NONLINEAR DOUBLE WELL SCHRÖDINGER EQUATIONS IN THE SEMICLASSICAL LIMIT

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Abstract. We consider time-dependent Schrödinger equations with a double well potential and an external nonlinear, both local and non-local, perturbation. In the semiclassical limit, the finite dimensional eigenspace associated to the lowest eigenvalues of the linear operator is almost invariant for times of the order of the *beating period* and the dominant term of the wavefunction is given by means of the solutions of a finite dimensional dynamical system. In the case of local nonlinear perturbation we assume the spatial dimension $d=1$ or $d=2$

1. Introduction

The theoretical analysis of time-dependent nonlinear Schrödinger (hereafter NLS) equations

$$\left\{ \begin{array}{l} i\hbar\dot{\psi} = H_0\psi + \epsilon W\psi, \quad \epsilon \in \mathrm{R},\ \dot{\psi} = \frac{\partial \psi}{\partial t}, \\ \psi(x,0) = \psi^0(x) \in L^2(\mathrm{R}^d),\ \|\psi^0\| = 1, \end{array} \right. \tag{1}$$

where

$$H_0 = -\frac{\hbar^2}{2m}\Delta + V, \quad \Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2},\ d \geq 1, \quad 2m = 1 \tag{2}$$

is the linear Hamiltonian and where $W = W(x, |\psi|)$ is a nonlinear perturbation, has attracted an increasing interest in these last few years (see, e.g., [13] and [17]).

In this paper we consider the case of symmetric potentials V with *double well* shape. In fact, such a potential appears in several fields, from Bose-Einstein condensate states weakly coupled [14] to localization of symmetric molecules [12]. If the nonlinear term is absent then the linear Hamiltonian H_0 has even–parity and odd-parity eigenstates and the state ψ generically performs a *beating motion*, hence the *beating period* plays the role of unit of time. When we restore the nonlinear term a symmetry breaking phenomenon occurs: that is, if the nonlinearity is larger than a threshold value then new asymmetric stationary states appears [1], [8], [19]. Furthermore, for higher nonlinearity, the beating motion is forbidden [9], [14], [18]. These results can be obtained by reducing the NLS equation to a finite dimensional dynamical system exactly solvable and proving the stability of this approximation for times of the order of the beating period with a rigorous estimate of the error in the semiclassical limit [9], [16].

Date: July 29, 2004.

This work is partially supported by the Italian MURST and INDAM-GNFM (project *Comportamenti Classici in Sistemi Quantistici*).

More precisely, in [9] has been considered the case on NLS in any dimension $d \geq 1$ where the nonlinear perturbation is *non-local*, that is it is given by

$$W = \langle \psi, g\psi \rangle g(x) \tag{3}$$

where $g(x)$ is a given odd function. In [16] has been considered the case of NLS in dimension $d = 1$ and with a nonlinear *local cubic* perturbation given by

$$W = |\psi|^{2\sigma}, \quad \sigma = 1 \tag{4}$$

Here, we consider NLS equation (1) in the semiclassical limit in the cases of both *local* (3) and *non-local* (4), with any $\sigma > 0$, nonlinearity, where we assume that $d = 1$ and $d = 2$ in the case of *local* nonlinear perturbation. Under some generic assumptions on the double-well potential, we give the asymptotic behavior of the solution ψ with a precise estimate of the error. In particular, as general results it follows that new asymmetric stationary states appear and the beating motion, between the two wells of a state initially prepared on the two lowest eigenstates, gradually disappears for increasing nonlinearity.

Hence, the results previously obtained by [9] and [16] can be seen as a particular case of the general treatment given here.

Our paper is organized as follows.

In Section 2 we introduce the assumptions on the potential. Moreover, we collect some semiclassical results concerning the spectrum of the linear Schrödinger operator.

In Section 3 we discuss the beating motion for the unperturbed problem and the choice of the parameters.

In Section 4 we give the existence results for equation (1), the conservation laws and a priori estimate. The global existence of the solution is proved for both repulsive and attractive nonlinear perturbation, where, in the second case, we have to assume that the strength of the nonlinear perturbation is small enough.

In Section 5 we introduce the two-level approximation and we discuss, in such an approximation, the appearance of new asymmetric stationary states for large nonlinearity strength. The two-level approximation, roughly speaking, consists in projecting equation (1) onto the two-dimensional space spanned by the eigenvectors of the linear Schrödinger operator associated to the two lowest eigenvalues. For practical purposes, it is more convenient to choose, as a basis of such a two-dimensional space, the two *single-well* states. The dynamical system we obtain is, in some cases (for instance for cubic local nonlinearity), exactly solvable.

In Section 6 we prove the stability of the two-level approximation in the semiclassical limit. We make use of the comparison criterion between ordinary differential equations and of a priori estimate of the solution of the NLS equation.

In Appendix we recall some useful inequalities.

We close this section by introducing some notations:

- Here $\| \cdot \|_p$ denotes the norm of the Banach space $L^p(\mathrm{R}^d)$, $p \in [1, +\infty]$, $\| \cdot \|$ usually denotes the norm of the space $L^2(\mathrm{R}^d)$ and sometimes (when this does not cause misunderstanding) it denotes the norm of a bounded operator, too;
- The notation

$$y = c_{p,d}$$

means that there exist $\hbar^\star > 0$ and a positive constant $C > 0$, independent of $\hbar$, such that

$$|y| \leq C\hbar^{-d\frac{p-2}{4p}}, \quad \forall \hbar \in (0, \hbar^\star)$$

- The notations $y = o(\hbar^\alpha)$, $y = O(\hbar^\alpha)$, $\alpha \geq 0$, and $y = O(e^{-\Gamma/\hbar})$ respectively mean that $y\hbar^{-\alpha} \to 0$ as $\hbar \to 0$ and that there exist $\hbar^\star > 0$ and a positive constant $C > 0$, independent of $\hbar$, such that

$$|y| \leq C\hbar^\alpha \quad \text{and} \quad |y| \leq Ce^{-\Gamma/\hbar}, \quad \forall \hbar \in (0, \hbar^\star)$$

- The notation $y = \tilde{O}(e^{-\Gamma/\hbar})$ means that for any Γ', $0 < \Gamma' < \Gamma$, then $y = O(e^{-\Gamma'/\hbar})$; that is, there exist $\hbar^\star > 0$ and a positive constant $C = C_{\Gamma'} > 0$, independent of $\hbar$, such that

$$|y| \leq Ce^{-\Gamma'/\hbar}, \quad \forall \hbar \in (0, \hbar^\star)$$

As usual, R denotes the set of real numbers, N denotes the set of positive integer numbers and $\mathrm{N}^\star = \mathrm{N} \cup \{0\}$; C denotes any positive constant indipendent of $\hbar$ and t.

2. Assumptions and preliminary results

2.1. **Linear operator.** Here, we introduce the assumptions on the double-well potential V and we collect some well known results on the linear operator H_0.

Hypothesis 1. *The potential $V(x)$ is a real valued function such that:*

i. *V is a symmetric potential; hence, the Hamiltonian H_0 is invariant under some space inversion $\mathcal{S}$: $[\mathcal{S}, H_0] = 0$;*
ii. *$V \in C^\infty(\mathrm{R}^d)$;*
iii. *$V(x)$ admits two minima at $x = x_\pm$, where $x_- = \mathcal{S}x_+$ and $x_+ \neq x_-$, such that*

$$V(x) > V_{min} = V(x_\pm), \quad \forall x \in \mathrm{R}^d, \ x \neq x_\pm; \tag{5}$$

iv. *finally we assume that*

$$V_\infty^- = \liminf_{|x|\to\infty} V(x) > V_{min} \tag{6}$$

and

$$V_\infty^+ = \limsup_{|x|\to\infty} V(x) < +\infty. \tag{7}$$

Remarks

- For the sake of definiteness we can always assume that, by means of a suitable choice of the coordinates, V is symmetric with respect to the spatial coordinate x_1, that is

$$V(-x_1, x_2, \ldots, x_d) = V(x_1, x_2, \ldots, x_d), \quad \forall (x_1, \ldots, x_d) \in \mathrm{R}^d; \tag{8}$$

- For the sake of simplicity, we assume also that

$$\nabla V(x_\pm) = 0 \quad \text{and} \quad \text{Hess } V(x_\pm) > 0;$$

The case of degenerate minima, that is Hess $V(x_\pm) = 0$, could be treated in a similar way; however, we don't dwell here on such details;
- In fact, we could replace assumption ii. with the weaker assumption $V \in C^2$;

- Assumption (7) is introduced in order to obtain the local existence of the solution of the Cauchy problem (1) by means of well known results [4] (see also [7]). However, by making use of some ideas contained in the paper by Carles [3], condition (7) could be partially removed and anharmonic potentials like, e.g. in dimension $d=1$, $V(x)=(x^2-a^2)(x^2-b^2)$ could be considered.

The operator H_0 formally defined by (2) admits a self-adjoint realization (still denoted by H_0) on $L^2(\mathrm{R})$ (Theorem III.1.1 in [2]).

Let $\sigma(H_0)=\sigma_d\cup\sigma_{ess}$ be the spectrum of the self-adjoint operator H_0, where σ_d denotes the discrete spectrum and σ_{ess} denotes the essential spectrum. It follows that (see Theorem III.3.1 in [2])

$$\sigma_d\subset(V_{min},V_\infty^-)\quad\text{and}\quad\sigma_{ess}=[V_\infty^-,+\infty)$$

Furthermore, the following two Lemmas hold:

Lemma 1. *Let σ_d be the discrete spectrum of H_0. Then, for any $\hbar\in(0,\hbar^\star)$, for some $\hbar^\star>0$ fixed, it follows that:*

i. *σ_d is not empty and, in particular, it contains two eigenvalues at least;*
ii. *let $\lambda_\pm$ be the lowest two eigenvalues of H_0, they are non-degenerate, in particular $\lambda_+<\lambda_-$ and there exists $C>0$, independent of $\hbar$, such that*

$$\inf_{\lambda\in\sigma(H_0)-\{\lambda_\pm\}}[\lambda-\lambda_\pm]\ge C\hbar. \tag{9}$$

Proof. The proof is an immediate consequence of the above assumptions and standard WKB arguments. In fact, by assuming, for the sake of simplicity, that

$$\mathrm{Hess}V(x_\pm)=2\mathrm{diag}(\mu_1,\mu_2,\dots,\mu_d),\quad \mu_j>0,\ j=1,\dots,d, \tag{10}$$

then the first two eigenvalues of H_0 are given by the semi-classical *single-well* non-degenerate eigenvalues (Theorem 2.3.1 [11])

$$\lambda_\pm=V_{min}+\left[\sum_{j=1}^d\sqrt{\mu_j}\right]\hbar+o(\hbar),\quad\text{as}\quad\hbar\to0, \tag{11}$$

Furthermore, from conditions (5) and (6) the estimate (9) follows (Corollary 2.3.5 [11]) □

Lemma 2. *Let $\varphi_\pm$ be the normalized eigenvectors associated to $\lambda_\pm$, then:*

i. *$\varphi_\pm$ can be chosen to be real-valued functions such that*

$$\varphi_\pm(-x_1,x_2,\dots,x_d)=\pm\varphi_\pm(x_1,x_2,\dots,x_d); \tag{12}$$

ii. *$\varphi_\pm\in H^1(\mathrm{R}^d)$;*
iii. *$\varphi_\pm\in L^p(\mathrm{R}^d)$ for any $p\in[2,+\infty]$;*
iv. *there exists a positive constant C, independent on $\hbar$, such that*

$$\|\varphi_\pm\|_p=c_{p,d}\le C\hbar^{-d\frac{p-2}{4p}},\quad\forall p\in[2,+\infty],\quad\forall\hbar\in(0,\hbar^\star). \tag{13}$$

Proof. Property i. immediately follows from (8). Property ii. follows from Lemma III.3.1 in [2]. In order to prove the statement iii. we recall that the eigenvectors $\varphi_\pm$ satisfy to the following global estimate: for any $\delta>0$ fixed there exists a positive constant $C_{\delta,\hbar}>0$ such that (Theorem III.3.2 and Corollary III.3.1 in [2])

$$|\varphi_\pm(x)|\le C_{\delta,\hbar}\exp\left[-\delta|x|/\hbar\right]$$

Hence $\varphi_\pm \in L^\infty(\mathrm{R}^d)$. From this fact and since $\varphi_\pm \in L^2(\mathrm{R}^d)$ then statement iii. immediately follows. Finally, in order to prove the statement iv. let

$$\varphi_\pm = \frac{1}{\sqrt{2}}\left[\varphi_R \pm \varphi_L\right]$$

where the vectors $\varphi_{R,L}$ satisfies to the following WKB estimates [11]

$$\varphi_R(x) = \varphi_L(-x) \sim [2\pi\hbar]^{-d/4} e^{-[\sum_{j=1}^d (x_j - x_{+,j})^2 \sqrt{\mu_j}]/2\hbar}, \quad \text{as} \quad \hbar \to 0, \tag{14}$$

in a neighborhood of the mimina $x_\pm = (x_{\pm,1}, \dots, x_{\pm,d})$ and where μ_j are defined in (10). Hence, property iv. follows for $p = +\infty$. From this fact, from the normalization of the eigenvectors and from the Hölder inequality then property iv. follows for any $p \in [2, +\infty]$:

$$\|\varphi_\pm\|_p = \left[\|\varphi_\pm^2 \varphi_\pm^{p-2}\|_1\right]^{1/p} \leq \|\varphi_\pm\|_2^{2/p} \|\varphi_\pm\|_\infty^{(p-2)/p} = \|\varphi_\pm\|_\infty^{(p-2)/p}.$$

□

From (11) it follows that the *splitting* between the two lowest eigenvalues

$$\omega = \frac{1}{2}(\lambda_- - \lambda_+) \tag{15}$$

vanishes as $\hbar$ goes to zero. In order to give a precise estimate of the splitting ω we make use of the fact that V is a symmetric double-well potential with non-zero barrier between the wells. That is, let

$$\Gamma = \inf_\gamma \int_\gamma \sqrt{V(x) - V_{min}}\, dx > 0, \tag{16}$$

be the Agmon distance between the two wells; where γ is any path connecting the two wells, that is $\gamma \in AC([0,1], \mathrm{R}^d)$ such that $\gamma(0) = x_-$ and $\gamma(1) = x_+$. From standard WKB arguments (see [11]) then it follows that the splitting is *exponentially small*, that is

$$\omega = \tilde{O}(e^{-\Gamma/\hbar}). \tag{17}$$

Furthermore, the vectors, usually called *single well states*,

$$\varphi_R = \frac{1}{\sqrt{2}}\left[\varphi_+ + \varphi_-\right] \quad \text{and} \quad \varphi_L = \frac{1}{\sqrt{2}}\left[\varphi_+ - \varphi_-\right]$$

are *localized on one well*, and

$$\|\varphi_R \varphi_L\|_\infty = \tilde{O}(e^{-\Gamma/\hbar}) \tag{18}$$

More precisely, from (12) it follows that

$$\varphi_R(-x_1, x_2, \dots, x_d) = \varphi_L(x_1, x_2, \dots, x_d)$$

and these functions are localized on only one of the two wells in the sense that for any $r > 0$ there exists $C > 0$ such that

$$\int_{D_r(x_+)} |\varphi_R(x)|^2 dx = 1 + O(e^{-C/\hbar})$$

and

$$\int_{D_r(x_-)} |\varphi_L(x)|^2 dx = 1 + O(e^{-C/\hbar})$$

where $D_r(x_\pm)$ is the ball with center $x_\pm$ and radius r. For such a reason we call them *single-well* (normalized) states.

Remark:

- We underline that, by assuming some further regularity properties on the potential V, it is possible to obtain the precise asymptotic behavior of the splitting as $\hbar$ goes to zero [10].

2.2. **Nonlinear perturbation.** Here we admit both *local* and *non-local* nonlinear perturbations

Hypothesis 2. *Let $g(x)$ be a given real-valued bounded and continuous function. We assume that*

i. **Nonlinear local perturbation.** *The perturbation W has the form*
$$W = W_\ell(x, |\psi|) = g(x)|\psi(x)|^{2\sigma}, \quad \sigma > 0 \tag{19}$$
ii. **Nonlinear non-local perturbation.** *The perturbation W has the form*
$$W = W_{n\ell}(x, |\psi|) = g(x)\langle \psi, g\psi\rangle \tag{20}$$

In the local perturbation case (19) we assume the dimension $d = 1$ or $d = 2$. In the non-local perturbation case (20) we don't introduce any assumption on the dimension d.

Remarks:

- If $\sigma = 1$ and $g \equiv 1$ then the NLS equation (1) with *local perturbation* (19) coincides with the one previously studied by Sacchetti [16] in dimension d=1. If $g(x)$ is an even function then the NLS equation (1) with *non-local perturbation* (20) coincides with the one previously studied by Grecchi, Martinez and Sacchetti [9].
- In the case of nonlinear local perturbation (19) we have to assume that the dimension d is not higher than 2. In fact, in the case of dimension $d > 2$ then, provided that $\sigma < \frac{2}{d-2}$, the existence results and the conservation laws (see Section 4) still hold, but the stability result fails (see Section 6).

2.3. **Assumption on the initial state.** Let
$$\Pi_c = 1 - [\langle \varphi_+, \cdot\rangle \varphi_+ + \langle \varphi_-, \cdot\rangle \varphi_-]$$
be the projection operator onto the eigenspace orthogonal to the bi-dimensional space associated to the doublet $\{\lambda_\pm\}$. Let ψ^0 be the initial wavefunction, we assume that

Hypothesis 3. $\Pi_c \psi^0 = 0$.

That is, we assume that
$$\psi^0 = c_+\varphi_+ + c_-\varphi_- = c_R\varphi_R + c_L\varphi_L$$
for some $c_\pm$ and $c_{R,L}$.

Remarks:

- In fact, we could assume that the initial state ψ^0 belongs to a finite dimensional eigenspace of H_0. More precisely, let $\sigma_1, \sigma_2 \subset \sigma(H_0)$ such that $\sigma(H_0) = \sigma_1 \cup \sigma_2$, $\sigma_1 \cap \sigma_2 = \emptyset$, $\sigma_1 \subset \sigma_{pp}(H_0)$, where σ_{pp} denotes the pure point spectrum of H_0, and σ_1 has a finite number of elements. Let $\mathcal{H}_1$ be the finite-dimensional spectral eigenspace associated to σ_1. Then we can replace the previous assumption by assuming that $\psi^0 \in \mathcal{H}_1$ and
$$d(\sigma_1, \sigma_2) = \inf_{\lambda\in\sigma_1,\ \mu\in\sigma_2} |\lambda - \mu| \geq C\hbar.$$

In such a case we have to define $\omega = \frac{1}{2}\inf_{\lambda,\mu\in\sigma_1,\ \lambda\neq\mu}|\lambda-\mu|$.

3. Beating motion and choice of parameters

Let us consider, for a moment, the time-dependent linear Schrödinger equation

$$\begin{cases} i\hbar\dot{\psi} = H_0\psi, \quad \epsilon\in \mathrm{R},\ \dot{\psi} = \frac{\partial\psi}{\partial t}, \\ \psi(x,0) = \psi^0(x)\in L^2(\mathrm{R}^d),\ \Pi_c\psi^0 = 0, \end{cases} \tag{21}$$

This equation has an explicit solution given by

$$\begin{aligned} \psi(x,t) &= e^{-i\lambda_+ t/\hbar}c_+\varphi_+ + e^{-i\lambda_- t/\hbar}c_-\varphi_- \\ &= \frac{1}{\sqrt{2}}\left[c_+e^{-i\lambda_+t/\hbar} + c_-e^{-i\lambda_-t/\hbar}\right]\varphi_R + \frac{1}{\sqrt{2}}\left[c_+e^{-i\lambda_+t/\hbar} - c_-e^{-i\lambda_-t/\hbar}\right]\varphi_L \\ &= \frac{1}{\sqrt{2}}e^{-i\Omega t/\hbar}\left[(\Delta\varphi_R + \delta\varphi_L)\cos(\omega t/\hbar) + i\left(\delta\varphi_R + \Delta\varphi_L\right)\sin(\omega t/\hbar)\right] \end{aligned}$$

where we set

$$\lambda_\pm = \Omega\mp\omega,\ \Delta = c_+ + c_-,\ \delta = c_+ - c_-$$

That is $\psi(x,t)$ performs a *beating motion* with *beating period*

$$T = \frac{2\pi\hbar}{\omega}$$

Such a period will play the role of unit of time.

Hypothesis 4. *Let ω be the splitting (15) satisfying to the asymptotic estimate (17). We assume that the real-valued parameter ϵ depends on $\hbar$ in such a way*

$$\frac{|\epsilon|\hbar^{-d\sigma/2}}{\omega} \le C, \quad \forall\hbar\in(0,\hbar^\star) \tag{22}$$

for some positive constant C, independent of $\hbar$, and for some $\hbar^\star$, where σ is defined in (19) for nonlinear local perturbations and where $\sigma = 0$ for nonlinear non-local perturbations (20).

Remarks:

- We underline that the strength of the perturbation is, roughly speaking, given by $|\epsilon|$ is the case of nonlinear non-local perturbation (20), and by $|\epsilon|\hbar^{-d\sigma/2}$ in the case of nonlinear local perturbation (19). The ratio
$$\eta = \frac{\epsilon\hbar^{-d\sigma/2}}{\omega}$$
plays the role of effective nonlinearity parameter. The above assumption implies that $|\eta|\le C$.
- Condition (22) implies that
$$\begin{gathered}(\text{Beating period})\times(\text{Perturbation strength}) = \\ = T\times|\epsilon|\hbar^{-d\sigma/2} \approx \frac{\hbar}{\omega}\times\omega = \hbar \approx \mathrm{dist}(\sigma(H_0),\lambda_\pm)\end{gathered}$$
Thus, intuitive arguments do not suggest us that the subspace $(1-\Pi_c)L^2$ is almost invariant for times of the order of the beating period. In fact, we will prove that $\|\Pi_c\psi\| = \tilde{O}(e^{-\Gamma/\hbar})$ for any $t\in[0,T]$ for $\hbar$ small enough.

4. Existence results and conservation laws

Here, making use of some results by [4], we prove that the solution of equation (1) globally exists. To this end we recall that $\psi^0 \in H^1 \cap L^p$ for any $p \in [2, +\infty]$ (see Lemma 2). The assumptions on the strength of the nonlinear perturbation, that is $\epsilon = \tilde{O}(e^{-\Gamma/\hbar})$ (see equations (17) and (22)), could, in order to prove the global existence result and the conservation laws, be relaxed; in fact, here we simply require that $\epsilon = O(\hbar^\alpha)$ for some $\alpha > 2$.

4.1. **Local existence.**

Theorem 1. *There exists $T^\star > 0$ and an unique solution $\psi \in C([0, T^\star), H^1) \cap C^1([0, T^\star), H^{-1})$ of (1), where $T^\star = +\infty$ or $\|\nabla \psi\| \to +\infty$ as $t \to T^\star$.*

Proof. This result is a consequence of Theorem 2.1 and Examples 1–3 by [4]. In fact, $V \in L^\infty$ and $\psi^0 \in H^1$; furthermore we show that both W_ℓ and $W_{n\ell}$ satisfy the conditions of [4]. To this end let, in the case of local perturbation,

$$f(x,z) = g(x)|z|^{2\sigma} z.$$

If we prove that

$$|f(x,z_1) - f(x,z_2)| \leq M \left[1 + |z_1|^\gamma + |z_2|^\gamma\right] |z_1 - z_2| \tag{23}$$

for some positive constant M and for some $\gamma \in [0, +\infty)$, then the local existence of the solution in the case of local nonlinear perturbation follows. In order to prove this inequality we assume, for the sake of definiteness, that $0 < |z_1| < |z_2|$ (if $|z_1| = 0$ or $|z_1| = |z_2|$ then (23) immediately follows) and we obtain that

$$\begin{aligned} |f(x,z_1) - f(x,z_2)| &= |g(x)| \cdot \left| |z_1|^{2\sigma} z_1 - |z_2|^{2\sigma} z_2 \right| \\ &\leq \tilde{g} \left[|z_2|^\gamma |z_1 - z_2| + |z_1| \left| |z_1|^\gamma - |z_2|^\gamma \right| \right] \end{aligned}$$

where $\gamma = 2\sigma$ and $\tilde{g} = \max_x |g(x)|$. If $\gamma \in (0, 1]$ then, recalling the inequality (66), it follows that

$$\begin{aligned} |f(x,z_1) - f(x,z_2)| &\leq \tilde{g} \left[|z_2|^\gamma |z_1 - z_2| + |z_1|^\gamma |z_1|^{1-\gamma} \left| |z_1|^\gamma - |z_2|^\gamma \right| \right] \\ &\leq \tilde{g} |z_1 - z_2| \left(|z_2|^\gamma + |z_1|^\gamma \frac{1}{\gamma} \right) \end{aligned}$$

from which (23) follows. In the case

$$2\sigma = \gamma > 1$$

then the inequality (66) gives that

$$\begin{aligned} |f(x,z_1) - f(x,z_2)| &\leq \tilde{g} \left[|z_2|^\gamma |z_1 - z_2| + \sigma |z_2|^\gamma \left| |z_2| - |z_1| \right| \right] \\ &\leq M \left[|z_2|^\gamma (1 + \gamma) \right] |z_2 - z_1| \end{aligned}$$

since

$$|z_1| < |z_2| \quad \text{and} \quad \left| |z_1| - |z_2| \right| \leq |z_1 - z_2|$$

obtaining (23). In the non-local case, where

$$W = W_{n\ell}(x, \psi) = g(x) \langle \psi, g\psi \rangle$$

the local existence result follows by means the same arguments. We simply have to check that

$$\|W_{n\ell}(x,u)u - W_{n\ell}(x,v)v\| \leq C \|u - v\|$$

Indeed,

$$\begin{aligned}
\|W_{n\ell}(x,u)u - W_{n\ell}(x,v)v\| &= \tilde{g}\|\langle u, gu\rangle u - \langle v, gv\rangle v\| \\
&= \tilde{g}\,\|(\langle u, gu\rangle - \langle v, gv\rangle)\, u + \langle v, gv\rangle (u-v)\| \\
&\leq \tilde{g}\|u\|\,|\langle u, gu\rangle - \langle v, gv\rangle| + \tilde{g}^2\|v\|^2\|u-v\| \\
&\leq \tilde{g}\|u\|\,|\langle u-v, gu\rangle + \langle v, g(u-v)\rangle| + \tilde{g}^2\|v\|^2\|u-v\| \\
&\leq \tilde{g}^2\|u\|\,[\|u\|\cdot\|u-v\| + \|v\|\cdot\|u-v\|] + \tilde{g}^2\|v\|^2\|u-v\| \\
&\leq C\|u-v\|
\end{aligned}$$

where

$$C = \tilde{g}^2\left[\|u\|^2 + \|v\|^2 + \|u\|\cdot\|v\|\right]$$

□

4.2. **Conservation laws.** By means of a direct computation the following first integral exists.

4.2.1. *Conservation of the norm.* Let

$$\mathcal{N}(\psi) = \|\psi\|^2$$

then

$$\mathcal{N}[\psi(x,t)] = \mathcal{N}[\psi^0(x)] = 1$$

4.2.2. *Conservation of the energy.* Let us consider the case of local nonlinear perturbation (19). Let

$$\mathcal{H}(\psi) = \mathcal{H}_\ell(\psi) = \langle \psi, H_0\psi\rangle + \frac{\epsilon}{\sigma+1}\langle \psi^{\sigma+1}, g\psi^{\sigma+1}\rangle \tag{24}$$

defined on $H^1(\mathrm{R}^d)\cap L^{2(\sigma+1)}(\mathrm{R}^d)$. Then a direct computation gives that

$$\mathcal{H}_\ell[\psi(x,t)] = \mathcal{H}_\ell[\psi^0(x)]$$

Similarly, in the case of non-local nonlinear perturbation (20) then it follows that

$$\mathcal{H}_{n\ell}[\psi(x,t)] = \mathcal{H}_{n\ell}[\psi^0(x)]$$

where the energy is defined as

$$\mathcal{H}(\psi) = \mathcal{H}_{n\ell}(\psi) = \langle \psi, H_0\psi\rangle + \frac{1}{2}\epsilon\langle \psi, g\psi\rangle^2 \tag{25}$$

on $H^1(\mathrm{R}^d)\cap L^2(\mathrm{R}^d)$.

4.3. **A priori estimates.**

Theorem 2. *Let*

$$\Lambda = \frac{\mathcal{H}[\psi^0] - V_{min}}{\hbar^2}$$

where $\mathcal{H}$ is the energy defined above in equations (24) and (25) for, respectively, local and non-local perturbations. The solution $\psi(x,t)$ of equation (1) satisfies to the following a priori estimates

$$\|\nabla\psi\| \leq C\sqrt{\Lambda} \tag{26}$$

and

$$\|\psi\|_p \leq C\Lambda^{d\frac{p-2}{4p}} \tag{27}$$

where

$$p \in [2,+\infty] \quad if \quad d=1, \quad and \quad p\in [2,+\infty) \quad if \quad d=2 \tag{28}$$

Proof. We consider, at first, the case of local perturbation (19). The conservation of the energy $\mathcal{H}_\ell(\psi)$ gives that

$$\begin{aligned} \hbar^2\|\nabla\psi\|^2 &= \mathcal{H}_\ell(\psi^0) - \frac{\epsilon}{\sigma+1}\langle g\psi^{\sigma+1},\psi^{\sigma+1}\rangle - \langle V\psi,\psi\rangle \\ &\le \mathcal{H}_\ell(\psi^0) - V_{min}\mathcal{N}(\psi^0) + \frac{\tilde g|\epsilon|}{\sigma+1}\|\psi\|^{2(\sigma+1)}_{2(\sigma+1)} \end{aligned}$$

where

$$V_{min} = \min_x V(x) = V(x_\pm) > -\infty \quad \text{and} \quad \tilde g = \max_x |g(x)|.$$

Hence

$$\|\nabla\psi\|^2 \quad \le \quad \Lambda + \rho^2\|\psi\|^{2(\sigma+1)}_{2(\sigma+1)}$$

where

$$\rho^2 = \frac{\tilde g|\epsilon|}{(\sigma+1)\hbar^2} \le C|\epsilon|\hbar^{-2} \ll 1 \quad \text{and} \quad \hbar|\Lambda| = C + o(1)$$

since ϵ is a small semiclassical estimate satisfying (22). Now, we make use of the Gagliardo-Nirenberg inequality (67) obtaining

$$\|\nabla\psi\|^2 \le \Lambda + C\rho^2\|\nabla\psi\|^{\sigma d}\|\psi\|^{2+\sigma(2-d)} \le \Lambda + C\rho^2\|\nabla\psi\|^{\sigma d} \tag{29}$$

from which and from the fixed point theorem the estimate (26) follows. Indeed, if $\sigma d \le 2$ then the result immediately follows. If $\sigma d > 2$ we recall that φ_R and φ_L satisfy to the asymptotic behavior (14) then $\|\nabla\psi^0\| \le C\hbar^{-1/2}$. If we set

$$y = \hbar^{1/2}\|\nabla\psi\|, \ \alpha = \hbar\Lambda, \ \beta = C\rho^2\hbar^{1-\sigma d/2}$$

where, initially $y \le C$, and where

$$C^{-1} \le \alpha \le C \quad \text{and} \quad \beta = o(1) \quad \text{as} \quad \hbar \to 0,$$

then (29) can be written as

$$y^2 \le \alpha + \beta y^{\sigma d}$$

Then fixed point arguments proved that $y \le C$ for any time, from which (26) follows. From this fact, and making use of the Gagliardo-Nirenberg inequality again, we obtain that

$$\|\psi\|_p \le C\|\nabla\psi\|^\delta\|\psi\|^{1-\delta} \le C\Lambda^{\frac12\delta}, \quad \delta = \frac{d(p-2)}{2p} \tag{30}$$

Now, we consider the case of non-local perturbation (20). In such a case we have that the conservation of the energy $\mathcal{H}_{n\ell}$ gives inequality (26) immediately. Indeed,

$$\begin{aligned} \|\nabla\psi\|^2 &= \frac{1}{k^2}\left[\mathcal{H}_{n\ell}(\psi^0) - \langle V\psi,\psi\rangle - \frac12\epsilon\langle\psi,g\psi\rangle^2\right] \\ &\le \Lambda + \rho^2\left[\|\psi\|_2^2\right]^2, \quad \rho^2 = \frac12\tilde g|\epsilon| \ll 1 \\ &\le \Lambda + C\rho^2\|\nabla\psi\| \end{aligned}$$

As above, the estimates $\|\nabla\psi\| \le C\sqrt{\Lambda}$ and (27) follow. □

Remarks:

- Since ψ^0 is prepared on the first two states and since (22) then it follows that $\Lambda \sim \hbar^{-1}$. Hence the above estimates take the form
$$\|\nabla \psi\| \le C\hbar^{-1/2} \quad \text{and} \quad \|\psi\|_p = c_{p,d} \le C\hbar^{-d\frac{p-2}{4p}} \tag{31}$$
for any p satisfying (28);
- In fact the above a priori estimates hold for any $\epsilon > 0$ (repulsive nonlinearity), and for any $\epsilon < 0$ (attractive nonlinearity) such that $\epsilon = O(\hbar^\alpha)$ for any $\alpha > 2$;

4.4. **Global existence.**

Theorem 3. *The solution ψ of (1) globally exists; that is $T^\star = +\infty$.*

Proof. The global solution immediately follows from Theorem 1 and from the estimate (25). □

5. Two-level approximation

5.1. **Two-level approximation.** Since the beating period $T = \frac{2\pi\hbar}{\omega}$ plays the role of the unit of time it is more convenient to consider the *slow time*
$$\tau = \frac{\omega t}{\hbar}$$
Therefore, if we consider the change of variable (with abuse of notation)
$$\psi(x,t) \to \psi(x,\tau) = e^{i\Omega t/\hbar}\psi(x,t), \quad \Omega = \frac{1}{2}\left[\lambda_+ + \lambda_-\right]$$
then equation (1) takes the form (here $'$ denotes the derivative with respect to τ)
$$\left\{ \begin{array}{l} i\psi' = \frac{1}{\omega}\left[H_0 - \Omega\right]\psi + \frac{\epsilon}{\omega} W\psi \\ \psi(x,0) = \psi^0(x) \in L^2(\mathrm{R}^d),\ \Pi_c\psi^0 = 0, \end{array} \right. \tag{32}$$
since $W = W(x, |\psi|)$. In order to study this equation for $\tau \in [0, \tau']$, for any fixed $\tau' > 0$, in the semiclassical limit we rewrite ψ in the following form
$$\psi(x,\tau) = \varphi(x,\tau) + \psi_c(x,\tau), \quad \varphi(x,\tau) = a_R(\tau)\varphi_R(x) + a_L(\tau)\varphi_L(x) \tag{33}$$
where
$$\psi_c(x,\tau) = \Pi_c\psi(x,\tau)$$
and where
$$a_R(\tau) = \langle \varphi_R, \psi\rangle \quad \text{and} \quad a_L(\tau) = \langle \varphi_L, \psi \rangle$$
are unknown complex-valued functions. Since
$$\begin{array}{rcl} H_0\psi &=& a_R H_0 \varphi_R + a_L H_0 \varphi_L + H_0 \psi_c \\ &=& a_R\left[\Omega\varphi_R - \omega\varphi_L\right] + a_L\left[-\omega\varphi_R + \Omega\varphi_L\right] + H_0\psi_c \end{array}$$
then, by substituting (33) in (32) and projecting the resulting equation onto the one-dimensional spaces spanned by the *single-well* states φ_R and φ_L, and on the space $\Pi_c L^2(\mathrm{R}^d)$ it follows that it takes the form
$$\left\{ \begin{array}{ll} ia_R' = -a_L + r_R & r_R = r_R(a_R, a_L, \psi_c) = \frac{\epsilon}{\omega}\langle \varphi_R, W\psi\rangle \\ ia_L' = -a_R + r_L & r_L = r_L(a_R, a_L, \psi_c) = \frac{\epsilon}{\omega}\langle \varphi_L, W\psi\rangle \\ i\psi_c' = \frac{1}{\omega}\left[H_0 - \Omega\right]\psi_c + r_c & r_c = r_c(a_R, a_L, \psi_c) = \frac{\epsilon}{\omega}\Pi_c W\psi \end{array} \right. \tag{34}$$
with initial conditions
$$a_{R,L}(0) = \langle \varphi_{R,L}, \psi^0\rangle, \quad \psi_c^0 = \psi_c(x,0) = \Pi_c\psi^0 = 0$$

Lemma 3. *Let*

$$r_{R,L} = r_{R,L}(a_R, a_L, \psi_c) = \frac{\epsilon}{\omega}\langle \varphi_{R,L}, W\psi\rangle$$

Then, it follows that

$$r_{R,L}(a_R, a_L, 0) = \eta \tilde r_{R,L}(a_R, a_L) + \tilde O(e^{-\Gamma/\hbar}) \tag{35}$$

where:

i. **Local perturbation:**

$$\tilde r_R = C_R |a_R|^{2\sigma} a_R, \quad \tilde r_L = C_L |a_L|^{2\sigma} a_L \quad \text{and} \quad \eta = \frac{\epsilon}{\omega}\hbar^{-d\sigma/2}$$

where

$$C_R = \hbar^{d\sigma/2}\langle \varphi_R, g|\varphi_R|^{2\sigma}\varphi_R\rangle \quad \text{and} \quad C_L = \hbar^{d\sigma/2}\langle \varphi_L, g|\varphi_L|^{2\sigma}\varphi_L\rangle$$

are such that $C_{R,L} = O(1)$ as $\hbar$ goes to zero.

ii. **Non-Local perturbation:**

$$\tilde r_R = C_R a_R |a_R|^2 + \tilde C a_R |a_L|^2, \quad \tilde r_L = C_L a_L |a_L|^2 + \tilde C a_L |a_R|^2$$

and

$$\eta = \frac{\epsilon}{\omega}$$

where

$$C_R = \langle \varphi_R, g\varphi_R\rangle^2, \quad C_L = \langle \varphi_L, g\varphi_L\rangle^2 \quad \text{and} \quad \tilde C = \sqrt{C_R C_L}$$

are such that $C_{R,L} = O(1)$ as $\hbar$ goes to zero.

Proof. In order to give the explicit expression of the terms $r_{R,L}$ we consider the local and non-local perturbations separately.

Local perturbation. In such a case we have that (where we set $\psi_c = 0$ inside W_ℓ)

$$\begin{aligned} r_R(a_R, a_L, 0) &= \frac{\epsilon}{\omega}\langle \varphi_R, g|a_R\varphi_R + a_L\varphi_L|^{2\sigma}(a_R\varphi_R + a_L\varphi_L)\rangle \\ &= \frac{\epsilon}{\omega}\left[|a_R|^{2\sigma} a_R \langle \varphi_R, g|\varphi_R|^{2\sigma}\varphi_R\rangle + \tilde O(e^{-\Gamma/\hbar})\right] \end{aligned}$$

since (18). Similarly, we obtain that

$$r_L(a_R, a_L, 0) = \frac{\epsilon}{\omega}\left[|a_L|^{2\sigma} a_L \langle \varphi_L, g|\varphi_L|^{2\sigma}\varphi_L\rangle + \tilde O(e^{-\Gamma/\hbar})\right]$$

If we set

$$C_R = \hbar^{d\sigma/2}\langle \varphi_R, g|\varphi_R|^{2\sigma}\varphi_R\rangle \quad \text{and} \quad C_L = \hbar^{d\sigma/2}\langle \varphi_L, g|\varphi_L|^{2\sigma}\varphi_L\rangle$$

then

$$r_{R,L} = \eta C_{R,L}|a_{R,L}|^{2\sigma} a_{R,L} + \tilde O(e^{-\Gamma/\hbar})$$

for any τ. Furthermore, from Lemma 2 it follows that

$$\begin{aligned} |C_{R,L}| &\le \tilde g \hbar^{d\sigma/2}\|\varphi_{R,L}^{\sigma+1}\|^2 = \tilde g \hbar^{d\sigma/2}\|\varphi_{R,L}\|_{2(\sigma+1)}^{2(\sigma+1)} = \tilde g \hbar^{d\sigma/2} c_{2(\sigma+1),d}^{2(\sigma+1)} \\ &\le C\hbar^{d\sigma/2}\left[\hbar^{-d\frac{2\sigma}{8(\sigma+1)}}\right]^{2(\sigma+1)} \le C, \quad \forall \hbar \in (0, \hbar^\star) \end{aligned}$$

where $\tilde g = \max_x |g(x)| < +\infty$.

Non-Local perturbation. In such a case we have that (where we set $\psi_c = 0$ inside $W_{n\ell}$)

$$\begin{aligned} & r_R(a_R, a_L, 0) = \\ & = \frac{\epsilon}{\omega} \langle \varphi_R, (a_R\varphi_R + a_L\varphi_L) g\rangle \cdot \langle (a_R\varphi_R + a_L\varphi_\ell), g(a_R\varphi_R + a_L\varphi_\ell)\rangle \\ & = \frac{\epsilon}{\omega}\left[a_R\langle\varphi_R, g\varphi_R\rangle + \tilde{O}(e^{-\Gamma/\hbar})\right] \times \\ & \times \left[|a_R|^2\langle\varphi_R, g\varphi_R\rangle + |a_L|^2\langle\varphi_L, g\varphi_L\rangle + \tilde{O}(e^{-\Gamma/\hbar})\right] \end{aligned}$$

since (18). If we set

$$C_R = \langle\varphi_R, g\varphi_R\rangle^2, \quad C_L = \langle\varphi_L, g\varphi_L\rangle^2, \quad \tilde{C} = \sqrt{C_R C_L}$$

then

$$\begin{aligned} r_R &= \eta C_R a_R |a_R|^2 + \eta\tilde{C} a_R |a_L|^2 + \frac{\epsilon}{\omega}\tilde{O}(e^{-\Gamma/\hbar}) \\ &= \eta C_R a_R |a_R|^2 + \eta\tilde{C} a_R |a_L|^2 + \tilde{O}(e^{-\Gamma/\hbar}) \end{aligned}$$

since (22) and, similarly

$$\begin{aligned} r_L &= \eta C_L a_L |a_L|^2 + \eta\tilde{C} a_L |a_R|^2 + \frac{\epsilon}{\omega}\tilde{O}(e^{-\Gamma/\hbar}) \\ &= \eta C_L a_L |a_L|^2 + \eta\tilde{C} a_L |a_R|^2 + \tilde{O}(e^{-\Gamma/\hbar}) \end{aligned}$$

Finally

$$|C_{R,L}| \le \tilde{g}\|\varphi_{R,L}\|^4 \le C$$

□

Definition 1. *We call* two-level approximation *the system of differential equations given by*

$$\left\{ \begin{array}{lcl} ib'_R & = & -b_L + \eta\tilde{r}_R(b_R, b_L) \\ ib'_L & = & -b_R + \eta\tilde{r}_L(b_R, b_L) \end{array} \right. , \quad b_{R,L}(0) = a_{R,L}(0) \tag{36}$$

Remarks:

- The *two-level approximation* is obtained, up to an exponentially small term, by substituting $\psi_c \equiv 0$ inside equation (34).
- The solution of equation (36) globally exists.

5.2. **First integrals.** As for the complete equation (1) a direct computation proves the following conservation laws.

5.2.1. *Conservation of the norm.* Let

$$\tilde{\mathcal{N}}(b_R, b_L) = |b_R|^2 + |b_L|^2$$

then

$$\tilde{\mathcal{N}}[b_R(\tau), b_L(\tau)] = \tilde{\mathcal{N}}[b_R(0), b_L(0)], \quad \forall \tau \in \mathrm{R}$$

In particular $\tilde{\mathcal{N}}(b_R, b_L) = 1$ since

$$\tilde{\mathcal{N}}(b_R, b_L) \equiv |b_R(0)|^2 + |b_L(0)|^2 = |a_R(0)|^2 + |a_L(0)|^2 = \|\psi^0\|^2 = 1 \tag{37}$$

5.2.2. *Conservation of the energy.* Let, in the case of local perturbation (19),

$$\begin{aligned}\tilde{\mathcal{H}}(b_R,b_L) &= \tilde{\mathcal{H}}_\ell(b_R,b_L) = -\omega\left(\bar{b}_R b_L + \bar{b}_L b_R\right) + \\ &\quad + \frac{\epsilon}{\sigma+1}\left[|b_R|^{2(\sigma+1)}\langle\varphi_R^{\sigma+1}, g\varphi_R^{\sigma+1}\rangle + |b_L|^{2(\sigma+1)}\langle\varphi_L^{\sigma+1}, g\varphi_L^{\sigma+1}\rangle\right] \\ &= -\omega\left[\left(\bar{b}_R b_L + \bar{b}_L b_R\right) - \frac{\eta}{\sigma+1}\left(|b_R|^{2(\sigma+1)}C_R + |b_L|^{2(\sigma+1)}C_L\right)\right]\end{aligned}$$

or, in the case of non-local perturbation (20),

$$\begin{aligned}\tilde{\mathcal{H}}(b_R,b_L) &= \tilde{\mathcal{H}}_{n\ell}(b_R,b_L) \\ &= -\omega\left(\bar{b}_R b_L + \bar{b}_L b_R\right) + \epsilon\frac{1}{2}\left[|b_R|^2\langle\varphi_R, g\varphi_R\rangle^2 + |b_L|^2\langle\varphi_L, g\varphi_L\rangle^2\right] \\ &= -\omega\left[\left(\bar{b}_R b_L + \bar{b}_L b_R\right) - \frac{\eta}{2}\left(|b_R|^2 C_R + |b_L|^2 C_L\right)\right]\end{aligned}$$

Then a direct computation gives that

$$\tilde{\mathcal{H}}[b_R(\tau), b_L(\tau)] = \tilde{\mathcal{H}}[b_R(0), b_L(0)], \quad \forall \tau \in \mathrm{R}$$

5.3. **Analysis of the two-level approximation.** Here, we perform the qualitative analysis of the two level approximation. In particular, we prove that.

Theorem 4. *There exists a threshold value $\eta^\star > 0$ such that the two-level system (36) admits just two stationary symmetric (that is $|b_R|^2 = |b_L|^2 = \frac{1}{2}$) solutions for any $\eta \in [0, \eta^\star]$. At $\eta = \eta^\star$ a bifurcation phenomenon occurs and for $\eta > \eta^\star$ new stationary asymmetric (that is $|b_{R,L}|^2 \neq \frac{1}{2}$) solutions appear.*

Proof. In order to prove this result we set

$$b_R = pe^{i\alpha}, \ b_L = qe^{i\beta}, \ z = p^2 - q^2, \ \theta = \alpha - \beta$$

where p and q are such that $p^2 + q^2 = 1$. The *imbalance function* z takes value in the interval $[-1, 1]$; when $z = 1$ then $|b_R| = 1$ and $|b_L| = 0$ and the wavefunction $\varphi = b_R\varphi_R + b_L\varphi_L$ is practically localized on the *right-side* well, in contrast, when $z = -1$ then $|b_R| = 0$ and $|b_L| = 1$ and the wavefunction φ is practically localized on the *left-side* well.

For the sake of definiteness we just consider the local perturbation case, the non-local case could be similarly treated. In such a case the *two-level approximation* (36) takes the form

$$\left\{\begin{array}{lcl} z' &=& 2\sqrt{1-z^2}\sin\theta \\ \theta' &=& \frac{-2z}{\sqrt{1-z^2}}\cos\theta - \eta\left[C_R\left(\frac{1+z}{2}\right)^\sigma - C_L\left(\frac{1-z}{2}\right)^\sigma\right]\end{array}\right. \tag{38}$$

and the conservation of the energy takes the form

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}(z,\theta) = -\omega\left\{\sqrt{1-z^2}\cos\theta - \frac{\eta}{\sigma+1}\left[C_R\left(\frac{1+z}{2}\right)^{\sigma+1} + C_L\left(\frac{1-z}{2}\right)^{\sigma+1}\right]\right\}$$

It is not hard to see that when the nonlinear perturbation is small enough then the above dynamical system has only two stationary solutions (θ_1, z_1) and (θ_2, z_2) where $\theta_1 = 0$ and $\theta_2 = \pi$ and where $z_1 = z_2 = 0$ is the unique solution of the equation

$$\mp\frac{2z}{\sqrt{1-z^2}} - \eta\left[C_R\left(\frac{1+z}{2}\right)^\sigma - C_L\left(\frac{1-z}{2}\right)^\sigma\right] = 0 \tag{39}$$

In contrast, when the strength of the nonlinear perturbation is larger than a threshold parameter $\eta^\star$ then new solutions $z \neq 0$ of equation (39) appear. $\square$

For instance, if g is an even function then $C_R = C_L$ and the above equation take the form (where we assume, for the sake of definiteness, that $C_R = 1$)

$$\frac{2z}{\sqrt{1-z^2}} - \eta \left[\left(\frac{1+z}{2} \right)^\sigma - \left(\frac{1-z}{2} \right)^\sigma \right] = 0 \tag{40}$$

which has solutions $z_2 = 0$, corresponding to $\theta_2 = \pi$, for any η. In particular, if η is larger than a threshold parameter $\eta^\star$ then new solutions appear. In particular:

5.3.1. *Cubic ($\sigma = 1$) and Quintic ($\sigma = 2$) Local Nonlinearity.* In these cases equation (40) takes the form

$$\frac{2z}{\sqrt{1-z^2}} - \eta z = 0$$

For η larger than the threshold value $\eta^\star = 2$ then the solution (z_2, θ_2) bifurcates (see Figure 1(a)) and two new solutions (θ_3, z_3) and (θ_4, z_4) appear, where $\theta_{3,4} = \pi$ and

$$z_{3,4} = \pm \frac{\sqrt{\eta^2 - 4}}{\eta}$$

5.3.2. *Higher Local Nonlinearity: $\sigma = 3$ and $\sigma = 4$.* In the case $\sigma = 3$ then we have a picture similar as before. That is, for η larger than the threshold value $\eta^\star = \frac{8}{3}$ then the solution (z_2, θ_2) bifurcates (see Figure 1(a)) and two new solutions (θ_3, z_3) and (θ_4, z_4) appear, where $\theta_{3,4} = \pi$ and

$$z_{3,4} = \pm \left[\frac{2\chi}{3\eta} + \frac{8\eta}{3\chi} - \frac{5}{3} \right]^{1/2}, \quad \eta^\star \leq \eta$$

where

$$\chi = \left[\left(8\eta^2 - 108 + 12\sqrt{81 - 12\eta^2} \right) \eta \right]^{1/3}$$

In the case $\sigma = 4$ we have a different picture, that is at η coinciding with the threshold value $\eta^\star = \sqrt{\frac{27}{2}}$ then four new solutions (θ_j, z_j), $j = 3, 4, 5, 6$ appear, where $\theta_j = \pi$ and $z_{3,5} = \frac{1}{\sqrt{3}}$ and $z_{4,6} = -\frac{1}{\sqrt{3}}$. In particular,

$$\begin{aligned} z_{3,4} &= \pm \left[\frac{2\chi}{3\eta} + \frac{2\eta}{3\chi} - \frac{1}{3} \right]^{1/2}, \quad \eta^\star \leq \eta \\ z_{5,6} &= \pm \left[\left(-\frac{\chi}{3\eta} - \frac{\eta}{3\chi} - \frac{1}{3} \right) - i\frac{\sqrt{3}}{2} \left(\frac{2\chi}{3\eta} - \frac{2\eta}{3\chi} \right) \right]^{1/2}, \quad \eta^\star \leq \eta \leq \eta^+ = 4 \end{aligned}$$

where

$$\chi = \left[\left(\eta^2 - 27 + 3\sqrt{81 - 6\eta^2} \right) \eta \right]^{1/3}$$

At $\eta = \eta^+ = 4$ the solutions $z_{5,6}$ collapse to the solution $z_2 = 0$ (see Figure 1(b)).

Remarks:

- We emphasize that equations (38) admits, when $\sigma = 1$ (cubic nonlinearity) and $C_R = C_L$, an explicit solution by means of Jacobian elliptic function (see [16] and the references therein).

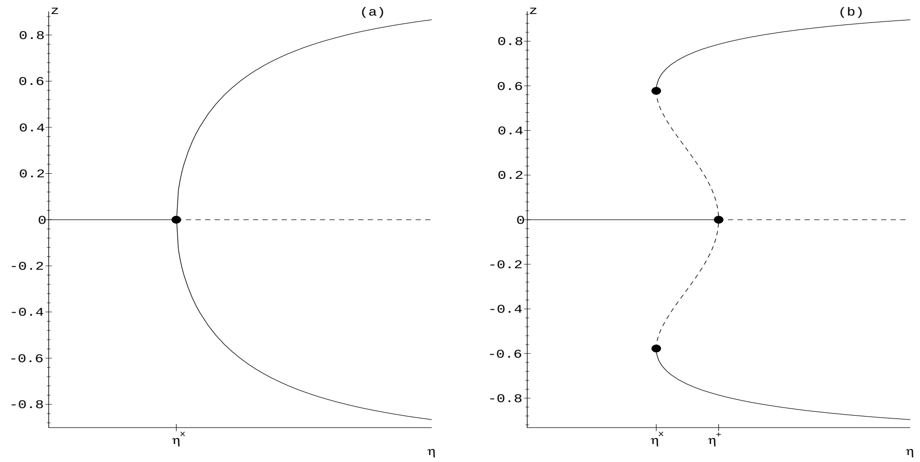

FIGURE 1. In this figure we plot the graph of the solutions (full lines represent stable centers, broken lines represent unstable centers) of equation (40) as function of the parameter η, for given nonlinearity. In figure (a) we consider the cases of $\sigma = 1, 2, 3$, where a pitch-fork bifurcation occurs at $\eta = \eta^\star$, where $\eta^\star = 2$ for $\sigma = 1, 2$ and $\eta^\star = \frac{8}{3}$ for $\sigma = 3$. In figure (b) we consider the case $\sigma = 4$, in such a case new solutions appear at $\eta = \eta^\star$ where $\eta^\star = \sqrt{27/2}$; at $\eta = \eta^+ = 4$ two of them collapses and, then, disappear.

- The qualitative behavior of the solutions of equation (38) could be easily studied by means of the conservation of the energy $\tilde{H}$ as done, for instance, by [9].

6. Stability of the two-level approximation

6.1. **Main result.** Our main result consists in proving the stability of the two-level approximation.

We prove that:

Theorem 5. *Let $\psi_c = \Pi_c \psi$, $a_R = \langle \psi, \varphi_R \rangle$ and $a_L = \langle \psi, \varphi_L \rangle$, where ψ is the solution of equation (32), let b_R and b_L be the solution of the system of ordinary differential equations (36). Then, for any fixed $\tau' > 0$*

$$|b_{R,L}(\tau) - a_{R,L}(\tau)| = \tilde{O}(e^{-\Gamma/\hbar}) \quad \text{and} \quad \|\psi_c(\cdot, \tau)\| = \tilde{O}(e^{-\Gamma/\hbar}) \tag{41}$$

for any $\tau \in [0, \tau']$, where $\Gamma > 0$ is given in equation (16).

Remark:

- From this theorem it follows that the time behaviour, at least for times of the order of the beating period, of the wavefunction ψ, initially prepared on the lowest states, is practically described by means of the solutions of the two-level approximation given in the previous Section.

6.2. **Proof of Theorem 5.** For the sake of simplicity, hereafter, let us drop out the parameters where this does not cause misunderstanding. The proof of the theorem is organized in several Lemmas.

Lemma 4. *Let*

$$\varphi(x,\tau) = a_R(\tau)\varphi_L(x) + a_L(\tau)\varphi_L(x) \quad \textit{and} \quad \psi = \psi_c + \varphi$$

Let $W\psi = W^I + W^{II}$ *where:*

i. **Local perturbation:**

$$W^I = W^I_\ell = g(x)|\varphi(x,\tau)|^{2\sigma}\varphi(x,\tau) \tag{42}$$

where W^I_ℓ *does not depend on* ψ_c*, and let*

$$W^{II}_\ell = W_\ell\psi - W^I_\ell$$

Then, it follows that

$$\|W^I_\ell\| \le C\hbar^{-\sigma d/2} \tag{43}$$

and

$$\|W^{II}_\ell\| \le C\hbar^{-d\sigma/2}\|\psi_c\| \tag{44}$$

for some positive constant C *independent on* τ *and* $\hbar$*.*

ii. **Non-Local perturbation:**

$$W^I = W^I_{n\ell} = \langle\varphi(\cdot,\tau), g(\cdot)\varphi(\cdot,\tau)\rangle g(x)\varphi(x,\tau) \tag{45}$$

where $W^I_{n\ell}$ *does not depend on* ψ_c*, and let*

$$W^{II}_{n\ell} = W_{n\ell}\psi - W^I_{n\ell}$$

Then, it follows that

$$\|W^I_{n\ell}\| \le C \quad \textit{and} \quad \|W^{II}_{n\ell}\| \le C\|\psi_c\| \tag{46}$$

for some positive constant C *independent on* τ *and* $\hbar$*.*

Proof. The proof of this Lemma is given in several steps. At first we estimate the terms $W^I_{n\ell}$ and $W^{II}_{n\ell}$, then we give the proof of the estimates (43) and (44) for the local perturbation case; for what concerns the estimate of the term W^{II}_ℓ we consider different cases depending on the dimension d.

Proof of estimates (46). Let us consider, at first, the non-local case where $W^I_{n\ell}$ is given by (45), then

$$\begin{aligned} \|W^I_{n\ell}\| &\le \tilde{g}|\langle\varphi(\cdot,\tau), g(\cdot)\varphi(\cdot,\tau)\rangle| \cdot \|\varphi\| \\ &\le \tilde{g}^2\|\varphi^2\|_1 \cdot \|\varphi\| \\ &\le \tilde{g}^2\|\varphi\|^3 \le C \end{aligned}$$

since $\|\varphi\| \le \|\psi\| \le 1$. For what concerns the other term we have that

$$\begin{aligned} W^{II}_{n\ell} &= W_{n\ell}\psi - W^I_{n\ell} \\ &= g(x)\left[\left(\langle\psi_c, g\psi_c\rangle + \langle\varphi, g\psi_c\rangle + \langle\psi_c, g\varphi\rangle\right) g(\psi_c+\varphi) + \langle\varphi, g\varphi\rangle\psi_c\right] \end{aligned}$$

Hence

$$\begin{aligned} \|W^{II}_{n\ell}\| &\le C\left\{\left[\|\psi_c\|^2 + 2\|\varphi\|\cdot\|\psi_c\|\right]\left(\|\psi_c\| + \|\varphi\|\right) + \|\varphi\|^2\|\psi_c\|\right\} \\ &\le C\|\psi_c\| \end{aligned}$$

Proof of estimate (43). The estimate of the term W_ℓ^I is immediate. Indeed

$$\|W_\ell^I\| \leq \tilde g \|\varphi^{2\sigma+1}\| = \tilde g \|\varphi\|_\infty^{2\sigma}\|\varphi\| \leq C\hbar^{-d\sigma/2}$$

since $\|\varphi\| \leq 1$ and (see Lemma 2)

$$\|\varphi\|_\infty \leq |a_R(\tau)| \cdot \|\varphi_R\|_\infty + |a_L(\tau)| \cdot \|\varphi_L\|_\infty \leq 2c_{\infty,d} = 2C\hbar^{-d/4} \tag{47}$$

Proof of the estimate (44) — Dimension $d=1$. In such a case the proof is simpler than the case of dimension 2, because we can make use of the inequality

$$\|\psi_c\|_\infty = c_{\infty,d} \leq C\hbar^{-d/4}, \quad d=1, \tag{48}$$

which immediately follows in the case $d=1$ for any σ from the Minkowski inequality and from (31):

$$c_{p,d} = C\hbar^{-d\frac{p-2}{4p}} \geq \|\psi\|_p \geq -\left(|a_R(\tau)|\|\varphi_R\|_p + |a_L(\tau)|\|\varphi_L\|_p\right) + \|\psi_c\|_p$$

for $p=+\infty$, where $|a_{R,L}(\tau)| \leq 1$ and where $\varphi_{R,L}$ satisfy the estimate (13). We assume, for a moment, that

$$\sigma = n \in \mathrm{N}$$

is a non negative integer number. Then

$$\begin{aligned} W_\ell^{II} &= W_\ell\psi - W_\ell^I = g(x)\left[|\psi_c+\varphi|^{2\sigma}(\psi_c+\varphi) - |\varphi|^{2\sigma}\varphi\right] \\ &= g(x)\left[\left(\bar\psi_c+\bar\varphi\right)^n(\psi_c+\varphi)^{n+1} - |\varphi|^{2n}\varphi\right] \\ &= g(x)\sum_{j=0,\ldots,n;l=0,\ldots,n+1;j+l>0}\binom{n}{j}\binom{n+1}{l}\bar\psi_c^j\psi_c^l\bar\varphi^{n-j}\varphi^{n+1-l} \end{aligned} \tag{49}$$

Hence,

$$\begin{aligned} \|W_\ell^{II}\| &\leq \tilde g \sum_{j=0,\ldots,n;l=0,\ldots,n+1;j+l>0}\binom{n}{j}\binom{n+1}{l}\|\bar\psi_c^{j+l}\varphi^{2n+1-j-l}\| \\ &\leq C\sum_{k=1}^{2n+1}\|\psi_c^k\|\cdot\|\varphi\|_\infty^{2n+1-k} \\ &\leq C\sum_{k=1}^{2n+1}\|\psi_c\|\cdot\|\psi_c\|_\infty^{k-1}\cdot\|\varphi\|_\infty^{2n+1-k} \\ &\leq C\hbar^{-\frac14 d(2n)}\|\psi_c\| = C\hbar^{-d\sigma/2}\|\psi_c\| \end{aligned} \tag{50}$$

since (47) and (48). If σ is not an integer number let

$$\sigma = n+\alpha, \quad n\in\mathrm{N}^\star \quad \text{and} \quad \alpha\in(0,1),$$

and let

$$W_\ell^{II} = W_{\ell,a}^{II} + W_{\ell,b}^{II}$$

where

$$W_{\ell,a}^{II} = g(x)|\psi_c+\varphi|^{2\sigma}\psi_c \quad \text{and} \quad W_{\ell,b}^{II} = g(x)\left[|\psi_c+\varphi|^{2\sigma} - |\varphi|^{2\sigma}\right]\varphi$$

As above

$$\|W_{\ell,a}^{II}\| \leq \tilde g\|\psi_c+\varphi\|_\infty^{2\sigma}\|\psi_c\| \leq C\hbar^{-\sigma d/2}\|\psi_c\|$$

since (47) and (48). For what concerns the other term $W_{\ell,b}^{II}$ it follows that

$$\begin{aligned}
\|W_{\ell,b}^{II}\| &\leq \tilde g \|\varphi\|_\infty \left\| |\psi_c+\varphi|^{2\sigma} - |\varphi|^{2\sigma} \right\| \\
&\leq \tilde g \|\varphi\|_\infty \left\| |\psi_c+\varphi|^{2n}|\psi_c+\varphi|^{2\alpha} - |\varphi|^{2n}|\varphi|^{2\alpha} \right\| \\
&\leq \tilde g \|\varphi\|_\infty \left\| \sum_{j,l=0,\ldots,n} \left(\begin{array}{c} n \\ j \end{array}\right) \left(\begin{array}{c} n \\ l \end{array}\right) \bar\psi_c^j \psi_c^l \bar\varphi^{n-j}\varphi^{n-l} |\psi_c+\varphi|^{2\alpha} - |\varphi|^{2n}|\varphi|^{2\alpha} \right\| \\
&\leq \tilde g \|\varphi\|_\infty \left[\left\| \sum_{j,l=0,\ldots,n;\ j+l>1} \left(\begin{array}{c} n \\ j \end{array}\right) \left(\begin{array}{c} n \\ l \end{array}\right) \bar\psi_c^j \psi_c^l \bar\varphi^{n-j}\varphi^{n-l} |\psi_c+\varphi|^{2\alpha} \right\| + \right. \\
&\quad \left. + \left\| |\varphi|^{2n} \left(|\psi_c+\varphi|^{2\alpha} - |\varphi|^{2\alpha}\right) \right\| \right] \qquad (51) \\
&\leq C\|\varphi\|_\infty \left[\sum_{k=1}^{2n} \left(\|\psi_c^k \varphi^{2n-k}\| \cdot \|\psi_c+\varphi\|_\infty^{2\alpha} \right) + \right. \\
&\quad \left. + \left\| |\varphi|^{2n-(1-\alpha)} |\varphi|^{1-\alpha} \left(|\psi_c+\varphi|^\alpha + |\varphi|^\alpha\right) \left(|\psi_c+\varphi|^\alpha - |\varphi|^\alpha \right) \right\| \right] \\
&\leq C\|\varphi\|_\infty \left[\sum_{k=1}^{2n} \left(\|\psi_c\| \cdot \|\psi_c\|_\infty^{k-1} \cdot \|\varphi\|_\infty^{2n-k} \left(\|\psi_c\|_\infty + \|\varphi\|_\infty \right)^{2\alpha} \right) + \right. \\
&\quad \left. + \|\varphi\|_\infty^{2n-(1-\alpha)} \left(\|\psi_c+\varphi\|_\infty^\alpha + \|\varphi\|_\infty^\alpha \right) \left\| |\varphi|^{1-\alpha} \left(|\psi_c+\varphi|^\alpha - |\varphi|^\alpha \right) \right\| \right] \\
&\leq C\hbar^{-\frac14 d} \left[\|\psi_c\| C \hbar^{-\frac14 d(2n+2\alpha-1)} + \alpha C \|\psi_c\| \hbar^{-\frac14 d(2n+2\alpha-1)} \right] \\
&\leq C\hbar^{-d\sigma/2}\|\psi_c\|
\end{aligned}$$

where we have make use of the inequality (66).

Proof of estimate (44) — Dimension $d=2$. In such a case the estimate (48) does not hold and we make use here of the Gagliardo-Nirenberg inequality. As above, we assume for a moment, that $\sigma = n \in \mathrm{N}$ is a non negative integer number. Then, from (50) it follows that

$$\|W_\ell^{II}\| \leq C \sum_{k=1}^{2n+1} \|\varphi\|_\infty^{2\sigma+1-k} \|\psi_c^k\|, \quad \|\varphi\|_\infty^{2\sigma+1-k} \leq C\hbar^{-\frac{d}{4}(2\sigma+1-k)} \qquad (52)$$

The Gagliardo-Nirenberg inequality (69) gives that

$$\|\psi_c^k\| \quad = \quad \|\psi_c\|_{2k}^k \leq C \|\nabla \psi_c\|^{k\delta} \|\psi_c\|^{(1-\delta)k}$$

where $\delta = \frac{d}{2} - \frac{d}{2k}$. From this fact and from the estimate (31) it follows that

$$\begin{aligned}
\|\psi_c^k\| &\leq C\hbar^{\frac{d}{4}(k-1)} \|\psi_c\|^{\frac{d}{2}+(1-\frac{d}{2})k} \\
&\leq C\hbar^{-d\frac{k-1}{4}} \|\psi_c\|, \quad d=2
\end{aligned} \qquad (53)$$

Hence, from the above estimate and from (52) it follows that

$$\begin{aligned}
\|W_\ell^{II}\| &\leq C \sum_{k=1}^{2n+1} \hbar^{-\frac14 d(2n+1-k)} \hbar^{-d(k-1)/4} \|\psi_c\| \\
&\leq C\hbar^{-dn/2}\|\psi_c\| = C\hbar^{-d\sigma/2}\|\psi_c\|
\end{aligned}$$

Now, if σ is not an integer number let $\sigma = n+\alpha$ where $n \in \mathrm{N}^\star$ and $\alpha \in (0,1)$. As above, let $W_\ell^{II} = W_{\ell,a}^{II} + W_{\ell,b}^{II}$ where

$$W_{\ell,a}^{II} = g(x)|\psi_c+\varphi|^{2\sigma}\psi_c \quad \text{and} \quad W_{\ell,b}^{II} = g(x)\left[|\psi_c+\varphi|^{2\sigma} - |\varphi|^{2\sigma}\right]\varphi$$

Now,

$$\|W_{\ell,a}^{II}\| \leq \tilde g \left\| |\psi_c+\varphi|^{2\sigma}\psi_c \right\| \tag{54}$$

recalling that

$$|\psi_c+\varphi|^{2\sigma} \leq (|\psi_c|+|\varphi|)^{2\sigma} \leq 2^{2\sigma}\left[\max(|\psi_c|,|\varphi|)\right]^{2\sigma} \leq 2^{2\sigma}\left[|\psi_c|^{2\sigma}+|\varphi|^{2\sigma}\right]$$

From this fact and from (53) then (54) takes the form

$$\begin{aligned} \|W_{\ell,a}^{II}\| &\leq C\left[\|\psi_c^{2\sigma+1}\| + \|\varphi\|_\infty^{2\sigma}\|\psi_c\|\right] \\ &\leq C\hbar^{-d\sigma/2}\|\psi_c\| \end{aligned} \tag{55}$$

In order to estimate the other term $W_{\ell,b}^{II}$ from (51) we have that

$$\begin{aligned}
\|W_{\ell,b}^{II}\| &\leq C\|\varphi\|_\infty \left[\sum_{k=1}^{2n}\left(\|\varphi\|_\infty^{2n-k}\left\|\psi_c^k|\psi_c+\varphi|^{2\alpha}\right\|\right) + \left\| |\varphi|^{2n}\left(|\psi_c+\varphi|^{2\alpha} - |\varphi|^{2\alpha}\right)\right\|\right] \\
&\leq C\|\varphi\|_\infty\left[\sum_{k=1}^{2n}\left(\|\varphi\|_\infty^{2n-k}\left(\left\|\psi_c^{k+2\alpha}\right\| + \|\psi_c^k\|\cdot\|\varphi\|_\infty^{2\alpha}\right)\right) + \right. \\
&\quad \left. + \left\| |\varphi|^{2n-(1-\alpha)}|\varphi|^{1-\alpha}\left(|\psi_c+\varphi|^\alpha + |\varphi|^\alpha\right)\left(|\psi_c+\varphi|^\alpha - |\varphi|^\alpha\right)\right\|\right] \\
&\leq C\sum_{k=1}^{2n}\left(\|\varphi\|_\infty^{2n-k+1}\left\|\psi_c^{k+2\alpha}\right\|\right) + C\sum_{k=1}^{2n}\left(\|\psi_c^k\|\cdot\|\varphi\|_\infty^{2n-k+1+2\alpha}\right) + \\
&\quad +C\|\varphi\|_\infty^{2n-(1-\alpha)+1}\left\|(|\psi_c|^\alpha + |\varphi|^\alpha)|\psi_c|\right\| \\
&\leq \sum_{k=1}^{2n}\left(\|\varphi\|_\infty^{2n-k+1}\|\psi_c^{k+2\alpha}\| + \|\varphi\|_\infty^{2n-k+1+2\alpha}\|\varphi\|_\infty^{2\alpha}\|\psi_c^k\|\right) + \\
&\quad +\|\varphi\|_\infty^{2n+\alpha}\left(\|\psi_c^{1+\alpha}\| + \|\varphi\|_\infty^\alpha\|\psi_c\|\right) \\
&\leq C\sum_{k=1}^{2n}\left[\hbar^{-\frac d4(2n-k+1)}\hbar^{-\frac d4(k+2\alpha-1)}\|\psi_c\|^{\frac d2+\left(1-\frac d2\right)(k+2\alpha)} + \right. \\
&\quad \left. +\hbar^{-\frac d4(2n+2\alpha-k+1)}\hbar^{-\frac d4(k-1)}\|\psi_c\|^{\frac d2+\left(1-\frac d2\right)k}\right] + \\
&\quad +C\hbar^{-\frac d4(2n+\alpha)}\left(\hbar^{-\frac d4\alpha}\|\psi_c\|^{\frac d2+\left(1-\frac d2\right)(1+\alpha)} + \hbar^{-\frac d4\alpha}\|\psi_c\|\right) \\
&\leq C\sum_{k=1}^{2n}\left[\hbar^{-\frac d4(2n+2\alpha)}\|\psi_c\|^{1+\left(1-\frac d2\right)(k+2\alpha)} + \hbar^{-\frac d4(2n+2\alpha)}\|\psi_c\|^{1+\left(1-\frac d2\right)(k-1)}\right] + \\
&\quad +\hbar^{-\frac d4(2n+2\alpha)}\left(\|\psi_c\|^{1+\left(1-\frac d2\right)\alpha} + \|\psi_c\|\right) \qquad (56) \\
&\leq C\hbar^{-d\sigma/2}\|\psi_c\|, \quad \text{since} \quad d=2.
\end{aligned}$$

obtaining the wanted result by means of the inequalities (52) and (53). □

Remarks:

- For what concerns the nonlinear local perturbation in dimension $d > 2$ we emphasize that from Theorem 2 then $\psi = \psi_c + \varphi \in L^p$ for any $p < \frac{2d}{d-2}$ when $d > 2$. Hence, in order to consider the L^2-norm of $W_\ell \psi$ we have to assume that $|\psi|^{2\sigma}\psi \in L^2$, that is $\sigma < \frac{1}{d-2}$. In such a case, then $W_\ell \psi \in L^2$ and we can apply the estimate (53) to (55) and (56) as in the previous case obtaining that

$$\|W^{II}_{\ell,a}\| \leq C\hbar^{-d\sigma/2}\|\psi_c\|^\gamma, \quad \gamma = 1 + (2-d)\sigma$$

since $\|\psi_c\| \leq 1$, and, similarly,

$$\|W^{II}_{\ell,b}\| \leq C\hbar^{-d\sigma/2}\|\psi_c\|^\gamma$$

Lemma 5. *$|a'_{R,L}| \leq C$ for any $\tau \geq 0$ for some positive constant C independent of τ and $\hbar$.*

Proof. In the local perturbation case from (34) and from the previous Lemma it follows that

$$\begin{aligned} |a'_R| &\leq |a_L| + |r_R| \leq |a_L| + \frac{|\epsilon|}{\omega}\|\varphi_R\| \cdot \|W\psi\| \\ &\leq |a_L| + \frac{|\epsilon|}{\omega}\|\varphi_R\| \left[\|W^I_\ell\| + \|W^{II}_\ell\|\right] \\ &\leq |a_L| + C\frac{|\epsilon|\hbar^{-\sigma d/2}}{\omega}\|\varphi_R\| \left[1 + \|\psi_c\|\right] \leq C \end{aligned}$$

since $|a_L| \leq 1$ and $\|\psi_c\| \leq 1$ for any τ, $\|\varphi_R\| = 1$ and (22). In the same way, the estimate $|a'_L| \leq C$ follows. The non-local perturbation case similarly follows. □

Lemma 6. *Let $W^I_\ell = W^I_\ell(x,\tau)$ and $W^{II}_{n\ell} = W^I_{n\ell}(x,\tau)$ be defined as in (42) and (45); then*

$$W^I_\ell, \ W^I_{n\ell} \in C^1(\mathrm{R}, L^2(\mathrm{R}^d))$$

and

$$\left\|\frac{\partial W^I_\ell}{\partial \tau}\right\|, \ \left\|\frac{\partial W^I_{n\ell}}{\partial \tau}\right\| = c^{2\sigma}_{\infty,d} \leq C\hbar^{-\sigma d/2}, \quad \forall \tau \geq 0 \ \text{ and } \ \forall \hbar \in (0, \hbar^\star)$$

Proof. Let us consider, for a moment, the local case where

$$\begin{aligned} W^I_\ell &= W^I_\ell(x,\tau) = g(x)\left|a_R(\tau)\varphi_R(x) + a_L(\tau)\varphi_L(x)\right|^{2\sigma}\left(a_R(\tau)\varphi_R(x) + a_L(\tau)\varphi_L(x)\right) \\ &= g(x)\left[\bar a_R(\tau)\bar\varphi_R(x) + \bar a_L(\tau)\bar\varphi_L(x)\right]^\sigma \left[a_R(\tau)\varphi_R(x) + a_L(\tau)\varphi_L(x)\right]^{\sigma+1} \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial W^I_\ell}{\partial \tau} &= g(x)\Big\{\sigma\left(\bar a_R\bar\varphi_R + \bar a_L\bar\varphi_L\right)^{\sigma-1}\left(a_R\varphi_R + a_L\varphi_L\right)^{\sigma+1}\left(\bar a'_R\bar\varphi_R + \bar a'_L\bar\varphi_L\right) + \\ &\quad +(\sigma+1)\left|\bar a_R\bar\varphi_R + \bar a_L\bar\varphi_L\right|^{2\sigma}\left(a'_R\varphi_R + a'_L\varphi_L\right)\Big\} \end{aligned}$$

Hence,

$$\left\|\frac{\partial W^I_\ell}{\partial \tau}\right\| \leq \tilde g(2\sigma+1)\max\left[|a'_R|, |a'_L|\right]\max\left[\|\varphi_R\|^{2\sigma}_\infty, \|\varphi_L\|^{2\sigma}_\infty\right] = c^{2\sigma}_{\infty,d}$$

where $|a'_{R,L}| \leq C$ from the previous Lemma. A similar estimate for the non-local perturbation case follows; where we can derivate with respect to τ under the integral $\langle\varphi, g\varphi\rangle$ since the integral converges uniformly with respect to τ. □

We give now an a priori estimate of the term ψ_c.

Lemma 7. *Let $\psi_c = \Pi_c \psi$ where ψ is the solution of equation (32); it satisfies to the following estimate*

$$e^{-C\tau} \|\psi_c\| = \tilde{O}(e^{-\Gamma/\hbar}), \quad \forall \tau \geq 0, \tag{57}$$

for some positive constant $C > 0$ independent of $\hbar$ and τ.

Remarks:

- In particular, from (57) it follows that
$$\|\psi_c\| = \tilde{O}(e^{-\Gamma/\hbar})$$
uniformly for any $\tau \in [0, \tau']$ for any τ' fixed, from which the second estimate (41) follows.

Proof. Now, in order to prove the estimate (57) we make use of the third equation of (34) from which follows that

$$\begin{aligned} \psi_c(\cdot, \tau) &= -i \int_0^\tau e^{-i(H_0 - \Omega)(\tau - s)/\omega} r_c ds \\ &= -i \frac{\epsilon}{\omega} \int_0^\tau e^{-i(H_0 - \Omega)(\tau - s)/\omega} \Pi_c W(\cdot, \psi) \psi(\cdot, s) ds \end{aligned} \tag{58}$$

since $\psi_c^0 = \Pi_c \psi^0 = 0$ from assumption Hyp.2. Therefore, we can write

$$\psi_c = -i \frac{\epsilon}{\omega} [I + II]$$

where

$$\begin{aligned} I &= \int_0^\tau e^{-i(H_0 - \Omega)(\tau - s)/\omega} \Pi_c W^I ds, \quad W^I = W^I(\varphi) \\ II &= \int_0^\tau e^{-i(H_0 - \Omega)(\tau - s)/\omega} \Pi_c W^{II} ds, \quad W^{II} = W^{II}(\varphi, \psi_c) \end{aligned}$$

For what concerns the first term we have that, by integrating by part,

$$\begin{aligned} I &= \left[-i\omega e^{-i(H_0 - \Omega)(\tau - s)/\omega} [H_0 - \Omega]^{-1} \Pi_c W^I \right]_0^\tau + \\ &\quad + i\omega \int_0^\tau e^{-i(H_0 - \Omega)(\tau - s)/\omega} [H_0 - \Omega]^{-1} \Pi_c g \frac{\partial W^I}{\partial s} ds \end{aligned}$$

Let us underline that

$$\left\| e^{-i(H_0 - \Omega)(\tau - s)/\omega} \right\| = 1$$

and from Lemma 1 it follows that

$$\|\hbar [H_0 - \Omega]^{-1} \Pi_c\| \leq C.$$

From these facts and from Lemmas 4 and 6 then we have that

$$\begin{aligned} \|I\| &\leq C \frac{\omega}{\hbar} \max_{s \in [0, \tau]} \left\{ \|W^I\| + \tau \left\| \frac{\partial W^I}{\partial s} \right\| \right\} \\ &\leq C \frac{\omega}{\hbar} \left\{ \hbar^{-d\sigma/2} + \tau c_{\infty,d}^{2\sigma} \right\} \leq C \frac{\omega}{\hbar} \hbar^{-\sigma d/2} (1 + \tau) \end{aligned}$$

For what concerns the other term we have that

$$\|II\| \leq \int_0^\tau \|W^{II}\| ds \leq C \hbar^{-\sigma d/2} \int_0^\tau \|\psi_c\| ds$$

Collecting all these results and denoting

$$h(\tau) = \|\psi_c(\cdot,\tau)\|$$

we have that $h(\tau)$ is a non-negative real-valued function satisfying the estimate

$$\begin{aligned} h(\tau) &\leq \frac{\epsilon}{\omega}\left\{ C\omega\hbar^{-1-\sigma d/2}(1+\tau) + C\hbar^{-\sigma d/2}\int_0^\tau h(s)ds \right\} \\ &\leq a\int_0^\tau h(s)ds + b(1+\tau) \end{aligned} \tag{59}$$

where

$$a = C\frac{\epsilon\hbar^{-\sigma d/2}}{\omega} = C\eta = O(1)$$

since (22) and

$$b = C\epsilon\hbar^{-1-\sigma d/2} = \tilde{O}\left(e^{-\Gamma/\hbar}\right)$$

since (22) and (17). Then, the Gronwall's Lemma gives that

$$h(\tau) \leq be^{a\tau} + \frac{b}{a}\left[e^{a\tau}-1\right] \leq Cbe^{C\tau}$$

proving so the estimate (57). □

Remark:

- In dimension $d > 2$ for local perturbation case and where $\sigma < \frac{1}{d-2}$, then (59) takes the form

  $$h(\tau) \leq a\int_0^\tau h^\gamma(s)ds + b(1+\tau), \quad \gamma = 1 + (2-d)\sigma \in (0,1),$$

  from which and by means of Gronwall's Lemma arguments the a priori weaker estimate follows

  $$h(\tau) \leq [b + a\tau]^{1/(1-\gamma)}.$$

  Unfortunately, this estimate is not useful in order to extend the result of Lemma 7 to the case of dimension $d > 2$.

Now, we are ready to complete the proof of the theorem. Let

$$J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; A = \begin{pmatrix} a_R \\ a_L \end{pmatrix}, \; R = \begin{pmatrix} -ir_R \\ -ir_L \end{pmatrix}$$

then the system (34) takes the form

$$\left\{ \begin{array}{lcl} A' &=& JA + R \\ \psi_c' &=& -\frac{i}{\omega}\left[H_0 - \Omega\right]\psi_c - ir_c \end{array} \right. , \quad R = R(A,\psi_c) \tag{60}$$

Let

$$B = \begin{pmatrix} b_R \\ b_L \end{pmatrix}, \; \tilde{R} = \begin{pmatrix} -i\eta\tilde{r}_R \\ -i\eta\tilde{r}_L \end{pmatrix}$$

then the *two-level approximation* (36) takes the form

$$B' = JB + \tilde{R}, \quad \tilde{R} = \tilde{R}(B) \tag{61}$$

We underline that

$$A, B \in S^2$$

where

$$S^2 = \left\{ A = \begin{pmatrix} a_R \\ a_L \end{pmatrix},\ a_R,\ a_L \in \mathrm{C}\ :\ |A| = \sqrt{|a_R|^2 + |a_L|^2} \leq 1 \right\}$$

Let now

$$F : S^2 \to C^2$$

defined as

$$F(A) = JA + \tilde{R}(A) \tag{62}$$

Hence, the first equation of (60) and equation (61) can be written as

$$B' = F(B) \quad \text{and} \quad A' = F(A) + \left[R(A, \psi_c) - \tilde{R}(A) \right] \tag{63}$$

Lemma 8. *The function F defined in equation (62) satisfies to the following Lipschitz condition*

$$|F(A) - F(B)| \leq C|A - B| \tag{64}$$

for some $C > 0$.

Proof. By definition

$$F(A) - F(B) = J(A - B) + \left[\tilde{R}(A) - \tilde{R}(B) \right]$$

where

$$\tilde{R}(A) - \tilde{R}(B) = \begin{pmatrix} \eta \left[\tilde{r}_R(A) - \tilde{r}_R(B) \right] \\ \eta \left[\tilde{r}_L(A) - \tilde{r}_L(B) \right] \end{pmatrix}$$

In the local perturbation case it follows that (we assume, for definiteness, that $|b_R| \leq |a_R|$)

$$\begin{aligned} \tilde{r}_R(A) - \tilde{r}_R(B) &= C_R \left[|a_R|^{2\sigma} a_R - |b_R|^{2\sigma} b_R \right] \\ &= C_R \left[|a_R|^{2\sigma} (a_R - b_R) + (|a_R|^{2\sigma} - |b_R|^{2\sigma}) b_R \right] \\ &= C_R \left[|a_R|^{2\sigma} (a_R - b_R) + (|a_R|^{\sigma} - |b_R|^{\sigma}) (|a_R|^{\sigma} + |b_R|^{\sigma}) b_R \right] \end{aligned}$$

hence

$$|\tilde{r}_R(A) - \tilde{r}_R(B)| \leq C \, |a_R - b_R|$$

since (66) and since $|a_R|, |b_R| \leq 1$. The estimate for the term $\tilde{r}_L(A) - \tilde{r}_L(B)$ similarly follows. In the non-local case it follows that

$$\begin{aligned} \tilde{r}_R(A) - \tilde{r}_R(B) &= C_R \left[|a_R|^2 a_R - |b_R|^2 b_R \right] + \tilde{C} \left[a_R |a_L|^2 - b_R |b_L|^2 \right] \\ &= C_R \left[|a_R|^2 (a_R - b_R) + \left(|a_R|^2 - |b_R|^2 \right) b_R \right] + \\ &\quad + \tilde{C} \left[(a_R - b_R) |a_L|^2 + b_R \left(|a_L|^2 - |b_L|^2 \right) \right] \end{aligned}$$

from which follows (64) immediately, similarly for $r_L(A) - r_L(B)$. □

From (63) it follows that

$$(A - B)' = F(A) - F(B) + \left[R(A, \psi_c) - \tilde{R}(A) \right], \quad A(0) = B(0) \tag{65}$$

where

Lemma 9. *For any Γ', $0<\Gamma'<\Gamma$, there exist $\hbar^\star>0$ and $C>0$, independent of $\hbar$ and τ, such that*

$$\left| R(A,\psi_c)-\tilde R(A)\right| \le Ce^{-\Gamma'/\hbar}e^{C\tau},\quad \forall\tau\ge 0,\ \forall\hbar\in(0,\hbar^\star)$$

Proof. Indeed, we have that (for the sake of definiteness we consider only one term)

$$\begin{aligned} & r_R(a_R,a_L,\psi_c)-\eta\tilde r_R(a_R,a_L)= \\ & =[r_R(a_R,a_L,\psi_c)-r_R(a_R,a_L,0)]+[r_R(a_R,a_L,0)-\eta\tilde r_R(a_R,a_L)] \\ & =[r_R(a_R,a_L,\psi_c)-r_R(a_R,a_L,0)]+\tilde O(e^{-\Gamma/\hbar}) \end{aligned}$$

since (35). Moreover

$$r_R(a_R,a_L,\psi_c)=\frac{\epsilon}{\omega}\langle\varphi_R,W\psi\rangle$$

where $W\psi=W^I+W^{II}$ and where W^I does not depend on ψ_c. From this fact and from Lemmas 4 and 7 it follows that

$$\begin{aligned} |r_R(a_R,a_L,\psi_c)-r_R(a_R,a_L,0)| &= \frac{|\epsilon|}{\omega}\left|\langle\varphi_R,W^{II}\rangle\right| \\ &\le \frac{|\epsilon|}{\omega}\|\varphi_R\|\cdot\|W^{II}\|\le\frac{|\epsilon|}{\omega}C\hbar^{-d\sigma/2}\|\psi_c\| \\ &\le Ce^{-\Gamma'/\hbar}e^{C\tau} \end{aligned}$$

for any $\Gamma'\in(0,\Gamma)$ since (57). □

Now, we are ready to complete the proof of the theorem. From (65) it follows that

$$A(\tau)-B(\tau)=\int_0^\tau [F[A(s)]-F[B(s)]]\,ds+\int_0^\tau\left[R(A,\psi_c)-\tilde R(A)\right]ds$$

If we set

$$q(\tau)=|A(\tau)-B(\tau)|$$

then from this equation and from Lemmas 8 and 9 it follows that

$$q(\tau)\le C\int_0^\tau q(s)ds+Ce^{-\Gamma'/\hbar}\left[e^{C\tau}-1\right]$$

for any $\Gamma'\in(0,\Gamma)$, from which and from Gronwall's Lemma the desired estimate (41) follows. The proof of the Theorem is so completed.

Appendix A. Inequalities

Basic Inequalities. Let $h(x)=x^\gamma$, defined for $\gamma>0$ and $x\ge 0$. Then, for any $y\ge x$, it follows that

$$x\,[h(y)-h(x)]\le\gamma(y-x)\begin{cases} x^\gamma & \text{if } 0<\gamma\le 1 \\ y^\gamma(x/y)^{\gamma 2^{-n}} & \text{if } 1<\gamma\end{cases}\tag{66}$$

where n is a positive integer such that $2^{n-1}<\gamma\le 2^n$.

In order to prove this inequality we consider, at first, the case $\gamma\le 1$. The Taylor expansion, up to the second term, with a remainder term gives that

$$\begin{aligned} h(y) &= h(x)+(y-x)h'(x)+\frac12(y-x)^2h''(\bar x) \\ &= h(x)+\gamma(y-x)x^{\gamma-1}-\gamma(1-\gamma)\frac12(y-x)^2\bar x^{\gamma-2} \end{aligned}$$

for some $\bar{x} \in (x, y)$. Hence, the wanted inequality follows since $y \geq x$.

If $\gamma > 1$ let n be a positive integer such that $2^{n-1} < \gamma \leq 2^n$, then

$$\begin{aligned} y^\gamma - x^\gamma &= \left(y^{\frac{1}{2}\gamma} + x^{\frac{1}{2}\gamma} \right) \cdot \left(y^{\frac{1}{2}\gamma} - x^{\frac{1}{2}\gamma} \right) \\ &= \left(y^{\frac{1}{2}\gamma} + x^{\frac{1}{2}\gamma} \right) \cdot \left(y^{\frac{1}{4}\gamma} + x^{\frac{1}{4}\gamma} \right) \cdot \left(y^{\frac{1}{4}\gamma} - x^{\frac{1}{4}\gamma} \right) \\ &= \left[\Pi_{j=1}^n \left(y^{\gamma 2^{-j}} + x^{\gamma 2^{j-1}} \right) \right] \cdot \left(y^{\gamma 2^{-n}} - x^{\gamma 2^{-n}} \right) \\ &\leq 2^n \left[\Pi_{j=1}^n y^{\gamma 2^{-j}} \right] x^{\gamma 2^{-n}-1} \frac{\gamma}{2^n} (y - x) \\ &= \gamma y^{\gamma(1-2^{-n})} x^{\gamma 2^{-n}-1} (y - x) \end{aligned}$$

Gagliardo-Nirenberg inequality. The Gagliardo-Nirenberg inequality states that [6]

$$\|f\|_{2\sigma+2}^{2\sigma+2} \leq C_{\sigma,d} \|\nabla f\|_2^{\sigma d} \|f\|_2^{2+\sigma(2-d)} \tag{67}$$

where

$$\sigma \in \begin{cases} [0, +\infty] & \text{if } d = 1 \\ [0, +\infty) & \text{if } d = 2 \\ [0, 2/(d-2)) & \text{if } d > 2 \end{cases} \tag{68}$$

and where C is a given constant. Such an inequality (67) can be also rewritten as

$$\|f\|_p \leq C_{p,d} \|\nabla f\|^\delta \|f\|^{1-\delta}, \quad \delta = \frac{\sigma d}{2(\sigma+1)} = \frac{(p-2)d}{2p} = \frac{d}{2} - \frac{d}{p} \tag{69}$$

where

$$p \in \begin{cases} [2, +\infty] & \text{if } d = 1 \\ [2, +\infty) & \text{if } d = 2 \\ [2, 2d/(d-2)) & \text{if } d > 2 \end{cases}$$

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