Conditional Intensity and Gibbsianness of Determinantal Point Processes

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Abstract

The Papangelou intensities of determinantal (or fermion) point processes are investigated. These exhibit a monotonicity property expressing the repulsive nature of the interaction, and satisfy a bound implying stochastic domination by a Poisson point process. We also show that determinantal point processes satisfy the so-called condition (Σ_λ) which is a general form of Gibbsianness. In the absence of percolation, the Gibbsian conditional probabilities can be identified explicitly.

Keywords. Determinantal point process, fermion point process, Gibbs point process, Papangelou intensity, stochastic domination, percolation
Running head. Determinantal point processes

1 Introduction

The aim of this paper is to investigate the dependence structure of determinantal (or fermion) point processes, abbreviated DPP's. These are point processes on $\mathbb{R}^d$ (or more general spaces) with a particular repulsive dependence structure induced by the fact that their correlation functions are given by suitable determinants. More explicitly, the correlation function $\rho(\alpha)$ at a finite configuration α is the determinant of the matrix obtained by evaluating a positive definite function at the points of α; see Subsection 2.1 for details.

Since their invention by Macchi [11], DPP's have attracted much interest from various viewpoints. Spohn [26, 27] investigated the dynamics of the so called Dyson's model, a model of interacting Brownian particles in one dimension with pair force $1/x$, or pair interaction $-\log x$. Its invariant measure is a typical DPP having the sine kernel as defining positive definite function. Rather recently, the theory of DPP's has been developed much further. Soshnikov [25] established the full existence theorem for DPP's and discussed many examples occurring in various fields of physics and mathematics. Shirai and Takahashi [21] also dealt with the existence theorem and extended the theory to some generalized point processes including boson processes. They also established some particular properties such as limit theorems, ergodic properties, and the Gibbs property of the corresponding discrete model [22]. In a series of

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papers, Borodin and Olshanski studied the DPP's appearing in the representation of the infinite symmetric group, see [2] and the references therein. Lyons and Steif [9, 10] investigated the ergodic and stochastic domination properties of DPP's on discrete lattices. The Glauber dynamics leaving some DPP's invariant was studied by Shirai and the second author [23].

In this paper we ask for the dependence properties of DPP's. Our leading questions are the following:

– What can be said about the repulsive nature of the interaction?

– When are determinantal point processes Gibbsian?

A key quantity we consider is the conditional intensity in the sense of Papangelou, which is a function $c(x, \xi)$ of points $x \in \mathbb{R}^d$ and configurations ξ. Its intuitive meaning is that $c(x, \xi)dx$ is the conditional probability of having a particle in dx when the configuration ξ is given. We will show that, locally on bounded regions, Papangelou intensities always exists, are given by ratios of determinants, and are decreasing functions of ξ (Theorem 3.2). This monotonicity is a natural expression of the repulsiveness of DPP's. In particular, it implies negative correlations of local vacuum events stating that some bounded regions contain no particles (Corollary 3.3). We also show that the local Papangelou intensities are bounded from above. This implies stochastic domination of DPP's by Poisson point processes and, in particular, the absence of percolation in the associated Boolean model when the underlying positive definite function is small enough (Corollaries 3.5 and 3.6).

Next we will show that the Papangelou intensities of DPP's exist not only locally but also globally (Theorem 3.7). This means that all DPP's are Gibbsian in a general sense, in that one can write down natural versions of their conditional distributions in bounded regions when the configuration outside of this region is fixed, and these conditional distributions are absolutely continuous with respect to the Poisson point process. It is a more difficult question to decide whether the associated conditional Hamiltonians can be expressed in terms of the underlying positive definite function in the natural way one expects. Here we only have a partial result, stating in particular that this holds whenever the underlying positive definite function has finite range and is small enough (Theorem 3.9).

In the next section we set up the stage. That is, in Subsection 2.1 we recall the definition of DPP's together with some basic facts, while Subsection 2.2 contains a discussion of the Papangelou intensity and its significance. Our results are stated in Section 3. Section 4 provides some auxiliary results on operators and determinants, and the proofs of our results follow in Section 5.

2 Preliminaries

2.1 Determinantal point processes

In this section we describe the general setting and recall the definition of determinantal point random fields (DPP). For a more complete account of DPP's we refer to the survey [25] and the articles [9, 10, 21, 22]. The state space of a DPP may be a quite general separable Hausdorff space, but in this paper we simply take $\mathbb{R}^d$. We write $\mathcal{B}$ for the Borel σ-algebra on $\mathbb{R}^d$, λ for Lebesgue measure on $(\mathbb{R}^d, \mathcal{B})$, and $\mathcal{B}_0$ for the system of all *bounded* Borel sets in $\mathbb{R}^d$.

Let $\mathcal{X}$ be the space of all locally finite subsets (configurations) in $\mathbb{R}^d$, i.e.,

$$\mathcal{X} := \{\xi \subset \mathbb{R}^d : |\xi \cap \Lambda| < \infty \text{ for all } \Lambda \in \mathcal{B}_0\},$$

where $|A|$ denotes the cardinality of a set A. Given any $\Lambda \subset \mathbb{R}^d$, we write $\mathcal{X}_\Lambda := \{\xi \in \mathcal{X} : \xi \subset \Lambda\}$ for the set of all configurations in Λ, and $r_\Lambda : \xi \to \xi_\Lambda := \xi \cap \Lambda$ for the corresponding projection from $\mathcal{X}$ onto $\mathcal{X}_\Lambda$. Also let $N_\Lambda : \xi \to |\xi \cap \Lambda|$ be the associated counting variable on $\mathcal{X}$, and $\mathcal{F}_\Lambda$ be the smallest σ-algebra on $\mathcal{X}$ such that N_Δ is measurable for all bounded Borel subsets $\Delta \subset \Lambda$. We write $\mathcal{F}$ for $\mathcal{F}_{\mathbb{R}^d}$. Each configuration $\xi \in \mathcal{X}$ can be identified with the integer-valued Radon measure $\sum_{x\in\xi} \delta_x$ on the Borel σ-algebra on $\mathbb{R}^d$. The vague topology for the latter then induces a topology on $\mathcal{X}$ turning $\mathcal{X}$ into a Polish space. It is well-known that $\mathcal{F}$ is the associated Borel σ-algebra [3, 20]. A *point process* (PP for short) is a probability measure μ on $(\mathcal{X}, \mathcal{F})$. We write $\mu_\Lambda := \mu \circ r_\Lambda^{-1}$ for its marginal on $\mathcal{X}_\Lambda$.

Next let $\mathcal{X}_0 = \{\alpha \in \mathcal{X} : |\alpha| < \infty\}$ be the set of all *finite* configurations in $\mathbb{R}^d$. $\mathcal{X}_0$ is equipped with the trace σ-algebra $\mathcal{F}_0 = \mathcal{F}|_{\mathcal{X}_0}$ and the *Lebesgue-Poisson measure* L defined by the identity

$$\int_{\mathcal{X}_0} f(\alpha)\, L(d\alpha) = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{(\mathbb{R}^d)^m} f(\{x_1, \ldots, x_m\})\, dx_1 \cdots dx_m$$

for any measurable function $f : \mathcal{X}_0 \to \mathbb{R}_+$. For any $\Lambda \subset \mathbb{R}^d$ we let $L_\Lambda(d\alpha) = 1_{\{\alpha\subset\Lambda\}} L(d\alpha)$ be the restriction of L to $\mathcal{X}_\Lambda$. Here we use the notation 1_A for the indicator function of a set A.

Recall that a PP μ is said to have the *correlation function* $\rho : \mathcal{X}_0 \to \mathbb{R}_+$ if ρ is measurable and satisfies

$$\int_{\mathcal{X}} \sum_{\alpha\in\mathcal{X}_0 :\, \alpha\subset\xi} u(\alpha)\ \mu(d\xi) = \int_{\mathcal{X}_0} u(\alpha)\rho(\alpha)\, L(d\alpha) \tag{2.1}$$

for any measurable $u : \mathcal{X}_0 \to \mathbb{R}_+$. The *Poisson point process* π^z *with intensity* $z > 0$ is the unique PP with correlation function $\rho(\alpha) = z^{|\alpha|}$. Equivalently, π^z is the unique PP such that, for each $\Lambda \in \mathcal{B}_0$, the projection π^z_Λ has the Radon-Nikodym density $\xi \to e^{-z\lambda(\Lambda)} z^{|\xi|}$ relative to L_Λ. The characteristic feature of the determinantal point processes to be considered here is that their correlation function is given by suitable determinants. Given a function $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, we write

$$K(\alpha, \alpha) = \big(K(x, y)\big)_{x,y\in\alpha} \tag{2.2}$$

for the matrix obtained by evaluating K at a finite configuration $\alpha \in \mathcal{X}_0$.

Definition 2.1 Let $K(x, y)$, $x, y \in \mathbb{R}^d$, be the integral kernel function of a positive Hermitian integral operator K on $L^2(\mathbb{R}^d)$. A PP μ with correlation function

$$\rho(\alpha) = \det K(\alpha, \alpha)$$

is called a *determinantal point process* (abbreviated DPP) with respect to K.

The DPP's defined above are also known as *fermion point processes*; see e.g. [21] for the boson case where the determinant is replaced by the permanent, and more general determinantal processes. For the existence of DPP's we state the following theorem from [25], see also [11, 3, 21]. I stands for the identity operator, and the ordering $S \le T$ between operators means that $T - S$ is a positive operator.

Theorem 2.2 (Macchi, Soshnikov) *A Hermitian locally trace class operator K on $L^2(\mathbb{R}^d)$ defines a DPP μ if and only if $0 \le K \le I$, and in this case μ is unique.*

Any DPP μ is locally absolutely continuous with respect to the Lebesgue-Poisson measure L and admits explicit expressions for the local densities. To be specific, for each $\Lambda \in \mathcal{B}_0$ let $P_\Lambda : L^2(\mathbb{R}^d) \to L^2(\Lambda)$ be the projection operator and $K_\Lambda := P_\Lambda K P_\Lambda$ the restriction of K onto $L^2(\Lambda)$. That is, K_Λ has the kernel $K_\Lambda(x, y) = 1_\Lambda(x)K(x, y)1_\Lambda(y)$. Suppose that μ is the unique DPP for an operator K as in Theorem 2.2. Then the density function of μ_Λ with respect to L_Λ is given by [21, 25]

$$\sigma_\Lambda(\xi) = \det(I - K_\Lambda) \det J_{[\Lambda]}(\xi, \xi), \quad \xi \in \mathcal{X}_\Lambda; \tag{2.3}$$

here $J_{[\Lambda]} := K_\Lambda(I - K_\Lambda)^{-1}$, and the normalization constant $\det(I - K_\Lambda)$ is to be understood as a Fredholm determinant [24]. Moreover, the correlation function ρ is recovered from the local densities (σ_Λ) by

$$\rho(\alpha) = \int_{\mathcal{X}_\Lambda} \sigma_\Lambda(\alpha\xi)\, L_\Lambda(d\xi)$$

for $\alpha \in \mathcal{X}_\Lambda$, where $\alpha\xi$ is shorthand for $\alpha \cup \xi$.

2.2 Papangelou intensities

Here we summarize some facts concerning the reduced (compound) Campbell measure of a PP μ as well as its Papangelou intensity which describes the local dependence of particles.

Definition 2.3 (a) The *reduced* (or modified) *Campbell measure* of a PP μ is the measure C_μ on the product space $(\mathbb{R}^d \times \mathcal{X}, \mathcal{B} \otimes \mathcal{F})$ defined by

$$C_\mu(A) = \int \mu(d\xi) \sum_{x \in \xi} 1_A(x, \xi \setminus x), \quad A \in \mathcal{B} \otimes \mathcal{F},$$

where $\xi \setminus x := \xi \setminus \{x\}$.

(b) The *reduced compound Campbell measure* of μ is the measure $\widehat{C}_\mu$ on the product space $(\mathcal{X}_0 \times \mathcal{X}, \mathcal{F}_0 \otimes \mathcal{F})$ satisfying

$$\widehat{C}_\mu(B) = \int \mu(d\xi) \sum_{\alpha \in \mathcal{X}_0 :\, \alpha \subset \xi} 1_B(\alpha, \xi \setminus \alpha), \quad B \in \mathcal{F}_0 \otimes \mathcal{F}.$$

It is well-known and easy to check that the reduced Campbell measure of the Poisson PP π^z is given by $C_{\pi^z} = z\lambda \otimes \pi^z$, and a classical result of Mecke [13] states that π^z is the only PP with this property. This fact suggests the following concept.

Definition 2.4 A PP μ is said to satisfy *condition* (Σ_λ) if $C_\mu \ll \lambda \otimes \mu$. Any Radon-Nikydom density c of C_μ relative to $\lambda \otimes \mu$ is then called (a version of) the *Papangelou (conditional) intensity* (abbreviated PI) of μ.

More explicitly, c is a PI of μ if

$$\int \mu(d\xi) \sum_{x\in\xi} f(x, \xi \setminus x) = \int dx \int \mu(d\xi)\, c(x,\xi) f(x,\xi) \tag{2.4}$$

for all measurable functions $f : \mathbb{R}^d \times \mathcal{X} \to \mathbb{R}_+$. Intuitively, $c(x,\xi)$ is the conditional probability for a particle in the differential region dx given the configuration ξ. Also, if

$$Gh(\xi) = \int dx\; c(x,\xi)\big[h(\xi\cup x) - h(\xi)\big] + \sum_{x\in\xi} \big[h(\xi\setminus x) - h(\xi)\big]$$

is the formal generator of a birth-and-death process with birth rate $c(x,\xi)dx$ for a particle in dx and death rate 1 for each particle, (2.4) is equivalent to the reversibility equation $\int g\, Gh\, d\mu = \int h\, Gg\, d\mu$; to see this it is sufficient to set $f(x,\xi) = g(\xi)h(\xi\cup x)$.

The following remark exhibits the significance of condition (Σ_λ). In particular, it shows that (Σ_λ) processes are Gibbsian in a general sense.

Remark 2.5 For any μ satisfying condition (Σ_λ) the following conclusions hold.

(a) The reduced compound Campbell measure $\widehat{C}_\mu$ is absolutely continuous with respect to $L \otimes \mu$ with a Radon-Nikodym density $\widehat{c}$ satisfying $\widehat{c}(\emptyset, \xi) = 1$ and

$$\widehat{c}(\alpha,\xi) = c(x_1,\xi) \prod_{i=2}^{n} c(x_i, x_1 \dots x_{i-1}\xi) \quad \text{when } \alpha = \{x_1,\dots,x_n\}; \tag{2.5}$$

here $x_1 \dots x_{i-1}\xi = \{x_1,\dots,x_{i-1}\} \cup \xi$. In particular, the right-hand side of (2.5) is almost surely symmetric in $x_1,\dots,x_n$. $\widehat{c}$ is called the *compound Papangelou intensity* (CPI). Explicitly, the relation $\widehat{c} = d\widehat{C}_\mu / d(L\otimes\mu)$ means that

$$\int \mu(d\xi) \sum_{\alpha\in\mathcal{X}_0:\,\alpha\subset\xi} f(\alpha, \xi\setminus\alpha) = \int L(d\alpha) \int \mu(d\xi)\, \widehat{c}(\alpha,\xi)\, f(\alpha,\xi) \tag{2.6}$$

for any measurable $f : \mathcal{X}_0 \times \mathcal{X} \to \mathbb{R}_+$, and follows easily from (2.4) by induction on $|\alpha|$.

(b) For an f depending only on α, a comparison of (2.6) and (2.1) shows that the correlation function and the CPI of μ are related to each other by

$$\rho(\alpha) = \int \mu(d\xi)\, \widehat{c}(\alpha,\xi) \quad \text{for } L\text{-almost all } \alpha.$$

(c) Let $\Lambda \in \mathcal{B}_0$. Applying (2.6) to $f(\alpha,\xi) = g(\alpha)h(\xi)1_{\{\alpha\subset\Lambda,\, N_\Lambda(\xi)=0\}}$ for any $\mathcal{F}_\Lambda$-measurable $g : \mathcal{X} \to \mathbb{R}_+$ and $\mathcal{F}_{\Lambda^c}$-measurable $h : \mathcal{X} \to \mathbb{R}_+$ and taking conditional expectations we find

$$\begin{aligned}\int \mathbb{E}_\mu(g|\mathcal{F}_{\Lambda^c})\, h\, d\mu &= \int gh\, d\mu = \int f\, d\widehat{C}_\mu \\ &= \int \mu(d\xi)\, h(\xi)\ \mu(N_\Lambda = 0|\mathcal{F}_{\Lambda^c})(\xi) \int L_\Lambda(d\alpha)\, g(\alpha)\, \widehat{c}(\alpha,\xi_{\Lambda^c})\end{aligned}$$

Hence

$$\mathbb{E}_\mu(g|\mathcal{F}_{\Lambda^c})(\xi) = \mu(N_\Lambda = 0|\mathcal{F}_{\Lambda^c})(\xi) \int g\, \widehat{c}(\cdot,\xi_{\Lambda^c})\, dL_\Lambda \tag{2.7}$$

for μ-almost all ξ and any $\mathcal{F}_\Lambda$-measurable g. In particular, for $g \equiv 1$ we find that $\mu(N_\Lambda = 0|\mathcal{F}_{\Lambda^c}) > 0$ almost surely for each $\Lambda \in \mathcal{B}_0$, a property introduced by Papangelou [17] as condition (Σ), and by Kozlov [8] as the condition of non-degenerate vacuum.

The observations in the preceding remark are due to Matthes, Warmuth and Mecke [12, Section 3] and give one part of their theorem below; cf. also [15, Theorem 2′].

Theorem 2.6 (Matthes, Warmuth and Mecke) *A point process μ satisfies condition (Σ_λ) with PI c if and only if, for each $\Lambda \in \mathcal{B}_0$, μ is absolutely continuous relative to $L_\Lambda \otimes \mu_{\Lambda^c}$ with a density γ_Λ satisfying $\gamma_\Lambda(x\xi) = 0$ whenever $\gamma_\Lambda(\xi) = 0$ and $x \in \Lambda \setminus \xi$. In this case, for $L_\Lambda \otimes \mu_{\Lambda^c}$-almost all $\xi \in \mathcal{X}$ we have*

$$\gamma_\Lambda(\xi) = Z_\Lambda(\xi_{\Lambda^c})^{-1}\, \widehat{c}(\xi_\Lambda,\xi_{\Lambda^c}) \tag{2.8}$$

with $0 < Z_\Lambda(\xi_{\Lambda^c}) = \int \widehat{c}(\cdot,\xi_{\Lambda^c})\, dL_\Lambda < \infty$, and $c(x,\xi) = \gamma_\Lambda(x\xi)/\gamma_\Lambda(\xi)$ for $\lambda \otimes \mu$-almost all $(x,\xi) \in \Lambda \times \mathcal{X}$.

More explicitly, Equation (2.8) means that for each bounded measurable function $f : \mathcal{X} \to \mathbb{R}$, the conditional expectation $\mathbb{E}_\mu(f|\mathcal{F}_{\Lambda^c})(\xi)$ has the version

$$G_\Lambda(f|\xi) := Z_\Lambda(\xi_{\Lambda^c})^{-1} \int f(\alpha\xi_{\Lambda^c})\, \widehat{c}(\alpha,\xi_{\Lambda^c}) L_\Lambda(d\alpha). \tag{2.9}$$

Remark 2.7 Theorem 2.6 has a counterpart for the restriction of a PP μ to a bounded region $\Lambda \in \mathcal{B}_0$. Let λ_Λ be the restriction of λ to the σ-algebra $\mathcal{B}_\Lambda$ of Borel subsets of Λ. Then μ_Λ satisfies condition $(\Sigma_{\lambda_\Lambda})$ with a PI c_Λ if and only if μ_Λ is absolutely continuous with respect to L_Λ with a density σ_Λ having an increasing zero-set $\{\sigma_\Lambda = 0\}$. In this case, $\sigma_\Lambda = \sigma_\Lambda(\emptyset)\, \widehat{c}_\Lambda(\cdot,\emptyset)$ L_Λ-almost surely, and $c_\Lambda(x,\xi) = \sigma_\Lambda(x\xi)/\sigma_\Lambda(\xi)$ for $\lambda_\Lambda \otimes \mu_\Lambda$ almost all $(x,\xi) \in \Lambda \times \mathcal{X}_\Lambda$. This follows from the above by noting that μ_Λ is trivial on $\mathcal{F}_{\Lambda^c}$; cf. also [5, Proposition 3.1].

3 Results

The DPP's considered in this paper are defined via some positive Hermitian integral operator J on $L^2(\mathbb{R}^d)$ satisfying the following condition of boundedness, continuity, and translation invariance.

Assumption 3.1 The kernel function $J(\cdot,\cdot)$ of J is given by $J(x,y) = j(x-y)$, $x,y \in \mathbb{R}^d$, where

$$j(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot t}\, \varphi(t)\, dt$$

is the Fourier transform of a nonnegative integrable function φ on $\mathbb{R}^d$. We call $z(J) := j(0) = (2\pi)^{-d/2}\|\varphi\|_1$ the *activity* of J.

To any such J we associate the operator

$$K^J := \frac{J}{I+J}. \tag{3.10}$$

By construction, K^J has the integral kernel $K^J(x,y) = k^J(x-y)$, $x,y \in \mathbb{R}^d$, where k^J is given by

$$k^J(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot t}\, \frac{\varphi(t)}{1+\varphi(t)}\, dt. \tag{3.11}$$

K^J then satisfies the conditions of Theorem 2.2 and hence defines a unique DPP μ^J. Our basic observations are the following; recall the notation (2.2).

Theorem 3.2 *For each $\Lambda \in \mathcal{B}_0$, μ^J_Λ satisfies condition $(\Sigma_{\lambda_\Lambda})$, and the associated PI c^J_Λ exhibits the following properties:*

(a) *For $L_\Lambda \otimes \mu_\Lambda$-almost all $(\alpha,\xi) \in \mathcal{X}_\Lambda \times \mathcal{X}_\Lambda$,*

$$\widehat{c}^J_\Lambda(\alpha,\xi) = \det J_{[\Lambda]}(\alpha\xi,\alpha\xi)/\det J_{[\Lambda]}(\xi,\xi)\,, \tag{3.12}$$

where the ratio is defined as zero if the denominator vanishes. As in (2.3), $J_{[\Lambda]}$ is a continuous kernel of the operator $J_{[\Lambda]} = K^J_\Lambda(I-K^J_\Lambda)^{-1}$ on $L^2(\Lambda)$ and K^J_Λ denotes the restriction of K^J to $L^2(\Lambda)$. In particular, $c^J_\Lambda(x,\xi) = \det J_{[\Lambda]}(x\xi,x\xi)/\det J_{[\Lambda]}(\xi,\xi)$ for $\lambda_\Lambda \otimes \mu_\Lambda$-almost all $(x,\xi) \in \Lambda \times \mathcal{X}_\Lambda$.

(b) *For any $\xi \subset \eta \in \mathcal{X}_\Lambda$ and $x \in \Lambda \setminus \eta$,*

$$c^J_\Lambda(x,\xi) \geq c^J_\Lambda(x,\eta) \quad \textit{and} \quad 0 \leq c^J_\Lambda(x,\xi) \leq z(J).$$

Here are some consequences of the theorem. First, let $\Lambda,\Delta \in \mathcal{B}_0$ with $\Lambda \subset \Delta$ and $f : \mathcal{X}_\Lambda \to \mathbb{R}_+$ be any measurable function. Then, combining (3.12) with Remark 2.7 and (2.9) we find that

$$\begin{aligned} \mathbb{E}_\mu(f|\mathcal{F}_{\Delta\setminus\Lambda})(\xi) &= G^J_{\Lambda,\Delta}(f|\xi) \\ &:= Z^J_{\Lambda,\Delta}(\xi_{\Delta\setminus\Lambda})^{-1} \int L_\Lambda(d\alpha)\, f(\alpha)\, \widehat{c}_\Delta(\alpha,\xi_{\Delta\setminus\Lambda}) \end{aligned} \tag{3.13}$$

for μ-almost all $\xi \in \mathcal{X}$. Note that the normalization constant

$$Z^J_{\Lambda,\Delta}(\xi) = \int L_\Lambda(d\alpha)\, \widehat{c}_\Delta(\alpha,\xi)$$

is always finite due to the bound in assertion (b), and non-zero whenever $\det J_{[\Delta]}(\xi,\xi) > 0$ because $L_\Lambda(\{\emptyset\}) = 1$. This means that $G^J_{\Lambda,\Delta}(f|\xi)$ is well-defined for *all* such ξ.

Statement (b) of Theorem 3.2, the monotonicity of the PI's $c^J_\Lambda(x,\cdot)$, expresses the repulsive nature of the particle interaction in an infinitesimal way. The following corollary provides an integral version of this repulsiveness.

Corollary 3.3 *For any $\Lambda \subset \Delta \in \mathcal{B}_0$ and any measurable function $f : \mathcal{X}_\Lambda \to \mathbb{R}_+$, the ratio $G^J_{\Lambda,\Delta}(f|\xi)/G^J_{\Lambda,\Delta}(N_\Lambda = 0|\xi)$ is decreasing in $\xi \in \mathcal{X}_{\Delta\setminus\Lambda}$. In particular, $G^J_{\Lambda,\Delta}(N_\Lambda = 0|\cdot)$ is increasing, and for $\Lambda \subset \Delta_1 \subset \Delta_2 \in \mathcal{B}_0$ we have*

$$\mu^J(N_\Lambda = 0|N_{\Delta_2\setminus\Lambda} = 0) \le \mu^J(N_\Lambda = 0|N_{\Delta_1\setminus\Lambda} = 0) \le \mu^J(N_\Lambda = 0)\,. \tag{3.14}$$

Remark 3.4 The statement of the corollary is weaker than one may hope. In fact, one might guess that $G^J_{\Lambda,\Delta}(f|\xi)$ is decreasing in ξ for any increasing $\mathcal{F}_\Lambda$-measurable function f. (The corollary implies this assertion only for $f = 1_{\{N_\Lambda\ge 1\}}$. This guess, however, cannot be expected to be true in general. For if f depends only on some part Λ_0 of Λ, then an increase of ξ may repel some particles from $\Lambda \setminus \Lambda_0$, giving a chance to some additional particles in Λ_0, so that $G^J_{\Lambda,\Delta}(f|\xi)$ will increase. So, the situation is less satisfactory for point processes than in the discrete case; cf. Theorem 6.5 of [9]. Nevertheless, (3.14) implies that, for disjoint $\Lambda, \Delta \in \mathcal{B}_0$, the events $\{N_\Lambda = 0\}$ and $\{N_\Delta = 0\}$ are negatively associated; see Proposition 2.7 of [22] for the corresponding result in the discrete case.

Next we exploit the domination bound $c^J_\Lambda \le z(J)$. Let μ, ν be two PP's. One says *μ is dominated by ν*, written $\mu \preceq \nu$, if

$$\int f\, d\mu \le \int f\, d\nu$$

for every increasing measurable function $f : \mathcal{X} \to \mathbb{R}$. We then have the following Poisson domination result.

Corollary 3.5 *For any J as above, $\mu^J \preceq \pi^{z(J)}$.*

The last result implies that for small activity there is no percolation in the Boolean model associated to μ^J. Let $b_R(x)$ denote the closed ball of radius $R < \infty$ in $\mathbb{R}^d$ centered at x, and for $\xi \in \mathcal{X}$ let

$$B_R(\xi) = \bigcup_{x\in\xi} b_R(x)$$

the associated Boolean set. $B_R(\xi)$ splits into connected components called *clusters*. A cluster is called *infinite* if it consists of infinitely many points of ξ, or equivalently, if its diameter is infinite. It is well-known [7, 14] that, for $d \ge 2$, there exists a critical threshold $0 < z_c \equiv z_c(d, R) < \infty$ for Poisson percolation:

$$\pi^z\big(\exists \text{ infinite cluster of } B_R(\cdot)\big) = \begin{cases} 0 & \text{if } z < z_c\,, \\ 1 & \text{if } z > z_c\,. \end{cases}$$

For $d = 1$ one has $z_c = +\infty$ [14, Theorem 3.1].

Corollary 3.6 *For $z(J) < z_c$, $\mu^J(\exists$ infinite cluster of $B_R(\cdot)) = 0$.*

We now address the question whether DPP's are Gibbsian. In view of Remark 2.5 and Theorem 2.6, the following result implies that this is the case at least in a general sense.

Theorem 3.7 *μ^J satisfies condition (Σ_λ). Its CPI is given by*

$$\widehat{c}^J(\alpha,\xi) = \lim_{n\to\infty} \widehat{c}^J_{\Delta_n}(\alpha,\xi_{\Delta_n}) \quad \text{for } L\otimes\mu\text{-almost all } (\alpha,\xi), \tag{3.15}$$

where $\widehat{c}^J_{\Delta_n}$ is given by (3.12) and (Δ_n) is any sequence in $\mathcal{B}_0$ that increases to $\mathbb{R}^d$. In particular, $\mu^J(N_\Lambda = 0|\mathcal{F}_{\Lambda^c}) > 0$ μ^J-almost surely for each $\Lambda\in\mathcal{B}_0$ and, for any $\mathcal{F}_\Lambda$-measurable $f : \mathcal{X}\to\mathbb{R}_+$, the ratio

$$\mathbb{E}_{\mu^J}(f|\mathcal{F}_{\Lambda^c})/\mu^J(N_\Lambda = 0|\mathcal{F}_{\Lambda^c})$$

of conditional expectations has a decreasing version.

Combining (3.15) with (3.12) and (2.9) we obtain at least an implicit formula for the conditional probabilities of μ^J given the events outside of a bounded region. But the question remains of whether the CPI, and thereby the conditional probabilities, of μ^J can be identified in a more specific way. In fact, there exists a natural candidate for $\widehat{c}^J$, namely

$$\widehat{c}^J_*(\alpha,\xi) = \lim_{\Delta\uparrow\mathbb{R}^d} \det J(\alpha\xi_\Delta,\alpha\xi_\Delta)/\det J(\xi_\Delta,\xi_\Delta)\,. \tag{3.16}$$

As a matter of fact, this limit exists because the expression on the righthand side is decreasing in Δ, as follows from (4.22) below. (As before, we set a ratio of determinants equal to zero if the denominator vanishes.) In contrast to (3.15), the determinants in (3.16) involve J itself rather than $J_{[\Delta_n]}$. We thus arrive at the following question.

Question 3.8 *When is it true that $\widehat{c}^J = \widehat{c}^J_*$ almost surely for $L\otimes\mu^J$?*

In fact, this question is closely related to the problem of whether μ^J is Gibbsian in the usual sense that its conditional probabilities are given by an interaction potential. We will discuss this point in Remark 3.10 below.

Unfortunately, we are unable to settle Question 3.8 in the same generality as this was done in the lattice case by Shirai and Takahashi [22, Theorem 6.2]. Their argument exploits the symmetry between occupied and empty lattice sites and therefore does not carry over to our continuous setting. The following theorem gives, at least, a partial answer. J is said to have range $R<\infty$ if $J(x-y)=0$ whenever $|x-y|>R$.

Theorem 3.9 (a) *In general, the inequality $\widehat{c}^J \le \widehat{c}^J_*$ holds $L\otimes\mu^J$-almost surely.*

(b) *If J has range $R<\infty$ and $\mu^J(\exists$ infinite cluster of $B_R(\cdot)) = 0$ (which is the case at least when $d=1$ or $z(J)<z_c$) then*

$$\widehat{c}^J(\alpha,\xi) = \widehat{c}^J_*(\alpha,\xi) = \det J(\alpha\xi^\sharp,\alpha\xi^\sharp)/\det J(\xi^\sharp,\xi^\sharp)\; 1_{\{\operatorname{diam} W(\alpha,\xi)<\infty\}} \tag{3.17}$$

for $L\otimes\mu^J$-almost all (α,ξ). Here $W(\alpha,\xi)$ is the union of the clusters of $B_R(\alpha\xi)$ hitting α, and $\xi^\sharp := \xi_{W(\alpha,\xi)}$ is the restriction of ξ to $W(\alpha,\xi)$.

Assertion (b) of this theorem has already been derived by the second author in [29] by different methods. In the situation of this statement, the CPI $\widehat{c}^J_*(\alpha,\cdot)$ is continuous at every configuration ξ for which $B_R(\xi)$ contains no infinite cluster. This implies that the conditional expectation $\mathbb{E}_\mu(f|\mathcal{F}_{\Lambda^c})$ has a version which is continuous at every configuration ξ for which $B_R(\xi)$ contains no infinite cluster hitting Λ, and the latter occurs with probability 1. This is Gibbsianness in the best sense one can expect. In particular, the conditional probabilities of μ^J can be expressed in terms of a suitable many-body potential. The following remark shows that such a potential exists whenever Question 3.8 can be answered in the positive.

Remark 3.10 For each $\Lambda \in \mathcal{B}_0$ and any $\xi \in \mathcal{X}_{\Lambda^c}$ satisfying $\det J(\zeta,\zeta) > 0$ for all finite $\zeta \subset \xi$, we can define a Gibbs distribution $G_\Lambda(\cdot|\xi)$ by inserting $\widehat{c}^J_*$ into (2.9). In fact, $G_\Lambda(\cdot|\xi)$ is a Gibbs distribution for the Hamiltonian $H_\Lambda(\cdot|\xi) = -\log \widehat{c}^J_*(\cdot,\xi)$ on $\mathcal{X}_\Lambda$, $\xi \in \mathcal{X}_{\Lambda^c}$. (If desired, one can express the Hamiltonian in terms of a many-body potential Φ, but this is not particularly useful.) These Gibbs distributions altogether form a Gibbsian specification $G = (G_\Lambda)_{\Lambda\in\mathcal{B}_0}$ in the sense of Preston [19, pp. 16, 17]; this has been proved by Glötzl [6] in a general setting, and in [29] for the particular case of DPP's. By construction, μ^J is a Gibbs measure for G whenever it admits the CPI $\widehat{c}^J_*$.

Conversely, suppose μ^J is a Gibbs measure for some Hamiltonian H. Then $\widehat{c}^H(\alpha,\xi)$:= $\exp[-H_\alpha(\alpha|\xi_{\alpha^c}]$ is a version of its CPI [15] and satisfies the continuity condition $\widehat{c}^H(\alpha,\xi) = \lim_{\Delta\uparrow\mathbb{R}^d} \widehat{c}^H(\alpha,\xi_\Delta)$. If one assumes that the function $c^J = \lim_{n\to\infty} c^J_{\Delta_n}$ in (3.15) has the same continuity property, one can conclude that $c^J = c^J_*$ almost surely. For, Proposition 4.5 implies that $\lim_{n\to\infty} c^J_{\Delta_n}(\alpha,\xi_\Delta) = c^J_*(a,\xi_\Delta)$ whenever $\alpha \in \mathcal{X}_0$, $\Delta \in \mathcal{B}_0$, and $\xi \in \mathcal{X}$ is such that $\det J(\xi_\Delta,\xi_\Delta) > 0$. But the last condition holds μ^J-almost surely.

4 Auxiliary results on operators and determinants

In this section we collect some results on the local versions of the operators J and K^J defined in Assumption 3.1 and Eq. (3.10), as well as their associated matrices. As before, let $P_\Lambda : L^2(\mathbb{R}^d) \to L^2(\Lambda)$ be the projection operator, and $K_\Lambda := P_\Lambda K P_\Lambda$ resp. $J_\Lambda := P_\Lambda J P_\Lambda$ the restrictions of K and J to $L^2(\Lambda)$. We also consider the local J-operator $J_{[\Lambda]} := K^J_\Lambda(I-K^J_\Lambda)^{-1}$ associated to K_Λ. Notice that $J_{[\Lambda]}$ is different from J_Λ.

Lemma 4.1 *For each $\Lambda \in \mathcal{B}_0$, the operator $J_{[\Lambda]}$ admits a continuous kernel function $J_{[\Lambda]}(x,y)$, $x,y \in \Lambda$.*

Proof: A constructive approach to the Radon-Nikodym theorem implies that $J_{[\Lambda]}$ admits a (jointly) measurable kernel function $J'_{[\Lambda]}(x,y)$, $x,y \in \Lambda$. Since

$$J_{[\Lambda]} = K^J_\Lambda + (K^J_\Lambda)^2 + K^J_\Lambda J_{[\Lambda]} K^J_\Lambda,$$

$J'_{[\Lambda]}(x,y)$ coincides for almost all $x,y \in \Lambda$ with

$$\begin{aligned} J_{[\Lambda]}(x,y) := k^J(x-y) &+ \int_\Lambda k^J(x-u)\,k^J(u-y)\,du \\ &+ \int_\Lambda\int_\Lambda k^J(x-u)\,J'_{[\Lambda]}(u,v)\,k^J(v-y)\,du\,dv\,. \end{aligned}$$

Since k^J is uniformly continuous by Eq. (3.11), it is easily checked that $J_{[\Lambda]}(x,y)$ is continuous, as required. $\Diamond$

Henceforth we fix any continuous kernel of $J_{[\Lambda]}$. We are interested in the matrices $J_{[\Lambda]}(\xi,\xi)$ obtained by evaluating this kernel at the points of arbitrary configurations $\xi \in \mathcal{X}_\Lambda$; cf. Definition 2.1. Our analysis relies on the following general fact which appears already in [16, p. 18]. Since no proof is given there, we provide a proof here for the convenience of the reader.

Lemma 4.2 *For any invertible positive operator T and any projection P,*

$$PT^{-1}P \geq P(PTP)^{-1}P.$$

Proof: Since T is positive and invertible, any restriction of T to some subspace is invertible in this subspace. This means that the operators $T_P^{-1} := P(PTP)^{-1}P$ and $T_{P^\perp}^{-1} := P^\perp(P^\perp T^{-1}P^\perp)^{-1}P^\perp$ are well-defined, where $P^\perp = I-P$. The key observation is the decomposition formula

$$PT^{-1}P = T_P^{-1} + PT^{-1}T_{P^\perp}^{-1}T^{-1}P\,. \tag{4.18}$$

To see this we observe that $P^\perp T^{-1}(P^\perp + P)TP = 0$. Multiplying with $T_{P^\perp}^{-1}$ from the left we find $P^\perp TP = -T_{P^\perp}^{-1}T^{-1}PTP$. Inserting this into the identity $PT^{-1}(PTP + P^\perp TP) = P$ we get

$$PT^{-1}\Big(I - T_{P^\perp}^{-1}T^{-1}\Big)PTP = P.$$

Multiplying with T_P^{-1} from the right and rearranging we arrive at (4.18). As the second operator on the right-hand side of (4.18) is positive, the lemma follows. ◊

Next we state a monotonicity result which is interesting in its own right; we will only need a weak form of it.

Lemma 4.3 *For any $\Lambda \subset \Delta \in \mathcal{B}_0$, $J_{[\Lambda]} \leq P_\Lambda J_{[\Delta]} P_\Lambda \leq J_\Lambda$ in the operator ordering. In particular,*

$$\det J_{[\Lambda]}(\xi,\xi) \leq \det J_{[\Delta]}(\xi,\xi) \leq \det J(\xi,\xi) \tag{4.19}$$

for all $\xi \in \mathcal{X}_\Lambda$.

Proof: We prove the inequality $J_{[\Lambda]} \leq J_\Lambda$ which implies in particular that $P_\Lambda J_{[\Delta]} P_\Lambda \leq P_\Lambda J_\Delta P_\Lambda = J_\Lambda$. The proof of the relation $J_{[\Lambda]} \leq P_\Lambda J_{[\Delta]} P_\Lambda$ is similar. Since $J = K^J(I-K^J)^{-1}$ and $L(I-L)^{-1} = -I + (I-L)^{-1}$ for any $0 \leq L < I$, the inequality $J_{[\Lambda]} \leq J_\Lambda$ is equivalent to

$$P_\Lambda(I-K^J_\Lambda)^{-1}P_\Lambda \leq P_\Lambda(I-K^J)^{-1}P_\Lambda.$$

But this follows from Lemma 4.2 as applied to $P = P_\Lambda$ and $T = I - K^J$, because $P_\Lambda(I-K^J_\Lambda)^{-1}P_\Lambda = P_\Lambda(P_\Lambda(I-K^J)P_\Lambda)^{-1}P_\Lambda$.

To prove inequality (4.19) we note that, for any two positive definite $n \times n$ matrices $0 \leq A \leq B$, the decreasingly ordered eigenvalues of A resp. B satisfy the inequalities

$$0 \leq \lambda_i^\downarrow(A) \leq \lambda_i^\downarrow(B), \quad i = 1, \ldots, n;$$

see [1, Corollary III.2.3]. The determinant is therefore an increasing function relative to the operator ordering. The inequalities (4.19) will thus be proved once we have shown that

$$J_{[\Lambda]}(\xi,\xi) \le J_{[\Delta]}(\xi,\xi) \le J(\xi,\xi)$$

as operators on $\mathbb{C}^n$, $n = |\xi|$. This, however, is a direct consequence of the continuity of the kernels $J_{[\Lambda]}(x,y)$ established in Lemma 4.1 together with the operator inequalities just proved; it is enough to approximate the Dirac delta-measures at the points $x \in \xi$ by $L^2(\Lambda)$-functions. $\Diamond$

We will also need the following inequalities for determinants; (4.21) is known as the Fischer inequality.

Lemma 4.4 *The following relations hold for determinants of positive definite Hermitian block matrices:*

$$\det \begin{pmatrix} A & D \\ D^* & B \end{pmatrix} = \det(A - DB^{-1}D^*) \det B \quad \textit{if } \det B > 0, \tag{4.20}$$

$$\det \begin{pmatrix} A & D \\ D^* & B \end{pmatrix} \le \det A \det B\,, \tag{4.21}$$

$$\det \begin{pmatrix} A & D & E \\ D^* & B & F \\ E^* & F^* & C \end{pmatrix} \det B \le \det \begin{pmatrix} A & D \\ D^* & B \end{pmatrix} \det \begin{pmatrix} B & F \\ F^* & C \end{pmatrix}. \tag{4.22}$$

Proof: Eqn. (4.20) follows from the identity

$$\begin{pmatrix} A & D \\ D^* & B \end{pmatrix} = \begin{pmatrix} A - DB^{-1}D^* & DB^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ D^* & B \end{pmatrix},$$

where I is the identity matrix. Since $DB^{-1}D^* \ge 0$ and the determinant is monotone with respect to the operator ordering, this implies (4.21) in the invertible case. For general B, one can approximate B by invertible matrices. To prove (4.22) one can assume that $\det B > 0$ and then divide both sides by $\det^2 B$. In view of (4.20), (4.22) then takes the form

$$\det \left[\begin{pmatrix} A & E \\ E^* & C \end{pmatrix} - \begin{pmatrix} D \\ F^* \end{pmatrix} B^{-1} \big(D^* F\big) \right] \le \det(A - DB^{-1}D^*) \det(C - F^*B^{-1}F)$$

which follows from (4.21). $\Diamond$

We conclude this section with a convergence result.

Proposition 4.5 *As $\Lambda \in \mathcal{B}_0$ increases to $\mathbb{R}^d$, $J_{[\Lambda]}(x,y)$ converges to $j(x-y)$ locally uniformly in the variables $x, y \in \mathbb{R}^d$.*

Proof: First we observe that $k^J \in L^2(\mathbb{R}^d)$. This follows from Eq. (3.11) and Plancherel's theorem because $\varphi/(1+\varphi) \in L^2(\mathbb{R}^d)$; see Example (c) on p. 483 of [4], for example. In particular, if we set $k_x(u) := k^J(u-x)$ for $u \in \mathbb{R}^d$ then $k_x(\cdot) \in L^2(\mathbb{R}^d)$ for each $x \in \mathbb{R}^d$.

Next, for each $N \geq 1$ we can write

$$J_{[\Lambda]} = \sum_{i=1}^{N+2} (K_\Lambda)^i + (K_\Lambda)^{N+2} J_{[\Lambda]} \quad \text{and} \quad J = \sum_{i=1}^{N+2} K^i + K^{N+2} J \,.$$

Since $0 \leq K_\Lambda \leq K < I$, the operator norms $\|(K_\Lambda)^N\| \leq \|K^N\|$ become arbitrarily small when N is large enough. Moreover, $\|J_\Lambda\|$ is bounded uniformly in Λ. Hence, if N is sufficiently large then the kernel

$$(K_\Lambda)^{N+2} J_{[\Lambda]}(x,y) = (k_x, (K_\Lambda)^N J_{[\Lambda]} k_y)_{L^2(\mathbb{R}^d)}$$

is arbitrarily small uniformly in x, y, Λ (including $\Lambda = \mathbb{R}^d$). It is therefore sufficient to show that, for each $i \in \mathbb{N}$,

$$(K_\Lambda)^i(x,y) \to K^i(x,y)$$

locally uniformly in x, y as $\Lambda \uparrow \mathbb{R}^d$. The case $i = 1$ is trivial. For $i = 2$ we have

$$(K_\Lambda)^2(x,y) = (P_\Lambda k_x, P_\Lambda k_y)_{L^2(\mathbb{R}^d)} \,,$$

and this converges to $(k_x, k_y)_{L^2(\mathbb{R}^d)} = K^2(x,y)$ locally uniformly in x, y. In the case $i \geq 3$ we write

$$(K_\Lambda)^i(x,y) = (k_x, (K_\Lambda)^{i-2} k_y)_{L^2(\mathbb{R}^d)}.$$

Since K_Λ converges strongly to K as $\Lambda \uparrow \mathbb{R}^d$, the inner product on the right-hand side converges to $(k_x, (K)^{i-2} k_y)_{L^2(\mathbb{R}^d)} = K^i(x,y)$ locally uniformly in x, y. $\Diamond$

5 Proofs

Let us start proving Theorem 3.2.

Proof of Theorem 3.2: (a) Fix any $\Lambda \in \mathcal{B}_0$. From Eq. (2.3) we know that μ_Λ is absolutely continuous relative to L_Λ with density $\sigma_\Lambda(\xi) \propto \det J_{[\Lambda]}(\xi, \xi)$. Inequality (4.21) implies that the zero set $\{\sigma_\Lambda = 0\}$ is increasing. Remark 2.7 thus tells us that μ_Λ satisfies condition $(\Sigma_{\lambda_\Lambda})$ with CPI given by (3.12). This proves statement (a) of the theorem.

(b) The first inequality in (b) is an immediate consequence of (4.22) which asserts that $\det J_{[\Lambda]}(x\eta, x\eta) \det J_{[\Lambda]}(\xi, \xi) \leq \det J_{[\Lambda]}(\eta, \eta) \det J_{[\Lambda]}(x\xi, x\xi)$ whenever $\xi \subset \eta \in \mathcal{X}_\Lambda$. In particular, we have $c^J_\Lambda(x, \xi) \leq c^J_\Lambda(x, \emptyset)$ for the empty configuration $\emptyset$. But the last quantity equals $\det J_{[\Lambda]}(x, x)$ by definition, which in turn is not larger than $z(J)$, by inequality (4.19). This proves the second inequality of assertion (b). $\Diamond$

We now turn to the proofs of Corollaries 3.3 through 3.6.

Proof of Corollary 3.3: In view of Eq. (3.13),

$$G^J_{\Lambda,\Delta}(f|\xi) \,/\, G^J_{\Lambda,\Delta}(N_\Lambda = 0|\xi) = \int L_\Lambda(d\alpha)\, f(\alpha)\, \widehat{c}^J_\Delta(\alpha, \xi)$$

for any $\xi \in \mathcal{X}_{\Delta\setminus\Lambda}$. Since $f \geq 0$, Theorem 3.2(b) together with equation (2.5) implies that the integrand on the right-hand side is a decreasing function of ξ. In particular, for $f \equiv 1$ we find that $G^J_{\Lambda,\Delta}(N_\Lambda = 0|\cdot)$ is increasing. To prove (3.14) we note that

$$\begin{aligned}
&\mu^J(N_\Lambda = 0|N_{\Delta_2\setminus\Lambda} = 0) = G^J_{\Lambda,\Delta_2}(N_\Lambda = 0|\emptyset) \\
&\leq \mathbb{E}_{\mu^J}\big(G^J_{\Lambda,\Delta_2}(N_\Lambda = 0|\cdot)\big|\mathcal{F}_{\Delta_1\setminus\Lambda}\big)(\emptyset) \\
&= \mu^J(N_\Lambda = 0|\mathcal{F}_{\Delta_1\setminus\Lambda})(\emptyset) = \mu^J(N_\Lambda = 0|N_{\Delta_1\setminus\Lambda} = 0)
\end{aligned}$$

because $G^J_{\Lambda,\Delta_2}(N_\Lambda = 0|\emptyset) \leq G^J_{\Lambda,\Delta_2}(N_\Lambda = 0|\cdot)$. The final inequality follows by setting $\Delta_1 = \Lambda$. $\Diamond$

Proof of Corollary 3.5: We show first that $\mu^J_\Lambda \preceq \pi^{z(J)}_\Lambda$ for any $\Lambda \in \mathcal{B}_0$. It is well-known and easy to check that the Poisson PP $\pi^{z(J)}_\Lambda$ has the constant PI $z(J)$ on Λ; cf. [13, 15]. The inequality $c^J_\Lambda(x,\xi) \leq z(J)$ established in Theorem 3.2(b) thus means that the PI of μ^J_Λ at a configuration $\xi \in \mathcal{X}_\Lambda$ is not larger than the PI of $\pi^{z(J)}_\Lambda$ evaluated at any larger configuration $\eta \in \mathcal{X}_\Lambda$, as long as $x \notin \eta$. This, however, is precisely the hypothesis of the point-process counterpart of the well-known FKG-Holley-Preston inequality for lattice systems. This continuous counterpart was obtained first by Preston [18]; an alternative simplified proof can be found in [5, Theorem 1.1]. It asserts that, under the above condition on the PI's, $\mu^J_\Lambda \preceq \pi^{z(J)}_\Lambda$, as required.

To get rid of the locality restriction we argue as follows. For any compact Λ, a celebrated theorem of Strassen [28] provides us with a probability measure m_Λ on $\mathcal{X}\times\mathcal{X}$ having marginals μ^J_Λ resp. $\pi^{z(J)}_\Lambda$ and being supported on the set $D := \{(\xi,\eta) \in \mathcal{X}\times\mathcal{X} : \xi \subset \eta\}$. Note that D is closed when $\mathcal{X}\times\mathcal{X}$ is equipped with the product of the vague topology on $\mathcal{X}$. By a standard compactness criterion for point processes [3, Proposition 9.1.V], the measures m_Λ admit a weak limiting measure m as Λ increases to $\mathbb{R}^d$. By construction, m has marginals μ^J resp. $\pi^{z(J)}$, and $m(D) = 1$ because D is closed. This implies that $\mu^J \preceq \pi^{z(J)}$. $\Diamond$

Proof of Corollary 3.6: As the event $\{\exists$ infinite cluster of $B_R(\cdot))\}$ is increasing, the result follows immediately from Corollary 3.5. $\Diamond$

Finally we provide the proofs of Theorems 3.7 and 3.9.

Proof of Theorem 3.7: We only need to show that μ^J satisfies condition (Σ_λ) with CPI (3.15); the remaining assertions then follow from Theorem 3.2 and Remark 2.5(c). Let (Δ_n) be any increasing sequence in $\mathcal{B}_0$ exhausting $\mathbb{R}^d$, $\Lambda \in \mathcal{B}_0$ a fixed set, and n so large that $\Lambda \subset \Delta_n$. Consider the product space $\mathcal{X}_\Lambda \times \mathcal{X}$, equipped with the probability measure $\nu^J_\Lambda := \pi^1_\Lambda \otimes \mu^J$ and the σ-algebras $\mathcal{G}_n = (\mathcal{F}|_{\mathcal{X}_\Lambda}) \otimes \mathcal{F}_{\Delta_n}$. Now, Theorem 3.2(a) asserts that, on $\mathcal{G}_n$, the restriction of $\widehat{C}_{\mu^J}$ to $\mathcal{X}_\Lambda \times \mathcal{X}$ is absolutely continuous with respect to ν^J_Λ with Radon-Nikodym density $R_n := e^{|\Lambda|}\,\widehat{c}^J_{\Delta_n}$. The sequence (R_n) is therefore a nonnegative martingale relative to ν^J_Λ. In view of Theorem 3.2(b), R_n satifies the uniform bound $R_n(\alpha,\xi) \leq S(\alpha) := e^{|\Lambda|}\,z(J)^{|\alpha|}$, and S is integrable relative to π^1_Λ. This means that R_n converges ν^J_Λ-almost surely and in $L^1(\nu^J_\Lambda)$-norm to a limit R. By the norm-convergence, R is a Radon-Nikodym density of $\widehat{C}_{\mu^J}$ relative to ν^J_Λ on the limiting σ-algebra $\sigma(\bigcup_n \mathcal{G}_n) = (\mathcal{F}|_{\mathcal{X}_\Lambda}) \otimes \mathcal{F}$. Finally we replace π^1_Λ with L_Λ by dropping the constant factor $e^{|\Lambda|}$, and use the fact that $\mathcal{X}_\Lambda \uparrow \mathcal{X}_0$ as $\Lambda \uparrow \mathbb{R}^d$. We then get the desired result that $\widehat{C}_{\mu^J} \ll L \otimes \mu^J$ with density (3.15). $\Diamond$

Proof of Theorem 3.9: (a) Consider the CPI

$$\widehat{c}^J(\alpha,\xi) = \lim_{n\to\infty} \widehat{c}^J_{\Delta_n}(\alpha,\xi_{\Delta_n}).$$

that occurs in (3.15); we know that this limit exists for $L \otimes \mu^J$-almost all $(\alpha,\xi) \in \mathcal{X}_0 \times \mathcal{X}$. The monotonicity result of Theorem 3.2(b) implies that, for each $\Lambda \in \mathcal{B}_0$,

$$\widehat{c}^J(\alpha,\xi) \le \liminf_{n\to\infty} \widehat{c}^J_{\Delta_n}(\alpha,\xi_\Lambda).$$

Now, Proposition 4.5 asserts that

$$\widehat{c}^J_{\Delta_n}(\alpha,\xi_\Lambda) \to \widehat{c}^J_*(\alpha,\xi_\Lambda) := \det J(\alpha\xi_\Lambda,\alpha\xi_\Lambda)/\det J(\xi_\Lambda,\xi_\Lambda)$$

as $n \to \infty$, provided that $\det J(\xi_\Lambda,\xi_\Lambda) > 0$. In view of (2.3) and (4.19), the last proviso holds true for μ^J-almost all ξ. Hence $\widehat{c}^J(\alpha,\xi) \le c^J_*(\alpha,\xi_\Lambda)$ for $L \otimes \mu^J$-almost all (α,ξ). As we have noticed after (3.16), $c^J_*(\alpha,\xi_\Lambda)$ is a decreasing function of Λ, and its limit is equal to $c^J_*(\alpha,\xi)$ by definition. This completes the proof of statement (a).

(b) Fix any $\Lambda \in \mathcal{B}_0$, $\alpha \in \mathcal{X}_\Lambda$, and consider the function $\widehat{c}^J_*(\alpha,\cdot)$. Let $W(\alpha,\cdot)$ be the union of all clusters of $B_R(\cdot)$ hitting α. If $\xi \in \mathcal{X}$ is such that $W(\alpha,\xi)$ is finite then $J(x,y) = 0$ for all $x \in \alpha\xi^\sharp$ and $y \in \xi^\flat$, where $\xi^\sharp = \xi_{W(\alpha,\xi)}$ and $\xi^\flat = \xi_{\mathbb{R}^d\setminus W(\alpha,\xi)}$. Hence, for any bounded $\Delta \supset W(\alpha,\xi)$, the matrix $J(\alpha\xi_\Delta,\alpha\xi_\Delta)$ consists of the two diagonal blocks $J(\alpha\xi^\sharp,\alpha\xi^\sharp)$ and $J(\xi^\flat_\Delta,\xi^\flat_\Delta)$. Thus

$$\det J(\alpha\xi_\Delta,\alpha\xi_\Delta) = \det J(\alpha\xi^\sharp,\alpha\xi^\sharp)\det J(\xi^\flat_\Delta,\xi^\flat_\Delta).$$

Dividing this by the analoguous equation for $\alpha = \emptyset$ we arrive at the second identity in (3.17). In particular, it follows that $c^J_*(\alpha,\cdot)$ is continuous on the set of configurations having no infinite cluster, and thus is continuous almost everywhere.

Consider now the DPP's μ^{J_Δ} relative to $J_\Delta = P_\Delta J P_\Delta$, $\Lambda \subset \Delta \in \mathcal{B}_0$. According to Theorem 6.17 of [21], μ^{J_Δ} converges weakly to μ^J as Δ increases to $\mathbb{R}^d$. Let f be any bounded continuous function on $\mathcal{X}_0 \times \mathcal{X}$ such that $f(\alpha,\xi) = 0$ unless $\alpha \subset \Lambda$ and $|\alpha\xi_\Lambda| \le k$ for some number k. The portmanteau theorem [3, Proposition A2.3.V] then implies that

$$\int \widehat{c}^J_*\, f\, d(L_\Lambda \otimes \mu^J) = \lim_{\Delta\uparrow\mathbb{R}^d} \int \widehat{c}^J_*\, f\, d(L_\Lambda \otimes \mu^{J_\Delta}).$$

But μ^{J_Δ} is supported on $\mathcal{X}_\Delta$, and $\widehat{c}^J_* = \widehat{c}^{J_\Delta}_*$ on $\mathcal{X}_\Lambda \times \mathcal{X}_\Delta$. Moreover, it is evident from the proof of statement (a) that $\widehat{c}^{J_\Delta}_*$ is a CPI of μ^{J_Δ}. Eq. (2.6) thus shows that

$$\int \widehat{c}^J_*\, f\, d(L_\Lambda \otimes \mu^{J_\Delta}) = \int \mu^{J_\Delta}(d\xi) \sum_{\alpha\in\mathcal{X}_0:\,\alpha\subset\xi} f(\alpha,\xi\setminus\alpha).$$

The integrand on the right-hand side is still a bounded continuous function of ξ. Letting $\Delta \uparrow \mathbb{R}^d$ we thus find that

$$\int \widehat{c}^J_*\, f d(L \otimes \mu^J) = \int f\, d\widehat{C}_{\mu^J}.$$

As f, k and Λ were arbitrarily chosen, it follows that $\widehat{c}_*^J$ is a CPI of μ^J. $\Diamond$

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