Renormalization of Hamiltonians for Diophantine frequency vectors

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We construct a rigorous renormalization scheme for analytic two degree-of-freedom Hamiltonians, which applies to the problem of the stability of invariant tori with Diophantine frequency vectors. We prove the existence of an attracting, integrable limit set of the renormalization.

1. INTRODUCTION

We define a sequence of renormalization operators $\mathcal{R}_n$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, between Banach spaces of analytic Hamiltonians of two degree-of-freedom systems. The Hamiltonians are functions on complex neighborhoods of $\mathbb{T}^2 \times \{0\}$, where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and 0 is the zero vector in $\mathbb{R}^2$. We call these neighborhoods the phase space. We refer to the coordinates of this space, q and p, as angles and momenta, respectively.

Each of the Banach spaces, indexed by n, contains an integrable Hamiltonian H_n^0 with an invariant torus of a given frequency $\omega_n \in \mathbb{R}^2$ at $p = 0$ and frequencies of nearby tori *twisted* in the direction of $\Omega_n \in \mathbb{R}^2$. The vectors ω_n and Ω_n are uniquely determined given $\omega = \omega_0 \in \mathbb{R}^2$ and $\Omega = \Omega_0 \in \mathbb{R}^2$. The domain of a renormalization operator $\mathcal{R}_n$ will be restricted to a small neighborhood of the integrable Hamiltonian H_n^0, keeping the analysis within the scope of classical KAM theory [1–3]. The renormalization operator $\mathcal{R}_n$ is a function from that neighborhood into another that acts on the winding ratio $\alpha_n = \omega_{n2}/\omega_{n1}$, $\omega_{n1} \neq 0$, of the frequency vector $\omega_n = (\omega_{n1}, \omega_{n2})^*$, as a shift of its continued fraction expansion. If $\alpha_n > 1$, then the inverse ratio α_n^{-1} is mapped according to the Gauss map. We refer to the renormalization operator $\mathcal{R}_n$ as the n^{th}-step renormalization operator and to the sequence of one-step operators $\mathcal{R}_0, \mathcal{R}_1, \ldots,$ as the sequence of renormalization operators. The renormalization consists of successive application of the operators from this sequence. Given a Hamiltonian H_0, we call the sequence of Hamiltonians H_n, $n \in \mathbb{N}_0$, consisting of H_0 and its images $H_{n+1} = \mathcal{R}_n \circ \cdots \circ \mathcal{R}_0(H_0)$, the orbit of the Hamiltonian H_0.

We show that the orbits of all Hamiltonians in a neighborhood of an integrable Hamiltonian H_0^0 associated with a Diophantine frequency vector ω_0, approach the orbit of the integrable Hamiltonian. In a separate publication [4], we apply this result to construct the Diophantine invariant tori for near-integrable Hamiltonians.

Renormalization group techniques originated in theoretical physics, where they have been extensively used since the pioneering work of Stueckelberg and Peterman [5] in quantum field theory and Kadanoff [6] in statistical mechanics. In the theory of dynamical systems, they were introduced by Feigenbaum [7, 8] and Coullet and Tresser [9] in studying the universality of period-doubling sequences in one-parameter families of one-dimensional maps. Since then, renormalization methods have been widely applied in investigations of a variety of dynamical phenomena including self-similarity of period-doubling bifurcations in two and higher-dimensional mappings [10, 11], bifurcations of invariant KAM curves in area-preserving maps [12–15] and bifurcations in circle maps [16–18]. In the framework of the breakup of invariant tori in Hamiltonian flows, renormalization ideas were first introduced by Escande and Doveil [19].

A rigorous renormalization scheme for analytic Hamiltonians was formulated by Koch [20, 21]. The renormalization considered here is an extension of that scheme, from quadratic irrational to a set of full Lebesgue measure Diophantine winding ratios, in the case of two degree-of-freedom Hamiltonians. Recently, such a generalization was done by Lopes-Dias for vector fields on a torus of dimension two [22]. It is also related to the approximate renormalization schemes of MacKay [23] and Chandre and Moussa [24] in the framework of the break-up of invariant tori in Hamiltonian systems (see also [25]).

Roughly speaking, renormalization techniques in dynamical systems are designed to study systems on progressively smaller spatial scales and longer time scales. Our renormalization operators basically consist of time-rescaling, composed with a nonlinear diffeomorphism homotopic to the identity, and a linear scaling of the (lifted) phase space.

The linear scaling transformation is intended to enlarge a region around the orbits of the integrable Hamiltonian flow while keeping the periodicity of the angle coordinates. It is a composition of a linear scaling of the momentum space and a linear canonical transformation of the phase space generated by a point transformation from $GL(2,\mathbb{Z})$.

The growth of the Fourier modes of the Hamiltonian in the direction of the dominant flow is prevented by eliminating these modes in each step. This is achieved by a nonlinear canonical transformation homotopic to the identity. The process of elimination (of 'irrelevant' modes of a Hamiltonian) and rescaling (of the Fourier lattice) is in spirit similar to block spin transformations in statistical mechanics - a standard tool in the theory of critical phenomena.

Additionally, time-rescaling, a nonlinear scaling of the momenta and a translation in momentum space are included. The latter transformation prevents the renormalization operators from having an expanding eigendirection. These transformations are also homotopic to the identity and do not change the winding number of the orbits of the Hamiltonian flow.

The sequence of renormalization transformations is associated to the sequence of vector pairs (ω_n, Ω_n), $n \in \mathbb{N}_0$. The construction of these vectors is based on the continued fraction expansion of the winding ratio α. In the next section, we recall some of the relevant properties of this expansion and construct the sequence of bases $\{\omega_n, \Omega_n\}$ in $\mathbb{R}^2$. In section 3, we define the Banach spaces of the Hamiltonians to be renormalized and show that the n^{th}-step renormalization operator is well-defined. The existence of a canonical transformation that eliminates the non-resonant modes of a near-integrable Hamiltonian is proved in section 4. In section 5, we show that if ω satisfies a Diophantine condition, then there is convergence of the renormalized Hamiltonians to the orbit of the integrable Hamiltonian. The main result is summarized in Theorem 5.1.

2. REVIEW OF THE THEORY OF CONTINUED FRACTION EXPANSION AND THE CONSTRUCTION OF PAIRS (ω_n, Ω_n)

The renormalization transformation for d degree-of-freedom Hamiltonians introduced by Koch is associated to a frequency vector ω whose components span an algebraic number field of degree $d \in \mathbb{N}$. For two degree-of-freedom systems, that renormalization can be applied to Hamiltonians associated to a frequency vector with a quadratic irrational slope. In that case, one can find a hyperbolic matrix $T \in GL(2,\mathbb{Z})$ with determinant ± 1, for which ω is an expanding eigenvector (Lemma 4.1 in [20]). That matrix is then used to perform linear scaling of the phase space at each renormalization step.

In order to extend the procedure to a larger set of frequency vectors, we use a different linear scaling transformation at each step. The n^{th}-step transformation is associated to the pair of vectors (ω_n, Ω_n). The frequency vector ω_n can be constructed from the continued fraction expansion of the winding ratio α of the frequency vector ω. We review first the relevant properties of this expansion.

The unique continued fraction expansion of an irrational $\alpha \in \mathbb{R}$ is given by

$$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \ldots}}\,, \tag{2.1}$$

where $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{Z}^+$, $n \geq 1$. We will also write $\alpha = [a_0, a_1, \ldots]$.

Let $[\alpha]$ be the integer part of α, i.e. $[\alpha] = \max\{k \in \mathbb{Z} : k \leq \alpha\}$. Also, let $\{\alpha\} = \alpha - [\alpha]$ be the fractional part of α.

For $x > 0$, the Gauss map is defined as

$$G : x \mapsto \left\{ \frac{1}{x} \right\}. \tag{2.2}$$

If we define $\alpha_n = [a_n, a_{n+1}, \ldots]$ and $x_n = \alpha_n - a_n$, for $n \geq 0$, we obtain $x_{n+1} = G(x_n)$. Thus, the shift $[a_n, a_{n+1}, \ldots] \mapsto [a_{n+1}, a_{n+2}, \ldots]$ of the continued fraction expansion of α_n corresponds to the Gauss map of its fractional part x_n.

The transformation $\alpha_n \mapsto \alpha_{n+1} = (\alpha_n - a_n)^{-1}$ is a special case of a modular transformation,

$$\alpha \mapsto \frac{T_{2,1} + \alpha T_{2,2}}{T_{1,1} + \alpha T_{1,2}}\,,$$

generated by the action of an integer matrix $T = (T_{i,j}) \in GL(2,\mathbb{Z})$ of determinant ± 1. Numbers related by such a transformation are called equivalent. Two numbers are equivalent if and only if they have the same tail in their continued fraction expansions (Theorem 175 in [26]).

At the n^{th} renormalization step, a linear coordinate change will be generated using the matrix

$$T_n = \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix} \in GL(2,\mathbb{Z})\,,$$

where, as before, $a_n = [\alpha_n]$. This matrix generates an equivalence relation between α_{n+1} and α_n.

We define $\omega_n = \ell(1,\alpha_n)^* \in \mathbb{R}^2$, where $\ell \in \mathbb{R}^+$. Thus, $\omega_{n+1} = \alpha_{n+1} T_n^{-1} \omega_n$.

If $\alpha > 1$, then, for all $n \geq 0$, $a_n \geq 1$ and T_n is a hyperbolic matrix with determinant (-1) and eigenvalues λ_n and $-1/\lambda_n$, where

$$\lambda_n = \frac{a_n + \sqrt{a_n^2 + 4}}{2}\,.$$

The expanding and contracting eigendirections corresponding to these eigenvalues are given by $(1,\lambda_n)^*$ and $(1,-1/\lambda_n)^*$, respectively. The expanding eigenvector is close to ω_n, in the sense that the absolute value of the angle between it and ω_n is smaller than $\pi/4$. The matrix T_n, therefore, provides the appropriate shift of Fourier modes and can be used for the above-mentioned linear coordinate change.

Now, define the sequence of convergent matrices associated to α,

$$P_n = \begin{bmatrix} q_{n-1} & p_{n-1} \\ q_n & p_n \end{bmatrix}, \text{for } n \geq 0 \text{ and } P_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

recursively, via $P_n = T_n P_{n-1}$. Thus, q_n and p_n are determined by the following recursion relations

$$\begin{aligned} q_n &= a_n q_{n-1} + q_{n-2}, \\ p_n &= a_n p_{n-1} + p_{n-2}, \end{aligned}$$

and $q_{-2} = 1, p_{-2} = 0, q_{-1} = 0, p_{-1} = 1$. Using the convergent matrices one can obtain the vectors $(1,\alpha_n)^* = (-1)^n (p_{n-1} - \alpha q_{n-1})^{-1} P_{n-1}^{*-1}(1,\alpha)^*$. Therefore,

$$\alpha = \frac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}}\,. \tag{2.3}$$

The convergent matrices define a sequence of convergents $p_n/q_n = [a_0,\ldots,a_n]$ that approaches α as $n \to \infty$. The convergents satisfy (Theorem 171 in [26]),

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}\,. \tag{2.4}$$

They are the best rational approximates of α, in the sense that for any $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, such that $0 < q \leq q_n$ and $p/q \neq p_n/q_n$, $n > 0$, one has $|p_n - \alpha q_n| < |p - \alpha q|$ (Theorem 182 in [26]).

From the equality (2.3), for $n \geq 0$, we find

$$x_n = -\frac{p_n - \alpha q_n}{p_{n-1} - \alpha q_{n-1}}\,.$$

Denoting the product of the first $(n+1)$ numbers x_i by $\beta_n = \prod_{i=0}^n x_i = \prod_{i=0}^n (1/\alpha_{i+1})$, we obtain that

$$\beta_n = (-1)^{n+1}(p_n - \alpha q_n)\,.$$

Notice that $\det P_n = (-1)^{n+1}$. Using the equality (2.3) again, one easily finds that

$$\beta_n = \frac{1}{q_{n+1} + q_n x_{n+1}}\,,$$

and thus,

$$\frac{1}{2q_{n+1}} < \beta_n < \frac{1}{q_{n+1}}\,.$$

Define $\tilde{A}_n = \prod_{i=0}^{n} \alpha_i$. As $\beta_n = \alpha_0/\tilde{A}_{n+1}$, the previous bound implies

$$\alpha_0 q_n < \tilde{A}_n < 2\alpha_0 q_n \,, \tag{2.5}$$

assuming that $\alpha_0 > 0$.

The values of q_n and p_n are associated to α. In the following, we will stress this fact by writing explicitly $q_n(\alpha)$ and $p_n(\alpha)$. Notice that $q_{n+1}(\alpha_0) = p_n(\alpha_1)$. The double inequality (2.5) for $(n+1)$ instead of n, can be written as

$$p_n(\alpha_1) < \frac{\tilde{A}_{n+1}}{\alpha_0} < 2p_n(\alpha_1) \,,$$

implying the following bounds on p_n,

$$p_n < \tilde{A}_n < 2p_n \,. \tag{2.6}$$

It is easy to show that $\tilde{A}_n$ grows at least exponentially with n. If $1 < \alpha_i \leq \gamma$, for some $i \in \{0, \ldots, n-1\}$, then $\alpha_i \alpha_{i+1} = \alpha_i/(\alpha_i - a_i) \geq \gamma/(\gamma - 1) = \gamma^2$, where $\gamma = (1+\sqrt{5})/2$ is the golden mean, the limit of the sequence of ratios F_{k+1}/F_k of successive Fibonacci numbers F_k, defined by $F_{k+2} = F_{k+1} + F_k$, for $k \in \mathbb{N}$, and $F_1 = F_2 = 1$. This implies that $\tilde{A}_n/\tilde{A}_{j-1} \geq \gamma^{n-j}$, for $0 < j \leq n$. If $\alpha_0 > 1$, then $\tilde{A}_n \geq \gamma^n$ and the previous inequality is also valid for $j = 0$, with $\tilde{A}_{-1} = 1$.

This growth can be controlled if α is a Diophantine number.

Definition 2.1 *An irrational number α will be called Diophantine of order $\beta \geq 0$ if there exists a constant $C > 0$, such that for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$,*

$$\left| \alpha - \frac{p}{q} \right| > \frac{C}{q^{2+\beta}} \,. \tag{2.7}$$

The set of all Diophantine numbers of order β will be denoted by $DC(\beta)$. Sometimes, less precisely, we will call Diophantine a vector $\omega \in \mathbb{R}^2$ with a Diophantine winding ratio. The Diophantine condition on ω states that there exists $C > 0$, such that for all $\nu \in \mathbb{Z}^2 \backslash \{0\}$, $|\omega \cdot \nu| > C\|\nu\|^{-(1+\beta)}$.

The Diophantine condition on α, together with the inequality (2.4), imposes an upper bound on the growth rate of the denominators of its convergents. For $\alpha \in DC(\beta)$, there exists a constant $K > 0$ such that $q_{n+1} < Kq_n^{1+\beta}$, for all $n \in \mathbb{N}_0$. Equivalently, using the inequalities (2.5), the Diophantine condition can be written as

$$\tilde{A}_{n+1} < \tilde{K}\tilde{A}_n^{1+\beta} \,, \tag{2.8}$$

where $\tilde{K} > 0$ is a constant.

In particular, constant-type numbers, which have bounded continued fraction expansion elements, are Diophantine of order zero. Among them are quadratic irrationals, i.e. the roots of quadratic equations with integer coefficients, whose continued fraction expansions are eventually periodic. Constant-type numbers have zero Lebesgue measure in the real numbers. Diophantine numbers of any order $\beta > 0$ are of measure one.

The bound (2.8) plays an essential role in the proof of our main Theorem concerning the existence of a trivial attracting orbit of the sequence of renormalization operators which is associated to a Diophantine vector $\omega \in \mathbb{R}^2$. The construction of a one-step renormalization transformation is, however, more general.

We assume that $\omega \in \mathbb{R}^2$ is of the form $\omega = \ell(1, \alpha)^*$, where $\alpha > 1$ and $\ell \in \mathbb{R}^+$. For the vectors $\omega = \ell(1, \alpha)^*$ with $\alpha < 1$, a canonical transformation of the phase space can be performed, generated by a matrix from $GL(2, \mathbb{Z})$, such that the 'new' vector ω is of the desired form. Thus, without loss of generality we can assume that $\alpha > 1$ and, consequently, for any $n \in \mathbb{N}_0$, $\omega_n = \ell(1, \alpha_n)^*$ with $\alpha_n > 1$.

We choose a vector $\Omega \in \mathbb{R}^2$ which is suitable in the sense of the following Definition.

Definition 2.2 *$\Omega \in \mathbb{R}^2$ will be called suitable if it is of the form $\Omega = (x, y)^*$, with $|x| \geq |y| \geq 0$, $xy \leq 0$ and $\|\Omega\| = 1$.*

Here and in what follows, $\|\cdot\|$ denotes the ℓ^1-norm of a vector in $\mathbb{R}^2$. Notice that a suitable vector Ω is close to the direction perpendicular to ω.

Proposition 2.3 *If $\Omega_0 \in \mathbb{R}^2$ is suitable, so is every $\Omega_n = T_{n-1}^{-1}\Omega_{n-1}/\|T_{n-1}^{-1}\Omega_{n-1}\|$, for all $n \in \mathbb{N}$. For a suitable choice of Ω_0, we have the following bounds,*

$$\frac{1+\alpha_0}{4\alpha_0}(\tilde{A}_n + \tilde{A}_{n-1}) < \prod_{i=0}^{n} \|T_i^{-1}\Omega_i\| < (\tilde{A}_n + \tilde{A}_{n-1})\,. \tag{2.9}$$

Proof: The first part of the claim can be proved by induction. Notice that $\|P_n^{*^{-1}}\Omega_0\| = \|T_n^{-1}\ldots T_0^{-1}\Omega_0\| = \prod_{i=0}^{n}\|T_i^{-1}\Omega_i\|$, as the matrices T_i are symmetric. One easily finds that for a suitable $\Omega_0 \in \mathbb{R}^2$, $1/2(p_n+p_{n-1}+q_n+q_{n-1}) \leq \|P_n^{*^{-1}}\Omega_0\| \leq p_n + p_{n-1}$. The bounds (2.9) follow from the inequalities (2.5) and (2.6). □

Thus, given a pair (ω, Ω) consisting of a vector $\omega = \ell(1,\alpha)^* \in \mathbb{R}^2$, with $\alpha > 1$ and $\ell \in \mathbb{R}^+$, and a suitable vector $\Omega \in \mathbb{R}^2$, one can construct a sequence of vector pairs (ω_n, Ω_n), with the same properties.

3. THE SPACES OF HAMILTONIANS AND THE n^{th}-STEP RENORMALIZATION OPERATOR

In this section, we define the spaces of Hamiltonians that will be considered and construct one step of the renormalization scheme.

For a given $n \in \mathbb{N}_0$, let $\omega_n, \Omega_n \in \mathbb{R}^2$ be two vectors with the properties described in the previous section. We introduce the normalized vector $\hat{\omega}_n = \omega_n/\|\omega_n\|$. Define ω_n' and Ω_n', by the following relations: $\omega_n' \cdot \Omega_n = 0$, $\omega_n' \cdot \hat{\omega}_n = 1$, $\Omega_n' \cdot \hat{\omega}_n = 0$, $\Omega_n' \cdot \Omega_n = 1$.

Definition 3.1 *Given a pair of positive numbers $\rho = (\rho_1, \rho_2)$, define*

$$\begin{aligned}
\mathcal{D}_{n,1}(\rho_1) &= \{q \in \mathbb{C}^2 : |\mathrm{Im}\,\omega_n' \cdot q| < \rho_1,\ |\mathrm{Im}\,\Omega_n' \cdot q| < \rho_1\}\,,\\
\mathcal{D}_{n,2}(\rho_2) &= \{p \in \mathbb{C}^2 : |\hat{\omega}_n \cdot p| < \rho_2,\ |\Omega_n \cdot p| < \rho_2\}\,,
\end{aligned}$$

and let $\mathcal{D}_n(\rho) = \mathcal{D}_{n,1}(\rho_1) \times \mathcal{D}_{n,2}(\rho_2)$.

The Hamiltonians that we will consider are analytic functions $H_n : \mathcal{D}_n(\rho) \to \mathbb{C}$, 2π-periodic in both q variables. These functions can be expanded in Fourier-Taylor series

$$H_n(q,p) = \sum_{(\nu,k)\in I} (H_n)_{\nu,k} (\hat{\omega}_n \cdot p)^{k_1} (\Omega_n \cdot p)^{k_2} e^{iq\cdot\nu}\,,$$

where $k = (k_1, k_2)$ and $I = \mathbb{Z}^2 \times \mathbb{N}_0^2$.

Definition 3.2 *Given $\rho > 0$, componentwise, define $\mathcal{A}_n(\rho)$ to be the Banach space of functions H_n that are analytic on $\mathcal{D}_n(\rho)$, extend continuously to the boundary of $\mathcal{D}_n(\rho)$, and have finite norm*

$$\|H_n\|_{n,\rho} = \sum_{(\nu,k)\in I} |(H_n)_{\nu,k}| \rho_2^{\|k\|} e^{\rho_1(|\hat{\omega}_n\cdot\nu| + |\Omega_n\cdot\nu|)}\,,$$

where, as before, $\|\cdot\|$ denotes ℓ^1-norm of a vector.

Define the projection operators $\mathbb{P}_n^k$ on $\mathcal{A}_n(\rho)$ by $\mathbb{P}_n^k H_n = (H_n)_{0,k}(\hat{\omega}_n \cdot p)^{k_1}(\Omega_n \cdot p)^{k_2}$, and let $\mathbb{E} = \sum_{k\in\mathbb{N}_0^2} \mathbb{P}_n^k$. Define also the functionals $\mathfrak{p}_n^k : \mathcal{A}_n(\rho) \to \mathbb{C}$, by $\mathfrak{p}_n^k H_n = (H_n)_{0,k}$.

In this paper, we will focus on near-integrable Hamiltonians of the form $H_n = H_n^0 + h_n$, where

$$H_n^0 = \omega_n \cdot p + \frac{1}{2}(\Omega_n \cdot p)^2\,,$$

is an integrable Hamiltonian, with an invariant torus of frequency ω_n at $p = 0$ and frequencies of nearby tori twisted in the direction of Ω_n, and $h_n \in \mathcal{A}_n(\rho)$ is a perturbation.

At the n^{th} renormalization step, the scaling of the (lifted) phase space is performed with a linear map on $\mathbb{C}^2 \times \mathbb{C}^2$, defined by $\mathcal{T}_n(q,p) = (T_n q, \mu_n T_n^{*^{-1}} p)$, where $T_n \in GL(2,\mathbb{Z})$ is the hyperbolic matrix introduced in section 2 and $\mu_n = \alpha_{n+1}^{-1}\|T_n^{-1}\Omega_n\|^{-2}$ is a positive number.

The composition $H_n \circ \mathcal{T}_n$ represents a singular operator, as the matrix T_n has an expanding eigendirection. The composition is, however, harmless when the domain of the operator is restricted to Hamiltonians which contain only the modes for which $\|k\|$ is large or ν is almost perpendicular to ω. These modes are called *resonant*. These are the modes that produce small denominators in the KAM theory.

Definition 3.3 *Given $\sigma, \kappa > 0$ and vectors $\omega_n, \Omega_n \in \mathbb{R}^2$, we define the non-resonant index set,*

$$I_n^- = \{(\nu, k) \in I : |\omega_n \cdot \nu| > \sigma |\Omega_n \cdot \nu|,\ |\omega_n \cdot \nu| > \kappa \|k\|\}.$$

The resonant index set is defined as its complement, $I_n^+ = I \backslash I_n^-$. The corresponding projection operators on $\mathcal{A}_n(\rho)$, $\mathbb{I}_n^-$ and $\mathbb{I}_n^+ = \mathbb{I} - \mathbb{I}_n^-$, are defined by setting

$$(\mathbb{I}_n^- H_n)(q,p) = \sum_{(\nu,k)\in I_n^-} (H_n)_{\nu,k} (\hat{\omega}_n \cdot p)^{k_1} (\Omega_n \cdot p)^{k_2} e^{iq\cdot\nu}.$$

Hamiltonians consisting only of resonant modes, will be called resonant. On the space of resonant Hamiltonians, the map $H_n \mapsto H_n \circ \mathcal{T}_n$ is analyticity improving.

Proposition 3.4 *Let $0 < \rho' < \rho$ with $3\rho' > 2\rho$, componentwise. If σ and κ are sufficiently small constants, then every Hamiltonian $H_n \in \mathbb{I}_n^+ \mathcal{A}_n(\rho')$, $n \in \mathbb{N}_0$, has an analytic extension to $\mathcal{T}_n \mathcal{D}_{n+1}(\rho)$. The linear map from $\mathbb{I}_n^+ \mathcal{A}_n(\rho')$ to $\mathcal{A}_{n+1}(\rho)$, given by : $H_n \mapsto H_n \circ \mathcal{T}_n$, is compact.*

Proof: As

$$H_n \circ \mathcal{T}_n(q,p) = \sum_{(\nu,k)\in I_n^+} (H_n)_{\nu,k} (\mu_n T_n^{-1} \hat{\omega}_n \cdot p)^{k_1} (\mu_n T_n^{-1} \Omega_n \cdot p)^{k_2} e^{iq\cdot T^* \nu},$$

one can obtain

$$\begin{aligned} \|H_n \circ \mathcal{T}_n\|_{n+1,\rho} &= \sum_{(\nu,k)\in I_n^+} |(H_n)_{\nu,k}| \left(\frac{\mu_n \|\omega_{n+1}\|}{\alpha_{n+1}\|\omega_n\|} \rho_2 \right)^{k_1} (\mu_n \|T_n^{-1}\Omega_n\| \rho_2)^{k_2} \\ &\quad \cdot e^{\rho_1 \left(\frac{\alpha_{n+1}}{\|\omega_{n+1}\|} |\omega_n \cdot \nu| + \frac{|\Omega_n \cdot \nu|}{\|T_n^{-1}\Omega_n\|} \right)} \\ &= \sum_{(\nu,k)\in I_n^+} |(H_n)_{\nu,k}| \left(\frac{\mu_n \|\omega_{n+1}\| \rho_2}{\alpha_{n+1}\|\omega_n\| \rho_2'} \right)^{k_1} \left(\frac{\mu_n \|T_n^{-1}\Omega_n\| \rho_2}{\rho_2'} \right)^{k_2} \rho_2'^{\|k\|} \\ &\quad \cdot e^{\left(\frac{\rho_1 \alpha_{n+1} \|\omega_n\|}{\|\omega_{n+1}\|} - \rho_1' \right) |\hat{\omega}_n \cdot \nu| + \left(\frac{\rho_1}{\|T_n^{-1}\Omega_n\|} - \rho_1' \right) |\Omega_n \cdot \nu|} e^{\rho_1' (|\hat{\omega}_n \cdot \nu| + |\Omega_n \cdot \nu|)} \\ &\le \|H_n\|_{n,\rho'}, \end{aligned} \tag{3.1}$$

provided that all of the modes contract. The terms with ν obeying the inequality $|\omega_n \cdot \nu| \le \sigma |\Omega_n \cdot \nu|$ contract for sufficiently small σ satisfying the second part of the double inequality

$$\frac{\sigma \frac{\alpha_{n+1}}{\|\omega_{n+1}\|} + \frac{1}{\|T_n^{-1}\Omega_n\|}}{1 + \frac{\sigma}{\|\omega_n\|}} \le \frac{\sigma}{\ell} + \frac{2}{3} < \frac{\rho_1'}{\rho_1}. \tag{3.2}$$

Given $\sigma > 0$, the first part is satisfied for any $n \in \mathbb{N}_0$. The other conditions are trivially satisfied as $\mu_n \|T_n^{-1}\Omega_n\| \le 2/3$.

The modes indexed by (ν, k) satisfying $|\omega_n \cdot \nu| \le \kappa \|k\|$ also contract if κ satisfies the second part of the double inequality

$$\mu_n \|T_n^{-1}\Omega_n\| e^{\kappa \left(\frac{\rho_1 \alpha_{n+1}}{\|\omega_{n+1}\|} - \frac{\rho_1'}{\|\omega_n\|} \right)} \le \frac{2}{3} e^{\kappa \rho_1 / \ell} < \frac{\rho_2'}{\rho_2}. \tag{3.3}$$

Given $\kappa > 0$, the first part is satisfied for any $n \in \mathbb{N}_0$.

These estimates show that $H_n \circ \mathcal{T}_n$ is analytic in $\mathcal{D}_{n+1}(\rho)$. Now, given $\sigma, \kappa > 0$ satisfying the second parts of the double inequalities (3.2) and (3.3), one can find $r > \rho$ satisfying $3\rho' > 2r$ componentwise, such that $H_n \circ \mathcal{T}_n$ is also analytic and bounded in $\mathcal{D}_{n+1}(r)$. The assertion now follows from the fact that the inclusion map from $\mathcal{A}_{n+1}(r)$ to $\mathcal{A}_{n+1}(\rho)$ is compact. $\square$

We will restrict the domain of the n^{th}-step renormalization operator to resonant Hamiltonians. The composition of a resonant Hamiltonian with $\mathcal{T}_n$ produces, in general, non-resonant modes. In the n^{th} renormalization step, we completely eliminate these modes such that the renormalized Hamiltonians are also resonant. We also include a translation in the variable $\Omega_n \cdot p$, to prevent the n^{th}-step renormalization operator from having an expanding eigendirection. Finally, we perform an additional, nonlinear, scaling of the action variables and time, in order to fix the coefficients of the $(\omega_{n+1} \cdot p)$ and $(\Omega_{n+1} \cdot p)^2$ modes of the renormalized Hamiltonians to 1 and 1/2, respectively.

Definition 3.5 *The n^{th}-step renormalization operator $\mathcal{R}_n$ is defined (formally) by the following action on a Hamiltonian $H_n \in \mathbb{I}^+_n \mathcal{A}_n(\rho')$, with $\rho' > 0$, componentwise,*

$$\mathcal{R}_n(H_n) = \frac{\tau_{\tilde{H}_{n+1}}}{z_{\tilde{H}_{n+1}}}[\tilde{H}_{n+1} \circ S_{\tilde{H}_{n+1}} - \mathbb{P}^{(0,0)}_{n+1} \tilde{H}_{n+1} \circ S_{\tilde{H}_{n+1}}], \quad \tilde{H}_{n+1} = \frac{\theta_n}{\mu_n} H_n \circ \mathcal{T}_n \circ V_{H'_{n+1}} \circ U_{H''_{n+1}},$$

where $H'_{n+1} = (\theta_n/\mu_n) H_n \circ \mathcal{T}_n$ and $H''_{n+1} = H'_{n+1} \circ V_{H'_{n+1}}$. The scaling parameter θ_n is α_{n+1}. The transformation $V_{H'_{n+1}}$ represents the translation $V_{H'_{n+1}}(q,p) = (q, p - v_{H'_{n+1}} \Omega'_{n+1})$, with $v_{H'_{n+1}} \in \mathbb{C}$ determined by the equation $\mathfrak{p}^{(0,1)}_{n+1} H'_{n+1} \circ V_{H'_{n+1}} = 0$. $U_{H''_{n+1}}$ is a canonical transformation that satisfies $\mathbb{I}^-_{n+1}(H''_{n+1} \circ U_{H''_{n+1}}) = 0$. The transformation $S_{\tilde{H}_{n+1}}$ represents a scaling $S_{\tilde{H}_{n+1}}(q,p) = (q, z_{\tilde{H}_{n+1}} p)$ of the action variables. The scaling parameter $z_{\tilde{H}_{n+1}} \in \mathbb{C}$ and the time renormalization parameter $\tau_{\tilde{H}_{n+1}} \in \mathbb{C}$ are determined such that $\mathfrak{p}^{(0,2)}_{n+1} H_{n+1} = 1/2$ and $\mathfrak{p}^{(1,0)}_{n+1} H_{n+1} = 1$, where $H_{n+1} = \mathcal{R}_n(H_n)$.

In the following, we show that the translations and scalings included in the n^{th}-step renormalization operator are well-defined on a sufficiently small open ball around H^0_{n+1}. Notice first that $H^0_{n+1} = \tilde{H}^0_{n+1} = H^{0'}_{n+1} = H^{0''}_{n+1} = \mathcal{R}_n(H^0_n)$. For $n \in \mathbb{N}_0$, $b > 0$ and $\rho > 0$, componentwise, define $B_{n,\rho}(b)$ to be the open ball of radius b, in $\mathcal{A}_n(\rho)$, centered at H^0_n. Define also $B^+_{n,\rho}(b)$ to be the open ball of radius b, in $\mathbb{I}^+_n \mathcal{A}_n(\rho)$, centered at H^0_n.

Proposition 3.6 *Given $\rho_2 > \varrho_2 > 0$ and $\rho_1 = \varrho_1 > 0$, the following holds for a sufficiently small constant $b > 0$. For every Hamiltonian $H_{n+1} \in B_{n+1,\rho}(b)$, $n \in \mathbb{N}_0$, there exists $v_{H_{n+1}} \in \mathbb{C}$, such that the translation map $V_{H_{n+1}} : \mathcal{D}_{n+1}(\varrho) \to \mathcal{D}_{n+1}(\rho)$, is well-defined by $V_{H_{n+1}}(q,p) = (q, p - v_{H_{n+1}} \Omega'_{n+1})$, where $\mathfrak{p}^{(0,1)}_{n+1} H_{n+1} \circ V_{H_{n+1}} = 0$. The derivative of the map $\mathcal{V}_{H_{n+1}} : \mathcal{A}_{n+1}(\rho) \to \mathcal{A}_{n+1}(\varrho)$, defined by $\mathcal{V}_{H_{n+1}}(H_{n+1}) = H_{n+1} \circ V_{H_{n+1}}$, at H^0_{n+1}, is the linear map $D\mathcal{V}_{n+1}(H^0_{n+1}) = \mathbb{I} - \mathbb{P}^{(0,1)}_{n+1}$.*

Proof: Define the function $F : \mathcal{A}_{n+1}(\rho) \times \mathbb{C} \to \mathbb{C}$, by setting $F(H_{n+1}, v) = \mathfrak{p}^{(0,1)}_{n+1} H_{n+1} \circ V$, where $V(q,p) = (q, p - v\Omega'_{n+1})$. The implicit equation $F(H^0_{n+1}, v) = 0$ has a unique solution $v = v_{H^0_{n+1}} = 0$. Moreover, $D_2 F(H^0_{n+1}, v)|_{v=0} = -1 \neq 0$. We can use this fact to solve the implicit equation, $F(H_{n+1}, v) = 0$, for a given Hamiltonian H_{n+1} in a ball $B_{n+1,\rho}(b)$ of sufficiently small, n-independent radius $b > 0$.

The problem of the existence of a solution $v = v_{H_{n+1}}$ of the implicit equation $F(H_{n+1}, v) = 0$, for a given Hamiltonian $H_{n+1} \in B_{n+1,\rho}(b)$, is equivalent to the problem of the existence of a fixed point of the function $G_{H_{n+1}} : v \mapsto v + F(H_{n+1}, v)$. Notice that $|G_{H_{n+1}}(0)| = |(h_{n+1})_{0,(0,1)}| < b/\rho_2$. We will show that, given $\lambda > 1$, for sufficiently small $b > 0$, $G_{H_{n+1}}$ is a contraction on a ball of radius $\lambda b/\rho_2$.

Let $|v| \leq \lambda b/\rho_2$, $\lambda > 1$. The norms of $G_{H_{n+1}}(v)$ and $G'_{H_{n+1}}(v)$ can be bounded by

$$|G_{H_{n+1}}(v)| \leq \sum_{k_2=1}^{\infty} |(h_{n+1})_{0,(0,k_2)}| k_2 |v|^{k_2-1} < \frac{b}{\rho_2(1 - \lambda b/\rho_2^2)^2} = \tilde{b}$$

and

$$|G'_{H_{n+1}}(v)| = |1 + D_2 F(H_{n+1}, v)| \leq \sum_{k_2=2}^{\infty} |(h_{n+1})_{0,(0,k_2)}| k_2(k_2-1)|v|^{k_2-2} < \frac{2b}{\rho_2^2(1 - \lambda b/\rho_2^2)^3},$$

respectively. If $b > 0$ is sufficiently small, then

$$\lambda b/\rho_2 < \rho_2 - \varrho_2\,, \qquad \frac{1}{(1-\lambda b/\rho_2^2)^2} < \lambda \qquad \text{and} \qquad \frac{2b}{\rho_2^2(1-\lambda b/\rho_2^2)^3} < 1 - \frac{1}{\lambda} < 1\,. \tag{3.4}$$

These bounds show that for sufficiently small $b > 0$, $G_{H_{n+1}}$ is a contraction on a closed ball of radius $\tilde{b}$, and thus, has a unique fixed point in that ball. As the bounds (3.4) are independent of n, so is b. The first of the bounds (3.4) shows that the translation map is well-defined from $\mathcal{D}_{n+1}(\varrho)$ to $\mathcal{D}_{n+1}(\rho)$ and that $H_{n+1} \circ V_{H_{n+1}}$ belongs to $\mathcal{A}_{n+1}(\varrho)$. $\square$

The map $H''_{n+1} \mapsto H''_{n+1} \circ U_{H''_{n+1}}$ is well-defined on a sufficiently small ball centered at H^0_{n+1}. Theorem 4.6, proved in the next section, guarantees that given $\varrho > \varrho' > 0$, componentwise, for every Hamiltonian H''_{n+1} sufficiently close to H^0_{n+1}, there exists a map $U_{H''_{n+1}} : \mathcal{D}_{n+1}(\varrho') \to \mathcal{D}_{n+1}(\varrho)$, that satisfies the equation $\mathbb{I}^-_{n+1}\,(H''_{n+1} \circ U_{H''_{n+1}}) = 0$.

Proposition 3.7 *Let $\varrho'_2 > \rho'_2 > 0$ and $\varrho'_1 = \rho'_1 > 0$. For sufficiently small constant $b > 0$ and for every Hamiltonian $H_{n+1} \in B_{n+1,\varrho'}(b)$, $n \in \mathbb{N}_0$, there exist $z_{H_{n+1}}, \tau_{H_{n+1}} \in \mathbb{C}$ such that the map $\mathfrak{S}_{H_{n+1}}$: $H_{n+1} \mapsto (\tau_{H_{n+1}}/z_{H_{n+1}})(H_{n+1} \circ S_{H_{n+1}} - \mathbb{P}^{(0,0)}_{n+1} H_{n+1} \circ S_{H_{n+1}})$, with $S_{H_{n+1}}(q,p) = (q, z_{H_{n+1}}p)$, is well-defined from $\mathcal{A}_{n+1}(\varrho')$ to $\mathcal{A}_{n+1}(\rho')$, and satisfies $\mathfrak{p}^{(0,2)}_{n+1}\mathfrak{S}_{H_{n+1}}(H_{n+1}) = 1/2$ and $\mathfrak{p}^{(1,0)}_{n+1}\mathfrak{S}_{H_{n+1}}(H_{n+1}) = 1$. The map $S_{H_{n+1}}$ maps $\mathcal{D}_{n+1}(\rho')$ into $\mathcal{D}_{n+1}(\varrho')$. The derivative of the map $\mathfrak{S}_{H_{n+1}}$ at the point H^0_{n+1} is given by $D\mathfrak{S}_{n+1}(H^0_{n+1}) = \mathbb{I} - \mathbb{P}^{(0,0)}_{n+1} - \mathbb{P}^{(1,0)}_{n+1} - \mathbb{P}^{(0,2)}_{n+1}$.*

Proof: Let $H_{n+1} \in B_{n+1,\varrho'}(b)$. For sufficiently small $b > 0$, the scaling parameters $\tau_{H_{n+1}}$ and $z_{H_{n+1}}$ that satisfy the equations $\mathfrak{p}^{(0,2)}_{n+1}\mathfrak{S}_{H_{n+1}}(H_{n+1}) = 1/2$ and $\mathfrak{p}^{(1,0)}_{n+1}\mathfrak{S}_{H_{n+1}}(H_{n+1}) = 1$, are given by $\tau_{H_{n+1}} = 1/(1+(h_{n+1})_{0,(1,0)})$ and $z_{H_{n+1}} = (1+(h_{n+1})_{0,(1,0)})/(1+2(h_{n+1})_{0,(0,2)})$. We have the bound

$$|z_{H_{n+1}}| \le |z_{H_{n+1}} - 1| + 1 \le \frac{|(h_{n+1})_{0,(1,0)} - 2(h_{n+1})_{0,(0,2)}|}{1 - 2|(h_{n+1})_{0,(0,2)}|} + 1 \le \frac{1 + b/\varrho'_2}{1 - 2b/\varrho'^2_2} < \frac{\varrho'_2}{\rho'_2}\,,$$

where the last inequality is satisfied for sufficiently small n-independent constant $b > 0$. The scaling map is well-defined from $\mathcal{D}_{n+1}(\rho')$ to $\mathcal{D}_{n+1}(\varrho')$ and the resulting Hamiltonian belongs to $\mathcal{A}_{n+1}(\rho')$.$\square$

We can now show that the n^{th}-step renormalization operator is well-defined on an open ball around H^0_n.

Theorem 3.8 *Given $\rho'_1 > 0$, for sufficiently small $\sigma, \kappa > 0$ and $\rho'_2 > 0$ satisfying $3\rho'_2/2 < \sigma < \ell/3$, there exists a constant $C' > 0$, such that the n^{th}-step renormalization operator $\mathcal{R}_n$ is a well-defined analytic map from an open ball $B^+_{n,\rho'}(\zeta_n)$, of radius $\zeta_n = C'/(\alpha_n\alpha_{n+1})^2$, into $\mathbb{I}^+_{n+1}\mathcal{A}_{n+1}(\rho')$. Also, $\|\mathcal{R}_n(H^0_n + h_n) - H^0_{n+1}\|_{n+1,\rho'} \le \zeta_n^{-1}\|h_n\|_{n,\rho'}$ and $\|\mathcal{R}_n(H^0_n + h_n) - H^0_{n+1} - D\mathcal{R}_n(H^0_n)h_n\|_{n+1,\rho'} \le [\zeta_n(\zeta_n - \|h_n\|_{n,\rho'})]^{-1}\|h_n\|^2_{n,\rho'}$.*

Proof: Let $3\rho'/2 > 3\rho'/(2 + 9\rho'_2/(2\ell)) > \rho > \varrho > \varrho' > \rho'$, componentwise. The bound (3.1) implies that there exists a constant $b_1 > 0$, such that $(\theta_n/\mu_n)B^+_{n,\rho'}(\zeta_n) \circ \mathcal{T}_n \subset B_{n+1,\rho}(b_1 C')$, where $\zeta_n = C'/(\alpha_n\alpha_{n+1})^2$ and $C' > 0$. Proposition 3.6 guarantees that for sufficiently small $C' > 0$,

$$\{H''_{n+1} : H''_{n+1} = H'_{n+1} \circ V_{H'_{n+1}},\ H'_{n+1} \in B_{n+1,\rho}(b_1 C')\} \subset B_{n+1,\varrho}(b_1 b_2 C'),$$

with $b_2 > 0$. If $C' > 0$ is chosen sufficiently small, there exists (by Theorem 4.6) a canonical transformation $U_{H''_{n+1}}$ for Hamiltonians H''_{n+1} in a neighborhood of H^0_{n+1} containing $B_{n+1,\varrho}(b_1 b_2 C')$, that satisfies the equation $\mathbb{I}^-_{n+1}\,(H''_{n+1} \circ U_{H''_{n+1}}) = 0$. Furthermore,

$$\{\tilde{H}_{n+1} : \tilde{H}_{n+1} = H''_{n+1} \circ U_{H''_{n+1}},\ H''_{n+1} \in B_{n+1,\varrho}(b_1 b_2 C')\} \subset B^+_{n+1,\varrho'}(b_1 b_2 b_3 C'),$$

where $b_3 > 0$ is a constant (dependent on σ, κ and ϱ). Finally, Proposition 3.7 guarantees that for sufficiently small $C' > 0$,

$$\{\mathfrak{S}_{\tilde{H}_{n+1}}(\tilde{H}_{n+1}) : \tilde{H}_{n+1} \in B^+_{n+1,\varrho'}(b_1 b_2 b_3 C')\} \subset B^+_{n+1,\rho'}(b_1 b_2 b_3 b_4 C'),$$

with $b_4 > 0$. This shows that for sufficiently small $C' > 0$, the n^{th}-step renormalization operator is well-defined from $B^+_{n,\rho'}(\zeta_n)$ to $\overset{+}{\mathbb{I}}_{n+1}\mathcal{A}_{n+1}(\rho')$. As the composition of analytic maps, it is an analytic map itself.

Define the map $g : z \mapsto \mathcal{R}_n(H^0_n + zh_n) - H^0_{n+1}$, where $h_n \in \overset{+}{\mathbb{I}}_n\mathcal{A}_n(\rho')$ is given such that $H_n = H^0_n + h_n \in B^+_{n,\rho'}(\zeta_n)$. This map is analytic from an open ball in $\mathbb{C}$, of radius $\zeta_n/\|h_n\|_{n,\rho'} > 1$, into $\overset{+}{\mathbb{I}}_{n+1}\mathcal{A}_{n+1}(\rho')$. As $g(0) = 0$, we have, by the maximum principle, the bound

$$\|g(1)\|_{n+1,\rho'} \leq \sup_{\|\bar{h}_n\|_{n,\rho'}=\zeta_n} \|\mathcal{R}_n(H^0_n + \bar{h}_n) - H^0_{n+1}\|_{n+1,\rho'} \frac{\|h_n\|_{n,\rho'}}{\zeta_n}.$$

From the construction of the renormalization operator, $\sup_{\|\bar{h}_n\|_{n,\rho'}=\zeta_n} \|\mathcal{R}_n(H^0_n + \bar{h}_n) - H^0_{n+1}\|_{n+1,\rho'}$ can be bounded by a constant $b_1b_2b_3b_4C'$ less than or equal to 1, if $C' > 0$ is sufficiently small. Then, we have $\|\mathcal{R}_n(H^0_n + h_n) - H^0_{n+1}\|_{n+1,\rho'} \leq \|h_n\|_{n,\rho'}/\zeta_n$. Cauchy's formula gives an estimate on the norm of the second-order remainder of the Taylor expansion of $\mathcal{R}_n$ about H^0_n,

$$\begin{aligned} F^2_n = \|g(1) - g(0) - g'(0)\|_{n+1,\rho'} &\leq \frac{1}{2\pi} \oint_{|z|=\zeta_n/\|h_n\|_{n,\rho'}} \frac{\|g(z)\|_{n+1,\rho'}}{|z^2(z-1)|} dz\,, \\ F^2_n &\leq \frac{\|h_n\|^2_{n,\rho'}}{\zeta_n(\zeta_n - \|h_n\|_{n,\rho'})}\,, \end{aligned} \tag{3.5}$$

for sufficiently small $C' > 0$. This provides the second desired bound. □

Remark 3.9 *The renormalization operator $\mathcal{R}_n$ is actually analyticity improving and can be defined from an open ball in $\overset{+}{\mathbb{I}}_n\mathcal{A}_n(\rho'')$ into $\overset{+}{\mathbb{I}}_n\mathcal{A}_n(\rho')$, with $\rho' > \rho''$, componentwise. The loss of analyticity in the transformations close to the identity can be reduced by restricting the domain of the renormalization operator to a smaller ball.*

Remark 3.10 *The n^{th}-step renormalization operator is well-defined on a space of resonant Hamiltonians. For a given $n \in \mathbb{N}$, this is not a restriction as, by construction, the renormalized Hamiltonians are always resonant. In order to apply the 0^{th}-step renormalization operator to Hamiltonians containing non-resonant modes also, one can include a pre-renormalization step consisting of a canonical transformation that eliminates the non-resonant modes of the Hamiltonian.*

4. ELIMINATION OF NON-RESONANT MODES

In this section, we prove Theorem 4.6 concerning the existence of a canonical transformation that eliminates the non-resonant modes of a near-integrable Hamiltonian. The proof is constructive and the construction is associated to a single renormalization step. The index of the renormalization step will be suppressed in this section, in order to simplify the notation. We emphasize, that the constants that appear in this section will be chosen independently of the renormalization step.

We begin by making a canonical change of coordinates $(q,p) \to (x,y)$ with $x_1 = \omega' \cdot q$, $x_2 = \Omega' \cdot q$, $y_1 = \hat{\omega} \cdot p$ and $y_2 = \Omega \cdot p$. We will simplify the notation further, by writing $H(x,y)$ instead of $H(q(x,y), p(x,y))$. Moreover, some of the symbols in this section will have different meaning than in other sections. The use of those symbols should be restricted to this section.

In the new coordinates, the Fourier-Taylor series and the norm of a function $H \in \mathcal{A}(\rho)$, analytic on

$$\mathcal{D}(\rho) = \{x \in \mathbb{C}^2 : |\mathrm{Im}\, x_1| < \rho_1,\ |\mathrm{Im}\, x_2| < \rho_1\} \times \{y \in \mathbb{C}^2 : |y_1| < \rho_2,\ |y_2| < \rho_2\}\,,$$

where $\rho = (\rho_1, \rho_2) > 0$, componentwise, are given by

$$H(x,y) = \sum_{(v,k)\in I} H_{v,k} y_1^{k_1} y_2^{k_2} e^{iv\cdot x}\,, \quad \|H\|_\rho = \sum_{(v,k)\in I} |H_{v,k}|\, \rho_2^{k_1+k_2} e^{\rho_1(|v_1|+|v_2|)}\,.$$

Here $I = \mathbb{M}^2 \times \mathbb{N}^2_0$, where $\mathbb{M}^2 = \{(\hat{\omega}\cdot\nu, \Omega\cdot\nu) \in \mathbb{R}^2 : \nu \in \mathbb{Z}^2\} \subset \mathbb{R}^2$ is a set bijective to $\mathbb{Z}^2$ that can be determined from the vectors ω and Ω.

The non-resonant index set is defined as

$$I^- = \{(v,k) \in I : |v_1| > \frac{\sigma}{\|\omega\|}|v_2| \ , |v_1| > \frac{\kappa}{\|\omega\|}\|k\|\} .$$

The resonant index set is its complement $I^+ = I \backslash I^-$.

We state without proof the following technical Proposition. In what follows, the norm of the functions $X = (X_1, X_2) \in \mathcal{A}^2(\rho)$ is defined as $\|X\|_\rho = \max\{\|X_1\|_\rho, \|X_2\|_\rho\}$. We denote by $\partial_i H$, for $i = 1,2$, the partial derivatives of $H(x,y)$ with respect to x_1 and x_2, and for $i = 3,4$, the partial derivatives of the same function with respect to y_1 and y_2, respectively.

Proposition 4.1 *Let $\rho = (\rho_1, \rho_2)$ and $\delta = (\delta_1, \delta_2)$ be given pairs of positive numbers and let $0 < \delta < \rho$, componentwise. If $f, g, h \in \mathcal{A}(\rho)$, and $X, Y \in \mathcal{A}^2(\rho)$ satisfy $\|X\|_\rho \le \delta_1$ and $\|Y\|_\rho \le \delta_2$, and $U : (x,y) \mapsto (x+X, y+Y)$ is a given change of variables, then*

(i) $|f(x,y)| \le \|f\|_\rho \, , \forall (x,y) \in \mathcal{D}(\rho)$,

(ii) $fg \in \mathcal{A}(\rho)$ *and* $\|fg\|_\rho \le \|f\|_\rho \|g\|_\rho$,

(iii) $\|\partial_i h\|_{\rho-(\delta_1,0)} \le \delta_1^{-1}\|h\|_\rho$ *for* $i = 1,2$,

(iv) $\|\partial_j h\|_{\rho-(0,\delta_2)} \le \delta_2^{-1}\|h\|_\rho$ *for* $j = 3,4$,

(v) $\|h \circ U\|_\rho \le \|h\|_{\rho+\delta}$.

Given a Hamiltonian H, close to H^0, our goal is to construct a canonical transformation U_H that satisfies the equation $\mathbb{I}^- H \circ U_H = 0$. Such a transformation is close to the identity as the Hamiltonian H is close to integrable. We would like to perform first a canonical transformation $U : (x,y) \mapsto (x', y')$, generated by a function ϕ as

$$x' = x + \nabla_{y'}\phi(x,y'), \qquad\qquad y' = y - \nabla_x \phi(x,y') ,$$

where $\nabla_x = (\partial_1, \partial_2)$ and $\nabla_y = (\partial_3, \partial_4)$, which satisfies the linearized version of the above equation, i.e. $\mathbb{I}^-(H + \{H,\phi\}) = 0$. Here, $\{H,\phi\}$ denotes the Poisson bracket of the functions H and ϕ, defined by $\{H,\phi\} = \nabla_x H \cdot \nabla_y \phi - \nabla_x \phi \cdot \nabla_y H$.

We introduce $\psi = \partial_1 \phi$ and define the operators $\mathcal{D}_i = \partial_i \partial_1^{-1}$, for $i = 1,2,3,4$, on $\mathbb{I}^- \mathcal{A}(\rho)$, $\rho > 0$, componentwise.

Proposition 4.2 *$\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$ are bounded linear operators on $\mathbb{I}^- \mathcal{A}(\rho)$, with operator norms satisfying*

$$\|\mathcal{D}_2\| \le \frac{\|\omega\|}{\sigma}, \qquad\qquad \|\mathcal{D}_3 + \mathcal{D}_4\| \le \frac{\|\omega\|}{\kappa\rho_2} .$$

Proof: Consider first the action of the operators $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$ on $H_{v,k}(x,y) = y_1^{k_1} y_2^{k_2} e^{iv\cdot x}$ with (v,k) belonging to I^-. We find that $\|\mathcal{D}_2 H_{v,k}\|_\rho \le \frac{\|\omega\|}{\sigma}\|H_{v,k}\|_\rho$ and $\|(\mathcal{D}_3 + \mathcal{D}_4)H_{v,k}\|_\rho \le \frac{\|\omega\|}{\kappa\rho_2}\|H_{v,k}\|_\rho$. These bounds extend by linearity to the whole $\mathbb{I}^- \mathcal{A}(\rho)$. □

Let $H = H^0 + h$, where $H^0 = \|\omega\| y_1 + y_2^2/2$ and $h \in \mathcal{A}(\varrho)$, with $\varrho > \eta > 0$, componentwise. On $\mathbb{I}^- \mathcal{A}(\varrho - \eta)$, define the operator $L(h)$ by the following action on an arbitrary $\psi \in \mathbb{I}^- \mathcal{A}(\varrho - \eta)$,

$$L(h)\psi = \frac{1}{\|\omega\|}(-y_2 \mathcal{D}_2 \psi + \partial_1 h \mathcal{D}_3 \psi + \partial_2 h \mathcal{D}_4 \psi - \partial_3 h \mathcal{D}_1 \psi - \partial_4 h \mathcal{D}_2 \psi) .$$

Using the bounds obtained in Proposition 4.1 and Proposition 4.2, we find that

$$\|L(h)\psi\|_{\varrho-\eta} \le \left[\frac{\varrho_2 - \eta_2}{\sigma} + \left(\frac{2}{\kappa\eta_1(\varrho_2 - \eta_2)} + \frac{1}{\sigma\eta_2} + \frac{1}{\|\omega\|\eta_2}\right)\|h\|_\varrho\right]\|\psi\|_{\varrho-\eta} . \tag{4.1}$$

As $\|\omega\| \ge 2\ell$, if $\varrho_2 - \eta_2 < \sigma$ and $\|h\|_\varrho$ is sufficiently small, then $\|L(h)\psi\|_{\varrho-\eta} \le A\|\psi\|_{\varrho-\eta}$ for every $\psi \in \mathbb{I}^- \mathcal{A}(\varrho - \eta)$ and some positive constant $A < 1$. Therefore, the operator norm $\|\mathrm{L}(h)\| \le A < 1$.

This enables us to solve the equation $\mathbb{I}^-(H + \{H,\phi\}) = 0$ for the generating function of the canonical transformation U.

Proposition 4.3 *Let H belong to $\mathcal{A}(\varrho)$, $\varrho > \eta > 0$, componentwise, and $\varrho_2 - \eta_2 < \sigma$. If $\|h\|_\varrho$ is sufficiently small such that, by the inequality (4.1), $L(h)$ is an operator on $\mathbb{I}^- \mathcal{A}(\varrho - \eta)$ bounded in norm by a positive constant $A < 1$, then the equation*

$$\mathbb{I}^- (H + \{H, \phi\}) = 0 \,, \qquad \mathbb{I}^+ \phi = 0 \,, \tag{4.2}$$

has a unique solution ϕ, such that $\psi = \partial_1 \phi$ belongs to $\mathbb{I}^- \mathcal{A}(\varrho - \eta)$, and satisfies

$$\|\psi\|_{\varrho-\eta} \leq (1 - A)^{-1} \frac{\|\mathbb{I}^- H\|_{\varrho-\eta}}{\|\omega\|} \,, \quad \|\{H, \phi\}\|_{\varrho-\eta} \leq (1 - A)^{-1} \|\mathbb{I}^- H\|_{\varrho-\eta} \,. \tag{4.3}$$

Proof: Let $L^\pm = \mathbb{I}^\pm L(h) \mathbb{I}^-$. Equation (4.2) can now be written in the form $(\mathbb{I} - L^-)\psi = \mathbb{I}^- h/\|\omega\|$. The assumptions guarantee that the operator norms of $L^\pm$ satisfy $\|L^\pm\| \leq A < 1$. Equation (4.2) can then be solved by inverting the operator $\mathbb{I} - L^-$ by means of a Neumann series. The solution ψ satisfies the first of the bounds (4.3). As $\{H, \phi\} = (L(h) - \mathbb{I})\psi\|\omega\| = L^+ \psi \|\omega\| - \mathbb{I}^- h$, using this bound one obtains the second one. □

The next Proposition shows that the function ϕ generates an explicit canonical transformation.

Proposition 4.4 *Let $0 < \delta_2'' = 2\sigma^{-1}\epsilon^3 < \epsilon r_2 = \delta_2' < \varrho_2 - \eta_2$ and $\sigma < 2\ell$. Let also $\delta' = (0, \delta_2')$. Define B to be the closed ball of radius $\delta_2''/2$ in $\mathcal{A}(\varrho - \eta - \delta')$ centered at zero. If $\psi \in \mathbb{I}^- \mathcal{A}(\varrho - \eta)$ satisfies $\|\psi\|_{\varrho-\eta} \leq \epsilon^3/\|\omega\|$, the equation*

$$K(g) = g \,, \qquad K(g) = \nabla_x \phi(x, y - g) \,, \qquad g = g(x, y) = y - y' \,, \tag{4.4}$$

has a unique solution $g \in B^2 = B \times B$ and $\|g\|_{\varrho-\eta-\delta'} \leq \sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta}$.

Proof: By assumption, for every $g \in B^2$, $\|g\|_{\varrho-\eta-\delta'} \leq \delta_2''/2$. Using the bounds obtained in Proposition 4.1 and Proposition 4.2, we find that for every $g, g' \in B^2$, there exists $g^* \in B^2$, such that

$$\|K(g)\|_{\varrho-\eta-\delta'} \leq \|\nabla_x \phi\|_{\varrho-\eta} \leq \max\{1, \sigma^{-1}\|\omega\|\}\|\psi\|_{\varrho-\eta} \leq \sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta} \leq \sigma^{-1}\epsilon^3 \tag{4.5}$$

and

$$\begin{aligned}
\|K(g') - K(g)\|_{\varrho-\eta-\delta'} &\leq \|\nabla_g K(g^*)\|_{\varrho-\eta-\delta'} \|g' - g\|_{\varrho-\eta-\delta'} \\
&\leq \|\nabla_{y'} \nabla_x \phi(x, y')\|_{\varrho-\eta-\delta'} \|g' - g\|_{\varrho-\eta-\delta'} \\
&\leq \|\nabla_y \nabla_x \phi\|_{\varrho-\eta-\delta'/2} \|g' - g\|_{\varrho-\eta-\delta'} \\
&\leq 2\delta_2'^{-1} \|\nabla_x \phi\|_{\varrho-\eta} \|g' - g\|_{\varrho-\eta-\delta'} \\
&\leq 2\delta_2'^{-1} \max\{1, \sigma^{-1}\|\omega\|\} \|\psi\|_{\varrho-\eta} \|g' - g\|_{\varrho-\eta-\delta'} \\
&\leq 2\delta_2'^{-1} \sigma^{-1} \|\omega\| \|\psi\|_{\varrho-\eta} \|g' - g\|_{\varrho-\eta-\delta'} \\
&\leq 2\delta_2'^{-1} \sigma^{-1} \epsilon^3 \|g' - g\|_{\varrho-\eta-\delta'} \,.
\end{aligned}$$

As, by assumption, $2\delta_2'^{-1}\sigma^{-1}\epsilon^3 < 1$, these inequalities show that K is a contraction on B^2, and thus has a unique fixed point $g \in B^2$. The inequality (4.5) provides the desired bound on the norm of g. □

Proposition 4.5 *Let $\varrho - \eta - 2\delta' > 0$, $\eta > 0$ and $\delta' = (\delta_1', \delta_2') = \epsilon r > 0$, componentwise, with $\delta_1' = \sigma\delta_2'/[\kappa(\varrho_2 - \eta_2)]$. Also let $0 < \delta_2'' = 2\sigma^{-1}\epsilon^3 < \delta_2' = \epsilon r_2 < \varrho_2 - \eta_2 < \sigma < 2\ell$. Assume further that $H \in \mathcal{A}(\varrho - \eta)$ satisfies $\|H - H^0\|_{\varrho-\eta} < b$ and that $b > 0$ is small enough such that the linear operator $L(H - H^0)$ is bounded on $\mathbb{I}^- \mathcal{A}(\varrho - \eta)$ by $\|L(H - H^0)\| \leq A < A' < 1$. If $\|\mathbb{I}^- H\|_{\varrho-\eta} \leq (1 - A')\epsilon^3$, the canonical transformation U exists and maps $\mathcal{D}(\varrho - \eta - 2\delta')$ into $\mathcal{D}(\varrho - \eta)$. The function $H \circ U$ belongs to $\mathcal{A}(\varrho - \eta - 2\delta')$ and $L(H \circ U - H^0)$ is a bounded operator on $\mathbb{I}^- \mathcal{A}(\varrho - \eta - 2\delta')$. They satisfy the bounds*

$$\begin{aligned}
\|\mathbb{I}^- (H \circ U)\|_{\varrho-\eta-2\delta'} &\leq C_2(\epsilon)\epsilon^4 \,, \\
\|H \circ U - H\|_{\varrho-\eta-2\delta'} &\leq (1 + \epsilon C_2(\epsilon))\epsilon^3 = \Delta b(\epsilon) \,, \\
\|L(H \circ U - H^0)\| &\leq \|L(H - H^0)\| + C_3(\epsilon)(1 + \epsilon C_1(\epsilon))\epsilon^2 = A + \Delta A(\epsilon) \,,
\end{aligned} \tag{4.6}$$

where

$$C_n(\epsilon) = \frac{\sigma n\epsilon r_2 + \varrho_2 n\epsilon r_2 + \frac{1}{2}(n\epsilon r_2)^2 + \|h\|_{\varrho-\eta}}{\sigma n r_2(\sigma n r_2 - \epsilon^2)}\,,$$

for $n = 1, 2$, *and*

$$C_3(\epsilon) = \frac{2(\varrho_2 - \eta_2)}{\sigma r_2(\varrho_2 - \eta_2 - 2r_2\epsilon)} + \frac{1}{\sigma r_2} + \frac{1}{\ell r_2}\,.$$

Proof: Our assumptions guarantee that there exists a canonical transformation U with a generating function ϕ that solves the linear equation (4.2). More specifically, if $b > 0$ has been chosen sufficiently small, then there exists $\psi \in \mathcal{A}(\varrho - \eta)$ satisfying $\|\psi\|_{\varrho-\eta} < \epsilon^3/\|\omega\|$ and $g \in B^2$ of norm $\|g\|_{\varrho-\eta-\delta'} < \sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta}$ that solves the equation (4.4).

Define the following one parameter ($s \in \mathbb{C}$) family

$$\begin{aligned} F(s) = -s(\|\omega\|\psi(x, y - sg) + y_2\mathcal{D}_2\psi(x, y - sg)) + \frac{1}{2}s^2(\mathcal{D}_2\psi(x, y - sg))^2 \\ +h(x + s\nabla_{y-sg}\phi(x, y - sg), y - s\nabla_x\phi(x, y - sg))\,, \end{aligned}$$

passing through $F(0) = H - H^0$ and $F(1) = H \circ U - H^0$, with $F'(0) = \{H, \phi\}$. Assuming that $|s| \le s_0 = \sigma\epsilon^{-3}n\delta'_2$, where $n \in \{1, 2\}$, and using the Proposition 4.1 and Proposition 4.2, we obtain the following bounds,

$$\begin{aligned} \|sg\|_{\varrho-\eta-n\delta'} &\le s_0\sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta} < n\delta'_2\,, \\ \|s\nabla_x\phi(x, y - sg)\|_{\varrho-\eta-n\delta'} &\le s_0\sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta} < n\delta'_2\,, \\ \|s\nabla_{y-sg}\phi(x, y - sg)\|_{\varrho-\eta-n\delta'} &\le s_0[\kappa(\varrho_2 - \eta_2)]^{-1}\|\omega\|\|\psi\|_{\varrho-\eta} < n\delta'_1\,. \end{aligned} \tag{4.7}$$

These bounds, together with Proposition 4.1, imply that $F(s)$ belongs to $\mathcal{A}(\varrho - \eta - n\delta')$, whenever $|s| \le s_0$. In fact, this is true on an open neighborhood of the disc $|s| \le s_0$, as the inequalities (4.7) are strict due to $A < A'$. Now, we have

$$\begin{aligned} \|H \circ U - H - \{H, \phi\}\|_{\varrho-\eta-n\delta'} &= \|F(1) - F(0) - F'(0)\|_{\varrho-\eta-n\delta'} \\ &= \left\| \frac{1}{2\pi i}\oint_{|s|=s_0} \frac{ds}{s^2(s-1)}F(s)\right\|_{\varrho-\eta-n\delta'} \\ &\le \frac{1}{s_0(s_0 - 1)}(s_0\|\omega\|\|\psi\|_{\varrho-\eta} + s_0\varrho_2\sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta} \\ &\qquad + \frac{1}{2}s_0^2\sigma^{-2}\|\omega\|^2\|\psi\|^2_{\varrho-\eta} + \|h\|_{\varrho-\eta}) \\ &\le \frac{\sigma n\delta'_2 + \varrho_2 n\delta'_2 + \frac{1}{2}(n\delta'_2)^2 + \|h\|_{\varrho-\eta}}{\sigma\epsilon^{-3}n\delta'_2(\sigma\epsilon^{-3}n\delta'_2 - 1)} = C_n(\epsilon)\epsilon^4\,. \end{aligned}$$

As $\mathbb{I}^-(H \circ U) = \mathbb{I}^-(H \circ U - H - \{H, \phi\})$, the first of the inequalities (4.6) immediately follows from this estimate for $n = 2$. From Proposition 4.3 and the inequality

$$\|H \circ U - H\|_{\varrho-\eta-2\delta'} \le \|H \circ U - H - \{H, \phi\}\|_{\varrho-\eta-2\delta'} + \|\{H, \phi\}\|_{\varrho-\eta-2\delta'}\,,$$

we find that $\|H \circ U - H\|_{\varrho-\eta-2\delta'} \le (1 - A)^{-1}\|\mathbb{I}^- H\|_{\varrho-\eta} + C_2(\epsilon)\epsilon^4$. This implies the second inequality in (4.6). The fact that the map U takes $\mathcal{D}(\varrho - \eta - 2\delta')$ into $\mathcal{D}(\varrho - \eta)$ follows from the bounds (4.7) and Proposition 4.1.

From the definition of the operator $L(h)$, one can find that

$$\|L(H \circ U - H^0)\| \le \|L(H - H^0)\| + \left(\frac{2}{\kappa\delta'_1(\varrho_2 - \eta_2 - 2\delta'_2)} + \frac{1}{\sigma\delta'_2} + \frac{1}{\|\omega\|\delta'_2}\right)\|H \circ U - H\|_{\varrho-\eta-\delta'}\,.$$

Using the inequality $\|H \circ U - H\|_{\varrho-\eta-\delta'} \le (1 + \epsilon C_1(\epsilon))\epsilon^3$, which can be obtained analogously to the second bound in (4.6), one can obtain the last desired inequality. □

Theorem 4.6 *Let $\varrho > \varrho' > 0$, componentwise, $\varrho_2 < \sigma < 2\ell$ and $0 < A' < 1$. Let B be an open set of Hamiltonians $H \in \mathcal{A}(\varrho)$, for which $\|H - H^0\|_\varrho < b$ and $\|\mathbb{I}^- H\|_\varrho < (1-A')\epsilon^3$. If $b > 0$ and $\epsilon > 0$ are sufficiently small, then for every Hamiltonian $H \in B$ there exists an analytic canonical transformation $U_H : \mathcal{D}(\varrho') \to \mathcal{D}(\varrho)$ that solves the equation $\mathbb{I}^- H \circ U_H = 0$. The map $H \mapsto H \circ U_H$ is analytic from B to $\mathbb{I}^+ \mathcal{A}(\varrho')$, and*

$$\|H \circ U_H - H\|_{\varrho'} \le \epsilon^3 + \mathcal{O}(\epsilon^4). \tag{4.8}$$

Proof: Let $\varrho > \varrho - \eta > \varrho' > 0$ and $r > 0$, componentwise, with $r_1 = \sigma r_2/[\kappa(\varrho_2 - \eta_2)]$. For sufficiently small $b > 0$, the norm of the operator $L(H - H^0) : \mathbb{I}^- \mathcal{A}(\varrho - \eta) \to \mathcal{A}(\varrho - \eta)$ is bounded by a positive constant $A < A' < 1$.

By Proposition 4.5, there exists a canonical transformation $U : \mathcal{D}(\varrho - \eta - 2\epsilon r) \to \mathcal{D}(\varrho - \eta)$ such that $\|\mathbb{I}^-(H \circ U)\|_{\varrho - \eta - 2\epsilon r} \le C_2(\epsilon)\epsilon^4$. We would like to iterate the map $H \mapsto H \circ U$, indefinitely. Introducing $f(\epsilon) = [\epsilon C_2(\epsilon)/(1-A')]^{1/3}\epsilon$, we obtain $\|\mathbb{I}^-(H \circ U)\|_{\varrho-\eta-2\epsilon r} \le (1-A')f(\epsilon)^3$. Let $\epsilon_i = f^i(\epsilon)$, for $i \in \mathbb{N}_0$. For sufficiently small ϵ, the sum $\sum_{i=0}^\infty 2\epsilon_i r$ converges to a limit $\delta = \mathcal{O}(\epsilon)$. The sums $\sum_{i=0}^\infty \Delta b(\epsilon_i)$ and $\sum_{i=0}^\infty \Delta A(\epsilon_i)$ converge to $\Delta b = \mathcal{O}(\epsilon^3)$ and $\Delta A = \mathcal{O}(\epsilon^2)$, respectively. Thus, for sufficiently small ϵ the map $(H, \epsilon, \varrho - \eta) \mapsto (H \circ U, f(\epsilon), \varrho - \eta - 2\epsilon r)$ can be iterated indefinitely and the iterations converge to a limit $(H \circ U_H, 0, \varrho - \eta - \delta)$. For sufficiently small ϵ, $\varrho - \eta - \delta > \varrho'$, componentwise.

As ϵ_i are summable, the sequence of canonical transformations U generates a uniformly convergent sequence on $\mathcal{D}(\varrho - \eta - \delta)$. The analyticity of the map $H \mapsto H \circ U_H$ follows from uniform convergence of our iteration scheme. The desired bound can be obtained from the second inequality in (4.6) and its iterations. □

In the following, $\hat{H}_n$ denotes the Hamiltonian vector field generated by H_n, i.e. $\hat{H}_n = (\mathbb{J}\nabla H_n) \cdot \nabla$, where $\mathbb{J}(q,p) = (p, -q)$ and $\nabla = (\nabla_q, \nabla_p)$.

The derivative of the map $\mathcal{N}_{H_n} : H_n \mapsto H_n \circ U_{H_n}$ at a resonant Hamiltonian H_n^+ is given by $D\mathcal{N}_n(H_n^+) = \mathbb{I}_n^+ - \mathbb{I}_n^+ \hat{H}_n^+ (\mathbb{I}_n^- \hat{H}_n^+ \mathbb{I}_n^-)^{-1} \mathbb{I}_n^-$.

Let $K_n^0 = \omega_n \cdot p$ and let $\mathbb{E}H_n = K_n^0 + f_n$, where $f_n = \frac{1}{2}(\Omega_n \cdot p)^2 + \mathbb{E}h_n$. The derivative of the map $\mathcal{N}_{H_n}$ at a q-independent Hamiltonian $\mathbb{E}H_n$ is

$$D\mathcal{N}_n(\mathbb{E}H_n) = \mathbb{I}_n^+ - \mathbb{I}_n^+ \hat{f}_n \hat{K}_n^{0^{-1}} \mathbb{I}_n^- \left(\mathbb{I} + \mathbb{I}_n^- \hat{f}_n \hat{K}_n^{0^{-1}} \mathbb{I}_n^-\right)^{-1} \mathbb{I}_n^-. \tag{4.9}$$

The norm of the operator $\hat{f}_n \hat{K}_n^{0^{-1}} \mathbb{I}_n^- : \mathcal{A}_n(\varrho) \to \mathcal{A}_n(\varrho')$ satisfies the bound

$$\|\hat{f}_n \hat{K}_n^{0^{-1}} \mathbb{I}_n^-\| \le \frac{\varrho_2'}{\sigma} + \frac{1}{\|\omega_n\|} \sum_{k \in \mathbb{N}_0^2} |(h_n)_{0,k}| \left(k_1 + k_2 \frac{\|\omega_n\|}{\sigma}\right) \varrho_2'^{\|k\|-1},$$

and thus

$$\|\hat{f}_n \hat{K}_n^{0^{-1}} \mathbb{I}_n^-\| \le \frac{\varrho_2}{\sigma} + \left(\frac{1}{\|\omega_n\|} + \frac{1}{\sigma}\right) \frac{\|\mathbb{E}h_n\|_{n,\varrho}}{\varrho_2(1 - \varrho_2'/\varrho_2)^2}.$$

As $\varrho_2 < \sigma$, if $\|\mathbb{E}h_n\|_{n,\varrho}$ is sufficiently small, then $\|\hat{f}_n \hat{K}_n^{0^{-1}} \mathbb{I}_n^-\| < 1$ and the operator $(\mathbb{I} + \mathbb{I}_n^- \hat{f}_n \hat{K}_n^{0^{-1}} \mathbb{I}_n^-)$ in (4.9) can be inverted by means of a Neumann series. If $\varrho_2 < \sigma/2$ and $\|\mathbb{E}h_n\|_{n,\varrho}$ is sufficiently small, then the operator norm $\|D\mathcal{N}_n(\mathbb{E}H_n)\|$ can be bounded by 1.

5. CONVERGENCE OF THE RENORMALIZATION SCHEME

Recall that the orbit of an integrable Hamiltonian $H_0^0 = \omega \cdot p + 1/2(\Omega \cdot p)^2$ under the renormalization consists of Hamiltonians $H_n^0 = \omega_n \cdot p + 1/2(\Omega_n \cdot p)^2$, where $n \in \mathbb{N}_0$. The maps $\omega_n \mapsto \omega_{n+1}$ and $\Omega_n \mapsto \Omega_{n+1}$ are induced by the Gauss map of the inverse winding ratio of ω.

The vectors ω with the winding ratio $\alpha = [a, a, \ldots]$, where $a \in \mathbb{N}$, are the fixed points of the first of these maps. A suitable vector Ω is not necessarily a fixed point of the dynamics. However, in this case, it is possible to make a particular choice of Ω, such that it is a fixed point of the dynamics. The corresponding integrable Hamiltonian H_0^0 is a fixed point of the renormalization.

Similarly, if the winding ratio of a frequency vector ω has a periodic continued fraction expansion, one can make a particular choice of a suitable vector Ω, such that the Hamiltonian H_0^0 generates a periodic orbit of the renormalization. More generally, if the winding ratio of ω is a quadratic irrational and a particular choice of a suitable vector Ω is made, the dynamics of H_0^0 eventually settles on a periodic orbit.

In the following, we show that if ω_0 is a Diophantine vector and Ω_0 is an arbitrary suitable vector, then the orbit of a resonant Hamiltonian H_0, sufficiently close to H_0^0, approaches the orbit of H_0^0 exponentially fast, under the action of the sequence of the renormalization operators $\mathcal{R}_0, \mathcal{R}_1, \ldots$. More precisely, our main result is summarized in Theorem 5.1.

Theorem 5.1 *Let $\omega_0 \in DC(\beta)$, with $0 \le \beta < (\sqrt{161}-11)/10$. Given $\rho_1' > 0$, there exist $\rho_2', \sigma, \kappa, \tilde{C} > 0$ and a non-empty open neighborhood $B^+_{0,\rho'} \subset \overset{+}{\mathbb{I}}_0 \mathcal{A}_0(\rho')$ of H_0^0, such that for all Hamiltonians $H_0 \in B^+_{0,\rho'}$ and $n \in \mathbb{N}_0$, $\|H_n - H_n^0\|_{n,\rho'} \le \tilde{C}\gamma^{-2n}$.*

In the context of Remark 3.10, the ball $B^+_{0,\rho'}$ in Theorem 5.1 could be replaced with a ball $B_{0,\rho'} \subset \mathcal{A}_0(\rho')$. The proof of this Theorem follows directly from Lemma 5.3 and the fact that $\tilde{A}_n$ grows at least exponentially with $n \ge 0$, i.e. $\tilde{A}_n \ge \gamma^n$, for $\alpha_0 > 1$. We devote the remaining part of this section to the proof of this Lemma.

We first present a description of the eigenspaces of the derivative of the n^{th}-step renormalization operator at H_n^0, given by its action on an arbitrary function $f_n \in \overset{+}{\mathbb{I}}_n \mathcal{A}_n(\rho')$, where $\rho' > 0$, componentwise,

$$D\mathcal{R}_n(H_n^0)f_n = \frac{\theta_n}{\mu_n}(\mathbb{I} - \mathbb{P}^{(0,0)}_{n+1} - \mathbb{P}^{(1,0)}_{n+1} - \mathbb{P}^{(0,1)}_{n+1} - \mathbb{P}^{(0,2)}_{n+1}) D\mathcal{N}_{n+1}(H^0_{n+1}) f_n \circ \mathcal{T}_n.$$

The space of q-independent Hamiltonians is an invariant subspace of the derivative operator. A constant Hamiltonian, and the Hamiltonians $(\hat{\omega}_n \cdot p)$, $(\Omega_n \cdot p)$ and $(\hat{\omega}_n \cdot p)^2$, are eigenvectors for the eigenvalue zero. For $k \in \mathbb{N}_0^2$ different from $(0,0)$, $(1,0)$, $(0,1)$ and $(0,2)$, the derivative operator maps the Hamiltonian $(\hat{\omega}_n \cdot p)^{k_1}(\Omega_n \cdot p)^{k_2}$ into

$$D\mathcal{R}_n(H_n^0)(\hat{\omega}_n \cdot p)^{k_1}(\Omega_n \cdot p)^{k_2} = \frac{\|\omega_{n+1}\|^{k_1}}{(\alpha_{n+1}\|T_n^{-1}\Omega_n\|)^{2(k_1-1)+k_2}\|\omega_n\|^{k_1}} (\hat{\omega}_{n+1} \cdot p)^{k_1}(\Omega_{n+1} \cdot p)^{k_2} \,.$$

Thus, the operator norm of $D\mathcal{R}_n(H_n^0)$ acting on q-independent Hamiltonians can be bounded by

$$\|D\mathcal{R}_n(H_n^0)\mathbb{E}\| \le \frac{\max\{1, \frac{\|\omega_{n+1}\|}{\|\omega_n\|}\}}{\alpha_{n+1}\|T_n^{-1}\Omega_n\|} \le 2/3 \,. \tag{5.1}$$

The derivative of the n^{th}-step renormalization operator $\mathcal{R}_n$ at a q-independent Hamiltonian $\mathbb{E}H_n$ is the linear operator $\mathcal{L}_n = D\mathcal{R}_n(\mathbb{E}H_n) : \overset{+}{\mathbb{I}}_n \mathcal{A}_n(\rho') \to \overset{+}{\mathbb{I}}_{n+1} \mathcal{A}_{n+1}(\rho')$, given by its action on an arbitrary $f_n \in \overset{+}{\mathbb{I}}_n \mathcal{A}_n(\rho')$,

$$\mathcal{L}_n f_n = \theta_n/\mu_n D\mathfrak{S}_{n+1}(\mathbb{E}\tilde{H}_{n+1}) D\mathcal{N}_{n+1}(\mathbb{E}H''_{n+1}) D\mathcal{V}_{n+1}(\mathbb{E}H'_{n+1}) f_n \circ \mathcal{T}_n \,.$$

The operators $\mathfrak{S}_{n+1}$ and $\mathcal{V}_{n+1}$ have been introduced in Proposition 3.7 and Proposition 3.6, respectively. The next Proposition shows that there is a super-exponential shrinking of the q-dependent modes.

Proposition 5.2 *Let $\omega_0 \in DC(\beta)$, for some $\beta \ge 0$. There exist $c_1, c_2 > 0$, such that*

$$\|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j(\mathbb{I} - \mathbb{E})\| \le c_1^{n-j+1} \frac{\tilde{A}^2_{n+1}\tilde{A}^2_n}{\tilde{A}^2_j \tilde{A}^2_{j-1}} e^{-c_2 \Lambda_{j,n}} \,, \tag{5.2}$$

for $n \ge 0$ and $j = 0, \ldots, n$, assuming that $\|h_i\|_{i,\rho'} \le \zeta_i/2$, for $i = j, \ldots, n$, where, as before, $\zeta_i = C'/(\alpha_i\alpha_{i+1})^2$, $C' > 0$. Here,

$$\Lambda^{2+\beta}_{j,n} = \frac{\tilde{A}_{n+1}}{\max\{\sigma, \kappa\}\tilde{A}^{1+\beta}_{j-1}} \,. \tag{5.3}$$

Proof: The following properties of the linear operator $\mathcal{L}_n$ will be useful to prove this Proposition. First, $\mathcal{L}_n = \mathbb{I}^+_{n+1}\,\mathcal{L}_n$. Second, when acting on a Fourier mode, this operator changes its index ν into $T_n^*\nu$, as the derivatives of the elimination, translation and scaling maps do not change the value of ν. We are interested in the action of the operator $\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j(\mathbb{I}-\mathbb{E})$ on the modes indexed by $\nu \neq 0$. A mode with a particular value of $k \in \mathbb{N}_0^2$ can in general produce modes with different k'-values in $\mathbb{N}_0^2$. We will denote by $\|k'\|_{\min}$ the minimum of the norms of the vectors k' generated from a mode index k.

For every $n \in \mathbb{N}_0$, let

$$\begin{aligned} I_n^{1+} &= \{(\nu,k) \in I_n^+ : |\omega_n \cdot \nu| \le \sigma |\Omega_n \cdot \nu|\}\,, \\ I_n^{2+} &= \{(\nu,k) \in I_n^+ : |\omega_n \cdot \nu| \le \kappa \|k\|\}\,, \end{aligned}$$

be the two subsets of I_n^+. We also define the following subset of I_j^{1+}, for $j = 0,\ldots,n$,

$$V_{j,n}^+ = \{(\nu,k) \in I_j^{1+} : (T_n^* \ldots T_j^* \nu, k') \in I_{n+1}^{1+}\}\,.$$

For every $(\nu,k) \in V_{j,n}^+$, ν must satisfy the resonant condition

$$|\omega_{n+1} \cdot T_n^* \ldots T_j^* \nu| \le \sigma |\Omega_{n+1} \cdot T_n^* \ldots T_j^* \nu|\,. \tag{5.4}$$

Notice that $\omega_{n+1} \cdot T_n^* \ldots T_j^* \nu = T_j \ldots T_n \omega_{n+1} \cdot \nu$. As $T_n^{-1}\omega_n = \alpha_{n+1}^{-1}\omega_{n+1}$, we obtain

$$T_j \ldots T_n \omega_{n+1} = \omega_j \prod_{i=j+1}^{n+1} \alpha_i\,. \tag{5.5}$$

Similarly, $\Omega_{n+1} \cdot T_n^* \ldots T_j^* \nu = T_j \ldots T_n \Omega_{n+1} \cdot \nu$. From $T_n^{-1}\Omega_n = \Omega_{n+1}\|T_n^{-1}\Omega_n\|$, we find

$$T_j \ldots T_n \Omega_{n+1} = \Omega_j \prod_{i=j}^{n} \frac{1}{\|T_i^{-1}\Omega_i\|}\,.$$

The condition (5.4) can then be written in the form

$$\left(\frac{1}{\sigma}\prod_{i=j+1}^{n+1} \alpha_i \|T_{i-1}^{-1}\Omega_{i-1}\| + \frac{1}{\|\omega_j\|}\right) |\omega_j \cdot \nu| \le |\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|\,. \tag{5.6}$$

A lower bound on $|\omega_j \cdot \nu|$ can be obtained by using the Diophantine property of ω_0, i.e. that there exists $C_0 > 0$, such that $|\omega_0 \cdot \nu| \ge C_0 \|\nu\|^{-(1+\beta)}$, for any $\nu \in \mathbb{Z}^2$. This property implies that

$$|\omega_j \cdot \nu| = |\omega_0 \cdot T_0^{*^{-1}} \ldots T_{j-1}^{*^{-1}} \nu| \prod_{i=1}^{j} \alpha_i \ge \frac{C_0 \prod_{i=1}^{j} \alpha_i}{\|T_0^{*^{-1}} \ldots T_{j-1}^{*^{-1}} \nu\|^{1+\beta}}\,,$$

for $j \ge 1$. As $\|T_0^{*^{-1}} \ldots T_{j-1}^{*^{-1}}\nu\| = \|P_{j-1}^{-1}\nu\| \le (p_{j-1}+q_{j-1})\|\nu\| \le (1+1/\alpha_0)\tilde{A}_{j-1}\|\nu\|$ and $\|\nu\| \le 2(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)$, there exists $c > 1$, such that

$$|\omega_j \cdot \nu| \ge \frac{C_0 \tilde{A}_j}{\alpha_0 (2c\tilde{A}_{j-1})^{1+\beta} (|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)^{1+\beta}}\,, \tag{5.7}$$

for $j \ge 0$. The bounds (5.6) and (5.7) imply that

$$|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu| \ge \frac{\left(\frac{1}{\sigma}\prod_{i=j+1}^{n+1} \alpha_i \|T_{i-1}^{-1}\Omega_{i-1}\| + \frac{1}{\|\omega_j\|}\right) C_0 \tilde{A}_j}{\alpha_0 (2c\tilde{A}_{j-1})^{1+\beta} (|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)^{1+\beta}}\,,$$

and, thus,

$$(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)^{2+\beta} \ge \frac{C_0}{\alpha_0 (2c)^{1+\beta} \sigma} \frac{\tilde{A}_{n+1} \prod_{i=j+1}^{n+1} \|T_{i-1}^{-1}\Omega_{i-1}\|}{\tilde{A}_{j-1}^{1+\beta}}\,.$$

Using the bounds obtained in Proposition 2.3, we find that there exists $c' > 0$, such that

$$(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|) \geq c' \Lambda'_{j,n} \,, \tag{5.8}$$

where

$$\Lambda'^{2+\beta}_{j,n} = \frac{\tilde{A}_{n+1}\tilde{A}_n}{\sigma \tilde{A}^{1+\beta}_{j-1}\tilde{A}_{j-1}} \,. \tag{5.9}$$

Now consider the modes indexed by $(\nu,k) \in \mathbb{Z}^2 \times \mathbb{N}_0^2$ that belong to $I^+_j \backslash V^+_{j,n}$. Let $(T^*_n \ldots T^*_j \nu, k') \in I^{2+}_{n+1}$ be the index of a mode generated by the action of the operator $\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j(\mathbb{I} - \mathbb{E})$ on such a mode. The following condition must be satisfied,

$$|\omega_{n+1} \cdot T^*_n \ldots T^*_j \nu| \leq \kappa \|k'\|_{\min} \leq \kappa \|k'\| \,. \tag{5.10}$$

From the inequality (5.10), using the identity (5.5) and the Diophantine bound (5.7), we find that

$$\|k'\| \geq \|k'\|_{\min} \geq \frac{C_0}{\kappa \alpha_0 (2c)^{1+\beta}} \frac{\tilde{A}_{n+1}}{\tilde{A}^{1+\beta}_{j-1}(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)^{1+\beta}} \,.$$

Define $W^+_{j,n} = \{(\nu,k) \in I^+_j : (T^*_n \ldots T^*_j \nu, k') \in I^{2+}_{n+1},\ |\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu| \geq \|k'\|_{\min}\}$. If $(\nu,k) \in (I^+_j \backslash V^+_{j,n}) \cap W^+_{j,n}$, then

$$(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|) \geq c'' \Lambda''_{j,n} \,, \tag{5.11}$$

where

$$\Lambda''^{2+\beta}_{j,n} = \frac{\tilde{A}_{n+1}}{\kappa \tilde{A}^{1+\beta}_{j-1}} \,, \tag{5.12}$$

and $c'' > 0$. If $(T^*_n \ldots T^*_j \nu, k') \in I^{2+}_{n+1}$ and $(\nu,k) \in (I^+_j \backslash V^+_{j,n}) \cap (I^+_j \backslash W^+_{j,n})$, then

$$\|k'\|_{\min} \geq c'' \Lambda''_{j,n} \,. \tag{5.13}$$

Let $\mathbb{V}^+_{j,n} : \mathcal{A}_j(\rho') \to \mathcal{A}_j(\rho')$ be the projection operator on $\mathcal{A}_j(\rho')$ over the indexes in $V^+_{j,n}$, defined by the action

$$\mathbb{V}^+_{j,n} f_j = \sum_{(\nu,k) \in V^+_{j,n}} (f_j)_{\nu,k} (\hat{\omega} \cdot p)^{k_1} (\Omega \cdot p)^{k_2} e^{iq\cdot\nu} \,,$$

on an arbitrary $f_j \in \mathcal{A}_j(\rho')$. Let $\bar{\mathbb{V}}^+_{j,n} : \mathcal{A}_j(\rho') \to \mathcal{A}_j(\rho'')$ be the same projection followed by an analytic inclusion in the q variables, obtained by restricting the domain of $f_j \in \mathcal{A}_j(\rho')$ to $\mathcal{D}_j(\rho'')$, where $\rho'_1 > \rho''_1 > 0$ and $\rho'_2 = \rho''_2$. As

$$\|\bar{\mathbb{V}}^+_{j,n} f_j\|_{j,\rho''} = \sum_{(\nu,k) \in V^+_{j,n}} |(f_j)_{\nu,k}| \rho'^{\|k\|}_2 e^{\rho'_1(|\hat{\omega}_j\cdot\nu| + |\Omega_j\cdot\nu|)} e^{-(\rho'_1-\rho''_1)(|\hat{\omega}_j\cdot\nu| + |\Omega_j\cdot\nu|)} \,,$$

we have $\|\bar{\mathbb{V}}^+_{j,n} f_j\|_{j,\rho''} \leq \|\bar{\mathbb{V}}^+_{j,n}\| \|f_j\|_{j,\rho'}$, with $\|\bar{\mathbb{V}}^+_{j,n}\| \leq e^{-(\rho'_1-\rho''_1)c'\Lambda'_{j,n}}$, as follows from the bound (5.8).

Similarly, let $\mathbb{W}^+_{j,n} : \mathcal{A}_j(\rho') \to \mathcal{A}_j(\rho')$, be the projection operator on $\mathcal{A}_j(\rho')$ over the indexes in $W^+_{j,n}$ and $\bar{\mathbb{W}}^+_{j,n} : \mathcal{A}_j(\rho') \to \mathcal{A}_j(\rho'')$ the same projection followed by an analytic inclusion in the q variables. Here, as before, $\rho'_1 > \rho''_1 > 0$ and $\rho'_2 = \rho''_2$. As

$$\|\bar{\mathbb{W}}^+_{j,n} f_j\|_{j,\rho''} = \sum_{(\nu,k) \in W^+_{j,n}} |(f_j)_{\nu,k}| \rho'^{\|k\|}_2 e^{\rho'_1(|\hat{\omega}_j\cdot\nu| + |\Omega_j\cdot\nu|)} e^{-(\rho'_1-\rho''_1)(|\hat{\omega}_j\cdot\nu| + |\Omega_j\cdot\nu|)} \,,$$

from the bound (5.11), we find that the operator norm of $\bar{\mathbb{W}}^+_{j,n}$ satisfies $\|\bar{\mathbb{W}}^+_{j,n}\| \leq e^{-(\rho'_1-\rho''_1)c''\Lambda''_{j,n}}$.

Let $\bar{\mathbb{I}}_n : \mathcal{A}_n(\varrho'') \to \mathcal{A}_n(\rho')$, be the inclusion map obtained by restricting the domain of the functions $f_n \in \mathcal{A}_n(\varrho'')$ to $\mathcal{D}_n(\rho')$, where $\varrho_2'' > \rho_2' > 0$ and $\varrho_1'' = \rho_1'$. As

$$\|\bar{\mathbb{I}}_n f_n\|_{n,\rho'} = \sum_{(\nu,k')\in I_n} |(f_n)_{\nu,k'}| \varrho_2''^{\|k'\|} e^{\varrho_1''(|\hat{\omega}_n\cdot\nu|+|\Omega_n\cdot\nu|)} e^{-(\ln\varrho_2''-\ln\rho_2')\|k'\|} \leq \|\bar{\mathbb{I}}_n\|\|f_n\|_{n,\varrho''} ,$$

the norm of the inclusion map $\bar{\mathbb{I}}_n$ acting on functions that are composed only of modes with $\|k'\| \geq \|k'\|_{\min}$ satisfying the inequality (5.13), can be bounded by $\|\bar{\mathbb{I}}_n\| \leq e^{-(\ln\varrho_2''-\ln\rho_2')c''\Lambda''_{j,n}}$.

To obtain the desired bound (5.2), we write the operator $\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j(\mathbb{I}-\mathbb{E})$ as

$$\begin{aligned}
\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j(\mathbb{I}-\mathbb{E}) &= \mathcal{L}_n \circ \cdots \circ \mathcal{L}_{j+1}(\mathbb{I}-\mathbb{E})\overline{\mathcal{L}}_j^{(1)}\bar{\mathbb{V}}_{j,n}^{+} \\
&\quad + \mathcal{L}_n \circ \cdots \circ \mathcal{L}_{j+1}(\mathbb{I}-\mathbb{E})\overline{\mathcal{L}}_j^{(1)}\bar{\mathbb{W}}_{j,n}^{+}(\mathbb{I}-\mathbb{V}_{j,n}^{+}) \\
&\quad + \bar{\mathbb{I}}_{n+1}\overline{\mathcal{L}}_n^{(2)} \circ \mathcal{L}_{n-1} \circ \cdots \circ \mathcal{L}_j(\mathbb{I}-\mathbb{E})(\mathbb{I}-\mathbb{W}_{j,n}^{+})(\mathbb{I}-\mathbb{V}_{j,n}^{+}) .
\end{aligned}$$

Here, $\overline{\mathcal{L}}_n^{(1)} : \overset{+}{\mathbb{I}}_n\mathcal{A}_n(\rho'') \to \overset{+}{\mathbb{I}}_{n+1}\mathcal{A}_{n+1}(\rho')$ and $\overline{\mathcal{L}}_n^{(2)} : \overset{+}{\mathbb{I}}_n\mathcal{A}_n(\rho') \to \overset{+}{\mathbb{I}}_{n+1}\mathcal{A}_{n+1}(\varrho'')$, $n \in \mathbb{N}_0$, are the derivatives of the n^{th}-step renormalization operator at $\mathbb{E}H_n$. With superscripts (1) and (2), we explicitly emphasize that these operators actually improve analyticity in q and p variables, respectively. For some $\rho'', \varrho'' > 0$ satisfying $0 < \rho_1'' < \rho_1'$, $\rho_2'' = \rho_2'$, $\varrho_1'' = \rho_1'$ and $\varrho_2'' > \rho_2'$, the construction of the analyticity improving operators is possible, as mentioned in Remark 3.9.

The norm $\|D\mathcal{R}_i(H_i^0)\|$, $i \in \mathbb{N}_0$, can be bounded by a constant times θ_i/μ_i. By Cauchy's estimate, we have

$$\|D\mathcal{R}_i(H_i) - D\mathcal{R}_i(H_i^0)\| \leq \frac{2\|h_i\|_{i,\rho'}}{(\zeta_i - \|h_i\|_{i,\rho'})^2} , \tag{5.14}$$

where $D\mathcal{R}_i(H_i)$ is the derivative of the i^{th}-step renormalization operator $\mathcal{R}_i$ at a Hamiltonian $H_i = H_i^0 + h_i$. In particular, the inequality (5.14) is satisfied by $\mathcal{L}_i$, the derivative of the i^{th}-step renormalization operator at $\mathbb{E}H_i$. As, by assumption, $\|h_i\|_{i,\rho'} \leq \zeta_i/2$, we have the bound $\|\mathcal{L}_i\| \leq c_3\theta_i/\mu_i$, with $c_3 > 0$.

Using the bounds on the operator norms $\|\bar{\mathbb{V}}_{j,n}^{+}\|$, $\|\bar{\mathbb{W}}_{j,n}^{+}\|$ and $\|\bar{\mathbb{I}}_n\|$, obtained above, we find that

$$\|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j(\mathbb{I}-\mathbb{E})\| \leq c_3 e^{-c_2\Lambda_{j,n}} \left(2\frac{\theta_j}{\mu_j}\|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_{j+1}(\mathbb{I}-\mathbb{E})\| + \frac{\theta_n}{\mu_n}\|\mathcal{L}_{n-1} \circ \cdots \circ \mathcal{L}_j(\mathbb{I}-\mathbb{E})\| \right) ,$$

where $\Lambda_{j,n} \leq \min\{\Lambda'_{j,n}, \Lambda''_{j,n}\}$, with $\Lambda'_{j,n}$ and $\Lambda''_{j,n}$ given by the expressions (5.9) and (5.12), respectively, and $c_2 = \min\{(\rho_1' - \rho_1'')c', (\rho_1' - \rho_1'')c'', (\ln\varrho_2'' - \ln\rho_2')c''\}$.

Using the bounds on the norms of $\mathcal{L}_i$, for $i = j, \ldots, n$, we obtain

$$\|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j(\mathbb{I}-\mathbb{E})\| \leq 3c_3^{n-j+1} e^{-c_2\Lambda_{j,n}} \prod_{i=j}^{n} \frac{\theta_i}{\mu_i} .$$

The bound (5.2) follows from this inequality and the bounds in Proposition 2.3. □

Let $h_0 \in \overset{+}{\mathbb{I}}_0\mathcal{A}_0(\rho')$, $\rho' > 0$, componentwise. After the first $(n+1)$ renormalization steps the perturbation can be separated into two parts,

$$h_{n+1} = \mathbb{E}h_{n+1} + (\mathbb{I}-\mathbb{E})h_{n+1} .$$

The q-independent part of the perturbation can be determined by

$$\mathbb{E}h_{n+1} = D\mathcal{R}_n(H_n^0)\mathbb{E}h_n + \mathbb{E}\mathcal{O}_n^0(\|h_n\|_{n,\rho'}^2) ,$$

or after applying this equality recursively,

$$\begin{aligned}
\mathbb{E}h_{n+1} &= D\mathcal{R}_n(H_n^0)\ldots D\mathcal{R}_0(H_0^0)\mathbb{E}h_0 \\
&\quad + \sum_{j=1}^{n} D\mathcal{R}_n(H_n^0)\ldots D\mathcal{R}_j(H_j^0)\mathbb{E}\mathcal{O}_{j-1}^0(\|h_{j-1}\|_{j-1,\rho'}^2) + \mathbb{E}\mathcal{O}_n^0(\|h_n\|_{n,\rho'}^2) .
\end{aligned} \tag{5.15}$$

Here $\mathcal{O}_n^0(\|h_n\|_{n,\rho'}^2)$ denotes the second-order remainder of the Taylor expansion of $\mathcal{R}_n(H_n)$ about H_n^0. The norm of $\mathcal{O}_n^0(\|h_n\|_{n,\rho'}^2)$ will be denoted by F_n^2 and can be estimated by the bound (3.5).

In order to estimate the q-dependent part of the perturbation, we perform the Taylor expansion of $\mathcal{R}_n(H_n)$ about $\mathbb{E}H_n$, the q-independent part of the Hamiltonian H_n. The q-dependent part of the perturbation is

$$(\mathbb{I}-\mathbb{E})h_{n+1} = \mathcal{L}_n(\mathbb{I}-\mathbb{E})h_n + (\mathbb{I}-\mathbb{E})\mathcal{O}_n(\|(\mathbb{I}-\mathbb{E})h_n\|_{n,\rho'}^2)\,,$$

where $\mathcal{O}_n(\|(\mathbb{I}-\mathbb{E})h_n\|_{n,\rho'}^2)$ denotes the second-order remainder of the Taylor expansion of $\mathcal{R}_n(H_n)$ about $\mathbb{E}H_n$. The norm of this remainder is of the order of $\|(\mathbb{I}-\mathbb{E})h_n\|_{n,\rho'}^2$ and will be denoted by G_n^2.

After successive applications of the previous recursion relation, we obtain

$$\begin{aligned}(\mathbb{I}-\mathbb{E})h_{n+1} \;=\; & \mathcal{L}_n\ldots\mathcal{L}_0(\mathbb{I}-\mathbb{E})h_0 + \sum_{j=1}^{n}\mathcal{L}_n\ldots\mathcal{L}_j(\mathbb{I}-\mathbb{E})\mathcal{O}_{j-1}(\|(\mathbb{I}-\mathbb{E})h_{j-1}\|_{j-1,\rho'}^2) \\ & +(\mathbb{I}-\mathbb{E})\mathcal{O}_n(\|(\mathbb{I}-\mathbb{E})h_n\|_{n,\rho'}^2)\,. \end{aligned} \tag{5.16}$$

We will use this identity to estimate the decrease of the norm of the q-dependent part of the perturbation in the following Lemma.

Lemma 5.3 *Let $\omega_0 \in DC(\beta)$, $0 \le \beta < \sqrt{2}-1$, and let $\rho_1' > 0$. There exist $\tau > 2$ and $\rho_2', \sigma, \kappa, C > 0$, such that if $\|h_0\|_{0,\rho'} \le C^2 < 1$, then $\|(\mathbb{I}-\mathbb{E})h_n\|_{n,\rho'} \le C\tilde{A}_n^{-\tau} < \zeta_n/2$, $n \ge 0$, where $\zeta_n = C'/(\alpha_n\alpha_{n+1})^2$, $C' > 0$. Furthermore, if $\beta < (\sqrt{161}-11)/10$, then $\|h_n\|_{n,\rho'} \le C\tilde{A}_{n-1}^{-2} < \zeta_n/2$.*

Proof: We find first, using the Diophantine bound (2.8), that if $\beta \le \sqrt{2}-1$ and $\tau > 2$, there exists $C > 0$, such that

$$\frac{\zeta_n}{2} = \frac{C'}{2\alpha_n^2\alpha_{n+1}^2} \ge \frac{C'}{2\tilde{K}^4\tilde{A}_{n-1}^{2\beta}\tilde{A}_n^{2\beta}} \ge \frac{C'}{2\tilde{K}^{4+2\beta}\tilde{A}_{n-1}^{2\beta}\tilde{A}_{n-1}^{2\beta(1+\beta)}} > \frac{C}{\tilde{A}_{n-1}^2} > \frac{C}{\tilde{A}_n^\tau}\,.$$

Let $\tau' = \ln c_1/\ln\gamma$, where $c_1 > 1$ is the constant from Proposition 5.2. As $\tilde{A}_n/\tilde{A}_{j-1} \ge \gamma^{n-j}$, $0 \le j \le n$, the inequality (5.2) implies

$$\|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j(\mathbb{I}-\mathbb{E})\| \le \frac{\tilde{A}_{n+1}^{4+\tau'}}{\tilde{A}_{j-1}^{4+\tau'}}e^{-c_2\Lambda_{j,n}}\,. \tag{5.17}$$

Using the inequality $e^{-t} \le (s/t)^s$, valid for any $t > 0$ and $s > 0$, we find that, for any $\tau'' > 0$,

$$\frac{\tilde{A}_{n+1}^{4+\tau'}}{\tilde{A}_{j-1}^{4+\tau'}}e^{-c_2\Lambda_{j,n}} \le (\max\{\sigma,\kappa\})^{\tau''}\left(\frac{\tau''(2+\beta)}{c_2}\right)^{\tau''(2+\beta)}\frac{\tilde{A}_{j-1}^{\tau''(1+\beta)-(4+\tau')}}{\tilde{A}_{n+1}^{\tau''-(4+\tau')}}\,.$$

Let $\tau'' > 0$ be given, such that $\tau = \tau'' - \tau' - 4 > 2$. There exist $\sigma, \kappa > 0$, such that

$$\frac{\tilde{A}_{n+1}^{4+\tau'}}{\tilde{A}_{j-1}^{4+\tau'}}e^{-c_2\Lambda_{j,n}} \le (\max\{\sigma,\kappa\})^{\tau''}\left(\frac{\tau''(2+\beta)}{c_2}\right)^{\tau''(2+\beta)}\frac{\tilde{A}_{j-1}^{\tau(1+\beta)+(4+\tau')\beta}}{\tilde{A}_{n+1}^{\tau}} \le \frac{\tilde{A}_{j-1}^{\tau(1+\beta)+(4+\tau')\beta}}{6\tilde{A}_{n+1}^{\tau}}\,,$$

and thus,

$$\|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j(\mathbb{I}-\mathbb{E})\| \le \frac{\tilde{A}_{j-1}^{\tau(1+\beta)+(4+\tau')\beta}}{6\tilde{A}_{n+1}^{\tau}}\,. \tag{5.18}$$

We will prove the Lemma by induction. There exists $0 < C < 1$, such that

$$\|(\mathbb{I}-\mathbb{E})h_0\|_{0,\rho'} \le \|h_0\|_{0,\rho'} \le C^2 < C\tilde{A}_0^{-\tau} < C < \zeta_0/2\,. \tag{5.19}$$

Therefore, for $n = 0$, the claim is true. Assume that that the claim holds for $0 < j \le n$. Thus, there exists $C > 0$, such that $\|(\mathbb{I}-\mathbb{E})h_j\|_{j,\rho'} \le C\tilde{A}_j^{-\tau} < \zeta_j/2$ and $\|h_j\|_{j,\rho'} \le C\tilde{A}_{j-1}^{-2} < \zeta_j/2$. We will show that the claim is true for $j = n+1$.

Using the identity (5.16), we obtain

$$\|(\mathbb{I}-\mathbb{E})h_{n+1}\|_{n+1,\rho'} \le \|\mathcal{L}_n \ldots \mathcal{L}_0(\mathbb{I}-\mathbb{E})h_0\|_{n+1,\rho'} + \sum_{j=1}^{n} \|\mathcal{L}_n \ldots \mathcal{L}_j(\mathbb{I}-\mathbb{E})\| \cdot G_{j-1}^2 + G_n^2 \,. \tag{5.20}$$

We will estimate the size of the terms on the right hand side of the inequality (5.20).

As $\|h_0\|_{0,\rho'} \le C^2 < 1$, the inequality (5.18), for $j=0$, implies a bound on the first term

$$\|\mathcal{L}_n \ldots \mathcal{L}_0(\mathbb{I}-\mathbb{E})h_0\|_{n+1,\rho'} \le \frac{C}{6\tilde{A}_{n+1}^{\tau}} \,. \tag{5.21}$$

Using Cauchy's formula, one can obtain an estimate on the norm of the second-order remainder $\mathcal{O}_n(\|(\mathbb{I}-\mathbb{E})h_n\|_{n,\rho'}^2)$, analogous to the bound (3.5),

$$G_n^2 \le \frac{\|(\mathbb{I}-\mathbb{E})h_n\|_{n,\rho'}^2}{(\zeta_n - \|\mathbb{E}h_n\|_{n,\rho'})(\zeta_n - \|\mathbb{E}h_n\|_{n,\rho'} - \|(\mathbb{I}-\mathbb{E})h_n\|_{n,\rho'})} \,.$$

Applying the inductive hypothesis and the Diophantine condition in the form $\tilde{A}_{n+1} \le \tilde{K}\tilde{A}_n^{1+\beta}$, or equivalently, $\alpha_{n+1} \le \tilde{K}\tilde{A}_n^{\beta}$, we further obtain that, for some $C>0$,

$$G_n^2 \le \frac{4C^2}{\zeta_n^2 \tilde{A}_n^{2\tau}} \le \frac{4C^2\alpha_n^4\alpha_{n+1}^4}{C'^2\tilde{A}_n^{2\tau}} \le \frac{4C^2\tilde{K}^{2\tau/(1+\beta)+8}\tilde{A}_n^{4\beta}\tilde{A}_{n-1}^{4\beta}}{C'^2\tilde{A}_{n+1}^{2\tau/(1+\beta)}} < \frac{C}{6\tilde{A}_{n+1}^{\tau}} \,, \tag{5.22}$$

if $0 \le \beta < 1$ and $\tau \ge 8\beta(1+\beta)/(1-\beta)$. Using the bounds (5.18) and (5.22), we can estimate the sum

$$\begin{aligned} \sum_{j=1}^{n} \|\mathcal{L}_n \ldots \mathcal{L}_j(\mathbb{I}-\mathbb{E})\| \cdot G_{j-1}^2 \;&\le\; \frac{2C^2}{3\tilde{A}_{n+1}^{\tau}} \sum_{j=1}^{n} \frac{\tilde{A}_{j-1}^{\tau(1+\beta)+(4+\tau')\beta}}{\zeta_{j-1}^2\tilde{A}_{j-1}^{2\tau}} \\ &\le\; \frac{2C^2\tilde{K}^8}{3C'^2\tilde{A}_{n+1}^{\tau}} \sum_{j=1}^{n} \frac{1}{\tilde{A}_{j-1}^{\tau(1-\beta)-(12+\tau')\beta}} \,. \end{aligned}$$

Since for $\alpha_0 > 1$, we have $\tilde{A}_{j-1} \ge \gamma^{j-1}$, the previous sum can be bounded by some positive constant, if $0 \le \beta < 1$ and $\tau > \beta(\tau'+12)/(1-\beta)$. Thus, there exists $C>0$, such that

$$\sum_{j=1}^{n} \|\mathcal{L}_n \ldots \mathcal{L}_j(\mathbb{I}-\mathbb{E})\| \cdot G_{j-1}^2 \le \frac{C}{6\tilde{A}_{n+1}^{\tau}} \,. \tag{5.23}$$

Finally, if $0 \le \beta < 1$ is given, and constants $\tau > 2$ and $\sigma, \kappa, C > 0$ are chosen according to the various conditions stated above, using the bounds (5.21), (5.22) and (5.23), we obtain

$$\|(\mathbb{I}-\mathbb{E})h_{n+1}\|_{n+1,\rho'} \le \frac{C}{6\tilde{A}_{n+1}^{\tau}} + \frac{C}{6\tilde{A}_{n+1}^{\tau}} + \frac{C}{6\tilde{A}_{n+1}^{\tau}} = \frac{C}{2\tilde{A}_{n+1}^{\tau}} < \frac{C}{\tilde{A}_{n+1}^{\tau}} < \frac{\zeta_{n+1}}{2} \,. \tag{5.24}$$

Concerning the q-independent part of the perturbation, from the identity (5.15), we find that

$$\begin{aligned} \|\mathbb{E}h_{n+1}\|_{n+1,\rho'} \;&\le\; \|D\mathcal{R}_n(H_n^0)\ldots D\mathcal{R}_0(H_0^0)\mathbb{E}h_0\|_{n+1,\rho'} \\ &\quad + \sum_{j=1}^{n} \|D\mathcal{R}_n(H_n^0)\ldots D\mathcal{R}_j(H_j^0)\mathbb{E}\| \cdot F_{j-1}^2 + F_n^2 \,. \end{aligned} \tag{5.25}$$

Using the bound (5.1) on the derivative of a one-step renormalization operator acting on q-independent Hamiltonians and the first of the inequalities (2.9), one obtains that there exists $C>0$, such that

$$\|D\mathcal{R}_n(H_n^0)\ldots D\mathcal{R}_0(H_0^0)\mathbb{E}h_0\|_{n,\rho'} \le \frac{4\alpha_0^2}{(1+\alpha_0)\tilde{A}_n^2}\|h_0\|_{0,\rho'} \le \frac{4\alpha_0^2}{(1+\alpha_0)\tilde{A}_n^2}C^2 < \frac{C}{6\tilde{A}_n^2} \,. \tag{5.26}$$

From the estimate (3.5) obtained in Theorem 3.8, we have the following bound on the norm of the second-order remainder, $F_n^2 \leq \zeta_n^{-1}(\zeta_n - \|h_n\|_{n,\rho'})^{-1}\|h_n\|^2_{n,\rho'}$. Using the inductive hypothesis and the Diophantine property of ω_0, we find that there exists $C > 0$, such that

$$F_j^2 \leq \frac{2C^2}{\zeta_j^2 \tilde{A}_{j-1}^4} \leq \frac{2C^2 \tilde{K}^{4/(1+\beta)} \alpha_j^4 \alpha_{j+1}^4}{C'^2 \tilde{A}_j^{4/(1+\beta)}} \leq \frac{2C^2 \tilde{K}^{4/(1+\beta)+8} \tilde{A}_{j-1}^{4\beta} \tilde{A}_j^{4\beta}}{C'^2 \tilde{A}_j^{4/(1+\beta)}} < \frac{C}{6\tilde{A}_j^2}, \tag{5.27}$$

for $0 \leq \beta < (\sqrt{41} - 5)/8$. Since, from the bound (5.1) and the inequalities (2.9), one has

$$\|D\mathcal{R}_n(H_n^0) \dots D\mathcal{R}_j(H_j^0)\mathbb{E}\| \leq \frac{8\alpha_0 \tilde{A}_j \tilde{A}_{j-1}}{(1+\alpha_0)\tilde{A}_n^2},$$

the sum on the right hand side of the inequality (5.25) can be estimated by

$$\begin{aligned} \sum_{j=1}^{n} \|D\mathcal{R}_n(H_n^0) \dots D\mathcal{R}_j(H_j^0)\mathbb{E}\| \cdot F_{j-1}^2 \; &\leq \; \frac{16\alpha_0 C^2 \tilde{K}^8}{(1+\alpha_0)C'^2 \tilde{A}_n^2} \sum_{j=1}^{n} \frac{\tilde{A}_j \tilde{A}_{j-1}^{1+4\beta} \tilde{A}_{j-2}^{4\beta}}{\tilde{A}_{j-2}^4} \\ &\leq \; \frac{16\alpha_0 C^2 \tilde{K}^{11+5\beta}}{(1+\alpha_0)C'^2 \tilde{A}_n^2} \sum_{j=1}^{n} \frac{1}{\tilde{A}_{j-2}^{4-(2+5\beta)(1+\beta)-4\beta}}. \end{aligned}$$

If $2 - 11\beta - 5\beta^2 > 0$, the sum on the right side can be bounded by a positive constant, as $\tilde{A}_j$ grows at least exponentially with j. Thus, there exists $C > 0$, such that

$$\sum_{j=1}^{n} \|D\mathcal{R}_n(H_n^0) \dots D\mathcal{R}_j(H_j^{(0)})\mathbb{E}\| \cdot F_{j-1}^2 \leq \frac{C}{6\tilde{A}_n^2}. \tag{5.28}$$

Using the bounds (5.26), (5.27) and (5.28), the inequality (5.25) implies

$$\|\mathbb{E}h_{n+1}\|_{n+1,\rho'} \leq \frac{C}{6\tilde{A}_n^2} + \frac{C}{6\tilde{A}_n^2} + \frac{C}{6\tilde{A}_n^2} = \frac{C}{2\tilde{A}_n^2}. \tag{5.29}$$

Finally, taking into account the estimates (5.24) and (5.29), we find that

$$\|h_{n+1}\|_{n+1,\rho'} = \|(\mathbb{I} - \mathbb{E})h_{n+1}\|_{n+1,\rho'} + \|\mathbb{E}h_{n+1}\|_{n+1,\rho'} \leq \frac{C}{2\tilde{A}_{n+1}^{\tau}} + \frac{C}{2\tilde{A}_n^2} \leq \frac{C}{\tilde{A}_n^2} < \frac{\zeta_{n+1}}{2}.$$

This completes the proof of the claim. □

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