QUASI-PERIODIC SOLUTIONS FOR COMPLETELY RESONANT NON-LINEAR WAVE EQUATIONS IN 1D AND 2D.

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Abstract. We provide quasi-periodic solutions with two frequencies $\omega \in \mathbb{R}^2$, for a class of completely resonant non-linear wave equations in one and two spatial dimensions and with periodic boundary conditions. This is the first existence result for quasi-periodic solutions in the completely resonant case. The main idea is to work in an appropriate invariant subspace, in order to simplify the bifurcation equation. The frequencies, close to that of the linear system, belong to an uncountable Cantor set of measure zero where no small divisor problem arises.

1. Introduction

We consider the completely resonant nonlinear wave equations in $d = 1$ and $d = 2$ spatial dimensions with 2π periodic boundary conditions:

$$\text{a)} \begin{cases} v_{tt} - v_{xx} = -v^3 + f(v) \\ v(x,t) = v(x+2\pi, t) \end{cases}, \qquad \text{b)} \begin{cases} v_{tt} - v_{xx} - v_{yy} = -v^3 + f(v) \\ v(x,y,t) = v(x+2h\pi, y+2k\pi, t) \quad \forall k,h \in \mathbb{Z} \end{cases} \tag{1.1}$$

where $f(v)$ is an odd analytic function at $v = 0$ of degree at least five.

In this paper we prove existence of small amplitude quasi-periodic solutions of Equations 1.1, with two frequencies $\omega \in \mathbb{R}^2$.

Existence of periodic solutions for the one dimensional equation 1.1a) has been proved in the papers: [LS], [BCP], [BB1], [BB2], [GMP], [BB3] both for periodic and Dirichlet boundary conditions.

Up to now quasi-periodic solutions for Equations 1.1 have not been found. One should remark that quasi-periodic solutions for non resonant (or partially resonant) nonlinear Hamiltonian PDE's have been widely studied, see for instance [KP], [P], [CY], [CW], [B1], [B2], [B3] and references therein. In these cases the linearized equation at zero already has quasi-periodic solutions (which arise from a foliation by invariant tori). Indeed, given $\omega \in \mathbb{R}^n$, the space of quasi-periodic solutions of frequency ω is always finite dimensional (possibly empty).

On the other hand the completely resonant equations 1.1, linearized in zero, possess infinite dimensional spaces of periodic solutions with the same period. Indeed, given $\omega \in \mathbb{R}^n$, the space of quasi-periodic solutions of frequency ω is always either infinite dimensional or empty.

Actually for $d = 1$ all the solutions of the associated linear equation are of the form:

$$v_0(x,t) = r(x+t) + s(x-t), \tag{1.2}$$

hence 2π periodic. When looking for a quasi-periodic solution $Q(x, \omega t)$ of equation 1.1a),with $\omega \in \mathbb{R}^N$ and $\omega_i \sim 1$, it is not at all clear from which solution of type 1.2, and from which frequency, should $Q(x, \omega t)$ branch off.

For $d = 2$ the picture is still more complicated as there are infinite dimensional spaces both of periodic and quasi-periodic solutions.

The main idea of this paper is to look for solutions in appropriate invariant subspaces of functions $u : \mathbb{T}^2 \to \mathbb{R}$. In such subspaces the problem is similar to that of finding periodic solutions for equation 1.1a).

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To motivate our choice of subspaces let us first consider the case $d = 1$; the simplest way to correct 1.2 obtaining a quasi-periodic function is to change by small and different quantities the velocity of a forward and a backward traveling wave:

$$v(x,t) = r(\omega_1 t + x) + s(\omega_2 t - x) + \text{ small corrections}, \qquad \omega_i \sim 1.$$

As equation 1.1a) has constant coefficients, looking for such a solution is equivalent to restricting $v(x,t)$ to the invariant subspace:

$$\begin{cases} v(x,t) = u(x + \omega_1 t, \omega_2 t - x) \\ u(\varphi_1 + 2k\pi, \varphi_2 + 2h\pi) = u(\varphi_1, \varphi_2), & \forall k,h \in \mathbb{Z} \end{cases} \tag{1.3}$$

In the case $d = 2$ again we consider the simplest possible solution:

$$v(x,y,t) = r(t + x) + s(t + y),$$

and correct the velocities of the two waves; this is equivalent to looking for solutions in the invariant subspace:

$$\begin{cases} v(x,y,t) = u(x + \Omega_1 t, y + \Omega_2 t)\,, & \Omega_i \sim 1 \\ u(\varphi_1 + 2k\pi, \varphi_2 + 2h\pi) = u(\varphi_1, \varphi_2), & \forall k,h \in \mathbb{Z} \end{cases} \tag{1.4}$$

We define the frequencies to be:

$$\omega = (1 + \varepsilon, 1 + a\varepsilon)\,, \qquad \Omega = (\sqrt{1 + \varepsilon}, \sqrt{1 + a\varepsilon}); \tag{1.5}$$

notice that for $\varepsilon = 0$ the subspaces 1.3 and 1.4 are spaces of periodic solutions in t.

In the subspaces defined in 1.3and 1.4, finding quasi-periodic solutions of Equations 1.1 of frequency respectively ω and Ω, is equivalent to finding doubly 2π periodic solutions for the equations:

$$d = 1: \quad \begin{cases} [(\omega_1^2 - 1)\partial^2_{\varphi_1} + (\omega_2^2 - 1)\partial^2_{\varphi_2} + 2(\omega_1\omega_2 + 1)\partial_{\varphi_1}\partial_{\varphi_2}]u(\varphi) = -u^3(\varphi) + f(u) \\ u(\varphi_1 + 2k\pi, \varphi_2 + 2h\pi) = u(\varphi_1, \varphi_2), & \forall k,h \in \mathbb{Z} \end{cases} \tag{1.6}$$

$$d = 2: \quad \begin{cases} [(\Omega_1^2 - 1)\partial^2_{\varphi_1} + (\Omega_2^2 - 1)\partial^2_{\varphi_2} + 2\Omega_1\Omega_2\partial_{\varphi_1}\partial_{\varphi_2}]u(\varphi) = -u^3(\varphi) + f(u) \\ u(\varphi_1 + 2k\pi, \varphi_2 + 2h\pi) = u(\varphi_1, \varphi_2), & \forall k,h \in \mathbb{Z}. \end{cases} \tag{1.7}$$

Equations 1.6 and 1.7 can be written as:

$$\begin{cases} L_\alpha[u(\varphi)] = -u^3(\varphi) + f(u) \\ u(\varphi_1 + 2k\pi, \varphi_2 + 2h\pi) = u(\varphi_1, \varphi_2) \end{cases}. \tag{1.8}$$

where:

$$L_\alpha[u] := \alpha_0(\varepsilon\alpha_1\partial_{\varphi_1} + \partial_{\varphi_2}) \circ (\varepsilon\alpha_2\partial_{\varphi_2} + \partial_{\varphi_1})u(\varphi), \tag{1.9}$$

for an appropriate choice of $\alpha := (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{R}^3$.

Having unified the notation, from now on we work on equation 1.8. We rescale equation 1.8 in order to highlight the relationship between the amplitude and the variation in frequency:

$$u(\varphi) \to \sqrt{\varepsilon}u(\varphi).$$

In the following we consider the scaled equation:

$$L_\alpha[u] = -\varepsilon u^3 + \varepsilon^2 f(u, \varepsilon). \tag{1.10}$$

We are now ready to state the main results of the paper.

***Definition* 1.** *Given a positive $\sigma \in \mathbb{R}$, let $\mathcal{H}_\sigma$ be the Hilbert space of odd analytic functions $\mathbb{T}^2 \to \mathbb{R}$, equipped with the norm:*

$$|f|^2_\sigma = \sum_{j \in \mathbb{Z}^2} |\hat{f}_j|^2(|j|^2 + 1)e^{2|j|\sigma}.$$

***Definition* 2.** *Let $r_0(\varphi_1,\alpha)\not\equiv 0$, $s_0(\varphi_2,\alpha)\not\equiv 0$ be 2π periodic solutions of:*

$$\begin{cases} -\alpha_0\alpha_1\ddot r_0 = r_0^3 + 3\langle s_0^2\rangle r_0 \\ -\alpha_0\alpha_2\ddot s_0 = s_0^3 + 3\langle r_0^2\rangle s_0 \end{cases} \qquad \langle f\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} d\tau f(\tau).$$

We prove - see Lemma 2.3 and Remark 1 - the existence of such solutions $r_0, s_0\in\mathcal{H}_\sigma$.Moreover we prove that, for appropriate values of α_1,α_2, such solutions are non degenerate.

***Definition* 3.** *Given ε_0,γ such that $0<\varepsilon_0\ll 1$ and $\varepsilon_0\ll\gamma<\frac{1}{6}$, let $\mathcal{C}_\gamma\subset(0,\varepsilon_0)\times(0,\varepsilon_0)$ be the set of badly approximable pairs:*

$$\mathcal{C}_\gamma := \Big\{(A_1,A_2)\in(0,\varepsilon_0)\times(0,\varepsilon_0):\ |n_1+A_2n_2|>\frac{\gamma}{|n_2|},\quad |n_2+A_1n_1|>\frac{\gamma}{|n_1|},\quad n_1,n_2\in\mathbb{Z}\setminus\{0\}\Big\}.$$

***Proposition* 1.** *There exist positive numbers $\rho,\varepsilon_0,C_1,C_2,C_3,\gamma$ such that, for any $\varepsilon\in\mathbb{R}$, and $\alpha\in\mathbb{R}^3$ satisfying the assumptions:*

$$H:\qquad (\varepsilon\alpha_1,\varepsilon\alpha_2)\in\mathcal{C}_\gamma,\qquad \varepsilon\in(0,\varepsilon_0),\qquad |\alpha_1/\alpha_2-1|\le\rho,\qquad C_1\le\alpha_i\le C_2,$$

Equation 1.10 admits a doubly periodic solution $u(\varphi,\varepsilon,\alpha)\in\mathcal{H}_\sigma$,satisfying:

$$|u(\varphi,\varepsilon,\alpha)-r_0(\varphi_1,\alpha)-s_0(\varphi_2,\alpha)|_\sigma < C_3\varepsilon.$$

This implies the following two Theorems:

***Theorem* 1.** *There exists a positive number A, a Cantor set $\mathcal{C}^{(1)}\subset(1-A,1+A)\times(0,\varepsilon_0)$ and a function $\alpha(a,\varepsilon):\mathbb{R}^2\to\mathbb{R}^3$ such that, for all $(a,\varepsilon)\in\mathcal{C}^{(1)}$, the pair $(\varepsilon,\alpha(a,\varepsilon))$ satisfies the assumptions H, and*

$$\kappa := \frac{1+\varepsilon}{1+a\varepsilon}\notin\mathbb{Q}\,.$$

Therefore, for all $(a,\varepsilon)\in\mathcal{C}^{(1)}$, Equation 1.6 admits a **quasi-periodic** *solution:*

$$v(x,t)\equiv u(x+(1+\varepsilon)t,(1+a\varepsilon)t-x,\varepsilon,\alpha(a,\varepsilon)),$$

where $u(\varphi,\varepsilon,\alpha)$ is defined in Proposition 1.

***Theorem* 2.** *There exists an appropriate positive number A, a Cantor set $\mathcal{C}^{(2)}\subset(1-A,1+A)\times(0,\varepsilon_0)$ and a function $\bar\alpha(a,\varepsilon):\mathbb{R}^2\to\mathbb{R}^3$ such that, for all $(a,\varepsilon)\in\mathcal{C}^{(2)}$, the pair $(\varepsilon,\bar\alpha(a,\varepsilon))$ respects the assumptions H, and*

$$\bar\kappa := \sqrt{\frac{1+\varepsilon}{1+a\varepsilon}}\notin\mathbb{Q}\,.$$

Therefore, for all $(a,\varepsilon)\in\mathcal{C}^{(1)}$, Equation 1.7 admits a **quasi-periodic** *solution:*

$$v(x,t)\equiv u(x+\sqrt{1+\varepsilon}t,y+\sqrt{1+a\varepsilon}t,\varepsilon,\bar\alpha(a,\varepsilon)),$$

where $u(\varphi,\varepsilon,\alpha)$ is defined in Proposition 1.

We now summarize the general strategy of the paper which is very similar to that of [LS] and [BCP]:

(1) Following the standard Lyapunov-Schmidt decomposition scheme we divide the space $\mathcal{H}_\sigma$ in two orthogonal, complementary subspaces P-Q such that Q is the (infinite dimensional) space of periodic solutions of 1.10 for $\varepsilon=0$. Using standard notation we write:

$$u(\varphi)=q(\varphi)+p(\varphi)$$

with $q(\varphi)\in Q$ and $p(\varphi)\in P$. We call the equations 1.10 projected on P, resp. Q) the p, resp.q, equations.

In general solving the p-equation implies a small divisor problem as the eigenvalues of the linear operator L_α accumulate to zero for $\varepsilon\alpha_1,\varepsilon\alpha_2$ in any set of positive measure.

Moreover as the amplitude and the variation in frequency are of the same order the q-equation is a non-trivial non-linear dynamical system even in the limit $\varepsilon\to 0$.

(2) To simplify the solution of the p-equation, we restrict $(\varepsilon\alpha_1, \varepsilon\alpha_2)$ to the uncountable zero measure "Cantor like" set $\mathcal{C}_\gamma$ where the eigenvalues of the linear operator L_α , restricted to the P subspace, are bounded from below by an order one constant; see Lemma 2.1.
(3) We solve the p-equations, keeping $q(\varphi)$ as a parameter, by the standard contraction Lemma, for $\alpha \in \mathcal{C}_\gamma$ and $\varepsilon \in (0, \varepsilon_0)$; see Lemma 2.2.
(4) Once the p-equations are solved, we consider the ∞-dimensional q-equations. As $p(q, \varepsilon = 0) = 0$ we first study the equations for $p(q, \varepsilon) \equiv 0$. For $\lambda \equiv \alpha_1/\alpha_2 \sim 1$ we prove the existence of a non-degenerate solution $r_0(\varphi_1) + s_0(\varphi_2) = q_0(\varphi, \lambda) \in \mathcal{H}_\sigma$. Moreover we prove that such solution depends non trivially on the two variables φ_1, φ_2; see Lemma 2.3.
By the implicit function theorem, this implies the existence of a solution $u(\varphi, \alpha, \varepsilon) \in \mathcal{H}_\sigma$ for all (α, ε) satisfying assumptions H; see Lemma 2.4. We have proved Proposition 1.
(5) Theorems 1-2 follow directly from Proposition 1. We define the functions $\alpha(a, \varepsilon)$, $\bar{\alpha}(a, \varepsilon)$ such that equation 1.8 coincides respectively with Equation 1.6 and 1.7. Then $u(\varphi, \alpha(a, \varepsilon), \varepsilon)$ is a solution of Equation 1.6 for all (a, ε) such that $(\varepsilon, \alpha(a, \varepsilon))$ satisfies the assumptions H. Finally such solution is a quasi-periodic solution of equation 1.1a) provided that $\omega_1/\omega_2 \notin \mathbb{Q}$. The same holds for Equation 1.7 substituting $\bar{\alpha}$ to α and Ω to ω.
(6) We remark that if either r_0 or s_0 were identically zero then the the solution found in Proposition 1 would depend on only one variable. In such case the solutions in Theorems 1-2 would be trivial.

2. Study of Equation 1.10

As said in the Introduction we divide the Hilbert space $\mathcal{H}_\sigma$ in two orthogonal, complementary subspaces P-Q such that Q is the (infinite dimensional) space of periodic solutions of 1.10 for $\varepsilon = 0$. Equations 1.10 at $\varepsilon = 0$ are:

$$\partial_{\varphi_1}\partial_{\varphi_2} q = 0$$

so the subspace Q is defined as:

$$Q := \big\{ q \in \mathcal{H}_\sigma : q(\varphi) = r(\varphi_1) + s(\varphi_2) \big\},$$

The subspace P, orthogonal complement of Q in $\mathcal{H}_\sigma$, is:

$$P := \Big\{ p \in \mathcal{H}_\sigma : \int_0^{2\pi} d\varphi_1 p(\varphi) = \int_0^{2\pi} d\varphi_2 p(\varphi) = 0 \Big\}.$$

Conventionally we write:

$$u(\varphi) = r(\varphi_1) + s(\varphi_2) + p(\varphi_1, \varphi_2) \equiv q(\varphi_1, \varphi_2) + p(\varphi_1, \varphi_2),$$

with $q(\varphi) \in Q$ and $p(\varphi) \in P$.
Equations 1.10 projected on the Q, P subspaces are:

$$\begin{aligned} -\alpha_0\alpha_1 \ddot{r} &= r^3 + 3\langle s^2 \rangle r + \Pi_{\varphi_1}[(u^3 - q^3) + \varepsilon f(u, \varepsilon)] && q_1) \\ -\alpha_0\alpha_2 \ddot{s} &= s^3 + 3\langle r^2 \rangle s + \Pi_{\varphi_2}[(u^3 - q^3) + \varepsilon f(u, \varepsilon)] && q_2) \\ L_\alpha[p] &= \Pi_P \varepsilon [u^3 + \varepsilon f(u, \varepsilon)] && p), \end{aligned} \tag{2.1}$$

Here and in the following given a $T-$periodic function $f(t)$ we set:

$$\langle f \rangle = \frac{1}{T} \int_0^T f(t) dt.$$

We first consider the p-equations and prove that the operator L_α is invertible. This implies the existence of a solution $p = p(q, \varepsilon) \sim O_\varepsilon(\varepsilon)$ for Equation 2.1p).

2.1. **The p-equations.** Following the scheme proposed in the Introduction, item 2., let us study the "Cantor like" set $\mathcal{C}_\gamma$ where L_α is invertible.

Lemma 2.1. *Let $\mathcal{C}_\gamma$ be the set in Definition 1(i), for all $(\varepsilon\alpha_1, \varepsilon\alpha_2) \in \mathcal{C}_\gamma \subset (0, \varepsilon_0) \times (0, \varepsilon_0)$ we have that:*

$$|D_n| \equiv |(\varepsilon\alpha_1 n_1 + n_2)(n_1 + (2 + \varepsilon\alpha_2 n_2)| > \gamma \sim O_\varepsilon(1),$$

for all $n_1, n_2 \neq 0$.

Therefore, for $(\varepsilon\alpha_1, \varepsilon\alpha_2) \in \mathcal{C}_\gamma$, $\alpha_0 > C_1$ the operator L_α restricted to the P subspace has bounded inverse:

$$|\Pi_p L_\alpha^{-1}[p]|_\sigma \leq \frac{2|p|_\sigma}{C_1\gamma},$$

for all $p \in P$.

Proof. By definition $x \in (0, \varepsilon_0)$ is badly approximable if:

$$|n_1 + xn_2| > \frac{\gamma}{|n_2|}\,, \qquad \forall (n_1, n_2) \in \mathbb{Z}^2 \quad n_2 \neq 0.$$

We denote this set by $\mathcal{B}_\gamma(\varepsilon_0)$ or $\mathcal{B}$ for short. $\mathcal{B}$ is known to be uncountable, zero measure and accumulating to zero see [BCP] for a proof.

Therefore $(\varepsilon\alpha_1, \varepsilon\alpha_2) \in \mathcal{C}_\gamma \subset (0, \varepsilon_0) \times (0, \varepsilon_0)$ if both $\varepsilon\alpha_1, \varepsilon\alpha_2 \in \mathcal{B}$ with $\gamma = O_\varepsilon(1)$.

Notice that Lemma 2.1 is trivially satisfied if

$$-n_1 \neq [\varepsilon\alpha_2 n_2] \text{ and } -n_2 \neq [\varepsilon\alpha_1 n_1].$$

Now, if ε_0 is small enough, when $-n_1 = [\varepsilon\alpha_2 n_2]$ then $|n_1| < \frac{1}{2}|n_2|$ so that

$$|n_2 + \varepsilon\alpha_1 n_1| > \frac{1}{2}|n_2|.$$

This implies that

$$|n_1 + \varepsilon\alpha_2 n_2||n_2 + \varepsilon\alpha_1 n_1| > \frac{\gamma|n_2|}{2|n_2|} > \frac{1}{2}\gamma.$$

The same (exchanging n_1 with n_2) holds if $-n_2 = [\varepsilon\alpha_1 n_1]$.

The eigenvalues of L_α restricted to P are $\alpha_0 D_n$ so

$$|L_\alpha^{-1}[p]|_\sigma^2 \equiv \sum_{\substack{n \in \mathbb{Z}^2 \\ n_1, n_2 \neq 0}} \frac{|p_n|^2 |n|^2 e^{2i\sigma|n|}}{|\alpha_0 D_n|^2} \leq \frac{4}{C_1^2\gamma^2}|p|_\sigma^2,$$

for all $p \in P$. □

Now we pass to item 3. and solve the p-equation keeping $q(\varphi)$ as a parameter.

Lemma 2.2. *Given $q(\varphi) \in Q$ such that*

$$\frac{\varepsilon|q|_\sigma^2}{C_1\gamma} \ll 1\,,$$

the p equations:

$$p(\varphi) = \varepsilon L_\alpha^{-1}\Pi_P[(q+p)^3 + \varepsilon f(q+p)],$$

can be solved with q as a parameter. The solution $p(q)$ respects the bounds:

$$|p(q)|_\sigma \leq C_3 \frac{\varepsilon|q|_\sigma^3}{C_1\gamma},$$

for C_3 an appropriate order one constant.

Proof. By Lemma 2.1 L_α^{-1} is bounded on P, moreover the operator:

$$p \to \Pi_P[(q+p)^3 + \varepsilon f(q+p)]$$

is well defined and regular on $\mathcal{H}_\sigma$; we can apply the standard contraction mapping theorem. We define the sequence:

$$p^{(h)} = \varepsilon L_\alpha^{-1}\Pi_P[(q+p^{(h-1)})^3 + \varepsilon f(q+p^{(h-1)})]\,, \qquad p^{(0)} = 0$$

The sequence defined above is a contraction if

$$\frac{3\varepsilon|q|_\sigma^2}{C_1\gamma} \ll 1$$

in such case we have[1]:

$$|p^{(h)} - p^{(h-1)}|_\sigma \le C_3\frac{|q|_\sigma^2\varepsilon}{\alpha_0}(|p^{(h-1)} - p^{(h-2)}|_\sigma) \le (C_3\frac{|q|_\sigma^2\varepsilon}{C_1\gamma})^h|q|_\sigma.$$

□

We now pass to point 4. of our scheme and solve the Equations 2.1q), with $p = p(q,\varepsilon)$ computed in the preceding Lemma:

$$\begin{cases} \alpha_0\alpha_1\ddot{r} + r^3 + 3\langle s^2\rangle r + \Pi_{\varphi_1}[(q+p(\varepsilon,q))^3 - q^3) + \varepsilon f(u,\varepsilon)] = 0 & q_1) \\ \alpha_0\alpha_2\ddot{s} + s^3 + 3\langle r^2\rangle s + \Pi_{\varphi_2}[(q+p(\varepsilon,q))^3 - q^3) + \varepsilon f(u,\varepsilon)] = 0. & q_2) \end{cases} \tag{2.2}$$

For compactness of notation let us write this equations as $F(r,s,\varepsilon) = 0$, where F is a well defined and regular operator.

2.2. **The q-equations.** The q-equations 2.2 are non trivial at $p(q,\varepsilon) \equiv 0$:

$$\begin{cases} -\alpha_0\alpha_1\ddot{r} = r^3 + 3\langle s^2\rangle r \\ -\alpha_0\alpha_2\ddot{s} = s^3 + 3\langle r^2\rangle s \end{cases}. \tag{2.3}$$

It is convenient to rescale the Equations 2.3 setting:

$$r(\varphi_1) = \sqrt{\alpha_0\alpha_1}x(\varphi_1)\,, \qquad s(\varphi_2) = \sqrt{\alpha_0\alpha_2}y(\varphi_2); \qquad \lambda = \frac{\alpha_1}{\alpha_2}$$

we obtain the equations:

$$\begin{cases} -\ddot{x} = x^3 + 3\lambda^{-1}\langle y^2\rangle x \\ -\ddot{y} = y^3 + 3\lambda\langle x^2\rangle y, \end{cases} \tag{2.4}$$

Lemma 2.3. *There exists an appropriate ρ such that for all λ in*

$$|1-\lambda| \le \rho$$

the Equation 2.4 has a **non-degenerate** *solution: $(x(t,\lambda), y(t,\lambda))$ such that both x, y are not identically zero.*

The proof of this Lemma is in the Appendix. Let us define

$$r_0(\varphi_1,\alpha) = \sqrt{\alpha_0\alpha_1}x(\varphi_1, \frac{\alpha_1}{\alpha_2})\,, \qquad s_0(\varphi_2,\alpha) = \sqrt{\alpha_0\alpha_2}y(\varphi_2, \frac{\alpha_1}{\alpha_2}),$$

where $x(t,\lambda), y(t,\lambda)$ are given by Lemma 2.3. By definition r_0, s_0 solve the q-equations for $p = 0$.

Lemma 2.4. *Equations 2.1 $(q_1 - q_2)$ for $p = p(q,\varepsilon)$ have a solution $q(\varphi,\alpha,\varepsilon) \in Q$, which is ε close to*

$$q_0 = r_0(\varphi_1) + s_0(\varphi_2).$$

[1] recall that $|q(\varphi)p(\varphi)|_\sigma \le |q|_\sigma|p|_\sigma$, by the Hilbert algebra property

Proof. By Lemma 2.3 the solutions of 2.1 for $\varepsilon = 0$ (and therefore $p(q,\varepsilon) = 0$) are non-degenerate, i.e. equation 2.4, linearized in $(x(t,\lambda), y(t,\lambda))$ does not have solutions in $\mathcal{H}_\sigma$ and the linear operator:

$$(2.5)\qquad O^{-1}[X,Y] := \begin{cases} \ddot{X}+3x^2(t,\lambda)X + 3\lambda^{-1}\langle y^2(t,\lambda)\rangle X + 6\lambda^{-1}\langle y(t,\lambda)Y\rangle x(t,\lambda) \\ \ddot{Y}+3y^2(t,\lambda)Y + 3\lambda\langle x^2(t,\lambda)\rangle Y + 6\lambda\langle x(t,\lambda)X\rangle y(t,\lambda) \end{cases}$$

is C^1 and invertible.

In the Appendix we have studied the operator O and given bounds on its norm:

$$|O[f,g]|_\sigma \le |x(t,\lambda=1)|_\sigma^8 \max(|f|_\sigma, |g|_\sigma)$$

By the Implicit Function Theorem 2.5 implies that for ε small enough we can solve the q-equations 2.2 obtaining $q = q(\varphi,\varepsilon) = r(\varphi_1,\varepsilon) + s(\varphi_2,\varepsilon)$.

□

We can now prove the existence of solutions for Equation 1.10, let us restate the proposition:

Proposition 1. *There exist positive numbers $\rho, \varepsilon_0, C_1, C_2, C_3, \gamma$ such that, for any $\varepsilon \in \mathbb{R}$, and $\alpha \in \mathbb{R}^3$ satisfying the assumptions:*

$$H: \qquad (\varepsilon\alpha_1, \varepsilon\alpha_2) \in \mathcal{C}_\gamma, \qquad \varepsilon \in (0,\varepsilon_0), \qquad |\alpha_1/\alpha_2 - 1| \le \rho, \qquad C_1 \le \alpha_0 \le C_2,$$

Equation 1.10 admits a doubly periodic solution $u(\varphi,\varepsilon,\alpha) \in \mathcal{H}_\sigma$, with the property:

$$|u(\varphi,\varepsilon,\alpha) - r_0(\varphi_1,\alpha) - s_0(\varphi_2,\alpha)|_\sigma < C_3\varepsilon.$$

Proof. The solution is simply

$$u(\varphi,\alpha,\varepsilon) = r(\varphi_1) + s(\varphi_2) + p(q,\varepsilon).$$

By definition $p(q(\varphi,\varepsilon))$ and $q(\varphi,\varepsilon) - q_0(\varphi)$ are of order ε. Rescaling Equation 1.10 we obtain that $\sqrt{\varepsilon}u(\varphi,\alpha,\varepsilon)$ is a solution of 1.8. □

3. Quasi-periodic solutions for the wave equations

As we have said in the Introduction, Thorems 1-2 are a simple consequence of proposition 1.

Theorem 1. *There exists a positive number A, a Cantor set $\mathcal{C}^{(1)} \subset (1-A, 1+A) \times (0,\varepsilon_0)$ and a function $\alpha(a,\varepsilon) : \mathbb{R}^2 \to \mathbb{R}^3$ such that, for all $a,\varepsilon \in \mathcal{C}^{(1)}$, the numbers $\varepsilon, \alpha(a,\varepsilon)$ satisfy the assumptions H, and*

$$\kappa := \frac{1+\varepsilon}{1+a\varepsilon} \notin \mathbb{Q}\,.$$

Therefore, for all $(a,\varepsilon) \in \mathcal{C}^{(1)}$, Equation 1.6 admits a **quasi-periodic** *solution:*

$$v(x,t) \equiv u(x+(1+\varepsilon)t, (1+a\varepsilon)t - x, \varepsilon, \alpha(a,\varepsilon)),$$

where $u(\varphi,\varepsilon,\alpha)$ is defined in Proposition 1.

Proof. Equations 1.6:

$$\begin{cases} [(\omega_1^2-1)\partial_{\varphi_1}^2 + (\omega_2^2-1)\partial_{\varphi_2}^2 + 2(\omega_1\omega_2+1)\partial_{\varphi_1}\partial_{\varphi_2}]u(\varphi) = -u^3(\varphi) + f(u) \\ u(\varphi_1+2k\pi, \varphi_2+2h\pi) = u(\varphi_1,\varphi_2) \end{cases}$$

are of the type 1.8 for a suitable function $\alpha(a,\varepsilon)$:

$$A_2 = \varepsilon\alpha_2 := \frac{\varepsilon a}{2+\varepsilon}, \quad A_1 = \varepsilon\alpha_1 := \frac{\varepsilon}{2+a\varepsilon}, \alpha_0 := (2+\varepsilon)(2+a\varepsilon), \qquad \lambda \equiv \frac{\alpha_1}{\alpha_2} = \frac{2+\varepsilon}{a(2+a\varepsilon)}.$$

Notice that

$$C_1 \le |\alpha_i| \le C_2\,, \qquad |\lambda - 1| \le \rho$$

provided that $\varepsilon, |a-1|$ are small enough.

Let

$$\mathcal{C}^{(1)} := \{(a,\varepsilon) \in (1-A, 1+A) \times (0,\varepsilon_0) : \quad (A_1(a,\varepsilon), A_2(a,\varepsilon)) \in \mathcal{C}_\gamma\,, \quad \text{and} \quad \kappa(a,\varepsilon) \notin \mathbb{Q}\}$$

The application

$$(a,\varepsilon) \to (A_1, A_2)$$

is invertible for $a, \varepsilon \in (1-A, 1+A) \times (0, \varepsilon_0)$. Let us call its inverse

$$\psi : (0, C\varepsilon_0) \times (0, C\varepsilon_0) \longrightarrow (1-A, 1+A) \times (0, \varepsilon_0).$$

ψ establishes a bijection between $\mathcal{C}^{(1)}$ and a subset $\bar{\mathcal{C}}$ of $\mathcal{C}_\gamma$. $\bar{\mathcal{C}}$ by definition is the subset of $\mathcal{C}_\gamma$ where $\kappa \circ \psi$ is irrational. In order to obtain $\bar{\mathcal{C}}$ we remove from $\mathcal{C}_\gamma$, for each $A_1 \in \mathcal{B}$ (badly approximable), only the countable set of pairs (A_1, A_2), such that:

$$A_2 = \frac{\kappa + 2A_1\kappa - 1}{1 + A_1(1+\kappa)}\,, \qquad \kappa \in \mathbb{Q}.$$

In particular this clearly shows that $\mathcal{C}^{(1)}$ is uncountable.

□

Theorem 2. *There exists an appropriate positive number A, a Cantor set $\mathcal{C}^{(2)} \subset (1-A, 1+A) \times (0, \varepsilon_0)$ and a function $\bar{\alpha}(a,\varepsilon) : \mathbb{R}^2 \to \mathbb{R}^3$ such that, for all $(a,\varepsilon) \in \mathcal{C}^{(2)}$, the numbers $\varepsilon, \alpha(a,\varepsilon)$ respect the assumptions H, and*

$$\sqrt{\frac{1+\varepsilon}{1+a\varepsilon}} \notin \mathbb{Q}\,.$$

Therefore, for all $(a,\varepsilon) \in \mathcal{C}^{(2)}$, Equation 1.7 admits a **quasi-periodic** *solution:*

$$v(x,t) \equiv u(x + \sqrt{1+\varepsilon}t, y + \sqrt{1+a\varepsilon}t, \varepsilon, \alpha(a,\varepsilon)),$$

where $u(\varphi, \varepsilon, \alpha)$ is defined in Proposition 1.

Proof. The proof is identical to that of the preceding Theorem, let us state the functions $\bar{\alpha}(a,\varepsilon)$:

$$A_1 = \varepsilon\bar{\alpha}_1 = \frac{\varepsilon}{\sqrt{1+\varepsilon+a\varepsilon+a\varepsilon^2} + \sqrt{1+\varepsilon+a\varepsilon}}\,, \qquad A_2 = \varepsilon\bar{\alpha}_2 = \frac{\varepsilon}{\varepsilon^2}(\sqrt{1+\varepsilon+a\varepsilon+a\varepsilon^2} - \sqrt{1+\varepsilon+a\varepsilon}),$$

$$\alpha_0 = \sqrt{1+\varepsilon+a\varepsilon+a\varepsilon^2} + \sqrt{1+\varepsilon+a\varepsilon}, \qquad \lambda = \frac{1}{a}.$$

Again the map $A_1(a,\varepsilon), A_2(a,\varepsilon)$ is invertible and we can obtain $\bar{\mathcal{C}}$ from $\mathcal{C}_\gamma$ by removing for each $A_1 \in \mathcal{B}$ a numberable set of pairs (A_1, A_2). □

Appendix A. Proof of Lemma 2.3

We search for odd 2π periodic solutions of equations 2.4 for λ close to one:

$$\begin{aligned} -\ddot{x} &= x^3 + 3\lambda^{-1}\langle y^2\rangle x \\ -\ddot{y} &= y^3 + 3\lambda\langle x^2\rangle y, \end{aligned} \tag{A.1}$$

let us first consider the equation

$$-\ddot{a} = a^3 + Ca \tag{A.2}$$

which is known to have a odd periodic solutions expressed in terms of elliptic functions.

Let us recall that the function $Sn(t,m)$ is by definition the solution of:

$$\dot{f} = \sqrt{(1-f^2)(1-mf^2)},$$

while

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-m\sin^2\varphi}}\,, \qquad E(m) = \int_0^{\frac{\pi}{2}} d\varphi\sqrt{1-m\sin^2\varphi},$$

are respectively the complete Elliptic integrals of first and second kind.

Rescaling appropriately A.1, we find the solution:

$$a(m,C,t) = V(m,C)Sn(\Omega(m,C)t,m), \quad V = \sqrt{-2m}\Omega\,, \quad \Omega = \sqrt{\frac{C}{m+1}},$$

of period $4K(m)/\Omega(m,C)$.
We set $a(m,C,t)$ to be 2π periodic by assuming $\Omega = 2K(m)/\pi$.

The equations A.1 are of the type A.2 for appropriate $m_1(\lambda)$, $m_2(\lambda)$ which solve the "compatibility" equations:

$$3\lambda\langle a(m_1,C_1,t)^2\rangle = C_2\,, \qquad 3\lambda^{-1}\langle a(m_2,C_2,t)^2\rangle = C_1.$$

It is known that

$$\langle Sn(t,m)^2\rangle = \frac{K(m)-E(m)}{mK(m)}$$

so the compatibility equations are:

$$\begin{aligned} 6\lambda\Omega^2(m_1)(\frac{E(m_1)}{K(m_1)}-1)-(m_2+1)\Omega^2(m_2) = 0 \\ 6\lambda^{-1}\Omega^2(m_2)(\frac{E(m_2)}{K(m_2)}-1)-(m_1+1)\Omega^2(m_1) = 0 \end{aligned} \quad . \tag{A.3}$$

This equations have a solution (see [GMP]) for $\lambda = 1, m_1 = m_2 = \mathbf{m} \sim -0.2544$.

Computer assisted calculations show that the Jacobian of Equation A.3 is invertible in

$$\lambda = 1, m_1 = m_2 = \mathbf{m} \sim -0.2544.$$

By the implicit function theorem there exists $\rho_1 > 0$ such that, for $|\lambda - 1| \le \rho_1$, equation A.3 has a solution[2]:

$$x(\lambda,t) = a(m_1(\lambda),C_1(\lambda),t)\,, \qquad y(\lambda,t) = a(m_2(\lambda),C_2(\lambda),t), \tag{A.4}$$

both not identically zero. Moreover, by construction, the solution is 2π periodic odd and therefore in $\mathcal{H}_\sigma$ for some appropriate σ. We now prove the non-degeneracy.

Lemma A.1. *Given $F(t), G(t) \in \mathcal{H}_\sigma$, the equation:*

$$\begin{aligned} -\ddot{X} =& 3x^2X + 3\lambda^{-1}\langle y^2\rangle X + 6\lambda^{-1}\langle yY\rangle x + F(t) \\ -\ddot{Y} =& 3y^2Y + 3\lambda\langle x^2\rangle Y + 6\lambda\langle xX\rangle y + G(t), \end{aligned}$$

where x, y are defined in A.4, has a unique solution $X, Y \in \mathcal{H}_\sigma$, for $|\lambda - 1| \le \rho$.

Proof. We first consider the operator

$$X \to L[m,X] = \ddot{X} + 3a^2(m)X + C(m)X$$

where $a(m) = a(m,C(m))$ is 2π periodic; notice that

$$a(m_1(\lambda)) = x, a(m_2(\lambda) = y,$$

by definition.

This operator has been studied and inverted in [GMP]. We have:

$$X = -L^{-1}[m_1,(6\lambda^{-1}\langle a(m_2)Y\rangle a(m_1) + F(t))]\,, \quad Y = -L^{-1}[m_2,(6\lambda\langle a(m_1)X\rangle a(m_2) + G(t))],$$

which yields a linear equation for $\langle xX\rangle, \langle yY\rangle$:

$$\begin{vmatrix} 1 & 6\lambda\langle a(m_1)L^{-1}[m_1,a(m_1)]\rangle \\ 6\lambda^{-1}\langle a(m_2)L^{-1}[m_2,a(m_2)]\rangle & 1 \end{vmatrix} \begin{vmatrix} \langle a(m_1)X\rangle \\ \langle a(m_2)Y\rangle \end{vmatrix} \equiv$$

$$M\begin{vmatrix} \langle xX\rangle \\ \langle yY\rangle \end{vmatrix} = -\begin{vmatrix} \langle a(m_1)L^{-1}[m_1,F]\rangle \\ \langle a(m_2)L^{-1}[m_2,G]\rangle \end{vmatrix}.$$

This determines uniquely X, Y provided that the matrix M in the left hand side is invertible.

In [GMP] the constant $6\langle a(\mathbf{m})L^{-1}[\mathbf{m},a(\mathbf{m})]\rangle \sim 2.5$ is computed explicitly (computer assisted calculations);

[2]Indeed, as stated in [LS], equationA.3 has solutions $(x(\lambda,t),y(\lambda,t))$ both non-identically zero, provided that $\lambda \in (\pi/6, 6/\pi)$

as $\langle a(m)L^{-1}[m,a(m)]\rangle$ is smooth and $m_1(\lambda), m_2(\lambda) \sim \mathbf{m}$ for $\lambda \sim 1$ there is a ρ neighborhood of $\lambda = 1$ where matrix is invertible.

We have defined an invertible linear operator on Q :

$$O[F(\varphi_1), G(\varphi_2)] = (L^{-1}[m_1, F(\varphi_1) + 6\lambda^{-1}\langle yY\rangle x(\varphi_1)], (L^{-1}[m_2, G(\varphi_2) + 6\lambda^{-1}\langle xX\rangle y(\varphi_2)]), \tag{A.5}$$

where naturally

$$M \begin{vmatrix} \langle xX\rangle \\ \langle yY\rangle \end{vmatrix} = -M^{-1} \begin{vmatrix} \langle a(m_1)L^{-1}[m_1, F]\rangle \\ \langle a(m_2)L^{-1}[m_2, G]\rangle \end{vmatrix}.$$

Finally let us give a bound on the σ analytic norm of the operator O, we can use the bound on L^{-1} provided in [GMP]

$$|L^{-1}[m]|_\sigma \le |a(m)|_\sigma^2 \quad \text{which implies} \quad |O|_\sigma \le C\max(|a(\mathbf{m})|^8),$$

for $|\lambda - 1| \le \rho'$. □

Remark 1. *It should be noticed that given $x(\lambda, t), y(\lambda, t)$ 2π solutions of A.1, the functions*

$$x_j(\lambda, t) = jx(\lambda, jt), \qquad y_h(\lambda, t) = hx(\lambda, ht),$$

are solutions of

$$\begin{aligned} -\ddot{x}_j &= x_j^3 + 3\bar{\lambda}^{-1}\langle y_h^2\rangle x_j \\ -\ddot{y}_h &= y_h^3 + 3\bar{\lambda}\langle x_j^2\rangle y_h, \end{aligned}$$

with $\bar{\lambda} = \frac{j^2}{h^2}\lambda$, for all natural values of j, h such solutions are still 2π periodic. This remark permits us to construct more solutions of A.1 namely for all couple of naturals j, h such that $|\lambda\frac{h^2}{j^2} - 1| \le \rho$,

$$jx(\lambda\frac{h^2}{j^2}, jt), hy(\lambda\frac{h^2}{j^2}, ht)$$

are solutions of A.1 and they are clearly different from $x(\lambda, t)$, $y(\lambda, t)$ studied in the proof of Lemma A.1 which have minimal period 2π.

This implies that if λ is not close to one one can still find solutions of A.1 but they will have minimal periods $2\pi/j$ and $2\pi/h$. where j, h are the smallest co-prime naturals such that

$$|\frac{j^2}{h^2}\lambda - 1| \le \rho.$$

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