ON INSTABILITY OF THE ABSOLUTELY CONTINUOUS SPECTRUM OF DISSIPATIVE SCHRÖDINGER OPERATORS AND JACOBI MATRICES

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Abstract. We prove the absence of the absolutely continuous spectrum for dissipative Schrodinger operators and Jacobi matrices with slowly decaying imaginary part of the potential.

1. Introduction

This paper addresses the question of stability of the absolutely continuous spectrum of one-dimensional Schrödinger operators under slowly decaying perturbations. In the selfadjoint case the absolutely continuous spectrum is known [1, 2] to coincide with the positive real axis, the spectrum of the unperturbed operator, as long as the potential is L^2. In particular, this result means that the absolutely continuous spectrum preserves when the conventional scattering theory is not applicable anymore, which, in terms of the L^p scale, is the case when the potential is not L^1. Our aim here is to show that the picture is very different for non-selfadjoint Schrödinger operators. Namely, the following two theorems hold[1],

Theorem 1. *Let $\{q_j\}$ be a sequence of complex numbers, $\operatorname{Im} q_j \geq 0$, and $\operatorname{Im} q_j \to 0$, and let l be the discrete Schrödinger operator in $l^2(\mathbb{N})$ defined by the matrix*

$$\begin{pmatrix} q_1 & 1 & 0 & \dots \\ 1 & q_2 & 1 & \ddots \\ 0 & 1 & q_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

If the absolutely continuous subspace of l is nontrivial, then $\operatorname{Im} q \in l^1$.

Theorem 2. *Let q be a locally bounded function on $\mathbb{R}_+$ with $\operatorname{Im} q \geq 0$, $\operatorname{Im} q(x) \to 0$ as $x \to \infty$. Assume that $\operatorname{Re} q$ is bounded below, $\sup_n \int_n^{n+1} |\operatorname{Re} q|^2\, dt$ is finite, and define l to be the Schrödinger operator in $L^2(\mathbb{R}_+)$ with the potential q, $lu = -u'' + qu$, on the domain fixed by a self-adjoint boundary condition at zero. If the absolutely continuous subspace of l is nontrivial, then $\operatorname{Im} q \in L^1$.*

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[1] See (2.1) for the definition of the absolutely continuous subspace for the class of operators under consideration.

The abstract local scattering theory for trace class perturbations of selfadjoint operators developed in [3, 4] shows that the absolutely continuous parts of arbitrary operators L, L_0 are quasi-similar by means of wave operators and their spectra coincide when L_0 is selfadjoint and $L - L_0$ is of trace class. In the situation under consideration this implies that the absolutely continuous spectrum of l coincides with that of the selfadjoint Schrödinger operator $\operatorname{Re} l$ when $\operatorname{Im} q \in L^1$ ($\operatorname{Im} q \in l^1$ in the discrete case). Thus, the theorems above effectively say that the existence of the absolutely continuous spectrum for Schrödinger operators in the dissipative case is equivalent to having a scattering theory. To the best of author's knowledge, no non-trivial *negative* results on the absolutely continuous spectrum of non-selfadjoint differential operators with non-discrete spectrum have been obtained previously.

Notice that although the absolutely continuous subspace of a dissipative operator have mostly been studied [3, 5, 6] in terms of the Szökefalvi-Nagy - Foias functional model, we avoid using the model explicitly. A model-oriented reader will, however, easily recognize the condition of isometricity a. e. on the real axis of the characteristic function of the operator under consideration in the basic criterion (2.2).

The structure of the paper is as follows. In section 2 we define the absolutely continuous subspace of a dissipative operator and provide necessary abstract preliminaries. Section 3 is devoted to the proof of theorem 1, section 4 to the proof of theorem 2.

NOTATION:

$\mathbb{C}_\pm = \{z\colon \pm \operatorname{Im} z > 0\}$.

$H_{ac}(L)$ - the absolutely continuous subspace of an operator L. The notation H_{ac} without an argument is used when it is clear from the context which operator it refers to.

$\mathbf{S}^1$ - trace class.

$T(x,k)$ - the transfer matrix for the differential equation $-y'' + py = ky$ on the semi-axis $x \geq 0$:

$$T(x,k)\begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}$$

for any solution y to the differential equation.

$\|\cdot\|_X = \|\cdot\|_{L^2(0,X)}$.

$\sigma_{ess}(L)$ - the essential spectrum of the operator L. The expression $\sigma_{ess}(L) \subset \Omega$ means that the spectrum of L in the complement of the set Ω consists of at most countably many eigenvalues of finite geometric multiplicity.

Unless otherwise specified, the abbreviation a. e. refers to the Lebesgue measure.

2. Absolutely Continuous Subspace

Let H be a Hilbert space and L be a maximal dissipative operator in H of the form $L = A + iV$, $A = A^*$, $V \geq 0$ is bounded. We shall assume that $\sigma_{ess}(L) \subset \mathbb{R}$. It is also assumed throughout that the operator L is completely non-selfadjoint, that is, has no reducing subspaces on which it induces a self-adjoint operator. Define the Hardy classes of vector functions $\mathbf{H}^2_\pm$ to be the collections of functions $f : \mathbb{C}_\pm \to H$ analytic in $\mathbb{C}_\pm$, respectively, and satisfying $\sup_{\varepsilon>0} \int_{\mathbb{R}} \|f(k \pm i\varepsilon)\|^2 \, dk < \infty$.

Definition. *The absolutely continuous subspace*, $H_{ac}(L)$, of the operator L is defined as follows [3, 5]

$$H_{ac}(L) = \operatorname{clos}\left\{\begin{matrix} u \in H : & 1)\,(L-z)^{-1}\,u \text{ is analytic in } \mathbb{C}_+ \\ & 2)\ V^{1/2}\,(L-z)^{-1}\,u \in \mathbf{H}^2_+ \end{matrix}\right\}. \tag{2.1}$$

The absolutely continuous subspace is regular invariant [3], that is, $\overline{(L-z)^{-1}H_{ac}} = H_{ac}$ for all $z \in \rho(L)$. This implies that the restriction $L|_{H_{ac}}$ with the domain $\mathcal{D}(L) \cap H_{ac}$ is a closed operator in H_{ac}.

Various motivations of this definition and its relation to scattering theory are given in [3, 4, 5, 6, 7]. We only mention here the following "weak" characterization of the subspace $H_{ac}(L)$ (it will not be used in the paper).

Theorem.[10]

$$H_{ac}(L) = \operatorname{clos}\left\{\begin{matrix} u \in H : & 1)\,(L-z)^{-1}\,u \text{ is analytic in } \mathbb{C}_+ \\ & 2)\ \left\langle (L-z)^{-1}\,u, v\right\rangle\big|_{\mathbb{C}_\pm} \in \mathbf{H}^2_\pm \text{ for all } v \in H \end{matrix}\right\}.$$

This theorem shows, in particular, that (2.1) can be considered as a generalization of the definition of the absolutely continuous subspace in the self-adjoint theory.

The triviality of the subspace H_{ac} in the examples considered in this paper will be established on the basis of the following criterion which is implicit in [7].

Proposition 2.1. $H_{ac}(L) = \{0\}$ *iff for a. e.* $k \in \mathbb{R}$ *we have* $(z = k + i\varepsilon)$

$$D(z) \equiv \sqrt{\varepsilon}\,(L^* - z)^{-1}\sqrt{V} \xrightarrow{\ s\ } 0 \tag{2.2}$$

as $\varepsilon \downarrow 0$.

Notice that it suffices to verify the strong convergence on a dense set, since the function $D(z)$ is bounded in $\mathbb{C}_+$, $\|D(z)\| \le 1/2$ (see [7] or the proof of lemma 2.3 below where a similar calculation is carried out). To make the exposition closed, we give here an elementary proof of the if part (the only if part is not used in this paper[2]). In the argument we shall use the following inequality from [3],

$$\varepsilon \int_{\mathbb{R}} \left\|(L^* - k - i\varepsilon)^{-1}\,v\right\|^2 dk \le \pi\,\|v\|^2\,, \tag{2.3}$$

which holds true of any maximal dissipative operator L and all $v \in H$ and $\varepsilon > 0$. This inequality easily follows from the existence of a self-adjoint dilation [9] of the maximal dissipative operator $-L^*$, see [3] for details.

Proof. Given a $u \in H$, define the functions $u_\pm(z)$, $z \in \mathbb{C}_\pm$, to be the restrictions of the function $\sqrt{V}\,(L-z)^{-1}\,u$, to $\mathbb{C}_\pm$, respectively. According to a fundamental property of dissipative operators, $u_- \in \mathbf{H}^2_-$ for all $u \in H$ [3, Theorem 1]. Suppose that (2.2) holds, and $u_+ \in \mathbf{H}^2_+$ for a certain $u \in H$. We shall show that $u = 0$. Define $u_\pm(k) \in L^2(\mathbb{R}, H)$ to be the boundary values of the functions $u_\pm$ on the real axis. The required assertion

[2]The proof of the only if part [7] depends on the functional model for the operator L.

will be established if we show that $u_+ = u_-$ in $L^2(\mathbb{R}, H)$. For all $z = k + i\varepsilon$, $\varepsilon > 0$, and $e \in H$ we have

$$|\langle u_+(z) - u_-(\overline{z}), e\rangle| = 2\varepsilon \left|\left\langle \sqrt{V}\,(L-\overline{z})^{-1}\,(L-z)^{-1}\,u, e\right\rangle\right| = \\ 2\sqrt{\varepsilon}\left|\left\langle (L-z)^{-1}\,u, D(z)e\right\rangle\right|.$$

Then, for any compact interval $\Delta \subset \mathbb{R}$

$$\int_\Delta |\langle u_+(k+i\varepsilon) - u_-(k-i\varepsilon), e\rangle|\, dk \le 2\left(\int_\Delta \|D(k+i\varepsilon)e\|^2\, dk\right)^{1/2} \\ \left(\varepsilon\int_\Delta \left\|(L-k-i\varepsilon)^{-1}\,u\right\|^2 dk\right)^{1/2}. \tag{2.4}$$

The first factor on the right hand side vanishes as $\varepsilon \to 0$ by the assumption and the Lebesgue theorem. Let us show that the second factor is bounded in ε. An application of the resolvent identity gives

$$\sqrt{\varepsilon}\left\|(L-z)^{-1}\,u\right\| \le \sqrt{\varepsilon}\left\|(L^*-z)^{-1}\,u\right\| + \|D(z)\| \cdot \left\|\sqrt{V}\,(L-z)^{-1}\,u\right\|.$$

Integrating and taking into account the inequality (2.3) we find

$$\varepsilon\int_{\mathbb{R}} \left\|(L-k-i\varepsilon)^{-1}\,u\right\|^2 dk \le 2\pi\,\|u\|^2 + \int_{\mathbb{R}} \left\|\sqrt{V}\,(L-k-i\varepsilon)^{-1}\,u\right\|^2 dk,$$

which is bounded by the assumption about u. We obtain that the left hand side of (2.4) converges to 0 which implies that $\langle u_+(k) - u_-(k), e\rangle = 0$ a. e. on Δ. From arbitrariness of Δ and e we now conclude that $u_+ = u_-$. □

In the case of trace class perturbations a local scattering theory exists for the pair (L, A). The following assertion is implicit in [3, 4].

Theorem 2.2. *If $V \in \mathbf{S}^1$ then $\sigma\left(L|_{H_{ac}}\right) = \sigma_{ac}(A)$ where the absolutely continuous spectrum of the self-adjoint operator A is defined in the standard way.*

Proof (sketch). According to the results in [3, 6], the assumption about V implies that to any Borel subset $\omega \subset \mathbb{R}$ there corresponds an invariant subspace H_ω of the operator $L|_{H_{ac}}$ such that $\sigma\left(L|_{H_\omega}\right) \subset \overline{\omega}$, and H_ω is a maximal invariant subspace with this property. Moreover, if $\omega = \cup_n \omega_n$, then $\sigma\left(L|_{H_\omega}\right) = \overline{\cup_n \sigma\left(L|_{H_{\omega_n}}\right)}$; $H_{\mathbb{R}} = H_{ac}$, and H_ω is trivial if $|\omega| = 0$. It is shown in the course of the proof of theorem 5 in [3] that there exists a scalar contractive analytic function, m, in the upper half plane such that for any $c > 0$ the bounded local wave operators $W_\pm^\omega(A, L)\colon H_\omega \to H$ for the set $\omega = \{k \in \mathbb{R} : |m(k)| \ge c\}$ exist and are complete in the sense that $\operatorname{Ran} W_\pm^\omega(A, L) = P_\omega H$. Here P_ω is the spectral projection of the a.c. part of A corresponding to the set ω. This means that $L|_{H_\omega}$ and $A_\omega = A|_{\operatorname{Ran} P_\omega}$ are similar, and thus $\sigma\left(L|_{H_\omega}\right) = \sigma_{ac}(A_\omega)$. Now, considering the exhausting sequence $\{\omega_n\}$ of the sets defined as above with $c = 1/n$ and taking into account that the set $\{k : m(k) = 0\}$ has Lebesgue measure zero, we conclude from the additivity property that $\sigma\left(L|_{H_{ac}}\right) = \sigma_{ac}(A)$. □

Notice that the statement of the theorem holds if the weaker condition $(L-z)^{-1} - (A-z)^{-1} \in \mathbf{S}^1$ is satisfied. The set $\sigma\left(L|_{H_{ac}}\right)$ is referred to as *the absolutely continuous spectrum of the operator* L.

In general, the question whether $L_1 - L_2 \in \mathbf{S}^1$ for a pair of dissipative operators L_1, L_2 implies $\sigma\left(L_1|_{H_{ac}(L_1)}\right) = \sigma\left(L_2|_{H_{ac}(L_2)}\right)$ appears to be open if $\operatorname{Im} L_{1,2}$ are not of $\mathbf{S}^1$ separately. We shall need a result of this type in a special situation. The original work of this paper starts here.

Lemma 2.3. *Let* L *and* $\widetilde{L}$ *be two dissipative operators such that* $\widetilde{L} - L = i\Gamma$ *with a* $\Gamma \in \mathbf{S}^1$. *Assume that* $\Gamma \geq 0$ *and there exists a bounded operator* $T \geq 0$ *such that* $\Gamma = \widetilde{V}T$, $\widetilde{V} = \operatorname{Im}\widetilde{L}$. *If* $H_{ac}(\widetilde{L}) = \{0\}$, *then* $H_{ac}(L) = \{0\}$.

Proof. Applying the criterion of proposition 2.1 to $\widetilde{L}$ and taking into account the assumed factorization of Γ we find that

$$\sqrt{\varepsilon}\left(\widetilde{L}^* - z\right)^{-1}\sqrt{\Gamma} \xrightarrow{s} 0$$

for a. e. $k \in \mathbb{R}$. Let us show that $\sqrt{\varepsilon}\,(L^* - z)^{-1}\sqrt{\Gamma} \xrightarrow{s} 0$ for a. e. k. We have,

$$\sqrt{\varepsilon}\,(L^* - z)^{-1}\sqrt{\Gamma} \cdot G(z) = \sqrt{\varepsilon}\left(\widetilde{L}^* - z\right)^{-1}\sqrt{\Gamma},$$

where

$$G(z) = I + i\sqrt{\Gamma}\left(\widetilde{L}^* - z\right)^{-1}\sqrt{\Gamma}.$$

Let $V = \operatorname{Im} L$. Then the following calculation shows that $G(z)$ is a contractive function in $\mathbb{C}_+$,

$$\begin{aligned} I - G^*(z)G(z) = I - &\left(I - i\sqrt{\Gamma}\left(\widetilde{L} - \overline{z}\right)^{-1}\sqrt{\Gamma}\right)\left(I + i\sqrt{\Gamma}\left(\widetilde{L}^* - z\right)^{-1}\sqrt{\Gamma}\right) = \\ &i\sqrt{\Gamma}\left(\widetilde{L} - \overline{z}\right)^{-1}\left(\left(\widetilde{L}^* - z\right) - \left(\widetilde{L} - \overline{z}\right) + i\Gamma\right)\left(\widetilde{L}^* - z\right)^{-1}\sqrt{\Gamma} = \\ &\sqrt{\Gamma}\left(\widetilde{L} - \overline{z}\right)^{-1}(2\varepsilon + 2V + \Gamma)\left(\widetilde{L}^* - z\right)^{-1}\sqrt{\Gamma} \geq 0. \end{aligned} \tag{2.5}$$

Notice that the condition on Γ implies that $\sqrt{\Gamma}\left(\widetilde{L}^* - z\right)^{-1}\sqrt{\Gamma} \in \mathbf{S}^1$. Thus, $G(z)$ is a contractive analytic operator-function in $\mathbb{C}_+$ such that $I - G(z) \in \mathbf{S}^1$. Moreover, the operator $G(z)$ is boundedly invertible for $\operatorname{Im} z$ large enough, for $G(z) \to I$ in the operator norm when $\operatorname{Im} z \to +\infty$ since $\widetilde{L}$ has a bounded imaginary part. Hence, by the Szokefalvi-Nagy theorem [9], the function $G(z)$ admits scalar multiple, that is, there exists a scalar contractive analytic function, $g \not\equiv 0$, such that $G(z)\Omega(z) = g(z) \cdot I$ for a certain bounded analytic function Ω. By the Fatou theorem [9] it follows that the strong boundary values, $G^{-1}(k) = \text{s-lim}_{\varepsilon\downarrow 0}\, G^{-1}(k + i\varepsilon)$ of the function G^{-1} exist for a. e. $k \in \mathbb{R}$. We infer that for a. e. $k \in \mathbb{R}$

$$\sqrt{\varepsilon}\,(L^* - z)^{-1}\sqrt{\Gamma} = \sqrt{\varepsilon}\left(\widetilde{L}^* - z\right)^{-1}\sqrt{\Gamma} \cdot G^{-1}(z) \xrightarrow[\varepsilon\downarrow 0]{s} 0.$$

One can now verify the condition (2.2) for the operator L. We have,

$$\sqrt{\varepsilon}\,(L^*-z)^{-1}\sqrt{V}=\sqrt{\varepsilon}\left(\widetilde{L}^*-z\right)^{-1}\sqrt{V}-i\sqrt{\varepsilon}\,(L^*-z)^{-1}\sqrt{\Gamma}\cdot Q(z), \tag{2.6}$$

where $Q(z)=\sqrt{\Gamma}\left(\widetilde{L}^*-z\right)^{-1}\sqrt{V}$ is a bounded analytic function in $\mathbb{C}_+$. The latter follows from (2.5) as applied to the dissipative operator $-\widetilde{L}^*$, using the fact that the left hand side is not greater than unity. Finally, the first term on the right hand side of (2.6) converges strongly to zero by the assumption, and an application of the Fatou theorem to the function $Q(z)$ shows that the second term converges strongly to zero as well. □

In the next section we establish triviality of the absolutely continuous subspace for an operator L by verifying the condition (2.2) for an operator $\widetilde{L}$ satisfying the assumptions of this lemma. As the *proof* of the lemma shows, the final result then does not depend on the "only if" part of the criterion of proposition 2.1.

3. Discrete Schrödinger Operator

Let $\{q_n\}$ be an arbitrary sequence of complex numbers such that $0\le \operatorname{Im} q_n\le C$, $C<\infty$. Define $\Theta(z)$ and $\Phi(z)$, $z\in\mathbb{C}$, to be the solutions of the difference equation

$$u_{n+1}+u_{n-1}+q_nu_n=zu_n \tag{3.1}$$

for $n>1$ with the initial conditions $\Theta_1=0$, $\Theta_2=-1$, $\Phi_1=1$, $\Phi_2=z-q_1$.

The discrete Schrödinger operator, l, acts in the Hilbert space $H=l^2(\mathbb{N})$ and has the form

$$l=S+S^*+Q,$$
$$(Su)_n=u_{n+1},\ (Qu)_n=q_nu_n.$$

This formula defines l on the domain of the multiplication operator Q as a maximal dissipative operator which is easily seen to be completely nonself-adjoint if $\operatorname{Im} q\not\equiv 0$. For $z\in\mathbb{C}_-$ let $u(z)=(l-z)^{-1}\delta_1$ where $H\ni\delta_1=(1,0,\dots)$, so $u(z)$ is an l^2 - solution to (3.1) for $n>1$. The function u satisfies

$$u(z)=-\Theta(z)+m(z)\Phi(z)$$

where $m(z)\stackrel{def}{=}\left\langle (l-z)^{-1}\delta_1,\delta_1\right\rangle$. Obviously, $-m(z)$ is a Nevanlinna function in $\mathbb{C}_-$ ($\operatorname{Im} m(z)\le 0$), and therefore the finite boundary values, $m(k)=\lim_{\varepsilon\downarrow 0} m(k-i\varepsilon)$, exist for almost all $k\in\mathbb{R}$. At such k's, the solution $u(k)$, $u_j(k)=\lim_{\varepsilon\downarrow 0}u_j(k-i\varepsilon)$, to (3.1) with $z=k$ is defined.

Lemma 3.1. *If the finite boundary value $m(k)$ exists, then the sum $\sum_j \operatorname{Im} q_j\,|u_j(k)|^2$ is finite, and*

$$\limsup_{\varepsilon\downarrow 0}\left(\varepsilon\,\|u(k-i\varepsilon)\|^2\right)\le -\operatorname{Im} m(k). \tag{3.2}$$

Proof. Both assertions result immediately from the identity ($z = k - i\varepsilon$, $\varepsilon > 0$)

$$\sum_{j=1}^{\infty} \operatorname{Im} q_j \, |u_j(z)|^2 + \varepsilon \sum_{j=1}^{\infty} |u_j(z)|^2 = -\operatorname{Im} m(z)$$

since each term on the left hand side is non-negative. To verify this identity we notice that the left hand side equals to

$$\operatorname{Im} \langle (l - z)u(z), u(z) \rangle = \operatorname{Im} \langle \delta_1, u(z) \rangle = \operatorname{Im} \overline{m(z)} = -\operatorname{Im} m(z).$$

□

Our aim now is to apply the criterion from proposition 2.1. For a function ψ analytic in $\mathbb{C}_-$ introduce the notation $\widetilde{\psi}(z) = \overline{\psi\,(\overline{z})}$. Let $\alpha_j = \sqrt{\operatorname{Im} q_j}$, $\alpha = \operatorname{diag}\{\alpha_j\}$, and let $D(z)$, $z = k + i\varepsilon$, be the operator defined in (2.2) with $L = l$.

Lemma 3.2. *Let $k \in \mathbb{R}$ be such that $m(k)$ exists finitely. If $\{\alpha_j \Phi_j(k)\} \notin l^2$, then $D(k + i\varepsilon) \xrightarrow[\varepsilon \downarrow 0]{s} 0$.*

Proof. We shall check the required strong limit on a certain dense set of compactly supported vectors. Let us consider the action of the resolvent of l^* on such vectors. A straightforward calculation shows that for any compactly supported f ($z = k + i\varepsilon$, $\varepsilon > 0$),

$$\left[(l^* - z)^{-1} f \right]_j = \widetilde{\Phi}_j(z) \sum_{n=j}^{\infty} \widetilde{u}_n(z) f_n + \widetilde{u}_j(z) \sum_{n=1}^{j-1} \widetilde{\Phi}_n(z) f_n.$$

The first term on the right hand side vanishes for j's to the right of the support of f. If k is such that $m(k)$ exists finitely, then $\widetilde{u}_n(z)$ converges when $\varepsilon \to 0$ for each n, and one concludes that both terms on the right hand side are bounded when $\varepsilon \to 0$ uniformly in j ranging over the support of f. Thus, for any such k, we have

$$(l^* - z)^{-1} f = c_\varepsilon[f] \widetilde{u}(z) + r(\varepsilon), \tag{3.3}$$

$$c_\varepsilon[f] = \sum_{n=1}^{\infty} \widetilde{\Phi}_n(k + i\varepsilon) f_n,$$

where $r(\varepsilon)$ vanishes to the right of the support of f and is bounded uniformly and, moreover, in the l^2-norm, when $\varepsilon \to 0$. On substituting (3.3) in the definition of $D(z)$ and taking into account that $\varepsilon^{1/2} \|\widetilde{u}(z)\|$ is bounded when $\varepsilon \to 0$ by (3.2), we find that for any compactly supported v

$$\|D(z)v\| \le C \, |c_\varepsilon[\alpha v]| + o(1), \quad \varepsilon \to 0.$$

Observe now that the linear set $\mathcal{D} = \Big\{ v \in l^2(\mathbb{N}) : v$ is compactly supported, and $\sum_j v_j \alpha_j \widetilde{\Phi}_j(k) = 0 \Big\}$ is dense in l^2 when $\{\alpha_j \Phi_j(k)\} \notin l^2$. Hence, it is enough to check that $D(k + i\varepsilon)v \to 0$ for $v \in \mathcal{D}$. Indeed, $c_\varepsilon[\alpha v] \to \sum_n \alpha_n \widetilde{\Phi}_n(k) v_n = 0$ as $\varepsilon \to 0$ for $v \in \mathcal{D}$, and the required limit follows. □

Theorem 3.3. *Let $\{q_j\}$ be a sequence of complex numbers such that $0 \le \operatorname{Im} q_n \le C$, $C < \infty$, and $\sigma_{ess}(l) \subset \mathbb{R}$. If $\operatorname{Im} q \notin l^1$, then $H_{ac} = \{0\}$.*

Proof. From the previous lemma and proposition 2.1, it is enough to show that if $\operatorname{Im} q \notin l^1$ then $\sum_j \operatorname{Im} q_j |\Phi_j(k)|^2 = \infty$ for a. e. $k \in \mathbb{R}$. For k such that $m(k)$ exists finitely consider the Wronskian of the solutions Φ and u,

$$1 = W[\Phi(k), u(k)] \equiv \Phi_{j-1}(k)u_j(k) - \Phi_j(k)u_{j-1}(k),\ j > 1. \tag{3.4}$$

Multiplying this by $\alpha_j\alpha_{j-1}$ and summing over j, we have,

$$\sum^n \alpha_j\alpha_{j-1} \le \sum^n \alpha_{j-1} |\Phi_{j-1}(k)| \cdot \alpha_j |u_j(k)| + \sum^n \alpha_j |\Phi_j(k)| \cdot \alpha_{j-1} |u_{j-1}(k)| \le$$
$$2\left(\sum^n \operatorname{Im} q_j |\Phi_j(k)|^2\right)^{1/2} \left(\sum^n \operatorname{Im} q_j |u_j(k)|^2\right)^{1/2}.$$

By lemma 3.1 the last factor on the right hand side is bounded in n. We thus obtain that if $\sum_j \alpha_j\alpha_{j-1} = \infty$ then $\sum_j \operatorname{Im} q_j |\Phi_j(k)|^2 = \infty$ for a. e. $k \in \mathbb{R}$, as required.

The general case is reduced to the one just considered, since if $\operatorname{Im} q \notin l^1$, then there exists a nonnegative $d \in l^1$ such that $\widetilde{q} = q + id$ satisfies $\sum_j \widetilde{\alpha}_j\widetilde{\alpha}_{j-1} = \infty$ with $\widetilde{\alpha}_j = (\operatorname{Im} \widetilde{q}_j)^{1/2}$. Indeed, by the uniform boundedness principle, the assumption that $\alpha \notin l^2$ implies that there exists a $\rho \in l^2$ such that $\sum_j \rho_j\alpha_j = \infty$, and, obviously, the ρ can be chosen to be nonnegative. Now one can set $d_j = \rho_{j-1}^2$, for $\widetilde{\alpha}_j\widetilde{\alpha}_{j-1} \ge \rho_{j-1}\alpha_{j-1}$. An application of lemma 2.3 completes the proof. ☐

Since $\sigma_{ess}(l) \subset \mathbb{R}$ when $\operatorname{Im} q_n \to 0$, this theorem contains theorem 1 from the Introduction. Considering that in the case $\operatorname{Im} q \in l^1$ the imaginary part of l is of the trace class, and therefore $\sigma\left(l|_{H_{ac}}\right) = \sigma_{ac}(\operatorname{Re} l)$, it gives a complete description of the a. c. spectrum of dissipative discrete Schrödinger operator.

The above proof extends straightforwardly to Jacobi matrices with selfadjoint nondiagonal part. Namely, given a bounded sequence, $\{\rho_n\}$, of positive numbers, define the Jacobi matrix J by

$$J = \Lambda S + S^*\Lambda + Q,$$

where $(\Lambda u)_n = \rho_n u_n$. Then J is a maximal dissipative operator, and the corresponding generalization of theorem 3.3 reads as follows.

Theorem 3.4. *Let $\{q_j\}$ be a sequence of complex numbers such that $0 \le \operatorname{Im} q_n \le C$, $C < \infty$, and $\sigma_{ess}(J) \subset \mathbb{R}$. If $\operatorname{Im} q \notin l^1$, then $H_{ac}(J) = \{0\}$.*

Indeed, assuming without loss of generality that $\rho_1 = 1$, one can define the functions Θ, Φ to be the solutions of the equation $\rho_{n+1}u_{n+1} + \rho_n u_{n-1} + q_n u_n = zu_n$ with the same boundary conditions as in the $\rho \equiv 1$ case, and let $u(z) = (J - z)^{-1}\delta_1$. The proof then proceeds verbatim as in the case $\rho \equiv 1$ until (3.4), where we will have

$$W_j[\Phi, u] \equiv \Phi_{j-1}(k)u_j(k) - \Phi_j(k)u_{j-1}(k) = \rho_{j-1}^{-1}$$

instead. Obviously, the left hand side is bounded away from zero, since the ρ_j's are bounded, and the rest of the proof is the same as above.

4. Continuous case

§1. **Preliminaries and the beginning of the proof of the main result.** Let q be a locally bounded function on $\mathbb{R}_+$ satisfying $\operatorname{Im} q(x) \geq 0$ a. e. and h be an arbitrary real number. We shall assume throughout that $\operatorname{Im} q \in L^\infty(\mathbb{R}_+)$, and $\operatorname{Re} q$ is bounded below. Define $\Theta(x, z)$ and $\Phi(x, z)$ to be the solutions to

$$-u'' + \overline{q}u = zu, \; z \in \mathbb{C}, \tag{4.1}$$

with the Cauchy data $\Phi(0, z) = \Theta'(0, z) = 0$, $\Phi'(0, z) = \Theta(0, z) = 1$. It is standard that the solutions Θ and Φ are continuous in z uniformly in $x \in I$ for any compact interval I.

The Schrödinger operator l, $ly = -y'' + qy$, is defined in the Hilbert space $H = L^2(\mathbb{R}_+)$ on the domain of its real part, the Schrödinger operator with the potential $\operatorname{Re} q$, fixed by the self-adjoint boundary condition $y'(0) = hy(0)$. With this definition, l is a completely nonself-adjoint maximal dissipative operator, provided that $\operatorname{Im} q \not\equiv 0$ [5].

The following lemma is a continuous analog of lemma 3.1.

Lemma 4.1. *For all $z \in \mathbb{C}_+$ there exists exactly one linear independent L^2-solution, $u(x, z)$, to (4.1), which can be chosen in the form*

$$u(x, z) = \Theta(x, z) + m(z)\Phi(x, z) \tag{4.2}$$

where m is an analytic function in $\mathbb{C}_+$. For almost all $k \in \mathbb{R}$ the function m has finite boundary values, $m(k) = \lim_{\varepsilon \downarrow 0} m(k + i\varepsilon)$, and the solution $u(x, k) = \lim_{\varepsilon \downarrow 0} u(x, k + i\varepsilon)$ satisfies

$$\int \operatorname{Im} q(x) \, |u(x, k)|^2 \, dx < \infty. \tag{4.3}$$

Moreover, for any $\varepsilon > 0$

$$\operatorname{Im} m(z) = \int_0^\infty (\operatorname{Im} q(t) + \varepsilon) \, |u(t, z)|^2 \, dt, \; z = k + i\varepsilon. \tag{4.4}$$

In particular,

$$\limsup_{\varepsilon \downarrow 0} \left(\varepsilon \, \|u(\cdot, k + i\varepsilon)\|^2 \right) \leq \operatorname{Im} m(k) \tag{4.5}$$

whenever $m(k)$ exists.

Proof. As is well-known, see e.g. [13], the assumption that $\operatorname{Re} q$ is bounded below implies that the differential expression corresponding to the potential $\operatorname{Re} q$ is in the limit-point case at infinity, that is, the minimal operator $\widetilde{l} = \operatorname{Re} l|_{C_0^\infty}$ has deficiency indices $(1, 1)$. It follows that the operator $l_0 = \widetilde{l} + i\operatorname{Im} q$ satisfies $\dim \ker (l_0^* - z) = 1$ for $\operatorname{Im} z$ large enough, and therefore for all $z \in \mathbb{C}_+$, because $\mathbb{C}_- \subset \widehat{\rho}(l_0)$. Thus, (4.1) has exactly one linear independent L^2-solution for all $z \in \mathbb{C}_+$.

Assume now that the representation (4.2) for the solution holds[3]. Consider the identity,

(4.6)
$$0=\int_0^x(-u''+(\overline{q}-z)u)\,\overline{u}\,dt=-u'(x)\overline{u(x)}+m(z)+\int_0^x(\overline{q}-z)\,|u|^2\,dt+\int_0^x|u'|^2\,dt.$$

Since $\mathrm{Re}\,q$ is bounded below, it shows that the function $|u'\overline{u}|$ has a limit, finite or infinite, when $x\to+\infty$. On taking the real part, we find

$$\frac{1}{2}\frac{d}{dx}\,|u(x)|^2=\mathrm{const}+\int_0^x\mathrm{Re}\,(q-z)\,|u|^2\,dt+\int_0^x|u'|^2\,dt,$$

which implies that $u'\in L^2$ (otherwise $|u(x)|\to\infty$ as $x\to+\infty$). This yields that $u'(x)\overline{u(x)}\to 0$, $x\to+\infty$. On taking the imaginary part of (4.6) and passing to the limit $x\to+\infty$, we obtain (4.4). It shows that $m(z)$ is a Nevanlinna function in $\mathbb{C}_+$ ($\mathrm{Im}\,m(z)\ge 0$), and therefore has finite boundary values for a. e. k on the real axis, and for any such k the integral $\int^x\mathrm{Im}\,q(t)\,|u(t,k)|^2\,dt$ is bounded in x (by $\mathrm{Im}\,m(k)$). It follows that the integral in (4.3) is finite.

To establish the representation (4.2) it is enough to notice that $\Phi(x,z)$ cannot be the L^2 solution, for otherwise a calculation similar to the above would lead to the formula $\int_0^\infty(\mathrm{Im}\,q+\varepsilon)\,|\Phi|^2\,dt=0$ implying that $\Phi\equiv 0$. The analyticity of the function m was established in [11]. $\square$

We shall often suppress the argument x in the notation and write $u(z)$ for the element of the Hilbert space H defined by the function $u(x,z)$ for each $z\in\mathbb{C}_+$.

From now on, we shall assume that $\mathrm{Im}\,q(x)\to 0$ as $x\to\infty$, which guarantees that $\sigma_{ess}(l)\subset\mathbb{R}$ by a relative compactness argument. Then, l is an operator of the class considered in section 2.

In the following lemma $D(z)$ is defined by (2.2) with $L=l$. Let $u_h=\Theta+h\Phi$.

Lemma 4.2. *Let $k\in\mathbb{R}$ be such that the finite boundary value $m(k)$ exists, and $m(k)\ne h$. Then*

(i) for any compactly supported $v\in H$ ($z=k+i\varepsilon$)

$$D(z)v=\sqrt{\varepsilon}\,C_\varepsilon[v]u(z)+o(1),\quad\varepsilon\to 0,\tag{4.7}$$

where

$$C_\varepsilon[v]=\frac{1}{h-m(k+i\varepsilon)}\int_0^\infty u_h(x,k+i\varepsilon)\sqrt{\mathrm{Im}\,q(x)}v(x)dx,\tag{4.8}$$

and the o-symbol refers to the L^2 - norm.

(ii) If $\sqrt{\mathrm{Im}\,q}\,u_h(\cdot,k)\notin L^2$, then $D(k+i\varepsilon)\xrightarrow[\varepsilon\downarrow 0]{s}0$.

[3]Once the uniqueness of an L^2 - solution is established, the representation (4.2) and the identity (4.4) are results of the nesting circles analysis of the equation (4.1) carried out in [11]. For completeness, we provide here a simple derivation of these facts, except for the analyticity of m.

Proof of this lemma is completely analogous to that of lemma 3.2, so we just sketch it. Let $f \in H$ be supported on a finite interval $[0, X]$. It is straightforward to check that for all $z \in \mathbb{C}_+$

$$\left((l^* - z)^{-1} f\right)(x) = \tag{4.9}$$
$$\frac{1}{h-m(z)} \left[u(x,z) \int_0^x u_h(s,z) f(s) ds + u_h(x,z) \int_x^\infty u(s,z) f(s) ds\right].$$

Obviously, the function in the square brackets, which will be denoted by g, coincides with

$$\left(\int_0^\infty u_h(s,z) f(s) ds\right) u(x,z)$$

for $x > X$. Moreover, $g(x,z)$ converges when $\varepsilon \to 0$ uniformly in $x \in [0, X]$. On taking into account that the condition $m(k) \neq h$ secures the boundedness of the common factor in (4.9), this gives

$$(l^* - z)^{-1} f = \left(\frac{1}{h - m(z)} \int_0^\infty u_h(s,z) f(s) ds\right) u(z) + r(z)$$

where the function $r(x,z)$ vanishes for $x > X$ and is uniformly bounded when $\varepsilon \to 0$. This implies (i).

Notice that, as (4.5) shows, $\varepsilon^{1/2} \|u(k + i\varepsilon)\|$ is bounded when $\varepsilon \to 0$, so that $\|D(z)v\| \leq C\,|C_\varepsilon[v]| + o(1)$ for any compactly supported v. Now, define $C_0[v]$ by the formula (4.8) with $\varepsilon = 0$. If v satisfies $C_0[v] = 0$, then $C_\varepsilon \to 0$, and thus $D(z)v \to 0$. Since the set of compactly supported v such that $C_0[v] = 0$ is dense in L^2 when $\sqrt{\operatorname{Im} q}\, u_h(\cdot, k) \notin L^2$, (ii) follows. $\square$

Since the set of k such that $m(k) = h$ has measure zero by the uniqueness theorem for Nevanlinna functions, we have the following

Corollary 4.3. *If* $\sqrt{\operatorname{Im} q}\, u_h(\cdot, k) \notin L^2$ *for a. e.* $k \in \mathbb{R}$, *then* $H_{ac}(L) = \{0\}$.

Before proving the main result (theorem 2 in the Introduction) we would like to give a motivation of it in terms of the WKB asymptotics for the solutions. As is shown in the asymptotic theory of ordinary differential equations [12], for a class of potentials q decaying at infinity in a sufficiently regular way the asymptotics of solutions of the equation (4.1) at $x \to \infty$ can be calculated explicitly. In particular, the theory asserts the existence of two solutions, $u_\pm$, of the equation $-u'' + \overline{q} u = ku$ for $k > 0$ with the asymptotics of the form (the WKB asymptotic)

$$u_\pm \sim \exp\left[\pm i \left(\sqrt{k} x - \frac{1}{2\sqrt{k}} \int^x \overline{q(\xi)} d\xi\right)\right], \quad x \to \infty, \tag{4.10}$$

if q is decaying and satisfies various additional assumptions. Obviously, for $\operatorname{Im} q \notin L^1$ the solution u_- is growing, $|u_-| \sim \exp\left(\frac{1}{2\sqrt{k}} \int^x \operatorname{Im} q \, d\xi\right)$, while u_+ decays. Notice that if the WKB asymptotics holds for a given $k > 0$, and v, a solution to (4.1) with

$z = k > 0$, satisfies $\sqrt{\operatorname{Im} q} v \in L^2$, then it must be a multiple of u_+, for

$$\int^{\infty} \operatorname{Im} q \, |u_-|^2 \, dx \sim \int^{\infty} \operatorname{Im} q(x) \exp\left(C \int^{x} \operatorname{Im} q(\xi) d\xi \right) dx = \exp\left(C \int^{\infty} \operatorname{Im} q(x) dx \right) = \infty.$$

Let us call the function q *a WKB-class potential* if the asymptotic (4.10) holds for a. e. $k > 0$, and $\sigma_{ess}(l) = \mathbb{R}_+$. It now follows that if q is a WKB-class potential, then the solutions $u_h(\cdot, k)$ and $u(\cdot, k)$ must coincide for a. e. $k > 0$ whenever $m(k)$ exists and $\sqrt{\operatorname{Im} q}\, u_h(\cdot, k) \in L^2$. This is, however, only possible on the set of k such that $m(k) = h$, hence $\sqrt{\operatorname{Im} q}\, u_h(\cdot, k) \notin L^2$ for a. e. $k > 0$. Applying corollary 4.3, we thus obtain theorem 2 for WKB-class potentials. A simple sufficient condition for a function q to be a WKB-class potential is $q(x) \to 0$, $q \in L^2$ and $q' \in L^1$ [12], which in fact secures the asymptotics (4.10) for all $k > 0$. In particular, this shows immediately that the operator with the potential $q(x) = ix^{-\beta}$, $0 < \beta \leq 1$, has trivial absolutely continuous subspace. Notice that the eigenvalues of this operator only accumulate at 0 by a complex scaling argument (a remark due to S. Naboko).

We now proceed to prove theorem 2. The argument becomes less direct than in the discrete case since estimating the Wronskian of solutions u and u_h requires a bound on their derivatives which is only available "in the mean" and leads to some mild conditions on the real part of the potential. The proof involves spectral averaging for the self-adjoint problem corresponding to the potential $\operatorname{Re} q$ [14] and the subordinacy theory [13]. The required part of the Simon-Last result reads as follows.

Theorem 4.4. [14, Theorem 2.1C.] *Let V be a real potential locally bounded and bounded below, and let $d\mu = \min\{d\rho^D, d\rho^N\}$ where $d\rho^{D/N}$ are the spectral measures for the self-adjoint Schrödinger operators with the potential V and the Dirichlet/Neumann boundary conditions at zero, respectively. Let Θ and Φ be the solutions to $-y'' + Vy = ky$ with the Cauchy data $\Phi(0, k) = \Theta'(0, k) = 0$, $\Phi'(0, k) = \Theta(0, k) = 1$. Then for any compact interval $I \subset \mathbb{R}$ there exist a constant $C(I)$ such that for all x*

$$\int_I \left(|\Theta(x, k)|^2 + |\Phi(x, k)|^2 \right) d\mu(k) \leq C(I), \tag{4.11}$$

$$\int_I \left[\int_{x-1}^{x+1} \left(|\Theta'(y, k)|^2 + |\Phi'(y, k)|^2 \right) dy \right] d\mu(k) \leq C(I). \tag{4.12}$$

If additionally $\sup_n \int_n^{n+1} |V(t)|^2 \, dt$ *is finite, then*

$$\int_I \left(|\Theta'(x, k)|^2 + |\Phi'(x, k)|^2 \right) d\mu(k) \leq C(I). \tag{4.13}$$

In terms of the transfer matrix $T(x, k)$ for the differential equation $-y'' + Vy = ky$, (4.11) and (4.13) together are equivalent to saying that

$$\int_I \|T(t, k)\|^2 \, d\mu(k) \leq C(I). \tag{4.14}$$

We shall need the following lemma of some interest of its own.

Lemma 4.5. *Let y_0 be any solution to the equation $-y'' + \overline{q}y = ky$ such that $r = y_0'(0)/y_0(0) \in \mathbb{R}$ (or $y_0(0) = 0$). Then for any solution y to this equation there exists a constant C such that $|y(x)| \le C|y_0(x)|$ for x large enough, provided that $\operatorname{Im} q \not\equiv 0$.*

Proof. As we shall see later, the solution $y_0(x)$ does not vanish for x large enough[4]. It is then enough to establish the estimate for the solution

$$y_1(x) = \left(\int^x \frac{d\xi}{y_0^2(\xi)}\right) y_0(x)$$

linear independent of y_0. On multiplying the equation for y_0 by $\overline{y_0}$ and integrating from 0 to x, we obtain,

$$- y_0'(x)\overline{y_0(x)}\Big|_0^x + \int_0^x |y_0'|^2 \, dt + \int_0^x (\overline{q} - k) \, |y_0|^2 \, dt = 0.$$

Taking the imaginary part and noticing that $y_0'(0)\overline{y_0(0)} = r\,|y_0(0)|^2$ is real, we find

$$-\operatorname{Im} \left[y_0'(x)\overline{y_0(x)}\right] = \int_0^x \operatorname{Im} q \; |y_0|^2 \, dt.$$

Let us denote the left hand side by $\rho(x)$. This identity shows that for x large enough $\rho(x) > 0$, so that $y_0(x) \ne 0$, and one can introduce the variables R, Ψ, $R > 0$, by $y_0 = Re^{-i\Psi}$. On calculating, $\rho = R^2\Psi'$, so that $\Psi' > 0$, and the phase Ψ can be chosen to be a continuous monotonic increasing function of x. We then have,

$$\int^x \frac{d\xi}{y_0^2(\xi)} = \int^x e^{2i\Psi(\xi)} \frac{\Psi'(\xi) d\xi}{R^2(\xi)\Psi'(\xi)} = \int^{\Psi(x)} \frac{e^{2i\Psi} d\Psi}{\rho(\xi(\Psi))}.$$

The nonnegative function $\zeta(s) = 1/\rho\,(\xi(s))$ is monotonic decreasing, which implies that the integral on the right hand side is uniformly bounded, and the result follows. □

This lemma shows, in particular, that the solution $u_h(x, k)$ estimates above the solution $u(x, k)$ whenever the latter exists, $u = O(|u_h|)$ at large x. In the following lemma we use the standard asymptotic perturbation theory [12] to compare the asymptotics of solutions corresponding to the potentials q and $\operatorname{Re} q$.

Lemma 4.6. *Let $k \in \mathbb{R}$ be such that $m(k)$ exists, and $m(k) \ne h$. If $\sqrt{\operatorname{Im} q}\, u_h(\cdot, k) \in L^2$, then the equation*

$$-y'' + \operatorname{Re} q \, y = ky \tag{4.15}$$

has solutions $y_{1,2}$ such that $y_1 = u_h(1 + o(1))$, $y_2 = u + o(|u_h|)$ as $x \to \infty$, and the Wronskian

$$W[y_1, y_2] = W[u, u_h] = m(k) - h. \tag{4.16}$$

Proof. Rewriting the equation (4.15) in the matrix form,

$$Y' = \begin{pmatrix} 0 & 1 \\ \operatorname{Re} q - k & 0 \end{pmatrix} Y, \;\; Y = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix},$$

[4]This can be also seen immediately from complete non-selfadjointness of l and the fact that a completely non-selfadjoint dissipative operator cannot have real eigenvalues.

and applying the variation of the parameter formula, we obtain the equation

$$Z' = \frac{i}{h - m(k)} \begin{pmatrix} \operatorname{Im} q\, u_h u & \operatorname{Im} q\, u^2 \\ -\operatorname{Im} q\, u_h^2 & -\operatorname{Im} q\, u_h u \end{pmatrix} Z, \tag{4.17}$$

where

$$Y = \begin{pmatrix} u_h & u \\ u_h' & u' \end{pmatrix} Z.$$

The fact that $\sqrt{\operatorname{Im} q}\, u(\cdot, k) \in L^2$ (see (4.3) and the assumption $\sqrt{\operatorname{Im} q}\, u_h(\cdot, k) \in L^2$ imply that

$$\begin{pmatrix} \operatorname{Im} q\, u_h u & \operatorname{Im} q\, u^2 \\ -\operatorname{Im} q\, u_h^2 & -\operatorname{Im} q\, u_h u \end{pmatrix} \in L^1.$$

Applying now the Levinson theorem to the equation (4.17), we find that it has a solution Z satisfying $Z(x) = I + o(1)$. On taking into account that $u = O(|u_h|)$, we obtain the existence of the solutions $y_{1,2}$. Calculating the determinant of Y at infinity, we obtain (4.16). □

Proof of theorem 2 (beginning). Suppose that H_{ac} is nontrivial. Then, according to and proposition 2.1, there exists a set $M \subset \sigma_{ess}(l)$ of positive Lebesgue measure such that $D(k + i\varepsilon)$ does not converge to 0 strongly when $\varepsilon \downarrow 0$ for $k \in M$, and therefore the function $\sqrt{\operatorname{Im} q}\, u_h(\cdot, k) \in L^2$ for a. e. $k \in M$ by lemma 4.2(ii). Let $y_{1,2}(\cdot, k)$ be the solutions to (4.15) defined in lemma 4.6 for a. e. $k \in M$. From the lemma it is immediate that the solutions $y_{1,2}(\cdot, k)$ are square-summable with the weight $\operatorname{Im} q$ for a.e. $k \in M$. Let μ be the a.c. measure given by $d\mu = \min\{d\rho^D, d\rho^N\}$ where $d\rho^{D/N}$ are the spectral measures for the self-adjoint Schrödinger operators with the potential $\operatorname{Re} q$ and the Dirichlet/Neumann boundary conditions at zero, respectively. Fix arbitrary c_0, N and A such that the set

$$I = \left\{ k \in M \colon\ |m(k)| \le c_0,\ \int_0^\infty \operatorname{Im} q \left(|y_1(t,k)|^2 + |y_2(t,k)|^2 \right) dt \le N \right\} \cap [-A, A]$$

has positive μ - measure. This is possible since, as we shall show later, $\mu(M) > 0$. Let $\rho(k) = 1 + \max\{|y_{1,2}(0,k)|, |y_{1,2}'(0,k)|\}$, and $T(x,k)$ be the transfer matrix for the equation (4.15). Then, $|y_{1,2}'(x,k)| \le \rho(k)\|T(x,k)\|$. Taking into account (4.16), we have,

$$|h - m(k)| = |W[y_1, y_2]| \le \rho(k)\|T(x,k)\| \left(|y_1| + y_2| \right).$$

Multiplying this by $(\rho(k))^{-1} \operatorname{Im} q$ and integrating, we obtain

$$\begin{aligned} \int_I \frac{|h - m(k)|}{\rho(k)} d\mu(k) \int^x \operatorname{Im} q\, dt \le C \left(\int_I d\mu(k) \int^x \operatorname{Im} q\ \left(|y_1|^2 + |y_2|^2 \right) dt \right)^{1/2} \\ \left(\int^x dt \operatorname{Im} q(t) \int_I \|T(t,k)\|^2 d\mu(k) \right)^{1/2}. \end{aligned} \tag{4.18}$$

Now, $\int_I \|T(t,k)\|^2\, d\mu(k)$ is bounded uniformly in t by theorem 4.4 (see (4.14)). At this point we used the assumption that $\operatorname{Re} q$ is uniformly locally L^2-bounded. This gives

$$C\left(\int^x \operatorname{Im} q\, dt\right)^{1/2} \le C\left(\int_I d\mu(k) \int^x \operatorname{Im} q\, \left(|y_1|^2 + |y_2|^2\right) dt\right)^{1/2} \le C\sqrt{N\mu(I)},$$

and so the integral on the left hand side is bounded.

Thus, it remains to show that $\mu(M) > 0$. This will occupy the rest of the paper and requires developing an analog of the subordinacy theory [13] for the dissipative Schrödinger operator l.

§2. **Subordinacy theory and the end of the proof of theorem 2.** Recall [13] that a solution u_1 to a second-order differential equation on the semi-axis $[0,\infty)$ is called *subordinate* if

$$\frac{\|u_1\|_N}{\|u_2\|_N} \to 0$$

as $N \to \infty$ for any solution u_2 linear independent of u_1. Here $\|\cdot\|_N \equiv \|\cdot\|_{L^2(0,N)}$.

Theorem 4.7. *Let* $\operatorname{Im} q \not\equiv 0$. *Then for a. e.* $k \in \mathbb{R}$ *the following assertions are equivalent*

(*i*) $D(k+i\varepsilon) \xrightarrow[\varepsilon\downarrow 0]{s} 0$.

(*ii*) *There exists a subordinate solution to* $-y'' + qy = ky$.

As the proof shows, the exceptional set in this theorem can be identified with the complement of the set of k such that $m(k)$ exists finitely and $m(k) \ne h$. We shall provide a complete proof for the implication (*ii*) $\Rightarrow$ (*i*) only, for the reverse implication is not used in this paper. The proof is subdivided into several assertions.

Lemma 4.8. *For any real k such that $m(k)$ exists and $m(k) \ne h$ the assertion* (*i*) *is equivalent to saying that*

$$\varepsilon \|u(\cdot, k+i\varepsilon)\|^2 \to 0. \tag{4.19}$$

Proof. From lemma 4.2(i) we obtain that for any compactly supported $v \in H$

$$\|D(k+i\varepsilon)v\| \le C\varepsilon^{1/2}\|u(k+i\varepsilon)\| + o(1)$$

since $C_\varepsilon[v]$ given by (4.8) converges when $\varepsilon \to 0$. This shows that if (4.19) is satisfied, then $D(k+i\varepsilon)v \to 0$ for any compactly supported $v \in H$, and (*i*) follows.

To establish the reverse implication, we choose v to be $\overline{u_h(x,k)}\chi(x)$ where χ is the indicator of an interval Δ. Then,

$$C_\varepsilon[v] \underset{\varepsilon\to 0}{\longrightarrow} \frac{1}{h - m(k)} \int_\Delta |u_h(s,k)|^2 \sqrt{\operatorname{Im} q}\, ds \ne 0$$

if Δ is large enough. Now, lemma 4.2(i) shows that if $D(z)v \to 0$, then (4.19) holds. $\square$

The proof of the following assertion, which is central in the subordinacy theory, carries over from the self-adjoint theory [13] without any modification but a slight simplification.

Proposition 4.9. *If the equation $-y''+\overline{q}y = ky$ has a subordinate solution for a certain $k\in\mathbb{R}$ such that $m(k)$ exists and $m(k)\neq h$, then this solution is a multiple of $u(x,k)$.*

Proof. Without loss of generality, one can assume that $u(\cdot,k)\notin L^2$. Let v be the subordinate solution with the normalization chosen so that $v=\Theta+c_0\Phi$, hence the Wronskian $W[\Phi,v]=1$. This is possible since, as lemma 4.5 shows, Φ cannot be a subordinate solution. We have to prove that $c_0=m(k)$. A straightforward calculation shows that the solution $u(x,z)$ satisfies the following integral equation,

$$g(x)=\Theta(x,k)+m(z)\Phi(x,k)-i\varepsilon\left[\Phi(x,k)\int_0^x v(s)g(s)ds-v(x)\int_0^x\Phi(s,k)g(s)ds\right].$$

Let us consider this equation in $L^2(0,A(\varepsilon))$ with a finite $A(\varepsilon)$ to be chosen so that $A(\varepsilon)\to\infty$ as $\varepsilon\to 0$. If

$$\varepsilon\left\|\Phi(\cdot,k)\right\|_{A(\varepsilon)}\left\|v\right\|_{A(\varepsilon)}<\frac{1}{2}, \tag{4.20}$$

then it can be iterated in $L^2(0,A(\varepsilon))$ to give

$$\begin{aligned}\left\|u(\cdot,z)\right\|_{A(\varepsilon)}\asymp\left\|\Theta(\cdot,k)+m(z)\Phi(\cdot,k)\right\|_{A(\varepsilon)}&=\\ \left\|v(\cdot,k)+(m(z)-c_0)\Phi(\cdot,k)\right\|_{A(\varepsilon)}&=\\ (|m(z)-c_0|+o(1))\left\|\Phi(\cdot,k)\right\|_{A(\varepsilon)}&\end{aligned}$$

where we used the fact that v is subordinate, and therefore $\left\|v\right\|_{A(\varepsilon)}/\left\|\Phi(\cdot,k)\right\|_{A(\varepsilon)}\to 0$. Define $A(\varepsilon)$ by

$$\left\|\Phi(\cdot,k)\right\|_{A(\varepsilon)}\left\|v\right\|_{A(\varepsilon)}=\tfrac{1}{3C}\left\|u(\cdot,k+i\varepsilon)\right\|^2,$$
$$C=\limsup_{\varepsilon\to 0}\operatorname{Im} m(k+i\varepsilon).$$

Then (4.20) is satisfied in view of (4.5), and $A(\varepsilon)$ tends to infinity as $\varepsilon\downarrow 0$ since otherwise $\|u(\cdot,k)\|\le\liminf\|u(\cdot,k+i\varepsilon)\|$ would be finite. We thus obtain,

$$|c_0-m(z)|\le\frac{\|u(\cdot,k+i\varepsilon)\|}{\left\|\Phi(\cdot,k)\right\|_{A(\varepsilon)}}+o(1)=\left(3C\frac{\left\|v\right\|_{A(\varepsilon)}}{\left\|\Phi(\cdot,k)\right\|_{A(\varepsilon)}}\right)^{1/2}+o(1)\to 0.\ \square$$

Proof of theorem 4.7 (ii) $\Rightarrow$ (i). To establish the implication it remains to show that if $u(\cdot,k)$ is a subordinate solution, then (4.19) is satisfied. It is clear from (4.4) that it suffices to show that

$$\int_0^\infty\operatorname{Im} q(x)\left|u(x,k)\right|^2dx=\operatorname{Im} m(k).$$

The argument is similar to that used in the proof of lemma 4.5. Multiplying the identity $-u''+(\overline{q}-k)u=0$ by $\overline{u}$, integrating and taking the imaginary part, we obtain,

$$\operatorname{Im}\left[u'(x,k)\overline{u(x,k)}\right]=\operatorname{Im} m(k)-\int_0^x\operatorname{Im} q(t)\ \left|u(t,k)\right|^2dt.$$

As (4.4) shows, the right hand side here is non-negative for all x, and so the required assertion is immediate if the left hand side has a zero. Suppose now that the left hand side does not vanish for all $x > 0$, so, at least, $u(x) \neq 0$, and one can introduce the variables R, Ψ, $R > 0$, by $u = Re^{i\Psi}$. Then the left hand side becomes $\rho(x) :=$ $R^2(x)\Psi'(x)$. Since $\rho(x)$ is positive, it follows that the phase Ψ can be chosen to be a continuous monotonic increasing function of x.

We then have for a solution v linear independent of u,

$$v(x) = \left(\int_a^x \frac{d\xi}{u^2(\xi)}\right) u(x) = \left(\int^x e^{-2i\Psi(\xi)} \frac{\Psi'(\xi)d\xi}{R^2(\xi)\Psi'(\xi)}\right) u(x) = \left(\int^{\Psi(x)} e^{-2i\Psi}\zeta(\Psi)d\Psi\right) u(x),$$

where $\zeta(s) = 1/\rho\left(\Psi^{-1}(s)\right)$ is a positive monotone increasing function. It follows that $\zeta(s) \to \infty$, otherwise the integral would be bounded and v would be bounded above by u, which contradicts the assumption that u is subordinate. Thus, $\int_0^x \operatorname{Im} q\, |u|^2\, dt \to \operatorname{Im} m(k)$ as $x \to \infty$, as required. □

It should be mentioned that in the selfadjoint theory the assertion corresponding to the last part of the proof of theorem 4.7 (if $u(\cdot, k)$ is a subordinate solution, then $\operatorname{Im} m(k) = 0$) is *trivial*, since a subordinate solution normalized by the condition $u(0) = 1$ must be real when $\operatorname{Im} q \equiv 0$.

Notice that we have left unproven only one assertion in the implication $(i) \Rightarrow (ii)$ of theorem 4.7: if $\varepsilon \left\|u(k + i\varepsilon)\right\|^2 \to 0$ for a certain $k \in \mathbb{R}$ such that $m(k)$ exists and $m(k) \neq h$, then $u(x, k)$ is a subordinate solution. The proof of this fact can be taken verbatim from the corresponding assertion in the self-adjoint theory [13].

Proof of theorem 2 (end). We have to show that $\mu(M) > 0$. Suppose, by contradiction, that $\mu(M) = 0$. Since μ is equivalent to the a.c. part of the spectral measure of a self-adjoint Schrödinger operator with the potential $\operatorname{Re} q$, the following assertion of the selfadjoint Gilbert - Pearson theory applies.

Proposition. [13] *Let V be a locally bounded real potential in the limit point case at infinity and let $d\nu$ be the a.c. part of the spectral measure for the self-adjoint Schrödinger operator with the potential V and a selfadjoint boundary condition at zero. If $\nu(S) = 0$, then the equation $-y'' + Vy = ky$ has a subordinate solution for Lebesgue-almost-all $k \in S$.*

This gives that the equation (4.15) has a subordinate solution for a. e. $k \in M$. Lemma 4.6 now shows that the equation (4.1) with $z = k$ has a subordinate solution for a. e. $k \in M$ as well. Indeed, let $y = c_0 y_1 + y_2$ be a subordinate solution to (4.15). Then, $y \sim u + c_0 u_h + o(u_h)$, hence

$$\frac{\|u + c_0 u_h\|_N}{\|u_h\|_N} \sim \frac{\|y\|_N}{\|y_1\|_N} \to 0,$$

so that $u + c_0 u_h$ is a subordinate solution to (4.1). Since y_1 cannot be a subordinate solution, we are done. Applying now theorem 4.7, we obtain that $D(k + i\varepsilon) \xrightarrow{s} 0$ for a. e. $k \in M$, a contradiction. □

Conditions on the real part of the potential in theorem 2 are required to apply spectral averaging. In particular, the condition that $\operatorname{Re} q$ is uniformly locally L^2-bounded allows to use the following standard estimate for solutions of a second-order differential equation (see, for instance, [15, Lemma 3.1]): if $\sup_n \int_n^{n+1} |V(t)|^2 dt$ is finite, then

$$|g'(x)|^2 \le C(1+k^2) \int_{x-1}^{x+1} |g(s)|^2 \, ds$$

for any solution g to $-y'' + Vy = ky$ with a constant C depending on the potential V only. As was noticed in [14], a combination of this bound with (4.11) allows to drop the additional integration in y in (4.12) leading to (4.13). According to the following theorem, if we omit the condition on the positive part of $\operatorname{Re} q$, then the non-triviality of H_{ac} implies that $\operatorname{Im} q$ is L^1 "in a local mean".

Theorem 4.10. *Let q be a locally bounded function on $\mathbb{R}_+$ with $\operatorname{Im} q \ge 0$, $\operatorname{Im} q(x) \to 0$ as $x \to \infty$. Assume that $\operatorname{Re} q$ is bounded below. If the absolutely continuous subspace of the Schrödinger operator l is nontrivial, then*

$$\sum_n \left(\int_n^{n+1} \sqrt{\operatorname{Im} q} \, dt \right)^2 < \infty.$$

Proof. The argument proceeds as before until (4.18) where we integrate in t over the interval $[n, n+1]$ instead of $[0, x]$ to obtain

$$\int_I \frac{|h - m(k)|}{\rho(k)} d\mu(k) \int_n^{n+1} \sqrt{\operatorname{Im} q} \, dt \le C \left(\int_I d\mu(k) \int_n^{n+1} \operatorname{Im} q \, \left(|y_1|^2 + |y_2|^2 \right) dt \right)^{1/2}.$$

Here we have taken (4.12) into account. Taking square and summing over n, we arrive at the result. □

We were not able to adapt the argument for an "individual" k from the proof of theorem 3.3 to the continuous case, for we do not know a proper continuous analog of the spread potential $\widetilde{q}$. If this were possible, then one could hope to relax conditions on $\operatorname{Re} q$ in theorem 2. It is also unclear whether it is possible to give a proof of the result in the discrete case analogous to the above proof of theorem 2. The obstacle here is that we do not know an appropriate discrete analog of the trigonometric substitution $u = Re^{i\Psi}$ in the proof of theorem 4.7.

Finally, it should be noticed that the condition of pointwise decay of $\operatorname{Im} q$ as $x \to \infty$ in theorem 2 is only used to ensure that $\sigma_{ess}(l) \subset \mathbb{R}$ and can be replaced by any condition guaranteeing this inclusion.

5. Conclusions

As is well-known [9], triviality of the space H_{ac} implies that the characteristic function of the operator L is inner. So far, the spectral analysis of operators with inner characteristic functions has only been developed effectively for the discrete part of the spectrum in the case of trace class perturbations [16]. The examples considered in this paper suggest, however, that the appearance of a nontrivial "singular" inner function is typical in applications where the trace class theory is not applicable.

Finally, one should mention that the instability phenomenon described in this paper is likely to be a specific feature of dissipative Schrödinger operators. In the non-dissipative case, a potential of the WKB - class will have all solutions of (4.1) bounded if its imaginary part is conditionally integrable. This suggests that in this case the absolutely continuous subspace can be nontrivial even if $\operatorname{Im} q \notin L^1$.

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