (5.13), we consider the following comparison equation:

$$\frac{dB}{dt} = (-\Gamma + \rho(t))B^5 + R(t), \qquad B(0) = \varepsilon, \tag{5.18}$$

where $B(t)$ is a real-valued function, while parameters $\Gamma > 0$, ε, $\rho(t)$, and $R(t)$ are given by (1.9), (1.10), (5.14), and (5.16), respectively.

Lemma 5.1. *There exists a T-independent constant $\mathcal{D}$ such that if ε satisfies the bound:*

$$\varepsilon < \varepsilon_0 \leq \frac{1}{2}\mathcal{D}^{-1}\mathcal{C}^{-9}, \tag{5.19}$$

then the initial-value problem (5.18) is well posed on $t \in [0, T]$. Moreover, the solution $B(t)$ satisfies the bounds for $t \in [0, T]$:

$$0 \leq B(t) \leq \frac{\mathcal{D}\varepsilon}{(1 + 4\Gamma\varepsilon^4 t)^{1/4}} + \frac{\mathcal{D}\mathcal{C}^9\varepsilon^2}{(1 + 4\Gamma\varepsilon^4 t)^{1/4}}. \tag{5.20}$$

Proof. We will treat the correction term $R(t)$ perturbatively and linearize around the solution of the dominant equation:

$$\frac{dB_0}{dt} = (-\Gamma + \rho(t))B_0^5, \qquad B_0(0) = \varepsilon. \tag{5.21}$$

Using separation of variables, we have the solution:

$$B_0(t) = \frac{\varepsilon}{(1 + 4\varepsilon^4\Omega(t))^{1/4}}, \qquad t \geq 0, \tag{5.22}$$

where

$$\Omega(t) = \int_0^t (\Gamma - \rho(s))\, ds. \tag{5.23}$$

Even though $\Omega(t)$ may take negative values, $B_0(t)$ is defined for all $t \geq 0$, if ε is sufficiently small. Indeed, since $\rho(t)$ is continuous and periodic on $\mathbb{R}$ with mean zero we have that $\Omega(t)$ is continuous on $\mathbb{R}$ and:

$$\lim_{t\to\infty} \frac{\Omega(t)}{t} = \Gamma > 0 \tag{5.24}$$

Therefore, there exists $N > 0$ such that $\Omega(t) \geq 0$ for $t > N$. But $\Omega(t)$ is continuous on $[0, N]$, hence bounded. Let us denote

$$\Omega_{max} = \sup_{0\leq t\leq N} \Omega(t), \qquad \Omega_{min} = \inf_{0\leq t\leq N} \Omega(t), \tag{5.25}$$

where the minimum of $\Omega(t)$ and the maximum of $\Omega(t) - \Gamma t$ are in fact attained on $t \in [0, 2\pi/\omega]$ because of the periodicity of $\rho(t)$. In the case $\Omega_{min} < 0$ we choose

$$\varepsilon < \varepsilon_0 \leq (-1/\Omega_{min})^{1/4},$$

and $B_0(t)$ is well defined for all $t \geq 0$. We note that $\Omega(t)$ can be estimated on $t \in [0, N]$ by using (5.17) and (5.23) in terms of $\sup_{t\in\mathbb{R}} |\gamma(t)|$ and the dispersive constant D_4 in the decay estimate (3.9). Hence the above restriction on ε_0 can be accommodated in the bound (5.19). Proceeding in a similar manner, we show that

$$B_0(t) \leq \frac{\mathcal{D}_1 \varepsilon}{(1 + 4\Gamma\varepsilon^4 t)^{1/4}} \tag{5.26}$$

where $\mathcal{D}_1$ satisfies the bounds:

$$\sup_{0\leq t<\infty} \left(\frac{1 + 4\Gamma\varepsilon^4 t}{1 + 4\varepsilon^4 \Omega(t)} \right)^{1/4} \leq \mathcal{D}_1 < \infty. \tag{5.27}$$

More precisely, we introduce the auxiliary function

$$M(t) = \frac{1 + 4\Gamma\varepsilon^4 t}{1 + 4\varepsilon^4 \Omega(t)} : [0, \infty) \to \mathbb{R}, \tag{5.28}$$

such that $M(0) = 1$ and

$$\lim_{t\to\infty} M(t) = \lim_{t\to\infty} \frac{\frac{1}{t} + 4\Gamma\varepsilon^4}{\frac{1}{t} + 4\varepsilon^4 \frac{\Omega(t)}{t}} = \frac{4\Gamma\varepsilon^4}{4\Gamma\varepsilon^4} = 1. \tag{5.29}$$

Hence there exist $N > 0$ such that $M(t) \leq 3/2$, for $t > N$. But $M(t)$ is continuous, hence bounded on the compact interval $[0, N]$. If we choose

$$\mathcal{D}_1 \geq \left(\max\left(\frac{3}{2}, \sup_{t\in[0,N]} M(t) \right) \right)^{1/4},$$

then the bound (5.27) holds. The latter implies (5.26) due to the exact solution (5.22). We represent solution of the comparison equation (5.18) as $B(t) = B_0(t) + B_1(t)$, where $B_1(t)$ satisfies the equation:

$$\frac{dB_1}{dt} = 5B_0^4(t)(-\Gamma + \rho(t))B_1(t) + (-\Gamma + \rho(t)) \sum_{k=2}^{5} \binom{5}{k} B_0^{5-k}(t) B_1^k(t) + R(t), \tag{5.30}$$

such that $B_1(0) = 0$. Since the right hand side of (5.30) is continuous in t and locally Lipschitz in the dependent variable B_1, the initial-value problem (5.30) is locally well posed. In order to show that it is also well posed on $t \in [0, T]$, we use the propagator of the linear part to write the solution of the equation (5.30) in the integral form:

$$B_1(t) = K[B_1](t) = \sum_{k=2}^{5} \binom{5}{k} \int_0^t U(t,s)(-\Gamma + \rho(s)) B_0^{5-k}(s) B_1^k(s) ds + \int_0^t U(t,s) R(s) ds, \tag{5.31}$$

where

$$U(t,s) = \left(\frac{1 + 4\varepsilon^4 \Omega(s)}{1 + 4\varepsilon^4 \Omega(t)} \right)^{5/4}. \tag{5.32}$$

Under the condition (5.19), we prove that the operator $K[B_1]$ is a contraction on the ball of radius $\mathcal{R}\varepsilon^2$, where $0 < \mathcal{R} \leq \mathcal{D}\mathcal{C}^9$, in the Banach space $Z = C([0,T],\mathbb{R})$ equipped with the norm:

$$\|f\|_Z = \sup_{t\in[0,T]} (1+4\Gamma\varepsilon^4 t)^{1/4}|f(t)|. \tag{5.33}$$

It is clear from the bound (5.27) and similar computations that there exists the constant $\mathcal{D}_2$, such that

$$\sup_{t\in[0,\infty)} \left(\frac{1+4\Gamma\varepsilon^4 t}{1+4\varepsilon^4\Omega(t)}\right)^{5/4} \sup_{s\in[0,\infty)} \left(\frac{1+4\varepsilon^4\Omega(s)}{1+4\Gamma\varepsilon^4 s}\right)^{5/4} \leq \mathcal{D}_2 < \infty \tag{5.34}$$

and

$$|U(t,s)| \leq \mathcal{D}_2 \left(\frac{1+4\Gamma\varepsilon^4 s}{1+4\Gamma\varepsilon^4 t}\right)^{5/4}, \qquad \text{for all } s,t\in[0,\infty). \tag{5.35}$$

Consider two arbitrary functions $B_2(t),\ B_3(t)$ in $C([0,T],\mathbb{R})$ such that $\|B_j\|_Z \leq \mathcal{R}\varepsilon^2$ where the constant $\mathcal{R} > 0$ will be determined later. Then

$$\begin{aligned}
&\|K[B_2]-K[B_3]\|_Z \leq \|B_2-B_3\|_Z \sup_{t\in[0,T]} |-\Gamma+\rho(t)| \\
&\times \sum_{k=2}^{5}\binom{5}{k} \sup_{t\in[0,T]} (1+4\Gamma\varepsilon^4 t)^{1/4} \int_0^t |U(t,s)||B_0^{5-k}(s)|k(\mathcal{R}\varepsilon^2)^{k-1}(1+4\Gamma\varepsilon^4 s)^{-k/4}ds \\
&\leq \mathcal{D}_3\|B_2-B_3\|_Z \left(\sum_{k=2}^{5} k\binom{5}{k}(\mathcal{R}\varepsilon)^{k-1}\right)\varepsilon^4(1+4\Gamma\varepsilon^4 t)^{1/4}\int_0^t \left(\frac{1+4\Gamma\varepsilon^4 s}{1+4\Gamma\varepsilon^4 t}\right)^{5/4}(1+4\Gamma\varepsilon^4 s)^{-5/4}ds,
\end{aligned}$$

where we used estimates (5.35) for $U(t,s)$, (5.26) for $B_0(t)$ and (5.17) for $(-\Gamma+\rho(t))$. The integral can be computed explicitly, such that

$$\|K[B_2]-K[B_3]\|_Z \leq \Gamma^{-1}\mathcal{D}_4\mathcal{R}\varepsilon(1+\mathcal{R}\varepsilon+(\mathcal{R}\varepsilon)^2+(\mathcal{R}\varepsilon)^3)\|B_2-B_3\|_Z. \tag{5.36}$$

Let $L \leq 1/2$ be the Lipschitz constant and assume that

$$\varepsilon \leq \mathcal{R}^{-1}\min\left(\frac{1}{2},\frac{1}{4}\Gamma\mathcal{D}_4^{-1}\right). \tag{5.37}$$

The ball of radius $\mathcal{R}\varepsilon^2$ is invariant under $K[B_1]$, if the inhomogeneous term in K satisfies:

$$\left\|\int_0^t U(t,s)R(s)ds\right\|_Z \leq (1-L)\mathcal{R}\varepsilon^2 \leq \frac{1}{2}\mathcal{R}\varepsilon^2, \tag{5.38}$$

which is proved from the estimates (5.16) and (5.35). As a result, we obtain:

$$\left\|\int_0^t U(t,s)R(s)ds\right\|_Z \leq \mathcal{D}_5\mathcal{C}^9\varepsilon^2. \tag{5.39}$$

If we choose that $\mathcal{R} = 2\mathcal{D}_5\mathcal{C}^9$ and $\mathcal{D} \geq \max\{2\mathcal{D}_5, 4\Gamma^{-1}\mathcal{D}_4\mathcal{D}_5\}$, the bounds (5.37) and (5.38) are satisfied provided that the bound (5.19) holds. By the contraction principle, applied to the amplitude equation (5.31), the solution $B_1(t)$ exists and is unique in the ball of radius $\mathcal{R}\varepsilon^2 \leq \mathcal{D}\mathcal{C}^9\varepsilon^2$. As a result, the initial-value problem (5.30) is well posed on $t\in[0,T]$ and the solution $B_1(t)$ belongs to the ball of radius smaller than $\mathcal{D}\mathcal{C}^9\varepsilon^2$ in the Banach space $Z = C([0,T],\mathbb{R})$ with the norm (5.33), such that

$$|B_1(t)| \leq \frac{\mathcal{D}\mathcal{C}^9\varepsilon^2}{(1+4\Gamma\varepsilon^4 t)^{1/4}}, \qquad \text{for all } t \in [0,T]. \tag{5.40}$$

Combining the bounds (5.26) and (5.40), we derive the estimate (5.20). □

Lemma 5.2. *Let the bound (5.19) be satisfied. Then, the amplitude $|A(t)|$ is bounded by the solution $B(t)$ as follows:*

$$|A(t)| \leq B(t), \qquad \text{for all } t \in [0,T].$$

Proof. By Lemma 5.1, there exists a unique solution $B(t)$ of the comparison equation (5.18) on $t \in [0,T]$. By relations (5.13) and (5.16), we have

$$\frac{d|A|}{dt} \leq \frac{dB}{dt}, \qquad \text{for all } t \in [0,T], \tag{5.41}$$

such that

$$|A(0)| \leq \varepsilon = B(0). \tag{5.42}$$

Then, the lemma follows from the comparison principle for ODE's, see [16, pp. 168]. □

Lemma 5.3. *The set τ is open in $[0, T_{\max})$.*

Proof. Combining Lemmas 5.1 and 5.2, we have:

$$|A(t)| \leq \frac{\mathcal{D}\varepsilon(1+\mathcal{C}^9\varepsilon)}{(1+4\Gamma\varepsilon^4 t)^{1/4}}, \qquad \forall t \in [0,T], \tag{5.43}$$

provided that the bound (5.19) holds. Due to inequality (3.15), we have: $\mathcal{D}(1+\mathcal{C}^9\varepsilon) < \mathcal{C}$, such that

$$|A(t)| < \frac{\mathcal{C}\varepsilon}{(1+4\Gamma\varepsilon^4 t)^{1/4}}, \qquad \forall t \in [0,T]. \tag{5.44}$$

Since $A(t)$ is continuous, there exists $\delta > 0$ such that the above inequality holds for all $t \in [0, T+\delta)$. Consequently, $(T-\delta, T+\delta) \subset \tau$, where the set τ is defined in (3.14). Since $T \in \tau$ is arbitrary, we infer that τ is open in $[0, T_{\max})$. □

6 H^1 estimates

We prove here the upper bound (3.17), which excludes the finite time blow-up at $T < \infty$.

Lemma 6.1. *There exists a constant $\mathcal{E} > 0$ such that*

$$\|u(t)\|_{H^1} \leq \mathcal{E} \log(1+4\Gamma\varepsilon^4 t), \qquad \text{for } t \in \tau. \tag{6.1}$$

Proof. Assume that $u_0 \in H^2(\mathbb{R}^3)$. By Theorem 2, the inner product of u_t with the NLS equation (1.1) is defined, such that

$$\mathrm{Re}\left[\langle u_t, (-\Delta + V)u + \gamma(t)|u|^2 u\rangle\right] = 0. \tag{6.2}$$

Since $u(t) \in H^2(\mathbb{R}^3)$, $u_t(t) \in L^2(\mathbb{R}^3)$, and $|u|^2 u(t) \in H^2(\mathbb{R}^3)$ by Theorem 2, an elementary computation transforms (6.2) to the form of the energy conservation equation:

$$\frac{d}{dt}\left[\langle u, (-\Delta + V)u\rangle + \frac{1}{2}\gamma(t)\|u\|_{L^4}^4\right] = -\|u\|_{L^4}^4 \dot{\gamma}(t). \tag{6.3}$$

Using the decomposition (3.1), we obtain:

$$\frac{d}{dt}\left[\lambda|A(t)|^2 + \langle u_d, (-\Delta + V)u_d\rangle + \gamma(t)\frac{1}{2}\|u\|_{L^4}^4\right] = \dot{\gamma}(t)\frac{1}{2}\|u\|_{L^4}^4. \tag{6.4}$$

As a result,

$$\begin{aligned}
\|\nabla u_d(t)\|_{L^2}^2 \;\leq\;& \|V\|_\infty(\|u(t)\|_{L^2}^2 + |A(t)|^2) + \|u(0)\|_{H^1}^2 + \|V\|_\infty \|u(0)\|_{L^2}^2 \\
-\;& \lambda(|A(t)|^2 + |A(0)|^2) + \frac{1}{2}(|\gamma_0| + |\gamma_1|)(\|u(t)\|_{L^4}^4 + \|u(0)\|_{L^4}^4) \\
+\;& 4\omega|\gamma_1| \int_0^t \left(|A(s)|^4 \|\psi\|_{L^4}^4 + \|u_d(s)\|_{L^4}^4\right) ds.
\end{aligned} \tag{6.5}$$

Since $\|u(t)\|_2 = \|u(0)\|_2$ by Theorem 2, and $t \in \tau$ by Lemma 5.3, we use the estimates (3.14) and (4.29) and derive the H^1 bound (6.1) in the case $u_0 \in H^2(\mathbb{R}^3)$. By the standard density argument for $H^2 \hookrightarrow H^1$ and the continuous dependence of u_0 in Theorem 3, the bound (6.1) still holds for $u_0 \in H^1(\mathbb{R}^3)$. □

7 Generalizations and open problems

The main Theorem 1 can be extended with obvious modifications to the case of almost periodic time-dependent nonlinearity coefficient:

$$\gamma(t) = \sum_{n=1}^{\infty} \gamma_n \cos(\omega_n t + \theta_n), \qquad \sum_{n=1}^{\infty} |\gamma_n| < \infty,$$

where $\gamma_n \in \mathbb{C}$, $\omega_n, \theta_n \in \mathbb{R}$, $\forall n \in \mathbb{N}$ and there exists $\delta > 0$ such that $|\lambda + \omega_n| \geq \delta$ whenever $\gamma_n \neq 0$. In particular, the main Theorem extends to the case of general periodic perturbations, when $\omega_n = n\omega$, $n \in \mathbb{N}$, provided that the resonance between the bound state and the lowest radiation mode is excluded, i.e. $\lambda + n\omega \neq 0$ whenever $\gamma_n \neq 0$. We need to exclude such resonances because the estimate (3.10), as opposed to the estimate (3.9), is not sufficient for our argument to work. Our preliminary calculations show that, for $\lambda + \omega < 0$ and $\lambda + n\omega > 0$, the decay of the bound state occurs with the rate:

$$\|u(t)\|_{L^2_{-\sigma}} \sim t^{-1/(4n)}.$$

There are three open problems beyond the scope of the current manuscript. The first one is to extend the main theorem to the case of resonance with the lowest radiation mode, as we have mentioned above. The second open problem is to consider the NLS equation (1.1) in the space of lower dimensions. Since the bound (3.7) on the dispersive part in L^p depends on the space dimension, there are obstacles in finding invariant subspaces for the integral operator $K_A[u_d]$ in the space of one and two dimensions. The obstacles could be removed by increasing the power of the nonlinearity term in the NLS equation (1.1) to make it supercritical [24]. However, applied problems involving dispersion-managed optical pulses [15] and nonlinearity-managed Bose-Einstein condensates [13, 14] require solution of the NLS equation in one dimension with cubic nonlinearity. The third open problem is to investigate the nonlinear Fermi Golden rule in the case of large-norm bound states. We use the smallness of the initial data to control the radiative part of the solution. The blow-up effect of attractive nonlinearity for large-norm solutions renders the treatment of such a case much more difficult.

A Appendix: Contraction principles and fixed points estimates in a hierarchy of Banach spaces

Let $(Y, \|\cdot\|_Y)$ be a Banach space and $K : Y \to Y$ a locally Lipschitz operator, i.e. for any $u,\ v \in Y$ such that $\|u\|_Y,\ \|v\|_Y \le R$ there exist $L_Y(R) > 0$ with the property

$$\|K[u] - K[v]\|_Y \le L_Y(R)\|u - v\|_Y. \tag{A.1}$$

Assume that

$$K[0] = 0 \tag{A.2}$$

and consider an abstract equation:

$$u = f + K[u] = K_1[u], \qquad f \in Y \tag{A.3}$$

The classical contraction principle for a locally Lipschitz operator is:

Proposition A.1. *Assume that there exist the constants $L < 1$ and $R > 0$ such that:*

$$L_Y(R) \le L \tag{A.4}$$

$$\|f\|_Y \le (1 - L)R. \tag{A.5}$$

Then equation (A.3) has an unique solution u in the ball of radius R of the Banach space Y. Moreover, for any $v \in Y$, $\|v\|_Y \le R$ and $n = 0, 1, \ldots$ we have:

$$\|u - K_1^n[v]\|_Y \le \frac{\|K_1^{n+1}[v] - K_1^n[v]\|_Y}{1 - L} \le \frac{L^n}{1 - L}\|K_1[v] - v\|_Y. \tag{A.6}$$

In particular, for $v = f$ we get:

$$\|u - K_1^n[f]\|_Y \le \frac{L^n}{1 - L}\|K[f]\|_Y \le \frac{L^{n+1}}{1 - L}\|f\|_Y. \tag{A.7}$$

Proof. Apply the contraction principle to K_1 in the ball of radius R of the Banach space Y, see for example [28]. □

Consider another abstract equation:

$$u = g + f + K[u] = K_2[u], \qquad g \in X,\ f \in Y \tag{A.8}$$

where $(X, \|\cdot\|_X)$ is a Banach space continuously embedded in Y, $X \hookrightarrow Y$. The embedding implies that both g and $g+f$ are in Y. Assuming that (A.5) holds with f replaced by $f+g$, the previous proposition applies and the solution u of (A.8) satisfies:

$$\|u - K_2^n[f+g]\|_Y \le \frac{L^{n+1}}{1-L}\|f+g\|_Y$$

for $n = 0, 1, \ldots$. However the above estimate does not use information that the forcing term g belongs to a "better" space. Nevertheless, assume that we can find a solution $v \in X \subset Y,\ \|v\|_Y \le R$ of the equation:

$$v = g + K[v] \tag{A.9}$$

Then

$$\|f + v\|_Y = \|f + g + K[v]\|_Y = \|K_2[v]\|_Y \le R$$

since v is in the ball of radius R in Y which is left invariant by K_2. Hence we can use $f+v$ in (A.6) and get:

$$\begin{aligned}
\|u - K_2^n[f+v]\|_Y &\le \frac{L^n}{1-L}\|K_2[f+v] - f - v\|_Y \\
&= \frac{L^n}{1-L}\|K[f+v] + g - v\|_Y \\
&= \frac{L^n}{1-L}\|K[f+v] - K[v]\|_Y \\
&\le \frac{L^{n+1}}{1-L}\|f\|_Y
\end{aligned} \tag{A.10}$$

for $n = 0, 1, \ldots$

Now the existence of v with the above properties could be inferred from assuming that the fixed point problem (A.9) satisfies the hypotheses of Proposition A.1 but in the Banach space X. We have thus proved the following result:

Proposition A.2. *In addition to (A.1) and (A.2), assume that K is locally Lipschitz in X, i.e. for any $u,\ v \in X$ such that $\|u\|_X,\ \|v\|_X \le Q$ there exist $L_X(Q) > 0$ with the property*

$$\|K[u] - K[v]\|_X \le L_X(Q)\|u - v\|_Y. \tag{A.11}$$

Futhermore, assume that there exist constants $L_i < 1,\ i = 1,2$ and $R_i > 0,\ i = 1,2$ such that:

$$\begin{aligned}
L_X(R_1) &\le L_1 && \text{(A.12)}\\
\|g\|_X &\le (1 - L_1)R_1, && \text{(A.13)}
\end{aligned}$$

and

$$\begin{aligned}
L_Y(R_2) &\le L_2 && \text{(A.14)}\\
\|g + f\|_Y &\le (1 - L_2)R_2. && \text{(A.15)}
\end{aligned}$$

and also assume that

$$B_1 = \{v \in X \ : \ \|v\|_X \le R_1\} \subset B_2 = \{u \in Y \ : \ \|u\|_Y \le R_2\}. \tag{A.16}$$

Then there exist a unique solution $u \in B_2$ of (A.8) and a unique solution $v \in B_1$ of (A.9) with the following properties:

$$\|v - g\|_X \le \frac{\|K[g]\|_X}{1 - L_1} \ \le \ \frac{L_1}{1 - L_1}\|g\|_X \tag{A.17}$$

$$\|u - K_2^n[f + v]\|_Y \ \le \ \frac{L_2^{n+1}}{1 - L_2}\|f\|_Y \tag{A.18}$$

for $n = 0, 1, 2\ldots$ In particular, for $n = 0, 1$ we have

$$\|u - v - f\|_Y \ \le \ \frac{L_2}{1 - L_2}\|f\|_Y \tag{A.19}$$

$$\|u - g - f\|_Y \ \le \ \frac{L_1}{1 - L_1}\|g\|_X + \frac{L_2}{1 - L_2}\|f\|_Y \tag{A.20}$$

$$\|u - g - f - K[v + f]\|_Y \ \le \ \frac{L_2^2}{1 - L_2}\|f\|_Y \tag{A.21}$$

Lemma 4.2 is a specific application of Proposition A.2. Moreover, an inductive procedure can generalize the bound (A.20) as follows:

$$\|u - f_1 - f_2 - \ldots - f_m\|_m \le \frac{L_1^2}{1 - L_1}\|f_1\|_1 + \frac{L_2^2}{1 - L_2}\|f_2\|_2 + \ldots + \frac{L_m^2}{1 - L_m}\|f_m\|_m$$

where u is the solution of

$$u = f_1 + f_2 + \ldots f_m + K[u], \qquad f_i \in X_i, \ i = 1, 2, \ldots m$$

with $(X_i, \|\cdot\|_i)$, $i = 1, 2, \ldots m$ Banach spaces that satisfy $X_1 \hookrightarrow X_2 \hookrightarrow \ldots \hookrightarrow X_m$ and on which K is contractive with corresponding Lipschitz constants L_i, $i = 1, 2, \ldots m$.

References

[1] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, vol. 10, New York, Courant Institute, 2003.

[2] O. Costin, R.D. Costin, J.L. Lebowitz, A. Rokhlenko, Evolution of a model quantum system under time periodic forcing: conditions for complete ionization, Comm. Math. Phys. 221 (2001) 1-26.

[3] M. Goldberg, W. Schlag, Dispersive estimates for Schrödinger operators in dimensions one and three, preprint 2004.

[4] S. Inouye, M.R. Andrews, J. Stenger, H.J. Miesner, D.M. Stemper-Kurn, W. Ketterle, Observation of Feshbach resonances in a Bose-Einstein condensate, Nature 392 (1998) 151-154.

[5] A. Jensen, R. Kato, Spectral properties of Schrödinger operators and time-decay of the wave functions, Duke Math. J. 46 (1979) 583-611.

[6] P.G. Kevrekidis, G. Theocharis, D.J. Franzeskakis, B.A. Malomed, Feshbach resonance management for Bose-Einstein condensates, Phys. Rev. Lett. 90 (2003) 230401.

[7] E. Kirr, M.I. Weinstein, Parametrically excited Hamiltonian partial differential equations, SIAM J. Math. Anal. 33 (2001) 16-52.

[8] E. Kirr, M.I. Weinstein, Metastable states in parametrically excited multimode Hamiltonian systems, Comm. Math. Phys. 236 (2003) 335-372.

[9] E. Kirr, M.I. Weinstein, Diffusion of power in randomly perturbed Hamiltonian partial differential equations, Comm. Math. Phys. (2004), to be published.

[10] F.Kh. Abdullaev, B.B. Baizakov, S.A. Darmanyan, V.V. Konotop, M. Salerno, Nonlinear excitations in arrays of Bose-Einstein condensates, Phys. Rev. A 64 (2001) 043606.

[11] A.J. O'Connor, Exponential decay of bound state wave functions, Comm. Math. Phys. 32 (1973) 319-340.

[12] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.

[13] D.E. Pelinovsky, P.G. Kevrekidis, D.J. Franzeskakis, Averaging for solitons with nonlinearity management, Phys. Rev. Lett. 91 (2003) 240201.

[14] D.E. Pelinovsky, P.G. Kevrekidis, D.J. Franzeskakis, V. Zharnitsky, Hamiltonian averaging of solitons with nonlinearity management, Phys. Rev. E 70 (2004), to be published.

[15] D.E. Pelinovsky, V. Zharnitsky, Averaging of dispersion-managed pulses: existence and stability, SIAM J. Appl. Math. 63 (2003) 745-776.

[16] L.C. Piccinini, G. Stampacchia, G. Vidossich, Ordinary differential equations in $\mathbf{R}^n$, Applied Mathematical Sciences, vol. 39, Springer-Verlag, New York, 1984.

[17] C.A. Pillet, C.E. Wayne, Invariant manifolds for a class of dispersive, Hamiltonian, partial differential equations, J. Diff. Eqs. 141 (1997) 310-326.

[18] I.M. Sigal, Nonlinear wave and Schrödinger equations. I: Instability of periodic and quasiperiodic solutions, Comm. Math. Phys. 153 (1993), 297-320.

[19] A. Soffer, M.I. Weinstein, Multichannel nonlinear scattering for nonintegrable equations, Comm. Math. Phys. 133 (1990) 119-146.

[20] A. Soffer, M.I. Weinstein, Multichannel nonlinear scattering for nonintegrable equations. II: The case of anisotropic potentials and data, J. Diff. Eqs. 98 (1992) 376-390.

[21] A. Soffer, M.I. Weinstein, Nonautonomous Hamiltonians, J. Statist. Phys. 93 (1998) 359-391.

[22] A. Soffer, M.I. Weinstein, Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations, Invent. Math. 136 (1999) 9-74.

[23] A. Soffer, M.I. Weinstein, Selection of the ground state in nonlinear Schrödinger equations, preprint (2001).

[24] C. Sulem, P.L. Sulem, The Nonlinear Schrödinger Equation, Springer-Verlag, New York, 1999.

[25] T.P. Tsai, H.T. Yau, Relaxation of excited states in nonlinear Schrödinger equations, Int. Math. Res. Not. 31 (2002) 1629-1673.

[26] K. Yajima, Scattering theory for Schrödinger equations with potentials periodic in time, J. Math. Soc. Japan 29 (1977) 729-743.

[27] K. Yajima, L^p-continuity of wave operators for Schrödinger operators and its applications, in Proceedings of the Korea-Japan Partial Differential Equations Conference (Taejon, 1996), Lecture Notes Series, vol. 39, Seoul, 1997.

[28] E. Zeidler, Nonlinear functional analysis and its applications. I: Fixed-point theorems, Springer-Verlag, New York, 1986.