Spatial birth-and-death processes in random environment

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Abstract

We consider birth-and-death processes of objects (animals) defined in $\mathbb{Z}^d$ having unit death rates and random birth rates. For animals with uniformly bounded diameter we establish conditions on the rate distribution under which the following holds for almost all realizations of the birth rates: (i) the process is ergodic with at worst power-law time mixing; (ii) the unique invariant measure has exponential decay of (spatial) correlations; (iii) there exists a perfect-simulation algorithm for the invariant measure. The results are obtained by first dominating the process by a backwards oriented percolation model, and then using a multiscale analysis due to Klein to establish conditions for the absence of percolation.
KEY WORDS: birth-and-death processes; random environment; backwards oriented percolation; multiscale analysis; random point processes; random loss networks.

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1 Introduction

A number of reasons explain the standing interest in the study of birth-and-death processes. First, they are important probabilistic constructions in their own right, with proven potential to generate and test new mathematical approaches. Second, they offer the correct framework to formalize and study several important statistical applications. Among the more recent ones, we mention point processes (Baddeley and van Lieshout, 1995; Strauss, 1995; Baddeley, Kendall and van Lieshout, 1996) and loss networks (Kelly, 1991, Ferrari and Garcia, 1998). Third, they have become useful tools for other mathematical endeavours. Examples of this are Kendall's (1998) perfect simulation scheme and its offsprings (see Møller, 2001 for a review) and the study of hard-core statistical mechanical systems presented in Fernández, Ferrari and Garcia (1998, 2001).

While the non-random version of these processes has a well developed theory (see, for instance, Møller and Waagepetersen, 2004), little is found in the literature for processes with random rates. Yet, in many situations a process with spatially homogeneous randomness is a more realistic model than a spatially homogeneous deterministic one. In general, the behavior of processes with "strong" disorder is expected to be qualitatively different from that of the non-disordered counterpart. In this paper, however, we tackle the complementary issue. We determine sufficient conditions for the disorder to be "weak" in the sense that it leads to processes exhibiting ergodic properties not far from those of the non-random versions.

We set our processes on a lattice that for convenience is taken to be $\mathbb{Z}^d$. The objects being born are called *animals*. They are supported on finite subsets of $\mathbb{Z}^d$, which, for technical reasons, are assumed to be uniformly bounded in size. Each animal γ has an associated birth rate $w^{\mathbf{J}}(\gamma)$ parametrized by a random variable $\mathbf{J}$ defined on a certain probability space. Each realization of $\mathbf{J}$ is a random environment. In addition, there is an *incompatibility relation* between animals which is also assumed to be finite-range. The birth-and-death process is defined as follows: for each fixed environment, each animal γ attempts to appear at rate $w^{\mathbf{J}}(\gamma)$ but to succeed, it must past a test, in general of stochastic nature, involving incompatible animals present at the moment of the attempt. Once born, animals disappear at a unit rate. Our main result, Theorem 2.15, establishes two conditions on the disorder [hypotheses (2.16) and (2.18) below] under which the process is ergodic and space-time mixing for almost all random environments. As in other models evolving in the presence of frozen-disorder (Klein, 1995; Gielis and Maes, 1996), while space mixing remains exponential, time relaxation is only proven to be faster than any power, with a bound of the form $\exp[-m \ln^q(1+t)]$, for some $m, q > 0$.

Two conditions are imposed on the disorder. On the one hand, in (2.16) the average local rate of *attempted* births is asked to have some finite logarithmic moment. This is a relatively mild condition which is independent of the incompatibility relation. Its role [see Lemma 5.92 and formulas (5.81)–(5.82)] is to ensure that there is a not-too-small probability that locally no animal is found to be alive throughout a sufficiently thin time slice, even in the absence of incompatibility restrictions. The second disorder condition (2.18) requires a weighted birth-rate of incompatible animals to have a sufficiently small mean. Weights are defined by a certain *size* function $S(\gamma)$ which in principle can be chosen in any way leading to the validity of the condition. This is a situation very much in the spirit of cluster-expansion formalisms; see, for instance, Dobrushin (1996). Usually —but not optimally— the size function is chosen as the number of sites in the support of the animal. This second disorder condition guarantees that large fluctuations in rate values are sufficiently sparse to have a negligible effect at sufficiently large scales.

In a subsequent corollary (Corollary 2.28), we determine a convenient sufficient criterium for both disorder hypotheses. We show that, barring very exceptional settings, there exists an $\widetilde{\varepsilon}$ (which we do not try to optimize) such that if

$$\mathbb{E}\Big[\sup_{x} \sum_{\theta \ni x} |H(\theta)| \; w^{\mathbf{J}}(\theta)\Big] \;\leq\; \widetilde{\varepsilon} \tag{1.1}$$

the process is almost-surely ergodic and mixing. Here $|H(\gamma)|$ is the cardinality of the *halo*, which is the region around γ that determines incompatibility with other animals [see (2.24)].

Our analysis is based on a construction of birth-and-death processes described in Fernández, Ferrari and Garcia (1998, 2001). In this approach, the process is constructed by resorting first to a *free* process obtained by turning off incompatibilities and allowing every attempted birth to succeed. This free process corresponds to a marked Poisson process, where the marks are the lifetime of the animal and a random variable to be used in the compatibility test. It is useful to visualize each realization of this free process as the collection of cylinders determined by the animals alive during a certain lifespan. These cylinders inherit the incompatibility relation of the animlas forming their sections. Of particular importance for the construction of the interacting process are those incompatible cylinders that are alive when a given cylinder is born. They are called *ancestors* of the latter. The interacting process can be constructed if each cylinder has only a finite number of generations of ancestors. In this case, one can descend down genealogical trees deciding, by means of succesive incompatibility tests, which cylinders of the free process remain for the interacting one. Formally, the relation "being ancestor of" defines a (backwards) oriented percolation model in the space of cylinder configurations. The construction scheme succeeds if this model does not exhibit infinite percolation.

A sufficient condition for this lack of percolation is, in turns, obtained by resorting to a dominant multitype branching process whose subcriticality allows the construction to work. The subcriticality condition is, precisely,

$$\sup_{\gamma} \frac{1}{S(\gamma)} \sum_{\theta \text{ incomp. with } \gamma} S(\theta)\, w(\theta) \;<\; 1\,. \tag{1.2}$$

The left-hand side of this inequality is the mean number of branches of the dominating branching process, and acts as a driving parameter: Time and space rates of convergence and mixing rates can be explicitly obtained in terms of it.

This approach can not be directly applied for random birthrates because, except in trivial cases, condition (1.2) is violated with probability one. An additional argument is needed showing that these violations are so sparse that the process is basically driven by the behavior in the overwhelmingly present "good" (regular) regions. As often in the study of disordered systems, this additional argument takes the form of a *multiscale analysis*. In this paper we adapt a time-tested multiscale argument whose origin can be traced to von Dreifus' (1987) dissertation. The argument was later adapted by Campanino and Klein (1991), and Campanino, Klein and Perez (1991) to the study of $(d+1)$-dimensional systems with d-dimensional disorder, which is our setting here. In the present work we follow the more general version due to Klein (1994) which has also been the basis of Gielis and Maes' (1996) study of spin-flip dynamics. Our proofs are similar to those of these references, and the informed reader will recognize many points in common, such as conditions (2.17), (5.43) and (5.44) below. But there is a number of small but frequent adaptations needed to accomodate "blob" rather than edge percolation. For the sake of clarity and completeness we have prefered to do a self-contained exposition of the proof, rather than a catalogue of differences with respect to preceding references. The paper size is comparable in both cases. We have, nevertheless, kept most of the notation, and of the architecture of the proof, adopted by Klein (1994) and Gielis and Maes (1996) for the benefit of readers familiar with these papers.

The initial ingredient of the multiscale scheme is a sequence of boxes of increasing size. The linear sizes of the boxes are the *scales*. Sites are then classified as *regular* or *singular* at a given scale according to whether there is an appropriate decay of the integrated connectivity function to the boundary of a box around the site. For the procedure to succeed there must exist a choice of scales that noticeably improves the probability of a site to be regular at succesive scales. This requires the singular regions to be sparsely distributed within a sea of regular sites; the distribution becoming more and more diluted as the observation scale grows. In the present setting of space-time (backwards) percolation, boxes are space-time parallelepipeds with a cubic spatial section. The space scales grow as a power law, but the time scale needs to grow faster. This is a consequence of the d-dimensional nature of the disorder. While regular and singular regions alternate in spatial directions, they are frozen in the time direction. Their effects are limited only by the deaths and lack of births of the concerned animals, which are processes of a more correlated nature than the (spatial) disorder distribution. Hence, the connectivity function decays much more slowly in time than in space. The time scale must satisfy two complementary requirements. First, it must grow fast enough to ensure a faster-than-power decay in the percolation probability from bottom to top inside a box. But, second, the growth must be suficiently slow to guarantee that time-like connections be mostly established inside a box, rather than by creeping through a sequence of spacially consecutive boxes. Mathematically, the first requirement is used in the proof of Theorem 5.29, to arrive to formula (5.38), while the second one appears explicitly in the proofs of Sublemma 5.63 and Lemma 5.92. The compromise is achieved by chosen a function $T(L)$ so the bottom-top and left-right percolation probabilities inside a box of size $L \times T(L)$ decrease at a comparable rate, namely exponentially with L. This corresponds to a stretched-exponential dependence [(see (5.114)–(5.116)].

In statistical mechanical terms, our results are, somehow, the low-temperature counterpart of those of Gielis and Maes (1996). While these authors study single-spin-flipping evolutions leading to high-temperature invariant measures, here we focus on gases of "deffects" characteristic of low-temperature invariant measures. The comparison is not fully valid, however, because we are able to study only families of uniformly bounded deffects.

2 Definitions, examples and results

2.1 Basic definitions

Animals We consider the lattice $\mathbb{Z}^d$, $d \geq 1$, (or, in general, the vertices of a graph with uniformly bounded coordination number) endowed with some norm. For concreteness we adopt $\|x\| = \max\{|x_i| : i = 1, 2, \ldots, d\}$, $x \in \mathbb{Z}^d$. An animal model in $\mathbb{Z}^d$ is defined by a countable family $\mathbf{G}$ of objects, the *animals*, for which there exists a map

$$\begin{array}{ccc} \mathbf{G} & \longrightarrow & \mathcal{P}(\mathbb{Z}^d) \\ \gamma & \mapsto & V(\gamma) \end{array} \tag{2.1}$$

such that (i) $V(\gamma)$ is a finite set —whose elements are called the *vertices* or *sites* of γ— and (ii) there is a finite number of animals associated to each fixed set of vertices: $V^{-1}(V(\gamma))$ is finite for every $\gamma \in \mathbf{G}$. Animal configurations are elements $\underline{\xi}$ of $\mathbb{N}^{\mathbf{G}}$.

Geometry is introduced through the map V in the obvious manner: The *diameter* of an animal γ is the diameter of its set $V(\gamma)$, the *distance* between two animals is the distance between the corresponding sets of vertices, etc. To abbreviate we shall often denote $x \in \gamma$, for $x \in \mathbb{Z}^d$, rather than $x \in V(\gamma)$. For a region $\Lambda \subset \mathbb{Z}^d$ (of finite or infinite cardinality), we denote $\mathbf{G}_\Lambda$ the set formed by all animals γ with $V(\gamma) \subset \Lambda$ (animals *in* Λ). The corresponding configuration will get a subscript Λ: $\underline{\xi}_\Lambda \in \mathbb{N}^{\mathbf{G}_\Lambda}$. The omission of a subscript Λ indicates $\Lambda = \mathbb{Z}^d$. Animals are also characterized by a *size function*. In general terms, a size function is any funcion $S : \mathbf{G} \to [1, \infty)$ that can be used in the convergence condition (2.18) below. In the examples of next section, this function is just the number of sites in $V(\gamma)$.

In this paper we suppose that there exists a number ℓ_1 such that

$$\operatorname{diam}(\gamma) \;\le\; \ell_1 \qquad \text{for all } \gamma \in \mathbf{G} \,. \tag{2.2}$$

Animal interactions We introduce an *interaction function* $M(\,\cdot\,|\,\cdot\,) : \mathbf{G} \times \mathbb{N}^{\mathbf{G}} \to [0,1]$, such that $M(\gamma|\underline{\xi})$ is the probability that an attempted birth of $\gamma \in \mathbf{G}$ actually occurs when the current configuration of animals is $\underline{\xi}$. If this function takes only the values 0 or 1 we refer the interaction as *deterministic*. This happens, for instance, for conditions such as volume- or perimeter-exclusion. The function M determines the (binary) interaction matrix

$$\operatorname{Int}(\gamma,\theta) \;:=\; \mathbf{1}\Bigl\{\sup_{\underline{\xi}} \Bigl| M(\gamma|\underline{\xi}) - M(\gamma|\underline{\xi}+\delta_\theta)\Bigr| \neq 0\Bigr\} \tag{2.3}$$

where $\mathbf{1}\{A\}$ is the indicator function of the set A, δ_θ is the configuration in $\mathbb{N}^{\mathbf{G}}$ having only θ present, and $\underline{\xi}+\delta_\theta$ is the configuration obtained by adding the animal θ to the configuration $\underline{\xi}$. We say that γ is *incompatible* with θ, and denote $\gamma \not\sim \theta$, iff $\operatorname{Int}(\gamma,\theta)=1$; otherwise we say that γ is compatible with θ. A family of animals $\Gamma \subset \mathbf{G}$ is a *compatible family* [*herd* in Dobrushin's (1996) nomenclature] if its elements are pairwise compatible. We assume that there exists a finite $\ell_2 \in \mathbb{R}$ such that

$$\operatorname{dist}(\gamma,\theta) > \ell_2 \implies \operatorname{Int}(\gamma,\theta) = 0 \,. \tag{2.4}$$

The random environment Each animal γ has an associated *birth rate* $w^{\mathbf{J}}(\gamma)$ which depends on a variable $\mathbf{J}$, the *random environment*, belonging to a certain probability space $(\mathcal{J},\mathbb{P})$. We assume that the *disorder measure* $\mathbb{P}$ is such that

$$\Gamma \text{ compatible} \implies \{w^{\mathbf{J}}(\gamma) \,:\, \gamma\in\Gamma\} \text{ independent.} \tag{2.5}$$

Birth-and-death processes in random environment For each fixed environment $\mathbf{J}$ and each $\Lambda \subset \mathbb{Z}^d$ we consider the interacting birth-and-death processes formally defined by the generator:

$$A^{\mathbf{J}}_\Lambda F(\underline{\eta}) \;=\; \sum_{\gamma\in\mathbf{G}_\Lambda} \Bigl\{ M(\gamma|\underline{\eta})\, w^{\mathbf{J}}(\gamma)\, [F(\underline{\eta}+\delta_\gamma) - F(\underline{\eta})] \;+\; \eta(\gamma)\, [F(\underline{\eta}-\delta_\gamma) - F(\underline{\eta})] \Bigr\} \,, \tag{2.6}$$

where F is a real continuous function on $\mathbb{N}^{\mathbf{G}_\Lambda}$.

In words, (2.6) says that when the current configuration of animals is $\underline{\eta}$, each animal γ appears at rate $M(\gamma|\underline{\eta})\, w^{\mathbf{J}}(\gamma)$ and disappears at rate 1 (if it is present). This is an interacting birth-and-death process of animals with quenched disorder. The factor $w^{\mathbf{J}}(\gamma)$ acts as the rate of an internal Poissonian clock marking possible birth instants. The actual birth takes place only if a further test, determined by $M(\gamma|\cdot)$, is passed. From the definition (2.3) of the interaction matrix we see that if $\mathrm{Int}(\gamma,\theta)$ takes the value 1 (resp. 0) the presence of θ may have (resp. does not have) an influence on the birth rate of γ.

Unit death-rates are no loss of generality. In general, our $w^{\mathbf{J}}(\gamma)$ stands for the ratio "birth rate / death rate" of γ. In fact, a similar treatment is possible for interacting-birth interacting-death processes. These are processes with a generator as in (2.6) but with a factor $M_{\mathrm{death}}(\gamma|\underline{\eta})\, w^{\mathbf{J}}_{\mathrm{death}}(\gamma)$ multiplying the last summand. Our theory can be easily adapted if the death rates $w^{\mathbf{J}}_{\mathrm{death}}(\gamma)$ are bounded below by a strictly positive number, uniformly in γ and $\underline{\eta}$.

In general, there may be forbidden cases for which the matrix M takes value 0. An extreme example of this is provided by deterministic interactions ($M = 0$ or 1), such as those defining fixed-routing loss networks and some statistical mechanical models (see examples in Section 2.2). In these cases, the configurations resulting from forbidden cases —that is those of the form $\underline{\xi} + \delta_\theta$ with $M(\theta|\underline{\xi}) = 0$— are *not* acceptable as initial configurations for the interacting process (thus, they will not be generated by it). The remaining configurations will be referred to as *acceptable configurations*.

Further notation The *support* of a function F on $\mathbb{N}^{\mathbf{G}}$ is the set

$$\mathrm{Supp}\,(F) \;=\; \Big\{x \in \mathbb{Z}^d : \exists\, \gamma \ni x\,,\, \underline{\eta} \in \mathbb{N}^{\mathbf{G}} \text{ such that } F(\underline{\eta}) \neq F(\underline{\eta} + \delta_\gamma)\Big\}\,. \tag{2.7}$$

In this paper we reserve the symbol $\mathbb{P}$ for the probability measure on the random environment, while the combination $\mathbb{Q}^{\mathbf{J}}$ (perhaps with further embellishments) will denote probability with respect to processes defined by a fixed environment $\mathbf{J}$. We shall use superindices for the time coordinate of space-time animal configurations: $\underline{\eta}^t_\Lambda \in \mathbb{N}^{\mathbf{G}_\Lambda} \times \mathbb{R}$. For finite sets A, the symbol $|A|$ will denote the cardinality of A. For animals, the notation $|\gamma|$ will mean $|V(\gamma)|$. We shall use a capital letter to denote a space-time point. As a default, we shall use the corresponding lowercase letter to denote its space component, and a subscript to identify its time component, ex. $X = (x, t_X) \in \mathbb{Z}^d \times \mathbb{R}$.

2.2 Examples

Some of the processes fitting our framework are the following.

Point processes

In these models the animals are subsets of $\mathbb{Z}^d$, hence $V(\gamma) = \gamma$. A large variety of point processes has been introduced in the literature. Our framework applies to their space-discretized version. The *area-interaction point processes* of Baddeley and van Lieshout (1995) involve identical animals defined by the translations of a fixed compact set $G \subset \mathbb{Z}^d$. An animal of the form $\gamma = x + G$ is called a *grain* of *germ* x. Animal configurations are labelled by the corresponding germs through the identification $\mathbf{G} \supset \{x + G : x \in A\} \longleftrightarrow A \subset \mathbb{Z}^d$. With this identification, the interaction function takes the form

$$M(x|A) \;=\; F\Big[\Big|(x+G) \cap (A \oplus G)\Big|\Big] \tag{2.8}$$

for some $F : \mathbb{N} \to [0,1]$, where $A \oplus G = \bigcup_{y \in A} \{y + G\}$. The process is attractive if F is increasing and repulsive otherwise. The former case corresponds to the *penetrable sphere model* introduced by Widom and Rowlison (1970).

In addition, the model is specified by internal-clock rates $w(x)$. In the non-random version they are usually independent of x. For the disordered version, the independence hypotheses (2.5) leads to *site* disorder:

$$\mathbf{J} \;=\; \{J_x : x \in \mathbb{Z}^d\} \tag{2.9}$$

for independent random variables $\{J_x\}$. The internal clock rates should be of the form

$$w^{\mathbf{J}}(x) \;=\; w(\{J_y : y \in x + G\}) \;. \tag{2.10}$$

As further examples of point processes within our framework we mention: (i) the *Strauss process* (Strauss, 1995), which does not involve grains and the interaction function depends on the number of pairs of points closer than a fixed threshold r, and (ii) the *perimeter-interaction process* where the interaction matrix depends on the overlapping of perimeters of grains. The latter is a particular instance of the generalization propossed by Baddeley, Kendall and van Lieshout (1996). Our results apply to the corresponding lattice versions with site disorder.

Fixed-routing loss networks

These models are defined on a graph $\mathcal{G}$ whose links define possible calls or connections. An example is $\mathbb{Z}^d$ with the usual nearest-neighbor links, or with a more general family of links $(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d$ with vertices x and y not necessarily at distance one. An animal γ is a connected finite subgraph possibly subjected to some further restriction, depending on the model (eg. not having a loop, or forming a closed circuit). The interaction matrix is often deterministic, preventing calls that would cause a link to be used beyond a predetermined maximal capacity. More generally, the interaction matrix embodies some penalization scheme for the multiple use of links.

The non-random rates $w(\gamma)$ usually decrease with the number of links in γ. A general type of disorder satisfying the independence condition (2.5) is *site-link* disorder:

$$\mathbf{J} \;=\; \{J_x : x \in V(\mathcal{G})\} \bigcup \{J_{x,y} : (x,y) \in L(\mathcal{G})\} \tag{2.11}$$

with all the random variables J_x, $J_{x,y}$ being independent. By $V(\mathcal{G})$ and $L(\mathcal{G})$ we denote, respectively, the set of vertices and of links of the graph $\mathcal{G}$. Condition (2.5) is satisfied by rates of the form

$$w^{\mathbf{J}}(\gamma) \;=\; w(\{J_x : x \in V(\gamma)\}\,,\, \{J_{x,y} : (x,y) \in L(\gamma)\})\;. \tag{2.12}$$

We emphasize that, due to our hypothesis (2.2), our disordered loss network must involve connections not exceeding a radius ℓ_1.

Models motivated by statistical mechanics

These are models of random geometrical objects coming from the study of spin systems in statistical mechanics. The associated birth-and-death processes have the statistical mechanical measure as invariant measure, and can be seen both as a convenient tool to study such measure and as a feasible simulation algorithm for it. We mention two of these models. Due to requirement (2.2), the disordered version considered here correspond to a "chopped" variant of the corresponding models.

Random cluster model This model plays an essential role in the study of the Potts model. We refer the reader to Grimmett (1995) for its detailed study as well as for references to the original articles. The animals, called *clusters*, are all finite subgraphs of a given graph $\mathcal{G}$ (whose vertices are usually the sites of $\mathbb{Z}^d$, but whose links may include non-nearest-neighbor pairs). The interaction matrix forbides the appearance of two clusters having a common vertix. Furthermore, the process has internal-clock rates of the form

$$w(\gamma) \;=\; \prod_{(x,y)\in L(\gamma)} \Bigl(\frac{J_{x,y}}{1-J_{x,y}}\Bigr) \prod_{x\in V(\gamma)} \Bigl(\frac{1}{J_x}\Bigr) \tag{2.13}$$

for appropiate parameters $J_{x,y}$ and J_x [often denoted $p(x,y)$ and $q(x)$]. A disordered version, satisfying the independence condition (2.5), is obtained by turning these parameters into independent random variables [site-link disorder (2.11)].

Ising contour model This is the model resulting from mapping into Peierls contours the typical spin configurations of one of the low-temperature pure phases of the Ising model. Contours are hypersurfaces formed by $(d-1)$-dimensional unit cubes centered at points of $\mathbb{Z}^d$ and perpendicular to the edges of the dual latice $\mathbb{Z}^d + (\frac{1}{2}, \cdots, \frac{1}{2})$. These cubes are called *plaquettes*. Two plaquettes are *adjacent* if they share a $(d-2)$-dimensional face. A contour is a finite family of plaquettes such that (i) the family can not be partitioned into two subfamilies with no adjacent plaquettes, and (ii) every $(d-2)$-dimensional face is covered by an even number of plaquettes of the family. Geometrically, a contour corresponds to a connected closed (hyper)surface. The set of vertices $V(\gamma)$ is the set of centers of the plaquettes forming γ.

The contour-model equilibrium measure is the invariant measure of a birth-and-death process with a deterministic interaction matrix which prevents the appearance of two contours with adjacent plaquettes. The clock rates take the form

$$w(\gamma) = e^{-\beta \sum_{x \in V(\gamma)} J(x)}, \tag{2.14}$$

where the parameter β is interpreted as inverse temperature and each $J(x)$ is a *coupling constant* associated to the plaquette centered at x. In the disordered version, these constants are independent random variables [site disorder (2.10)].

2.3 Main result

Theorem 2.15 *Consider an animal model as defined above, in particular satisfying (2.2), (2.4) and (2.5). Assume the disorder satisfies*

(i)

$$\aleph \;:=\; \mathbb{E}\Big[\ln^a\Big(1 + \sup_x \sum_{\gamma \ni x} w^{\mathbf{J}}(\gamma)\Big)\Big] \;<\; \infty \tag{2.16}$$

for some

$$a \;>\; 2d^2\Big(1 + \sqrt{1 + \frac{1}{d} + \frac{1}{2d}}\Big)\,. \tag{2.17}$$

Then there exists an $\varepsilon = \varepsilon(d, a, \aleph) > 0$, with a monotonically decreasing dependence $\aleph \to \varepsilon(d, a, \aleph)$, such that if there exists a function $S : \mathbf{G} \to [1, \infty)$ with

(ii)

$$\mathbb{E}\Big[\sup_{\gamma \in \mathbf{G}} \frac{1}{S(\gamma)} \sum_{\theta : \theta \not\sim \gamma} S(\theta)\, w^{\mathbf{J}}(\theta)\Big] \;\leq\; \varepsilon\,, \tag{2.18}$$

then there exist constants $m > 0$, $q > 0$ such that for $\mathbb{P}$-almost all configurations $\mathbf{J}$ the following is true. There exist constants $C_x^{\mathbf{J}} \geq 0$, $x \in \mathbb{Z}^d$, such that:

1. **Existence and uniqueness.** *For each $\Lambda \subset \mathbb{Z}^d$ there exists a unique time-invariant process $\overline{\mathbb{Q}}_\Lambda^{\mathbf{J}}$ on $\mathbb{N}^{\mathbf{G}_\Lambda} \times \mathbb{R}$ with generator (2.6). The process has a unique invariant measure $\mu_\Lambda^{\mathbf{J}}$.*

2. **Time convergence.** *For each $\Lambda \subset \mathbb{Z}^d$ and each acceptable configuration $\underline{\xi}_\Lambda \in \mathbb{N}^{\mathbf{G}_\Lambda}$, there exists a unique process $\overline{\mathbb{Q}}^{\mathbf{J}}(\,\cdot\,|\,\underline{\xi}_\Lambda)$ on $\mathbb{N}^{\mathbf{G}_\Lambda} \times \mathbb{R}_+$ defined by the generator (2.6) and the initial configuration $\underline{\xi}_\Lambda$. Furthermore, the process converges at a superpolynomial rate to the measure $\mu_\Lambda^{\mathbf{J}}$ on $\mathbb{N}^{\mathbf{G}_\Lambda}$. Explicitly, the following inequality holds for any function f on $\mathbb{N}^{\mathbf{G}_\Lambda}$:*

$$\sup_{\underline{\xi}_\Lambda} \Big| \mu_\Lambda^{\mathbf{J}} f - \overline{\mathbb{Q}}^{\mathbf{J}}(\, f(\underline{\eta}_\Lambda^t)\,|\,\underline{\xi}_\Lambda) \Big| \;\leq\; \|f\|_\infty \, C^{\mathbf{J}}(\mathrm{Supp}\,(f)) \,\exp\Big[-m\, \ln^q(1+t)\Big] \tag{2.19}$$

with $C^{\mathbf{J}}(\mathrm{Supp}\,(f)) = \sum_{x\in \mathrm{Supp}\,(f)} C_x^{\mathbf{J}}$. The supremum is taken over all acceptable configurations in $\mathbb{N}^{\mathbf{G}_\Lambda}$.

3. **Space convergence.** *As $\Lambda \to \mathbb{Z}^d$, $\mu_\Lambda^{\mathbf{J}}$ converges weakly to $\mu^{\mathbf{J}} := \mu_{\mathbb{Z}^d}^{\mathbf{J}}$ at an exponential rate. More precisely, for any function f depending only on animals contained in a finite set Λ:*

$$\Big| \mu^{\mathbf{J}} f - \mu_\Lambda^{\mathbf{J}} f \Big| \;\leq\; \|f\|_\infty \sum_{x\in \mathrm{Supp}\,(f)} C_x^{\mathbf{J}} \,\exp\Big[-m\,\mathrm{dist}(x,\Lambda^{\mathrm{c}})\Big]\,. \tag{2.20}$$

4. **Exponential mixing.** *For any functions f and g depending on animals contained in an arbitrary set $\Lambda \subset \mathbb{Z}^d$:*

$$\Big| \mu_\Lambda^{\mathbf{J}}(fg) - \mu_\Lambda^{\mathbf{J}} f\, \mu_\Lambda^{\mathbf{J}} g \Big| \;\leq\; \|f\|_\infty \, \|g\|_\infty \sum_{\substack{x\in \mathrm{Supp}\,(f),\\ y\in \mathrm{Supp}\,(g)}} C_x^{\mathbf{J}}\, C_y^{\mathbf{J}} \exp\Big[-m\,\|x-y\|/2\Big]\,. \tag{2.21}$$

5. **Perfect simulation.** *The invariant measure $\mu^{\mathbf{J}}$ (or any of the $\mu_\Lambda^{\mathbf{J}}$) can be perfectly simulated using the ancestors algorithm introduced by Fernández, Ferrari and Garcia (2002).*

These properties are similar to those of the deterministic process obtained by a fixed translation-invariant asignment of $\mathbf{J}$. The major difference is the possible slowing-down of the time convergence due to the subexponential time dependence in the right-hand side of (2.19).

We remark that while the size function S involved in hypothesis (2.18) is completely arbitrary, it is clear that it conveys some idea of the "mass" or "might" (Dobrushin's, 1996, terminology) of animals, in the sense that the larger $S(\gamma)$ the larger the set of animals incompatible with γ. In fact, the natural choice for the examples presented in Section 2.2 is the size of the relevant set of vertices of each animal (area of the grain, length of the call, perimeter of the contour, etc.). But, of course, "natural" is not a synonym of "optimal".

In most cases of interest, condition (2.18) is morally more limitant than (2.16). To formalize this fact, let us denote

$$\Upsilon^{\mathbf{J}} \;=\; \sup_x \sum_{\gamma \ni x} w^{\mathbf{J}}(\gamma) \tag{2.22}$$

and

$$\Psi^{\mathbf{J}} \;=\; \sup_{\gamma \in \mathbf{G}} \frac{1}{S(\gamma)} \sum_{\theta : \theta \not\sim \gamma} S(\theta)\, w^{\mathbf{J}}(\theta)\;, \tag{2.23}$$

and let us introduce the *halo* of an animal, namely the region of $\mathbb{Z}^d$ around it beyond which no incompatible is possible:

$$H(\gamma) \;:=\; \bigcap \Bigl\{ W \subset \mathbb{Z}^d : V(\theta) \subset W \Longrightarrow \theta \sim \gamma \Bigr\}^c\;. \tag{2.24}$$

This halo is a finite set because of assumption (2.4). In the examples of Section 2.2 $H(\gamma) = V(\gamma)$. In addition we denote

$$u_1 \;=\; \inf_{\gamma \in \mathbf{G} : \gamma \neq \emptyset} \frac{|H(\gamma)|}{S(\gamma)} \quad , \quad u_2 \;=\; \sup_{\gamma \in \mathbf{G}} \frac{|H(\gamma)|}{S(\gamma)} \tag{2.25}$$

and define

$$\Xi^{\mathbf{J}} \;=\; \sup_x \sum_{\theta \ni x} |H(\theta)|\; w^{\mathbf{J}}(\theta)\;. \tag{2.26}$$

In terms of these quantities we have the bounds

$$\Upsilon^{\mathbf{J}} \;\leq\; \Xi^{\mathbf{J}} \qquad \text{and} \qquad \Psi^{\mathbf{J}} \;\leq\; \frac{u_2}{u_1}\, \Xi^{\mathbf{J}} \tag{2.27}$$

which lead to the following corollary of our main result.

Corollary 2.28 *Consider an animal model as defined in Section 2.1 such that*

$$u_1 > 0 \quad \text{and} \quad u_2 < \infty\,. \tag{2.29}$$

Then, there exists $\widetilde{\varepsilon} > 0$ such that if $\mathbb{E}(\Xi^{\mathbf{J}}) \le \widetilde{\varepsilon}$ there exist constants $m, q > 0$ such that properties 1 to 5 of Theorem 2.15 hold for $\mathbb{P}$-almost all disorder configurations $\mathbf{J}$.

The conditions (2.29) are trivially satisfied by all the examples in Section 2.2, for which, in fact, $u_1 = u_2 = 1$. In our bounded-diameter setting, they are also satisfied if the function size is chosen translation invariant.

Proof. We first observe that for any $a \ge 1$ there exists C_a such that $\ln^a(1+x) \le C_a\, x$ for all $x \ge 0$. Therefore,

$$\mathbb{E}\Big[\ln^a(1+\Upsilon^{\mathbf{J}})\Big] \;\le\; C_a\,\mathbb{E}\Big[(\Upsilon^{\mathbf{J}})\Big]\,. \tag{2.30}$$

Pick an a as in (2.17) and take

$$\widetilde{\varepsilon} \;=\; \min\Big\{\frac{1}{C_a}\,,\,\frac{u_1}{u_2}\,\varepsilon(d,a,1)\Big\}\,, \tag{2.31}$$

where $\varepsilon(d,a,\aleph)$ is the function given in Theorem 2.15. From (2.30) and the leftmost inequality in (2.27) we obtain

$$\aleph \;=\; \mathbb{E}\Big[\ln^a(1+\Upsilon^{\mathbf{J}})\Big] \;\le\; C_a\,\widetilde{\varepsilon} \;\le\; 1\,. \tag{2.32}$$

From the rightmost inequality in (2.27) and the monotonicity of the function $\aleph \to \varepsilon(d,a,\aleph)$ we obtain

$$\mathbb{E}[\Psi^{\mathbf{J}}] \;\le\; \frac{u_2}{u_1}\,\widetilde{\varepsilon} \;\le\; \varepsilon(d,a,1) \;\le\; \varepsilon(d,a,\aleph)\,. \tag{2.33}$$

Thus, both hypothesis (2.16) and (2.18) of Theorem 2.15 are satisfied. $\square$

3 The basic construction and the key lemma

The proof of Theorem (2.15) relies on two ingredients: a graphical construction of birth-and-death processes propossed by Fernández, Ferrari and Garcia (1998, 2001), and a multiscale argument following Klein (1994). In this section we review the graphical construction and state the key result obtained from Klein's argument.

3.1 Graphical construction

The construction is performed for each *fixed* environment $\mathbf{J}$.

The free process We associate to each animal γ an independent marked Poisson process N_γ with rate $w^{\mathbf{J}}(\gamma)$. We call $T_k(\gamma)$, $\gamma \in \mathbf{G}$, the ordered time-events of N_γ with the convention that $T_0(\gamma) < 0 < T_1(\gamma)$. For each occurrence time $T_i(\gamma)$ of the process N_γ we choose independent marks $S_i(\gamma)$ exponentially distributed with mean 1 and $Z_i(\gamma)$ uniformly distributed in $[0,1]$. The *free animal process* is the process in which at each Poisson time-event $T_i(\gamma)$ (a copy of) the animal γ appears and lasts $S_i(\gamma)$ time units. It is convenient to identify each marked point $(\gamma, T_k(\gamma), S_k(\gamma), Z_k(\gamma))$ with $(\gamma \times [T_k(\gamma), T_k(\gamma) + S_k(\gamma)]\,,\, Z_k(\gamma))$, the *cylinder* with *basis* γ, *birth time* $T_k(\gamma)$ *lifetime* $S_k(\gamma)$ and *mark* $Z_k(\gamma)$. This identification turns the free process into a measure on the space of cylinders. Denoting $C = (\gamma, t, s, z)$, we use the notation

$$\mathrm{Basis}\,(C) = \gamma, \;\; \mathrm{Birth}\,(C) = t, \;\; \mathrm{Life}\,(C) = [t, t+s], \;\; \mathrm{Mark}\,(C) = z$$

The marks will be used later on to define the (interacting) birth-and-death process with generator (2.6). Let us denote $\mathbb{Q}^{\mathbf{J}}$ the probability measure on $\mathbb{Z}^d \times \mathbb{R}$ corresponding to the free process (it always exist, being a countable product of marked Poisson processes).

Backwards oriented percolation We first extend our definition of incompatibility to cylinders in the natural way: Two cylinders C and C' are incompatible if they have incompatible bases and they are simultaneously alive at some instant of time. That is, $C \not\sim C'$ if and only if $\mathrm{Basis}\,(C) \not\sim \mathrm{Basis}\,(C')$ and $\mathrm{Life}\,(C) \cap \mathrm{Life}\,(C') \neq \emptyset$; otherwise $C \sim C'$.

Let us now fix a family of cylinders $\underline{C}$ (for instance, obtained as a realization of the free process of the previous paragraph). We define the *ancestors* of a cylinder $C \in \underline{C}$ as the set

$$\mathbf{A}_1^C = \Big\{ C' \in \underline{C} : C' \not\sim C \text{ and } \mathrm{Birth}\,(C') \leq \mathrm{Birth}\,(C) \Big\}\,. \tag{3.1}$$

Recursively for $n \geq 1$, the nth generation of ancestors of C (in $\underline{C}$) is

$$\mathbf{A}_n^C = \Big\{ C'' : C'' \in \mathbf{A}_1^{C'} \text{ for some } C' \in \mathbf{A}_{n-1}^C \Big\}\,. \tag{3.2}$$

[In fact, $\mathbf{A}_n^C = \mathbf{A}_n^C(\underline{C})$, but we shall omit this dependence except if it is crucially needed.] Likewise, for an arbitrary space-time point $(x,t) \in \mathbb{Z}^d \times \mathbb{R}$ we define its set of ancestors as the set of cylinders that contain it

$$\mathbf{A}_1^{(x,t)} = \Big\{ C \in \underline{C} : \mathrm{Basis}\,(C) \ni x, \mathrm{Life}\,(C) \ni t \Big\}\,, \tag{3.3}$$

and, recursively,

$$\mathbf{A}_n^{(x,t)} = \Big\{ C'' : C'' \in \mathbf{A}_1^{C'} \text{ for some } C' \in \mathbf{A}_{n-1}^{(x,t)} \Big\} . \tag{3.4}$$

Definition 3.5 *We say that* there exists backwards oriented percolation *in* $\underline{C}$ *if there exists a space-time point* (x,t) *such that* $\mathbf{A}_n^{(x,t)} \neq \emptyset$ *for all* n*, that is, there exists a point with infinitely many generations of ancestors.*

The *clan* of a space-time point (x,t), resp. of a cylinder C, is the union of the corresponding ancestors:

$$\mathbf{A}^{(x,t)} = \bigcup_{n\geq 1} \mathbf{A}_n^{(x,t)} \qquad , \qquad \mathbf{A}^{C} = \bigcup_{n\geq 1} \mathbf{A}_n^{C} . \tag{3.6}$$

Other quantities that will be used later are the time-length, space-diameter and space-size of the clan of a point (x,t):

$$\mathrm{TL}\,(\mathbf{A}^{(x,t)}) \;=\; t - \inf\Big\{ s : \mathrm{Life}\,(C) \ni s, \text{ for some } C \in \mathbf{A}^{(x,t)} \Big\} \tag{3.7}$$

$$\mathrm{SD}\,(\mathbf{A}^{(x,t)}) \;=\; \mathrm{diam}\Big(\bigcup_{C\in\mathbf{A}^{(x,t)}} \mathrm{Basis}\,(C) \Big) \tag{3.8}$$

$$\mathrm{SS}\,(\mathbf{A}^{(x,t)}) \;=\; \Big| \bigcup_{C\in\mathbf{A}^{(x,t)}} \mathrm{Basis}\,(C) \Big| . \tag{3.9}$$

The interacting birth-and-death process If

$$\mathbb{Q}^{\mathbf{J}}(\{\text{no backwards percolation}\}) \;=\; 1 , \tag{3.10}$$

the stationary process with generator (2.6) can be constructed by "cleaning" the free process defined above. For completeness we present a summary of this construction here. The reader is referred to Fernández, Ferrari and Garcia (2001) for details. The idea is to start from first ancestors ("Eves") and classify cylinders into *kept* or *erased* according to the test determined by the interaction function. Cylinders that are born in presence of a kept ancestor and that fail the test are erased, all the others are kept.

Explicitly, let $\underline{C}$ be a cylinder configuration in the set

$$\mathbf{C} = \{\underline{C} \text{ without backwards percolation}\} . \tag{3.11}$$

Since all clans in $\underline{C}$ are finite, each cylinder has a well defined, finite number of ancestors. Therefore the configuration can be decompossed in the form $\underline{C} = \cup_{n\geq 0}\underline{C}_n$, where $\underline{C}_n :=$

$\{C \in \underline{C} : \mathbf{A}_n^C \neq \emptyset,\ \mathbf{A}_{n+1}^C = \emptyset\}$. The sets $\underline{K}$ and $\underline{D}$ of kept and erased cylinders are defined inductively as follows. Starting with $\underline{K}_0 = \underline{C}_0$ and $\underline{D}_0 = \emptyset$ (cylinders without ancestors are kept, as they do not need to pass any test), we define, recursively,

$$\begin{aligned} \underline{K}_n &= \Big\{C \in \underline{C}_n \setminus \cup_{i=0}^{n-1}(\underline{D}_i \cup \underline{K}_i) \ :\ \mathrm{Mark}\,(C) \leq \widetilde{M}(C| \cup_{i=0}^{n-1} \underline{K}_i)\Big\} \\ \underline{D}_n &= \underline{C}_n \setminus [\underline{K}_n \cup \cup_{i=0}^{n-1}(\underline{D}_i \cup \underline{K}_i)] \end{aligned} \tag{3.12}$$

where $\widetilde{M}(C|\underline{K}') = M(\mathrm{Basis}\,(C)|\{\mathrm{Basis}\,(C') : C' \in \underline{K}', \mathrm{Birth}\,(C) \in \mathrm{Life}\,(C')\})$. We denote the set of kept cylinders as $\underline{K}(\underline{C}) = \cup_n \underline{K}_n$ and the set of erased cylinders as $\underline{D}(\underline{C}) = \cup_n \underline{D}_n$. Clearly

(C1) $\underline{C}$ is the disjoint union of $\underline{K}(\underline{C})$ and $\underline{D}(\underline{C})$, and

(C2) The event $\{C \in \underline{K}(\underline{C})\}$ is measurable with respect to the sigma field generated by $\mathbf{A}^C$, in fact $\underline{K}(\mathbf{A}^C) = \underline{K}(\underline{C}) \cap \mathbf{A}^C$. In words, it is sufficient to know the (finite) clan of C to know whether C is kept or erased.

The stationary animal process $\overline{\mathbb{Q}}^{\mathbf{J}}$ is defined by the sections of the kept cylinders:

$$\eta^{\mathbf{J},t}(\gamma, \underline{C}) = \sum_{C \in \underline{K}(\underline{C})} \mathbf{1}\Big\{\mathrm{Basis}\,(C) = \gamma,\ \mathrm{Life}\,(C) \ni t\Big\}\,. \tag{3.13}$$

If $\mathbb{Q}^{\mathbf{J}}(\mathbf{C}) = 1$, the process $\overline{\mathbb{Q}}^{\mathbf{J}}$ is Markovian and has generator (2.6); that is,

$$\frac{d}{dt}\overline{\mathbb{E}}^{\mathbf{J}} f(\eta^t(\underline{C})) = \overline{\mathbb{E}}^{\mathbf{J}}\, A_\Lambda^{\mathbf{J}} f(\eta^t(\underline{C})) \tag{3.14}$$

This fact is proven in Fernández, Ferrari and Garcia (2001) for the homogeneous case. This proof extends to the inhomogeneous case in an obvious manner. Let us denote $\mu^{\mathbf{J}}$ the distribution of any t-section $\underline{\eta}^{\mathbf{J},t}$, which, by construction, is independent of t. We shall determine the properties of $\mu^{\mathbf{J}}$ by studying the law of $\eta^{\mathbf{J},0}$, the stationary (interacting) birth-and-death process at time zero.

As observed in (C2) above, the presence or absence of contours intersecting a region $V \subset \mathbb{Z}^d$ at time t depends only on the clans of the cylinders alive at time t whose bases intersect V, that is, on

$$\mathbf{A}^{V,t} := \Big\{C' \in \mathbf{A}^C \ :\ \mathrm{Basis}\,(C) \cap V \neq \emptyset,\ \mathrm{Life}\,(C) \ni t\Big\}\,. \tag{3.15}$$

In particular the function $\eta^{\mathbf{J},t}(\gamma,\,\cdot\,)$ defined by (3.13) is in fact a (deterministic) function only of $\mathbf{A}^{\gamma,t}$ $[=\mathbf{A}^{V(\gamma),t}]$. More precisely, if $\underline{C}$ and $\underline{C}'$ are two cylinder configurations such that $\mathbf{A}^{\gamma,t}(\underline{C})=\mathbf{A}^{\gamma,t}(\underline{C}')$, then $\eta^{\mathbf{J},t}(\gamma,\underline{C})=\eta^{\mathbf{J},t}(\gamma,\underline{C}')$. We code this fact as the identity (slightly abusive from the notational point of view):

$$\eta^{\mathbf{J},t}(\gamma,\underline{C})=\eta^{\mathbf{J},t}(\gamma,\mathbf{A}^{\gamma,t})\,. \tag{3.16}$$

3.2 The key lemma

Let us call a sequence of cylinders $C_1,C_2,\ldots,C_n$ an *open path* if $C_2\not\sim C_1,\ldots,C_n\not\sim C_{n-1}$ and $\text{Birth}\,(C_{i+1})<\text{Birth}\,(C_i)$ for all i. Given a cylinder configuration $\underline{C}$ and two space-time points $X=(x,t)$ and $Y=(y,s)$, with $s\le t$, we say that X and Y are *connected* (in the configuration $\underline{C}$) if there exists an open path $C_1,C_2,\ldots,C_n$ such that $x\in\text{Basis}\,(C_1)$, $t\in\text{Life}\,(C_1)$, $y\in\text{Basis}\,(C_n)$, $s\in\text{Life}\,(C_n)$. [Equivalently, Y is in (the interior of a cylinder belonging to) the clan of ancestors of X.] The existence of such a connection defines an event denoted $X\to Y$.

For a given realization of the environment, the *connectivity function* is defined by

$$G^{\mathbf{J}}(X,Y)=\mathbb{Q}^{\mathbf{J}}\{X\to Y\}\,. \tag{3.17}$$

Many of the properties stated in the main theorem are a direct consequence of the following result

Lemma 3.18 *Under the hypotheses of Theorem 2.15 [namely, (2.2), (2.4), (2.5) and (2.17)], but replacing (2.16) by an inequality*

$$\mathbb{E}\Big[\ln^{a}\Big(1+\sup_{x}\sum_{\gamma\ni x}w^{\mathbf{J}}(\gamma)\Big)\Big]\;\le\;\aleph\,, \tag{3.19}$$

there exist $m>0$ and $q_0(a,d)>1$ such that for all $q\in(1,q_0)$ there is a value $\varepsilon(d,a,m,q,\aleph)>0$ so that the validity of (2.18) implies that for every $x\in\mathbb{Z}^d$

$$G^{\mathbf{J}}\Big((x,t),(y,s)\Big)\;\le\;C_x^{\mathbf{J}}\,\exp\Big\{-m\max\Big[\|x-y\|\,,\,\ln^{q}(1+|t-s|)\Big]\Big\} \tag{3.20}$$

for all $y\in\mathbb{Z}^d$ and $t,s\in\mathbb{R}$, where the constants $C_x^{\mathbf{J}}=C_x^{\mathbf{J}}(\ell_1+\ell_2)$ are finite for $\mathbb{P}$-almost every environment $\mathbf{J}$.

The fact that the inequality (3.19) is used implies that the function $\aleph\to\varepsilon(\cdots,\aleph)$ can be chosen to be decreasing, as stated in Theorem 2.15.

We will prove this lemma by performing a multiscale analysis similar to the one used by Klein (1994) in his work on extintion of contact process in a random environment. That will be done in Section 5. We first discuss how Theorem 2.15 follows from the bound (3.20) for the connectivity function.

4 How the theorem follows from the key lemma

4.1 Relation with percolation properties

The following theorem relates properties of the measure $\mu^{\mathbf{J}}$ with properties of the percolation model.

Theorem 4.1 *Assume that for a given* $\mathbf{J}$ *there is no backwards oriented percolation with* $\mathbb{Q}^{\mathbf{J}}$*-probability one. Then,*

1. **Uniqueness.** *The measure* $\mu^{\mathbf{J}}$ *is the unique invariant measure for the process* $\eta^{\mathbf{J},t}$*.*

2. **Time convergence.** *For any function* f *with finite support,*

$$\sup_{\underline{\xi}_\Lambda} \left| \mu^{\mathbf{J}}_\Lambda f - \overline{\mathbb{E}}^{\mathbf{J}} \Big(f(\underline{\eta}^t_\Lambda) \;\Big|\; \underline{\xi}_\Lambda \Big) \right| \leq 2 \|f\|_\infty \sum_{x \in \mathrm{Supp}\,(f)} \Big[\mathbb{Q}^{\mathbf{J}} \Big\{ \mathrm{TL}\,(\mathbf{A}^{(x,0)}) > bt \Big\} + e^{-(1-b)t}\, \mathbb{E}^{\mathbf{J}} \Big(\mathrm{SS}\,(\mathbf{A}^{(x,0)}) \Big) \Big] \tag{4.2}$$

   for any $b \in (0,1)$*.*

3. **Space convergence.** *As* $\Lambda \to \mathbb{Z}^d$, $\mu^{\mathbf{J}}_\Lambda$ *converges weakly to* $\mu^{\mathbf{J}}$*. More precisely, if* $\mathrm{Supp}\,(f) \subset \Lambda$*, then*

$$\left| \mu^{\mathbf{J}} f - \mu^{\mathbf{J}}_\Lambda f \right| \leq 2 \|f\|_\infty \sum_{x \in \mathrm{Supp}\,(f)} \mathbb{Q}^{\mathbf{J}} \Big\{ \mathrm{SD}\,(\mathbf{A}^{(x,0)}) \geq \mathrm{dist}\,(\{x\}, \Lambda^c) \Big\} . \tag{4.3}$$

4. **Mixing.** *For* f *and* g *with finite support,*

$$\left| \mu^{\mathbf{J}}_\Lambda (fg) - \mu^{\mathbf{J}}_\Lambda f \, \mu^{\mathbf{J}}_\Lambda g \right| \leq 4 \|f\|_\infty \|g\|_\infty \times \sum_{\substack{x \in \mathrm{Supp}\,(f), \\ y \in \mathrm{Supp}\,(g)}} \Big[\mathbb{Q}^{\mathbf{J}} \Big\{ \mathrm{SD}\,(\mathbf{A}^{(x,0)}) \geq \|x-y\|/2 \Big\} + \mathbb{Q}^{\mathbf{J}} \Big\{ \mathrm{SD}\,(\mathbf{A}^{(y,0)}) \geq \|x-y\|/2 \Big\} \Big] . \tag{4.4}$$

Proof. Items 1, 2 and 3 follow from displays (4.6), (4.7) and (4.9) of Theorem 4.1 of Fernández, Ferrari and Garcia (2001). The analogous of item 4 is stated in that theorem with the extra assumption that there exists a time h such that there is no space-time percolation in $(0, h)$. We provide here a proof without this hypothesis.

We consider functions f and g such that $\mathrm{Supp}\,(f) = \{x\}$ and $\mathrm{Supp}\,(g) = \{y\}$, the general case follows by telescoping. Let's fix a partition $\{\Gamma, \Gamma'\}$ of $\mathbf{G}_\Lambda$. Below we choose Γ formed by animals "closer to x". Let $\underline{A}$ and $\underline{B}$ be $\mathbb{Q}^{\mathbf{J}}$-independent realizations of $\{C \in \mathbf{C} : \mathrm{Basis}\,(C) \in \Gamma\}$ and $\underline{A}'$ and $\underline{B}'$ independent realizations of $\{C \in \mathbf{C} : \mathrm{Basis}\,(C) \in \Gamma'\}$. Then $\underline{A} \cup \underline{A}'$, $\underline{A} \cup \underline{B}'$, $\underline{B} \cup \underline{A}'$, $\underline{B} \cup \underline{B}'$ have the same law as $\mathbf{C}_\Lambda$ and $\underline{A} \cup \underline{B}'$ is independent of $\underline{B} \cup \underline{A}'$. Let $\mathbf{A}^{(x,t)}(\underline{C})$ be the random variable defined in (3.6) and

$$X(\underline{A}, \underline{A}') \;=\; f\Big(\eta^{\mathbf{J},0}(\,\cdot\,, \underline{A} \cup \underline{A}')\Big) \;=\; f\Big(\eta^{\mathbf{J},0}(\,\cdot\,, \mathbf{A}^{(x,0)}(\underline{A} \cup \underline{A}'))\Big)\;.$$

Analogously

$$Y(\underline{A}, \underline{A}') \;=\; g\Big(\eta^{\mathbf{J},0}(\,\cdot\,, \underline{A} \cup \underline{A}')\Big) \;=\; g\Big(\eta^{\mathbf{J},0}(\,\cdot\,, \mathbf{A}^{(x,0)}(\underline{A} \cup \underline{A}'))\Big)\;.$$

With these definitions, we obtain

$$\mu^{\mathbf{J}}_\Lambda(fg) - \mu^{\mathbf{J}}_\Lambda f\,\mu^{\mathbf{J}}_\Lambda g \;=\; \widetilde{\mathbb{E}}\Big[X(\underline{A}, \underline{A}')\,Y(\underline{A}, \underline{A}') - X(\underline{A}, \underline{B}')\,Y(\underline{B}, \underline{A}')\Big]\;, \tag{4.5}$$

where $\widetilde{\mathbb{E}}$ corresponds to a four-fold product of the measure $\mathbb{Q}^{\mathbf{J}}$. This expression leads us to the bound

$$\Big|\mu^{\mathbf{J}}_\Lambda(fg) - \mu^{\mathbf{J}}_\Lambda f\,\mu^{\mathbf{J}}_\Lambda g\Big| \;\le\; 2\,\|f\|_\infty\,\|g\|_\infty\,\widetilde{\mathbb{P}}(\mathcal{A}^c) \tag{4.6}$$

with

$$\mathcal{A} = \Big\{X(\underline{A}, \underline{A}') = X(\underline{A}, \underline{B}') \text{ and } Y(\underline{A}, \underline{A}') = Y(\underline{B}, \underline{A}')\Big\}\;. \tag{4.7}$$

We now choose Γ as the set of animals intersecting $\{z \in \mathbb{Z}^d : \|z - x\| \le \|z - y\|\}$ and Γ' as its complement. Then the event $\mathcal{A}$ is verified whenever the bases of the cylinders in both $\mathbf{A}^{(x,0)}(\underline{A} \cup \underline{A}')$ and $\mathbf{A}^{(x,0)}(\underline{A} \cup \underline{B}')$ are contained in Γ and those of $\mathbf{A}^{(y,0)}(\underline{A} \cup \underline{A}')$ and $\mathbf{A}^{(y,0)}(\underline{B} \cup \underline{A}')$ are contained in Γ'. The complement of the intersection of these four events yields

$$\widetilde{\mathbb{P}}(\mathcal{A}^c) \;\le\; 2\,\mathbb{Q}^{\mathbf{J}}\Big\{\mathrm{SD}\,(\mathbf{A}^{(x,0)}) \ge \|x - y\|/2\Big\} \;+\; 2\,\mathbb{Q}^{\mathbf{J}}\Big\{\mathrm{SD}\,(\mathbf{A}^{(y,0)}) \ge \|x - y\|/2\Big\}\,. \;\square$$

For comparison purposes, let us present an alternative mixing bound.

Proposition 4.8 *Assume that for a given* $\mathbf{J}$ *there is no backwards oriented percolation with* $\mathbb{Q}^{\mathbf{J}}$*-probability one and consider functions* f *and* g *with finite support. Then,*

$$\begin{aligned}\Big|\mu^{\mathbf{J}}_\Lambda(fg) - \mu^{\mathbf{J}}_\Lambda f\, \mu^{\mathbf{J}}_\Lambda g\Big| \;&\leq\; 2\,\|f\|_\infty\,\|g\|_\infty \\ &\times \sum_{\substack{x\in\mathrm{Supp}\,(f),\\ y\in\mathrm{Supp}\,(g)}} \widehat{\mathbb{Q}}^{\mathbf{J}}\Big\{\mathrm{SD}\,(\mathbf{A}^{(x,0)}) + \mathrm{SD}\,(\mathbf{A}^{(y,0)}) \geq \|x-y\|\Big\}\;, \end{aligned} \tag{4.9}$$

where $\widehat{\mathbb{Q}}^{\mathbf{J}}$ *is the free process obtained doubling the birth rates of* $\mathbb{Q}^{\mathbf{J}}$.

The bound (4.9) corresponds to standard high-temperature results in statistical mechanics (see, for instance, the main result in Bricmont and Kupiainen, 1996). Its proof relies on the very popular technique of "duplication of variables". In contrast, (4.4) is proven by "tetra-plication of variables". In our general setting, however, we are able to exploit better our first bound (4.4).

Proof. As above, it is enough to assume $\mathrm{Supp}\,(f) = \{x\}$ and $\mathrm{Supp}\,(g) = \{y\}$. Let $\underline{C}$ and $\underline{C}'$ be $\mathbb{Q}^{\mathbf{J}}$-independent realizations of $\mathbf{C}_\Lambda$. We denote

$$X(\underline{C}) \;=\; f\Big(\eta^{\mathbf{J},0}(\,\cdot\,,\underline{C})\Big) \qquad\text{and}\qquad Y(\underline{C}) \;=\; g\Big(\eta^{\mathbf{J},0}(\,\cdot\,,\underline{C})\Big)\;.$$

The duplication-of-variables identity is:

$$\mu^{\mathbf{J}}_\Lambda(fg) - \mu^{\mathbf{J}}_\Lambda f\, \mu^{\mathbf{J}}_\Lambda g \;=\; \frac{1}{2}\,\widetilde{\mathbb{E}}\Big[\Big(X(\underline{C}) - X(\underline{C}')\Big)\Big(Y(\underline{C}) - Y(\underline{C}')\Big)\Big]\;, \tag{4.10}$$

where $\widetilde{\mathbb{E}}$ corresponds to the measure $\mathbb{Q}^{\mathbf{J}} \times \mathbb{Q}^{\mathbf{J}}$. Let us now consider the event

$$\mathcal{B} \;=\; \Big\{\mathrm{Basis}\,\Big(\mathbf{A}^{(x,0)}(\underline{C}\cup\underline{C}')\Big) \bigcap \mathrm{Basis}\,\Big(\mathbf{A}^{(y,0)}(\underline{C}\cup\underline{C}')\Big) \;=\; \emptyset\Big\}\;, \tag{4.11}$$

and the transformation $T : (\underline{C},\underline{C}') \to (T\underline{C}, T\underline{C}')$, that interchanges the $(x,0)$-ancestors in $\underline{C}$ and $\underline{C}'$:

$$\begin{aligned} T\underline{C} \;&=\; \Big(\underline{C} \setminus \mathbf{A}^{(x,0)}(\underline{C})\Big) \cup \mathbf{A}^{(x,0)}(\underline{C}') \\ T\underline{C}' \;&=\; \Big(\underline{C}' \setminus \mathbf{A}^{(x,0)}(\underline{C}')\Big) \cup \mathbf{A}^{(x,0)}(\underline{C})\;. \end{aligned} \tag{4.12}$$

Conditioned to $\mathcal{B}$ being true, the distribution of $(\underline{C},\underline{C}')$ coincides with that of $(T\underline{C}, T\underline{C}')$ (the processes inside and outside each realization of $\mathrm{Basis}\,\Big(\mathbf{A}^{(x,0)}(\underline{C}\cup\underline{C}')\Big)$ are independent). Furthermore, the event $\mathcal{B}$ is T-invariant. Hence,

$$\widetilde{\mathbb{E}}\Big[F(\underline{C},\underline{C}')\,\mathbf{1}\{\mathcal{B}\}\Big] \;=\; \widetilde{\mathbb{E}}\Big[F(T\underline{C},T\underline{C}')\,\mathbf{1}\{\mathcal{B}\}\Big] \tag{4.13}$$

for each local function F on $\mathbf{G}_\Lambda \times \mathbf{G}_\Lambda$. But, in the presence of $\mathcal{B}$, the function F involved in (4.10) is odd under this transformation. We conclude that

$$\widetilde{\mathbb{E}}\Big[\Big(X(\underline{C}) - X(\underline{C}')\Big)\Big(Y(\underline{C}) - Y(\underline{C}')\Big)\,\mathbf{1}\{\mathcal{B}\}\Big] \;=\; 0\;, \tag{4.14}$$

which, by (4.10), implies

$$\Big|\mu_\Lambda^{\mathbf{J}}(fg) - \mu_\Lambda^{\mathbf{J}} f\, \mu_\Lambda^{\mathbf{J}} g\Big| \;\leq\; 2\,\|f\|_\infty\,\|g\|_\infty\,\widetilde{\mathbb{P}}(\mathcal{B}^c)\;. \tag{4.15}$$

The proof follows from the observation that

$$\widetilde{\mathbb{P}}(\mathcal{B}^c) \;\leq\; \widehat{\mathbb{Q}}^{\mathbf{J}}\Big\{\mathrm{SD}\,(\mathbf{A}^{(x,0)}) + \mathrm{SD}\,(\mathbf{A}^{(y,0)}) \geq \|x-y\|\Big\}\;. \quad \square$$

4.2 Bounds for the size of the clan

The key inequality (3.20) leads to the following bounds for the probabilities of the time-length and space diameter and size of the clan of a space-time point.

Proposition 4.16 *Under the hypotheses of Theorem 2.15 there exist $\overline{m}, \widetilde{m} > 0$ and constants $\overline{C}_x^{\mathbf{J}}$, $\widetilde{C}_x^{\mathbf{J}}$, and $\widehat{C}_x^{\mathbf{J}}$, $x \in \mathbb{Z}^d$, which are finite for $\mathbb{P}$-almost every environment, such that*

$$\mathbb{Q}^{\mathbf{J}}\Big\{\mathrm{TL}\,(\mathbf{A}^{(x,0)}) > T\Big\} \;\;\leq\;\; \overline{C}_x^{\mathbf{J}}\,\exp\Big\{-\overline{m}\,\ln^q(1+T)\Big\} \tag{4.17}$$

$$\mathbb{Q}^{\mathbf{J}}\Big\{\mathrm{SD}\,(\mathbf{A}^{(x,0)}) > L\Big\} \;\;\leq\;\; \widetilde{C}_x^{\mathbf{J}}\,\exp\{-\widetilde{m}L\} \tag{4.18}$$

$$\mathbb{E}^{\mathbf{J}}\Big\{\mathrm{SS}\,(\mathbf{A}^{(x,0)})\Big\} \;\;\leq\;\; \widehat{C}_x^{\mathbf{J}}\;. \tag{4.19}$$

Proof. As $\max(a,b) \geq (a+b)/2$, inequality (3.20) leads to

$$G^{\mathbf{J}}\Big((x,t),(y,s)\Big) \;\leq\; C_x^{\mathbf{J}}\,\exp\Big\{-\frac{m}{2}\Big[\|x-y\| + \ln^q(1+|t-s|)\Big]\Big\}\;. \tag{4.20}$$

Inequality (4.17) is then a straightforward consequence of the fact that

$$\mathbb{Q}^{\mathbf{J}}\Big\{\mathrm{TL}\,(\mathbf{A}^{(x,0)}) > T\Big\} \;\leq\; \sum_y G^{\mathbf{J}}\Big((x,0) \to (y,-T)\Big)\;, \tag{4.21}$$

while (4.18) follows from

$$\mathbb{Q}^{\mathbf{J}}\Big\{\mathrm{SD}\,(\mathbf{A}^{(x,0)}) > L\Big\} \;\leq\; \sum_{y:\|x-y\|\geq L} \int_0^\infty G^{\mathbf{J}}\Big((x,0) \to (y,-t)\Big) dt\;. \tag{4.22}$$

To obtain (4.19) we use (4.18) to bound the right-hand side of the inequality

$$\mathbb{Q}^{\mathbf{J}}\Big\{\mathrm{SS}\,(\mathbf{A}^{(x,0)}) > L\Big\} \;\leq\; \mathbb{Q}^{\mathbf{J}}\Big\{\mathrm{SD}\,(\mathbf{A}^{(x,0)}) > L^{1/d}\Big\} \tag{4.23}$$

and sum over L. $\square$

4.3 Proof of Theorem 2.15

The bound (3.20) provided by the key Lemma 3.18 implies, by Borel-Cantelli, the absence of backwards oriented percolation. Thus, we can apply Theorem 4.1 for almost-all realizations of the disorder.

1. **Existence and uniqueness.** Under the hypothesis of no percolation in any region Λ, the process (3.13) defines a stationary process η^t in Λ by considering those cylinders in $\mathbf{C}$ with basis contained in Λ. Uniqueness follows from the next item.

2. **Time convergence.** We use (4.17) and (4.19) to bound the right hand side of (4.2) by

$$\begin{aligned} 2\,\|f\|_\infty \sum_{x\in\mathrm{Supp}\,(f)} \Big(\overline{C}_x^{\mathbf{J}} \exp\Big\{-\overline{m}\,\ln^q(1+bt)\Big\} + \widehat{C}_x^{\mathbf{J}}\, e^{-(1-b)t}\Big) \\ \leq\; 2\,\|f\|_\infty \sum_{x\in\mathrm{Supp}\,(f)} C_x^{\mathbf{J}}\, \exp\{-m\ln^q(1+bt)\}, \end{aligned} \tag{4.24}$$

   for some constant $C_x^{\mathbf{J}} < \infty$.

3. **Space convergence.** Inequality (2.20) follows immediately from (4.3) and (4.18).

4. **Exponential mixing.** The bound (2.21) is just the combination of (4.4) and (4.18).

5. **Perfect simulation.** The construction proposed in Fernández, Ferrari and Garcia (2001) works, for almost all disorder realizations, due to the absence of backwards oriented percolation.

5 Proof of Lemma 3.18

The lemma is proven by adapting Klein's (1994) multiscale analysis. The analysis involves several steps that will be studied separately.

5.1 General scheme and notation

The scheme is based on a sequence of linear sizes, called *scales*, defining boxes of increasing size. Sites are classified on *regular* or *singular* according to the decay of connectivity functions on a surrounding box. The definitions must be tunned up so that the probability for a site to be regular increase sufficiently fast with the scale. The presentation of this scheme is organized as follows:

(i) In Section 5.2 we recopilate a number of inequalities needed to prove next inductive step. In particular, the Hammersley-Lieb-Simon inequality is the key to pass from local to global decay of the connectivity function. The uniform bound on the size of the animals is needed to ensure its validity.

(ii) In Section 5.3 we define regularity and determine "good enough" probabilities for sites to be regular at each scale. Regularity at all scales with such probabilities guarantee, by Borel Cantelli, almost sure regularity.

(iii) Section 5.4 shows the crucial inductive step: A "good enough" probability of being regular at a given scale implies a "good enough" probability of being regular at the next scale. This is the heart of the argument.

(iv) The last step of the proof is the determination of conditions so the origin has a "good enough" probability of being regular at some initial scale. This is done in Section 5.5.

We introduce some notation. We fix $\ell_0 = \ell_1 + \ell_2$, the maximal size of the animals plus the radius of incompatibility – to avoid trivialities we assume $\ell_0 > d + 1$. For $L > 0$ and $x \in \mathbb{Z}^d$, we denote

$$\Lambda[x; L] \;=\; \Big\{ y \in \mathbb{Z}^d : \|x - y\| \leq L \Big\} \tag{5.1}$$

and define the ℓ_0-boundary of $\Lambda[x; L]$ as the set

$$\partial_{\ell_0} \Lambda[x; L] \;=\; \Lambda[x; L + \delta] \setminus \Lambda[x; L], \tag{5.2}$$

where $\delta > 1$ is chosen so the following is true: For any sequence of animals $\gamma_1, \gamma_2, \ldots, \gamma_n$ with $\gamma_i \not\sim \gamma_{i+1}$, $i = 1, 2, \ldots, n-1$ connecting a point inside $\Lambda[x; L]$ with a point outside $\Lambda[x; L+\delta]$, at least two animals in the sequence are contained in $\partial_{\ell_0}\Lambda[x; L]$. This condition is necessary to satisfy the Hammersley-Lieb-Simon (see next). A possible choice is $\delta = \delta(\ell_0) = \frac{3(\ell_0-2)}{2(d-1)}$.

For $X = (x,t) \in \mathbb{Z}^d \times \mathbb{R}$, $L > 0$ and $T > 0$ we set

$$B_{L,T}(X) \;=\; \Lambda[x; L] \times [t - T, t] \; . \tag{5.3}$$

The vertical, horizontal and complete boundaries of the box $B_{L,T}(X)$ are defined respectively as:

$$\begin{aligned} \partial_V B_{L,T}(X) &= \partial_{\ell_0}\Lambda[x; L] \times [t - T, t], & (5.4)\\ \partial_H B_{L,T}(X) &= \Lambda[x; L + \delta] \times \{t - T\}, & (5.5)\\ \partial B_{L,T}(X) &= \partial_V B_{L,T}(X) \cup \partial_H B_{L,T}(X). & (5.6) \end{aligned}$$

Note that, as we are considering backwards oriented percolation, the face $\Lambda[x; L+\delta] \times \{t\}$ is excluded from the boundary.

For any integrable function $H : \mathbb{Z}^d \times \mathbb{R} \to \mathbb{R}$ we denote

$$\sum_{Z \in \partial B_{L,T}(X)} H(Z) = \sum_{z \in \Lambda[x; L+\delta]} H(z, t - T) + \sum_{z \in \partial_{\ell_0}\Lambda[x;L]} \int_{t-T}^{t} H(z, s) ds. \tag{5.7}$$

We introduce the notion of connection *within a box*. Given a cylinder configuration $\underline{C}$, its *restriction* to a box $B = \Lambda \times I$, with $\Lambda \subset \mathbb{Z}^d$ and I a real interval, is the family $\underline{C}_B$ of cylinders obtained by "restricting" to I those cylinders of $\underline{C}$ with bases inside Λ. That is, each $C = (\gamma, t, s, z) \in \underline{C}$ with $\gamma \in \mathbf{G}_\Lambda$ defines a cylinder $C_B = (\gamma, t_I, s_I, z) \in \underline{C}_B$ with $t_I = \max(t, \inf I)$ and $s_I = \min(t + s, \sup I) - t_I$. Given two space-time points, $X = (x,t)$ and $Y = (y,s)$, $X, Y \in B$ with $s \le t$, the event $X \to_B Y$ is formed by all cylinder configurations $\underline{C}$ such that the configuration $\underline{C}_B$ exhibits an open path connecting X with Y. If this event is true, we say that X and Y are connected *in* B. The event defines the connectivity function in the region B:

$$G^{\mathbf{J}}_B(X, Y) = \mathbb{Q}^{\mathbf{J}}\{X \to_B Y\} \; . \tag{5.8}$$

As usual, we shall omit the subscript if $B = \mathbb{Z}^d \times \mathbb{R}$. We write

$$G^{\mathbf{J}}_{B_{L+\delta,T}(X)}(X, \partial) \;=\; \sum_{Z \in \partial B_{L,T}(X)} G^{\mathbf{J}}_{B_{L+\delta,T}(X)}(X, Z) \; . \tag{5.9}$$

5.2 A toolbox: Inequalities

The first two inequalities we need have been basically proven in the literature. The minor adaptations needed for our setting do not justify a detailed exposition of their proofs. We content ourselves with providing appropriate references and indications.

We consider the natural partial order in the space of cylinder configurations: $\underline{C} \leq \underline{C}'$ if $\underline{C}'$ contains all the cylinders in $\underline{C}$. Events are said *increasing*, resp. *decreasing*, if their characteristic functions are nondecreasing, resp. nonincreasing with respect to this partial order.

Proposition 5.10 (Harris-Fortuin-Kasteleyn-Ginibre inequality) *If $\mathcal{A}$ and $\mathcal{B}$ are both increasing or both decreasing events,*

$$\mathbb{Q}^{\mathbf{J}}(\mathcal{A} \cap \mathcal{B}) \ \geq \ \mathbb{Q}^{\mathbf{J}}(\mathcal{A})\, \mathbb{Q}^{\mathbf{J}}(\mathcal{B}) \,. \tag{5.11}$$

In the context of continuous-time percolation, an inequality of this sort was first proven by Bezuidenhout and Grimmett (1991). They did so by showing that the process is the weak limit of discrete-time processes that satisfy the corresponding inequality. Their approach provides at the same time a proof of the van den Berg-Kesten inequality, but has the inconvenience of imposing an additional topological condition on events, namely that their boundary —in some suitable metric topology— have measure zero. While such requirement is satisfied for the events of interest to us, a different approach, based on the martingale convergence of expectations of the relevant events, yields the result as stated above. A proof along this line is provided in the book by Meester and Roy (1996) (see their Theorem 2.2) for the percolation of spheres of random radii on a continuous space (Poisson Boolean model). The proof is easily adaptable to our setting of percolating cylinders.

The second inequality refers to increasing events happening in a disjoint manner. For brevity, we state it only for the type of events needed in the sequel. Let us consider a box $B = \Lambda \times I$, with $\Lambda \subset \mathbb{Z}^d$ and I a closed finite interval in $\mathbb{R}$, and space-time regions $B_1, \ldots, B_n \subset B$. We denote $\{B_1 \to_B B_2\} \circ \cdots \circ \{B_{n-1} \to_B B_n\}$ the event of having $n-1$ open paths, respectively connecting in B some point in B_i to some point in B_{i+1}, $1 \leq i \leq n-1$, such that no two paths share the same cylinder.

Proposition 5.12 (particular case of the van den Berg-Kesten inequality) *For B, $B_1, \ldots, B_n$ as above,*

$$\mathbb{Q}^{\mathbf{J}}\Big\{\{B_1 \to_B B_2\} \circ \cdots \circ \{B_{n-1} \to_B B_n\}\Big\} \ \leq \ \mathbb{Q}^{\mathbf{J}}\Big\{B_1 \to_B B_2\Big\} \cdots \mathbb{Q}^{\mathbf{J}}\Big\{B_{n-1} \to_B B_n\Big\} \,. \tag{5.13}$$

This inequality is a consequence of the more general inequality proven by van den Berg (1996) for increasing events of marked Poisson processes. The only subtlety is that van den Berg's result requires the events to depend on Poisson clocks ringing within a *bounded* region of $\mathbb{R}^{d+1}$. The connectivity events in (5.13) do not seem to fit in this framework because they may be determined also by cylinders born in an arbitrarily remote past. To obtain (5.13) we must, therefore, apply van den Berg's result to the marked process in B obtained by adding, independently, an invariant initial distribution of cylinders. This corresponds to a further, independent, spatial Poisson marked process on $\Lambda \times \{\inf I\}$ with rates $\{w^{\mathbf{J}}(\gamma) : \gamma \in \mathbf{G}_\Lambda\}$. Less direct proofs are also possible either by adapting the Bezuidenhout and Grimmett (1991) approach mentioned above, or the proof of Theorem 2.3 of Meester and Roy (1996).

Our last inequality is obtained in Klein (1994) as a corollary of van den Berg's inequality. In our oriented setting, we can present a totally different argument.

Proposition 5.14 (Hammersley-Lieb-Simon inequality) *Let $L, T \in \mathbb{R}_+$, δ as defined below (5.2) and $W \subset \mathbb{Z}^d \times \mathbb{R}$. Then, for every $X = (x, t_X) \in W$ and $Y = (y, t_Y) \in W \setminus B_{L+\delta,T}(X)$, with $t_Y \leq t_X$, we have*

$$G^{\mathbf{J}}_W(X,Y) \;\leq \sum_{Z \,\in\, \partial B_{L,T}(X) \,\cap\, W} G^{\mathbf{J}}_{B_{L+\delta,T}(X)}(X,Z)\; G^{\mathbf{J}}_W(Z,Y_Z) \tag{5.15}$$

with $Y_Z = (y, \min(t_Y, t_Z))$.

Proof. The enclosing character of $B_{L+\delta,T}(X)$ and the choice of δ, implies that every connection from X to Y includes a connection within $B_{L+\delta,T}(X)$ to a point $Z \in \partial B_{L,T}(X) \cap W$. It is here where the assumption of bounded animal sizes is critically needed. This point Z is subsequently joined by an open path to a final cylinder whose section contains y. If $t_Y \leq t_Z$, this open path determines the event $Z \to_W Y$ (recall that only connections that are backwards in time are considered). If on the contrary $t_Z \leq t_Y$, then this final cylinder must have been born before t_Z and stood alive at least up to t_Y. In particular, it contains the point (y, t_Z). Both situations are summed up in the inequality

$$G^{\mathbf{J}}_W(X,Y) \;\leq \sum_{Z \,\in\, \partial B_{L,T}(X) \,\cap\, W} \mathbb{Q}^{\mathbf{J}}\Big\{X \to_{B_{L+\delta,T}(X)} Z\,;\, Z \to_W Y_Z\Big\}\,. \tag{5.16}$$

(The sum in the right-hand side involves, in fact, a time integral. Its justification requires, therefore, a limit of conectivities involving time-discretizations of $\partial_V B_{L,T}(X)$. These connectivities are continuous functions, so the limit converges to the integral. We omit the details.)

The desired inequality (5.15) follows from (5.16) and the fact that the events $X \to_{B_{L+\delta,T}(X)} Z$ and $Z \to_W Y_Z$ are *independent*. Indeed, they are determined, respectively, by which cylinders are alive for $t < t_Z$ and which are alive for $t > t_Z$. Such events are independent due of the exponential character of cylinder lives. Perhaps the simpler way to see this is by resorting to an alternative construction of the cylinder process, where for each animal γ the birth and death events are generated independently with respective exponential rates $w^{\mathbf{J}}(\gamma)$ and 1. Cylinders are born at a birth event and live up to the next death event, neglecting intermediate events of the wrong type. With this construction, connectivity events before and after a given time t_Z are determined by different, hence independent, birth and death events. $\square$

Corollary 5.17 *Let $L, T \in \mathbb{R}_+$, δ as defined below (5.2) and $W \subset \mathbb{Z}^d \times \mathbb{R}$. Then, for every $X = (x, t_X)$ and $Y = (y, t_Y)$ in W with $t_Y \le t_X$, we have that,*

$$\begin{aligned} G^{\mathbf{J}}_W(X,Y) \;\le\; & \sum_{Z_1 \,\in\, \partial B_{L,T}(X) \cap W} G^{\mathbf{J}}_{B_{L+\delta,T}(X)}(X, Z_1) \sum_{Z_2 \,\in\, \partial B_{L,T}(Z_1) \cap W} G^{\mathbf{J}}_{B_{L+\delta,T}(Z_1)}(Z_1, Z_2) \cdots \\ & \cdots \sum_{Z_N \,\in\, \partial B_{L,T}(Z_{N-1}) \cap W} G^{\mathbf{J}}_{B_{L+\delta,T}(Z_{N-1})}(Z_{N-1}, Z_N)\, G^{\mathbf{J}}_W(Z_N, Y_{Z_N}) \end{aligned} \tag{5.18}$$

for every

$$N \;\le\; \text{integer part of } \max\left(\frac{\|x-y\|}{L+\delta}, \frac{|t_X - t_Y|}{T}\right) . \tag{5.19}$$

Proof. This is just an iteration of (5.15). The number of times such iteration can be performed is at least equal to the right-hand side in (5.19). $\square$

5.3 Regularity and "good enough" probabilities

In this section we introduce the main notions defining the multiscale approach.

¿From now on we consider boxes as in (5.3) where the temporal height T is an increasing function of L. In Section 5.4 the function $T(L)$ will be eventually chosen as a stretched exponential, but the following results do not depend on such a particular choice. To simplify the notation we characterize the boxes by its spacial lenght L and denote

$$B_L(X) = B_{L+\delta,T(L)}(X). \tag{5.20}$$

Following Klein (1994) we separate $\mathbb{Z}^d$ in regular and singular regions for a fixed realization of the environment $\mathbf{J}$.

Definition 5.21 *Let $m > 0$ and $L > 1$. A site $x \in \mathbb{Z}^d$ is said to be $(m, L)-$***regular** *if*

$$G^{\mathbf{J}}_{B_L(x,0)}\Big((x, 0)\,,\,\partial\Big) \;\leq\; e^{-m(L+\delta)}. \tag{5.22}$$

*Otherwise x is called $(m, L)-$***singular**. *A set $\Lambda \subset \mathbb{Z}^d$ is said $(m, L)-$regular if every $x \in \Lambda$ is $(m, L)-$regular; otherwise it is $(m, L)-$singular.*

Regularity will be used in conjunction with the Hammersley-Lieb-Simon inequality through the following crucial result.

Lemma 5.23 *Let Λ be a (m, L)-regular region, $W \supset \Lambda \times \mathbb{R}$. Then, for every $X = (x, t_X)$ and $Y = (y, t_Y)$ in W,*

$$G^{\mathbf{J}}_W(X, Y) \;\leq\; \exp\Big\{-m\,(L + \delta)\,N_\Lambda(X, Y)\Big\} \tag{5.24}$$

with

$$N_\Lambda(X, Y) \;=\; \text{integer part of } \min\left\{\frac{\operatorname{dist}(x, \Lambda^c)}{L + \delta}\;,\;\max\left[\frac{\|x - y\|}{L + \delta}\,,\,\frac{|t_X - t_Y|}{T}\right]\right\}. \tag{5.25}$$

Proof. This follows from Corollary 5.17. The value of $N_\Lambda(X, Y)$ satisfies the constraint (5.19) and guarantees that all the intermediate sites z_i are in Λ and, hence, they are regular. We then bound the right-hand side of (5.18) starting from the right: $G^{\mathbf{J}}_W(Z_N, Y_{Z_N})$ is bounded by one, and each of the preceding sums by $\exp\{-m\,(L + \delta)\}$. $\square$

We also formalize the notion of *scale*

Definition 5.26 *A* **scaled sequence** *is a triple (L_0, α, T) where $L_0, \alpha > 1$ and $T : (0, \infty) \to (0, \infty)$ is a function that grows faster than any power. Each such triple defines an increasing sequence of sizes $L_{k+1} = L_k^\alpha$ for $k = 0, 1, 2, \ldots$. The length L_k is the k-th scale of the sequence.*

Finally, we associate "good probabilities" to scales.

Definition 5.27 *A scaled sequence (L_0, α, T) has* m_∞**-good-enough probabilities** *if there exists $p > \alpha d$ such that*

$$\mathbb{P}\Big\{x \text{ is } (m_\infty, L_k) - \text{regular}\,\Big\} \;\geq\; 1 - \frac{1}{L_k^p} \tag{5.28}$$

for all $k = 0, 1, 2, \ldots$, for all $x \in \mathbb{Z}^d$.

We end this section with the proof of the "easy part" of the multiscale argument.

Theorem 5.29 *If a scaled sequence (L_0, α, T) has m_∞-good-enough probabilities, then, for any $m \in (0, m_\infty)$ there exist constants $\{C_x^{\mathbf{J}}(m) : x \in \mathbb{Z}^d\}$ with $\mathbb{P}\{C_x^{\mathbf{J}}(m) < \infty : x \in \mathbb{Z}^d\} = 1$, such that*

$$G^{\mathbf{J}}(X,Y) \;\le\; C_x^{\mathbf{J}}(m)\,\exp\Big\{-m\max\Big[\|x-y\|\;,\;T^{-1}(t_X-t_Y)\Big]\Big\} \tag{5.30}$$

for all $X, Y \in \mathbb{Z}^d \times \mathbb{R}$ with $t_X \ge t_Y \in \mathbb{R}$.

Proof. We follow Klein (1994). Take $b > 1$ (to be determined later) and consider

$$\mathcal{S}_k \;:=\; \Big\{\Lambda[x; b(L_{k+1}+\delta)] \text{ is not a } (m_\infty, L_k) - \text{regular region}\Big\}. \tag{5.31}$$

This event is verified if at least one of the sites in $\Lambda[x; b(L_{k+1}+\delta)]$ is (m_∞, L_k)-singular. As, by hypothesis, the probability for a given site to be singular is at most L_k^{-p}, we obtain

$$\begin{aligned}\mathbb{P}\{\mathcal{S}_k\} \;&\le\; \sum_{y\in\Lambda[x;b(L_{k+1}+\delta)]} \mathbb{P}\Big\{y \text{ is } (m_\infty, L_k) - \text{singular }\Big\}\\ &\le\; \frac{[2b(L_{k+1}+\delta)]^d}{L_k^p} \;\le\; \frac{(4bL_k^\alpha)^d}{L_k^p} \;=\; \frac{(4b)^d}{L_0^{\alpha^k(p-\alpha d)}}\,.\end{aligned}$$

As $p > \alpha d$, this bound shows that the probabilities $\mathbb{P}\{\mathcal{S}_k\}$ are summable in k. Therefore, by Borel-Cantelli, with probability 1 there exists $k_1 = k_1(x, \ell_0, b, \mathbf{J}) < \infty$, such that $\Lambda[x; b(L_{k+1}+\delta)]$ is (m_∞, L_k)- regular region for all $k \ge k_1$.

Now fix X, and classify the sites Y into regions

$$\overline{\mathcal{R}} \;=\; \Big\{Y :\; \max\Big[\|x-y\|\;,\;T^{-1}(t_X-t_Y)\Big] \;<\; b(L_{k_1}+\delta)\Big\} \tag{5.32}$$

and

$$\mathcal{R}_k \;=\; \Big\{Y :\; b(L_k+\delta) \;\le\; \max\Big[\|x-y\|\;,\;T^{-1}(t_X-t_Y)\Big] \;<\; b(L_{k+1}+\delta)\Big\} \tag{5.33}$$

for $k \ge k_1$.

Let us first consider $Y \in \mathcal{R}_k$ for some $k \ge k_1$. In this case we have $\|x-y\| < b(L_{k+1}+\delta)$ and so $y \in \Lambda[x; b(L_{k+1}+\delta)]$ which is a (m_∞, L_k)-regular region. It follows, from Lemma 5.23 (for $W = \mathbb{Z}^d \times \mathbb{R}$ and $\Lambda = \Lambda[x; b(L_{k+1}+\delta)]$), that

$$G^{\mathbf{J}}(X,Y) \le e^{-m_\infty(L_k+\delta)\,N} \tag{5.34}$$

with

$$N \;=\; \text{integer part of } \min\left\{\frac{b(L_{k+1}+\delta)}{L_k+\delta}, \max\left\{\frac{\|x-y\|}{L_k+\delta}, \frac{t_X-t_Y}{T(L_k)}\right\}\right\}. \tag{5.35}$$

A first bound of (5.34) comes from the observation that, as $Y \in \mathcal{R}_k$ implies $\|x-y\| \leq b(L_{k+1}+\delta)$,

$$N \;\geq\; \text{integer part of } \frac{\|x-y\|}{L_k+\delta}\,.$$

Therefore,

$$G^{\mathbf{J}}(X,Y) \;\leq\; \exp\left\{-m_\infty(L_k+\delta)\left(\frac{\|x-y\|}{L_k+\delta}-1\right)\right\} \tag{5.36}$$

(we bounded the integer part of a number by the number minus one).

This bound can be improved in cases where the temporal part dominates in the sense that $T^{-1}(t_X-t_Y) > \|x-y\|$. In this case we use that $b(L_k+\delta) \leq T^{-1}(t_X-t_Y)$ for $Y \in \mathcal{R}_k$, so that, as T grows faster than any power,

$$\frac{t_X-t_Y}{T(L_k)} \;\geq\; \frac{T(b(L_k+\delta))}{T(L_k)} \;\geq\; \frac{b(L_{k+1}+\delta)}{L_k+\delta} \;\geq\; \frac{\|x-y\|}{L_k+\delta}$$

if k_1, and hence k, is chosen large enough. Hence

$$N \;\geq\; \text{integer part of } \frac{b(L_{k+1}+\delta)}{L_k+\delta} \;\geq\; \text{integer part of } \frac{T^{-1}(t_X-t_Y)}{L_k+\delta} \tag{5.37}$$

where the last inequality comes from the definition of $\mathcal{R}_k$. From this and (5.34) we obtain that

$$G^{\mathbf{J}}(X,Y) \;\leq\; \exp\left\{-m_\infty(L_k+\delta)\left(\frac{T^{-1}(t_X-t_Y)}{L_k+\delta}-1\right)\right\} \tag{5.38}$$

whenever $T^{-1}(t_X-t_Y) > \|x-y\|$.

Inequalities (5.36) and (5.38) can be combined in the expresion

$$G^{\mathbf{J}}(X,Y) \leq \exp\Big\{-m_\infty\Big[\max\Big(\|x-y\|\;,\;T^{-1}(t_X-t_Y)\Big)-(L_k+\delta)\Big]\Big\}\,. \tag{5.39}$$

The additive correction $(L_k+\delta)$ can be turned into a factor $(1-b^{-1})$ because $(L_k+\delta) \leq (1/b)\max\{\|x-y\|\,, T^{-1}(t_X-t_Y)\}$, by definition of $\mathcal{R}_k$. In addition we choose $b = \frac{m_\infty}{m_\infty-m}$ so that $m = m_\infty(1-\frac{1}{b})$. In this way (5.39) yield

$$G^{\mathbf{J}}(X,Y) \;\leq\; \exp\Big\{-m\max\Big[\|x-y\|\;,\;T^{-1}(t_X-t_Y)\Big]\Big\} \tag{5.40}$$

uniformly in $k \geq k_1$.

Finally we consider $Y \in \overline{\mathcal{R}}$. They satisfy $\max\{\|x-y\|, T^{-1}(t_X - t_Y)\} < b\,(L_{k_1} + \delta)$, so we can write

$$\begin{aligned} G^{\mathbf{J}}(X,Y) &\leq G^{\mathbf{J}}(X,Y)\, \exp\Big\{mb\,(L_{k_1}+\delta)\Big\}\, \exp\Big\{-m\max\Big[\|x-y\|\,,\, T^{-1}(t_X-t_Y)\Big]\Big\} \\ &\leq C_x^{\mathbf{J}}(\ell_0, m)\, \exp\Big\{-m\max\Big[\|x-y\|\,, T^{-1}(t_X-t_Y)\Big]\Big\} \end{aligned} \tag{5.41}$$

where $C_x^{\mathbf{J}}(\ell_0, m) = e^{mb(L_{k_1}+\delta)}$. The desired bound (5.30) follows from (5.40) and (5.41). $\square$

5.4 The change of scale

5.4.1 The change-of-scale theorem

This section contains the heart of the multiscale argument. Its main result is the following theorem, related to Theorem 3.2 of Klein (1994), which establishes sufficient conditions for a scaled sequence to have good-enough probabilities.

Theorem 5.42 *Let $d \geq 1$ and choose a satisfying (2.17). Assume that the disorder is such that $\aleph < \infty$ satisfies (3.19) for the chosen a. Let $\alpha = d + \sqrt{d^2+d}$ and take ν and p such that*

$$\frac{\alpha d\,[\alpha + a + 1]}{a\,[\alpha - d + \alpha d]} < \nu \tag{5.43}$$

and

$$\alpha\, d \;<\; p \;<\; \frac{a(\nu[\alpha - d + \alpha d] - \alpha d) - \alpha d}{\alpha}\;. \tag{5.44}$$

Finally, consider m_0 and m_∞ such that $0 < m_\infty < m_0$. Then, there exists $\widetilde{L} = \widetilde{L}(d, \delta, a, \aleph, \nu, p, m_0, m_\infty) < \infty$ such that if for some $L_0 > \widetilde{L}$

$$\mathbb{P}\Big\{x \text{ is } (m_0, L_0) - \text{regular}\Big\} \;\geq\; 1 - \frac{1}{L_0^p}\;, \tag{5.45}$$

for all $x \in \mathbb{Z}^d$, then the scaled sequence (L_0, α, e^{L^ν}) has m_∞-good-enough probabilities.

Note that condition (2.17) guarantees the existence of ν satisfying (5.43), which in turns implies the existence of p as in (5.44).

The combination of this theorem and Theorem 5.29, implies that the key Lemma 3.18 —and hence all the properties stated in Theorem 4.1— follow once (5.45) is satisfied. Note that the logarithmic time-dependence in the key lemma is a consequence of the choice

$$T(L) \;=\; \exp(L^\nu)\;, \tag{5.46}$$

$0 < \nu < 1$, for the time-height of the scaled sequence.

The proof of Theorem 5.42 is inductive. We show that it is possible to arrange the parameters in such a way that, if

$$\mathbb{P}\Big\{x \text{ is } (m,l) - \text{regular}\Big\} \;\geq\; 1 - \frac{1}{l^p} \tag{5.47}$$

for $x \in \mathbb{Z}^d$, $m > 0$ and l sufficiently large, then we also have

$$\mathbb{P}\Big\{x \text{ is } (M,L) - \text{regular}\Big\} \;\geq\; 1 - \frac{1}{L^p} \tag{5.48}$$

where $L = l^\alpha$ and M is an appropiate function of m. The inductive step has two ingredients. First, in Subsection 5.4.2 we show that the inductive regularity holds in the presence of two events $\mathcal{A}$ and $\mathcal{B}_\Delta$. This part of the argument is based on geometric considerations and the inequalities of Section 5.2. Subsequently, in Subsection 5.4.3 these events are proven to hold with high-enough probability. The success of the approach relies on the careful definition of the events $\mathcal{A}$ and $\mathcal{B}_\Delta$.

5.4.2 Geometrical estimates: Good events imply good behavior

Lemma 5.49 *Consider some fixed $x \in \mathbb{Z}^d$. Let ν, α, l be such that $0 < \nu < 1$, $\alpha > 1$ and $l > \delta$. Set*

$$\theta_0 \;:=\; \min\{\alpha - 1, \alpha(1-\nu)\} \tag{5.50}$$

and take m_0, θ, m with $m_0 > 0$, $0 < \theta < \theta_0$ and $l^{-\theta} < m < m_0$. Put $L = l^\alpha$, pick a positive integer R and define the event

$$\begin{aligned}\mathcal{A} \;=\; & \Big\{\mathbf{J} : \textit{there exists } x_1, x_2, \ldots, x_R \in \Lambda[x; L+\delta] \textit{ such that} \\ & \Lambda[x; L+\delta] \setminus \bigcup_{j=1}^{R} \Lambda[x_j; 2(l+\delta)+1] \textit{ is a } (m,l) - \textit{regular region}\Big\}\,. \end{aligned} \tag{5.51}$$

Assuming $\mathcal{A}$ is true, take $\Delta > 0$, b such that $0 < b < \alpha\nu$ and $\kappa > \max\{1, \nu + \theta_0\}$. Define the event

$$\mathcal{B}_\Delta \;=\; \Big\{\mathbf{J} : \prod_{\gamma \in \mathbf{G}_{\tilde{\Lambda}}} K_\Delta^{\mathbf{J}}(\gamma) \;\geq\; e^{-l^b}\Big\}\,, \tag{5.52}$$

where

$$\widetilde{\Lambda} \;=\; \bigcup_{j=1}^{R} \Big(\Lambda[x_j; l^\kappa] \bigcap \Lambda[x; L+\delta]\Big) \tag{5.53}$$

—the union being taken on the points involved in the definition of the event $\mathcal{A}$— and

$$K^{\mathbf{J}}_{\Delta}(\gamma) \;=\; e^{-(1+\Delta)w^{\mathbf{J}}(\gamma)} + (1-e^{-w^{\mathbf{J}}(\gamma)})(1-e^{-\Delta})\, e^{-\Delta w^{\mathbf{J}}(\gamma)}\,. \tag{5.54}$$

Then, there exist $a_0 = a_0(d,\delta,\alpha,\nu,m_0,R) \ge 0$ and $l_0 = l_0(d,\delta,\alpha,\nu,\kappa,m_0,\theta,\theta_0,R) < \infty$ such that if $l > l_0$ the following holds: If $\mathcal{A}$ and $\mathcal{B}_\Delta$ are true for some $\Delta \in (0,1]$ then the site $x \in \mathbb{Z}^d$ is (M,L)– regular with

$$M \;\ge\; m - \frac{a_o}{l^{\theta_0}} \;\ge\; \frac{1}{L^{\theta}}\,. \tag{5.55}$$

This lemma is a direct consequence of the following Sublemas 5.56 and 5.63, where we analyze separately the connectivity of the site $X = (x,0)$ to sites in the vertical and horizontal boundary. The regularity stated below (5.54) follows immediately by summing the corresponding bounds over both boundaries.

Sublemma 5.56 involves estimations purely in the spatial direction, hence only the event $\mathcal{A}$ is relevant and the choice of the time scale $T(L)$ actually plays no role (though, for concreteness, we stick to the subexponential dependence). Event $\mathcal{B}_\Delta$ is required to control the time-like percolation studied in Sublemma 5.63. It provides a lower bound for the probability of not having long towers of ancestors based around "deffective" sites [see (5.82)–(5.83) below]. The stretched-exponential choice (5.46) becomes essential for this sublemma which is in fact the hardest estimation of the paper.

Sublemma 5.56 *There exist $l_1 = l_1(d,\delta,\alpha,\nu,m_0,\theta,R) < \infty$ and $a_1 = a_1(d,\delta,\alpha,\nu,m_0,R) > 0$ such that for all $l > l_1$, all $Y \in \partial_V B_L(X)$ and all $\mathbf{J} \in \mathcal{A}$*

$$G^{\mathbf{J}}_{B_L(X)}(X,Y) \;\le\; e^{-M_1(L+\delta)} \tag{5.57}$$

with

$$M_1 \;=\; m - \frac{a_1}{l^{\theta_0}} \;\ge\; \frac{1}{L^{\theta}}\,. \tag{5.58}$$

Proof. We work with a fixed environment $\mathbf{J} \in \mathcal{A}$. First we group the deffect-sets $\Lambda[x_j; 2(l+\delta)+1]$ into larger cubes which absorb "pockets" totally surrounded by original deffects. This leaves us with a much simpler geometrical situation, where the regular region is the complement of a finite family of cubes. More precisely, elementary geometrical considerations show that there exists a possibly smaller collection of sites $y_1, y_2, \ldots, y_{R'} \in \Lambda[x; L+\delta]$, with $R' \le R$, $n_1, n_2, \ldots, n_{R'} \in \{1,2,\ldots,R\}$ and $n_1 + n_2 + \ldots + n_{R'} \le R$, such that the sets $\Lambda[y_i; n_i(2(l+\delta)+1)]$, $i = 1,2,\ldots,R'$, are at a distance strictly larger than one, $\bigcup_{j=1}^{R} \Lambda[x_j; 2(l+\delta)+1] \subset \bigcup_{i=1}^{R'} \Lambda[y_i; n_i(2(l+\delta)+1)]$, and $\Lambda' := \Lambda[x; L+\delta] \setminus \bigcup_{i=1}^{R'} \Lambda[y_i; n_i(2(l+\delta)+1)]$ is a *connected* (m,l)-regular region.

These problematic regions define cylinders of radius $n_i(2(l+\delta)+1)$ and temporal height e^{L^ν} centered at the points y_i:

$$B_i \;=\; B_{n_i(2(l+\delta)+1),e^{L^\nu}}((y_i,0))\;, \quad i=1,2,\ldots,R'$$

[recall the notation in (5.3)]. We shall control the connectivity function in the spacial direction, on the (m,l)-regular region

$$B' \;=\; B_L(X)\setminus\bigcup_{i=1}^{R'} B_i \tag{5.59}$$

[we use the notation resulting from (5.20) and (5.46)].

If $X\in B'$, we denote $\partial B_0=\{X\}$; otherwise $X\in B_{i'}$ for some i' and we put $\partial B_0=\partial B_{i'}$. Similarly, if $Y\in B'$, we denote $\partial B_{R+1}=\{Y\}$; otherwise $Y\in B_{i''}$ for some i'' and we put $\partial B_{R+1}=\partial B_{i''}$. Every connection from X to Y can be decomposed into disjoint connections among some of the cylinders B_i. Therefore,

$$\begin{aligned}\Big\{X\to_{B_L(X)}Y\Big\}\;\subseteq\;\bigcup_{r=1}^{R'}\;\bigcup_{\{i_1,i_2,\ldots,i_r\}\subset\{1,2,\ldots,R'\}}&\Big\{\partial B_0\to_{B'}\partial B_{i_1}\Big\}\circ\Big\{\partial B_{i_1}\to_{B'}\partial B_{i_2}\Big\}\\ &\circ\cdots\circ\Big\{\partial B_{i_r}\to_{B'}\partial B_{R'+1}\Big\}\;.\end{aligned}\tag{5.60}$$

As $B'\subset\Lambda'\times\mathbb{R}$ with Λ' a (m,l)-regular region, we can apply (5.24) to obtain

$$\mathbb{Q}^{\mathbf{J}}\{\partial B_{j_1}\to_{B'}\partial B_{j_2}\}\;\le\;\Big[(R[2(l+\delta)+1])^d e^{L^\nu}\Big]^2\exp\Big\{-m(l+\delta)\Big(\frac{D_{j_1,j_2}}{l+\delta}-1\Big)\Big\}\;. \tag{5.61}$$

We have denoted

$$D_{j_1,j_2}\;=\;\min\Big\{\|x_1-x_2\|\;:\;(x_1,t)\in\partial B_{j_1}\,,\,(x_2,s)\in\partial B_{j_2}\text{ for some }t,s\Big\}$$

and used the bounds

$$\max\left\{\frac{\|x_{j_1}-x_{j_2}\|}{l+\delta},\frac{|t-s|}{e^{L^\nu}}\right\}\;\ge\;\frac{\|x_{j_1}-x_{j_2}\|}{l+\delta}\;\ge\;\frac{D_{j_1,j_2}}{l+\delta}\;.$$

¿From (5.60), (5.61) and the van den Berg-Kesten inequality (5.13) we have

$$\begin{aligned}G^{\mathbf{J}}_{B_L(X)}(X,Y)\;\le\;\sum_{r=1}^{R'}\;\sum_{\{i_1,i_2,\ldots,i_r\}\subset\{1,2,\ldots,R'\}}&\Big[(R[2(l+\delta)+1])^d e^{L^\nu}\Big]^{2(R'+1)}\\ &\times\exp\left\{-m(l+\delta)\left(\frac{D_{0,i_1}+D_{i_1,i_2}+\ldots+D_{i_r,R'+1}}{l+\delta}-(r+1)\right)\right\}.\end{aligned}$$

This inequality, together with the bounds $D_{0,i_1} + D_{i_1,i_2} + \ldots + D_{i_r,R'+1} \geq L - 2[2(l+\delta) + 1][n_{i_1} + n_{i_2} + \ldots + n_{i_r}] \geq L - 2[2(l+\delta)+1]R$ and $\sum_{r=1}^{R'} \sum_{\{i_1,i_2,\ldots,i_r\}\subset\{1,2,\ldots,R'\}} 1 \leq (R+1)!$, yield

$$
\begin{aligned}
G^{\mathbf{J}}_{B_L(X)}(X,Y) &\leq (R+1)!\Big[(R[2(l+\delta)+1])^d e^{L^\nu}\Big]^{2(R+1)} \\
&\quad \times \exp\left\{-m(l+\delta)\left(\frac{L-2[2(l+\delta)+1]R}{l+\delta} - (R+1)\right)\right\} . \quad (5.62)
\end{aligned}
$$

Therefore

$$
G^{\mathbf{J}}_{B_L(X)}(X,Y) \leq e^{-M_1'(L+\delta)};
$$

with

$$
M_1' \;=\; m - \frac{c_1 m_0}{l^{\alpha-1}} - \frac{c_2}{l^{\alpha(1-\nu)}}
$$

for some fixed constants $c_1, c_2 > 0$ depending on d, δ, α, ν and R. This yields the proposed bound (5.57) because, as $m \geq \frac{1}{l^\theta}$,

$$
M_1' \;\geq\; M_1 = m - \frac{a_1}{l^{\theta_0}} \;\geq\; \frac{1}{L^\theta}
$$

for some $a_1 = a_1(d,\delta,\alpha,\nu,m_0,R) > 0$ and for l large enough (the meaning of "large enough" depends on $d,\delta,\alpha,\nu,m_0,\theta$ and R). $\square$

Sublemma 5.63 *Suppose the events $\mathcal{A}$ and $\mathcal{B}_\Delta$ in (5.51) and (5.52) are true. Pick τ such that $\nu < \tau < \min\{\kappa - \theta_0, \alpha\nu\}$. Then there exists $l_2 = l_2(d,\delta,\alpha,\nu,m_0,\theta,\kappa,b,\tau,R) < \infty$, such that for $l > l_2$ we have*

$$
G^{\mathbf{J}}_{B_L(X)}(X,Y) \leq exp\{-M_2 e^{\frac{l^\tau}{4}}\}
$$

for all $Y \in \partial_H B_L(X)$, with

$$
M_2 \;=\; m - e^{-\frac{l^\tau}{3}} \;\geq\; \frac{1}{L^\theta} \, . \quad (5.64)
$$

Note that it is possible to choose τ satisfying the hypothesis because $\kappa > \nu + \theta_0$ and $\nu < \alpha\nu$.

Proof. Let us fix an environment $\mathbf{J} \in \mathcal{A} \cap \mathcal{B}_\Delta$. The proof relies on the introduction of additional space and time scales:

$$
\widehat{\delta}(l) = l^\kappa \quad \text{and} \quad \widehat{\tau}(l) = \exp(l^\tau) \, . \quad (5.65)
$$

The time scale $\widehat{\tau}(l)$ is intermediate between $T(l)$ and $T(L) = T(l^\alpha)$ because $1 < \tau < \alpha\nu$. The space scale $\widehat{\delta}$ is used to construct the region $\widetilde{\Lambda}$ defined in (5.53). For the time being it is only required to grow faster than l ($\kappa > 1$), but the hypotheses of Lemma 5.92 below will force κ to be strictly smaller than α [see condition (5.102)], making $\widehat{\delta}$ a scale intermediate between l and $L = l^\alpha$. The intermediate time scale defines a partition of the box $B_L(X)$ into

$$N \;=\; \text{integer part of } \frac{T(L)}{\widehat{\tau}(l)} \tag{5.66}$$

"slices" of time-heigth $\widehat{\tau}(l)$:

$$S_j \;=\; B_{L+\delta,\widehat{\tau}(l)}\Big((x, -(j-1)\widehat{\tau}(l))\Big)\;, \quad j = 1, 2, \ldots, N\;. \tag{5.67}$$

We note that events in different S_j are independent (see the paragraph preceding Corollary 5.17).

We introduce the notation $B_\Lambda = \Lambda \times [-T(L), 0]$ for any $\Lambda \subset \mathbb{Z}^d$. Let

$$\widehat{\Lambda} \;=\; \Big(\bigcup_{j=1}^{R} \Lambda[x_j; 2(l+\delta)+1]\Big) \bigcap \Lambda[x; L+\delta] \tag{5.68}$$

be the set of "deffective" or "irregular" sites. The region $\widetilde{\Lambda}$ defined in (5.53) corresponds to a "buffer zone" around, and including, these irregular sites. We decompose the event $\{X \to_{B_L(X)} Y\}$ according to whether there exists at least one slice such that no deffective point within it is connected to a preceding slice. That is, we write

$$\Big\{X \to_{B_L(X)} Y\Big\} \;=\; \Big(\Big\{X \to_{B_L(X)} Y\Big\} \cap A\Big) \bigcup \Big(\Big\{X \to_{B_L(X)} Y\Big\} \cap A^c\Big) \tag{5.69}$$

with

$$A \;=\; \bigcup_{j=1}^{N} A_j \;=\; \bigcup_{j=1}^{N} \Big\{B_{\widehat{\Lambda}} \cap S_j \longrightarrow_{S_j} S_{j+1}\Big\}^c\;. \tag{5.70}$$

The probability of the first event on the right of (5.69) is damped by regularity. Indeed, the occurrence of $\{X \to_{B_L(X)} Y\} \cap A$ implies the existence of a connection in $B_{\Lambda[x;L+\delta]\setminus\widehat{\Lambda}}$ of vertical height at least equal to $\widehat{\tau}(l)$. Therefore,

$$\{X \to_{B_L(X)} Y\} \cap A \;\subset\; \bigcup \Big\{(y_1, s_1) \longrightarrow_{B_{\Lambda[x;L+\delta]\setminus\widehat{\Lambda}}} (y_2, s_2)\Big\}\;, \tag{5.71}$$

the union being taken over all $(y_1, s_1), (y_2, s_2) \in B_{\Lambda[x;L+\delta]\setminus\widehat{\Lambda}}$ with $| s_1 - s_2 | \geq \widehat{\tau}(l)$. By regularity (Lemma 5.23) we get, after simple computations

$$\mathbb{Q}^{\mathbf{J}}\Big(\{X \to_{B_L(X)} Y\} \cap A\Big)$$
$$\leq \exp\Big\{-m(l+\delta)\Big[\frac{\widehat{\tau}(l)}{T(l)} - 1\Big] + 2d\ln(2L+2\delta) + 2\ln T(L)\Big\} . \tag{5.72}$$

Therefore,

$$\begin{aligned}\mathbb{Q}^{\mathbf{J}}\Big(\{X \to_{B_L(X)} Y\} \cap A\Big) & \\ \leq \; & \exp\Big\{-m(l+\delta)\Big[e^{l^\tau - l^\nu} - 1\Big] + 2d\ln(2l^\alpha + 2\delta) + 2l^{\alpha\nu}\Big\} \\ \leq \; & \exp\Big\{-(m - e^{-\frac{l^\tau}{3}})\, e^{\frac{l^\tau}{2}}\Big\},\end{aligned} \tag{5.73}$$

for l large enough (recall that $\nu < \tau$).

To prove that under our hypotheses A^c is sufficiently improbable, we bound

$$A_j \;\supset\; F_j \cap D_{-(j-1/2)\widehat{\tau}(l)} \;, \tag{5.74}$$

where F_j is the event that there is no connection inside the slice S_j from points in the deffective set to regular points outside the buffer zone $\widetilde{\Lambda}$:

$$F_j \;=\; \Big\{\partial B_{\widehat{\Lambda}} \cap S_j \longrightarrow_{S_j \setminus B_{\widehat{\Lambda}}} B_{\Lambda[x;L+\delta]\setminus\widetilde{\Lambda}} \cap S_j\Big\}^c \;, \tag{5.75}$$

and D_s represents the lack of connection within a strip of height Δ involving deffective sites:

$$D_s \;=\; \Big\{\widetilde{\Lambda} \times \{s\} \longrightarrow_{\widetilde{\Lambda}\times[s-\Delta,s]} \widetilde{\Lambda} \times \{s-\Delta\}\Big\}^c \;. \tag{5.76}$$

We are assumming that l is sufficiently large so that $\widehat{\tau}(l)/2 > 1$, and hence $\widehat{\tau}(l)/2 > \Delta$ for all $\Delta \in (0, 1]$.

Since each F_j^c involves connections within regular regions, we can apply Lemma 5.23 to obtain

$$\mathbb{Q}^{\mathbf{J}}\{F_j^c\} \leq e^{-h(l)}, \tag{5.77}$$

with

$$h(l) \;=\; m\{\widehat{\delta}(l) - (3l + 4\delta + 1)\} - 2\ln\widehat{\tau}(l) - d\ln\widehat{\delta}(l) - d\ln(2l + 3\delta + 1) - 2\ln R \;. \tag{5.78}$$

Hence

$$h(l) \;\geq\; m\, c_1\, l^{\kappa} - c_2\, l^{\tau} \;\geq\; c_3\, l^{\kappa-\theta} \;\geq\; c_3\, l^{\kappa-\theta_0}\ , \tag{5.79}$$

where c_1, c_1 and c_3 are positive constants —depending on $d, \delta, \kappa, \tau, \theta$ and R— and l is large enough. We have used that $m \geq l^{-\theta}$, $\theta < \theta_0$ and $\tau < \kappa - \theta$.

On the other hand,

$$D_s \;\supset\; \bigcap_{\gamma \in \mathbf{G}_{\widetilde{\Lambda}}} \{[\mathcal{E}_1(\gamma) \cup \mathcal{E}_2(\gamma)] \cap \mathcal{E}_3(\gamma)\}\ , \tag{5.80}$$

where

$$\begin{aligned}
\mathcal{E}_1(\gamma) &= \{\gamma \text{ is not present at time } s - \Delta\}\ ,\\
\mathcal{E}_2(\gamma) &= \{\gamma \text{ is present at time } s - \Delta \text{ but it does not survive until time } s\}\ ,\\
\mathcal{E}_3(\gamma) &= \{\text{ there is no birth of } \gamma \text{ in the interval } [s-\Delta, s]\}\ .
\end{aligned}$$

We observe that

$$\begin{aligned}
\mathbb{Q}^{\mathbf{J}}\Big([\mathcal{E}_1(\gamma) \cup \mathcal{E}_2(\gamma)] \cap \mathcal{E}_3(\gamma)\Big) &= \Big(\mathbb{Q}^{\mathbf{J}}[\mathcal{E}_1(\gamma)] + \mathbb{Q}^{\mathbf{J}}[\mathcal{E}_2(\gamma)]\Big)\, \mathbb{Q}^{\mathbf{J}}[\mathcal{E}_3(\gamma)]\\
&= \Big[e^{-w^{\mathbf{J}}(\gamma)} + (1 - e^{-w^{\mathbf{J}}(\gamma)})(1 - e^{-\Delta})\Big] e^{-\Delta w^{\mathbf{J}}(\gamma)}\\
&= K_{\Delta}^{\mathbf{J}}(\gamma)\ .
\end{aligned} \tag{5.81}$$

This result follows from well-known properties on Poisson processes and from the fact that the events in which an animal γ is activated at time $-t$ and survives until time 0 define an inhomogeneous Poisson process of rate $w^{\mathbf{J}}(\gamma)e^{-t}$. Therefore, due to the $\mathbb{Q}^{\mathbf{J}}$-independence of events pertaining to different animals,

$$\mathbb{Q}^{\mathbf{J}}\{D_s\} \;\geq\; \prod_{\gamma \in \mathbf{G}_{\widetilde{\Lambda}}} K_{\Delta}^{\mathbf{J}}(\gamma) \;\geq\; e^{-l^b}\ . \tag{5.82}$$

By the FKG-inequality we get, from (5.74), (5.77) and (5.79)–(5.81),

$$\begin{aligned}
\mathbb{Q}^{\mathbf{J}}\{A_j\} &\geq \mathbb{Q}^{\mathbf{J}}\{F_j\}\, \mathbb{Q}^{\mathbf{J}}\{D_{-(j-1/2)\widehat{\tau}(l)}\}\\
&\geq \Big(1 - e^{-c_3 l^{\kappa-\theta_0}}\Big)\, e^{-l^b}\\
&\geq e^{-2l^b}
\end{aligned} \tag{5.83}$$

for l large enough. Thus, the independence of the events A_j, $j = 1, 2, \ldots, N$ leads to the bound

$$\mathbb{Q}^{\mathbf{J}}(A^c) \;=\; \prod_{j=1}^{N}\Big(1 - \mathbb{Q}^{\mathbf{J}}(A_j)\Big) \;\leq\; (1 - e^{-2l^b})^N \;\leq\; \exp\Big(-N\, e^{-2l^b}\Big)\,. \tag{5.84}$$

The condition $\tau < \alpha\nu$ implies that, for l large enough, $N \geq \exp(l^{\alpha\nu}/2)$, and as, besides, $b < \alpha\nu$, we obtain from (5.84) that

$$\mathbb{Q}^{\mathbf{J}}(A^c) \;\leq\; \exp\Big(-e^{l^{\alpha\nu}/4}\Big)\,, \tag{5.85}$$

for l sufficiently large.

Finally, inequalities (5.73) and (5.85) together with the decompostion (5.69) yield (recall that $\tau < \alpha\nu$)

$$\begin{aligned} G^{\mathbf{J}}_{B_L(X)}(X,Y) \;&\leq\; \mathbb{Q}^{\mathbf{J}}\Big(\{X \to_{B_L(X)} Y\} \cap A\Big) + \mathbb{Q}^{\mathbf{J}}(A^c) \\ &\leq\; \exp\Big\{-(m - e^{-\frac{l^\tau}{3}})\, e^{\frac{l^\tau}{2}}\Big\} + \exp\Big\{-e^{\frac{l^{\alpha\nu}}{4}}\Big\} \\ &\leq\; \exp\Big\{-(m - e^{-\frac{l^\tau}{3}}) e^{\frac{l^\tau}{4}}\Big\}\,, \end{aligned} \tag{5.86}$$

with $m - e^{-l^\tau/3} \geq 1/L^\theta$, for l larger than a certain threshold that depends on d, δ, α, ν, m_0, θ, κ ,b, τ and R. $\square$

5.4.3 Probabilistic estimates: Good events have high probability

In this section we show that, assuming (5.47), the events $\mathcal{A}$ and $\mathcal{B}_\Delta$ introduced in (5.51) and (5.52) have good enough probability for l sufficiently large. Hypothesis (3.19) is used to bound the probability of the second event.

Lemma 5.87 *Let $m > 0$, $l > \delta$, $\alpha > 1$ and $p > \alpha d$. Put $L = l^\alpha$. Pick a positive integer $R > \alpha p/(p - \alpha d)$ and define the event $\mathcal{A}$ as in (5.51). If*

$$\mathbb{P}\Big\{x \text{ is } (m,l) - \text{regular}\Big\} \;\geq\; 1 - \frac{1}{l^p} \tag{5.88}$$

for all $x \in \mathbb{Z}^d$, then

$$\mathbb{P}\{\mathcal{A}\} \;\geq\; 1 - \frac{1}{2L^p}, \tag{5.89}$$

assuming l large exceed a certain minimum value that depends on d, α, p and R.

Proof. Let us say that two sites $x_1, x_2 \in \mathbb{Z}^d$ are l-nonoverlapping if the distance between the boxes $\Lambda[x_1; l+\delta]$ and $\Lambda[x_2; l+\delta]$ is strictly larger than one. In this case, the events $\{x_i \text{ is } (m,l) - \text{regular}\}$, $i = 1, 2$, are independent. We have

$$\mathcal{A}^c \subset \bigcup_{x_1,\ldots,x_{R+1} \in \Lambda[x;L+\delta]}^{(l)} \bigcap_{i=1}^{R+1} \Big\{ x_i \text{ is } (m,l) - \text{singular} \Big\}, \tag{5.90}$$

where the index (l) reminds us that the union is over collections of $R+1$ sites which are l-nonoverlapping. Hence

$$\begin{aligned} \mathbb{P}\{\mathcal{A}^c\} &\leq \sum_{x_1,\ldots,x_{R+1} \in \Lambda[x;L+\delta]}^{(l)} \Big[\mathbb{P}\Big\{ x \text{ is } (m,l) - \text{singular} \Big\} \Big]^{R+1} \\ &\leq \frac{[2(L+\delta)]^{d(R+1)}}{l^{p(R+1)}} \leq \frac{1}{L^p} \frac{4^{d(R+1)}}{l^{(p-\alpha d)}} \leq \frac{1}{2L^p}, \end{aligned} \tag{5.91}$$

for l large enough. $\square$

Lemma 5.92 *Let $l > \delta$, $\alpha > 1$ and $p > 0$. Put $L = l^\alpha$. For κ and b such that $0 < \kappa < b/d$ take a collection of sites $\{x_i : i = 1, 2, \ldots, R\}$ and introduce the event $\mathcal{B}_\Delta$ as in (5.52) for $\Delta = e^{-l^\eta}$, with $0 < \eta < b - \kappa d$. If $\aleph < \infty$ for some $a > \alpha(p+d)/\eta$, then*

$$\mathbb{P}\{\mathcal{B}_\Delta\} \geq 1 - \frac{1}{2L^p} \tag{5.93}$$

for l larger than a certain threshold that depends on α, p, κ, b, d, η, $\aleph$.

Proof. We start with the following observations:

(i) For any collection of sites $\{x_i : i = 1, 2, \ldots, R\}$

$$\mathbf{G}_{\cup_{i=1}^R \Lambda[x_i, l^\kappa]} \subset \bigcup_{i=1}^R \mathbf{G}_{\Lambda[x_i; l^\kappa + \delta]} . \tag{5.94}$$

(ii) As a consequence of (i),

$$\mathcal{B}_\Delta^c \subseteq \bigcup_{y \in \Lambda[x;L+\delta]} \Big\{ \sum_{\gamma \in \mathbf{G}_{\Lambda[y;l^\kappa+\delta]}} \ln \Big[\frac{1}{K_\Delta^{\mathbf{J}}(\gamma)} \Big] > \frac{l^b}{R} \Big\}. \tag{5.95}$$

(iii) For any $\Delta > 0$,

$$K^{\mathbf{J}}_{\Delta}(\gamma) \;\geq\; (1-e^{-\Delta})\, e^{-\Delta w^{\mathbf{J}}(\gamma)} \;>\; 0. \tag{5.96}$$

¿From (5.96) and the inequality $\ln(1-e^{-\Delta}) \geq \ln(\Delta/2)$, valid for Δ small enough, we obtain

$$\begin{aligned}
\sum_{\gamma\in \mathbf{G}_{\Lambda[y;l^{\kappa}+\delta]}} \ln\Big[\frac{1}{K^{\mathbf{J}}_{\Delta}(\gamma)}\Big] \;&\leq\; \sum_{\gamma\in \mathbf{G}_{\Lambda[y;l^{\kappa}+\delta]}} \Big[\Delta w^{\mathbf{J}}(\gamma) - \ln(1-e^{-\Delta})\Big] \\
&\leq\; \sum_{z\in\Lambda[y;l^{\kappa}+\delta]} \sum_{\gamma\ni z} \Big[\Delta w^{\mathbf{J}}(\gamma) - \ln(1-e^{-\Delta})\Big] \\
&\leq\; 4^{d} l^{\kappa d}\Big[\Delta \sup_{x} \sum_{\gamma\ni x} w^{\mathbf{J}}(\gamma) - \Theta_{\mathbf{G}} \ln(\frac{\Delta}{2})\Big].
\end{aligned} \tag{5.97}$$

with $\Theta_{\mathbf{G}} := \sup_x |\{\gamma\ni x\}| < \infty$. Therefore, for $\Delta = e^{-l^{\eta}}$, $0<\eta<b-\kappa d$, and assuming l large enough in every step we get

$$\begin{aligned}
\mathbb{P}\Big\{\sum_{\gamma\in \mathbf{G}_{\Lambda[y;l^{\kappa}+\delta]}} \ln\Big[\frac{1}{K^{\mathbf{J}}_{\Delta}(\gamma)}\Big] \;>\; \frac{l^{b}}{R}\Big\} \;&\leq\; \mathbb{P}\Big\{\Delta\,\sup_{x}\sum_{\gamma\ni x} w^{\mathbf{J}}(\gamma) > \frac{l^{b-\kappa d}}{4^{d}\,R} + \Theta_{\mathbf{G}}\ln(\frac{\Delta}{2})\Big\} \\
&\leq\; \mathbb{P}\Big\{\sup_{x}\sum_{\gamma\ni x} w^{\mathbf{J}}(\gamma) > \frac{l^{b-\kappa d}}{4^{d+1}\,R\,\Delta}\Big\} \\
&\leq\; \mathbb{P}\Big\{\ln\Big(1+\sup_{x}\sum_{\gamma\ni x} w^{\mathbf{J}}(\gamma)\Big) > \frac{l^{\eta}}{2}\Big\} \\
&\leq\; \frac{2^{a}}{l^{a\eta}}\,\mathbb{E}\Big[\ln^{a}\Big(1+\sup_{x}\sum_{\gamma\ni x} w^{\mathbf{J}}(\gamma)\Big)\Big],
\end{aligned} \tag{5.98}$$

where in the last step we have used Chebyshev inequality. Finally, (5.95), (5.98) and the fact that $a\eta > \alpha(p+d)$ imply that

$$\mathbb{P}\{\mathcal{B}^{c}_{\Delta}\} \;\leq\; \frac{1}{L^{p}}\,\frac{2^{(a+2d)}\aleph}{l^{a\eta-\alpha(p+d)}} \;\leq\; \frac{1}{2L^{p}}, \tag{5.99}$$

for l large enough. □

5.4.4 Discussion on optimal choices

We pause to discuss the possible choices for the parameters involved in Lemmas 5.49, 5.87 and 5.92. The parameter a in Lemma 5.92 involves explicitly the birth rates of the animals. We observe that the smaller the value of a the weaker the condition on the birth rates. Thus, the goal is to choose the smallest possible value of this parameter compatible with the hypotheses. Let us first list the requirements on the different parameters:

$$\alpha > 1 ; \tag{5.100}$$

$$0 < \nu < 1 ; \tag{5.101}$$

$$\max\{1, \nu + \theta_0\} < \kappa < \frac{b}{d} < \frac{\alpha\nu}{d} ; \tag{5.102}$$

$$0 < \eta < b - \kappa d ; \tag{5.103}$$

$$p > \alpha d ; \tag{5.104}$$

$$a > \frac{\alpha(p+d)}{\eta} \tag{5.105}$$

Inequalities (5.102) tell us that for choice of κ and b to be possible we must have that $\alpha\nu > d$ and hence that [see (5.50)]

$$\theta_0 = \alpha(1-\nu) . \tag{5.106}$$

This identity, in combination with the extreme and leftmost inequalities in (5.102), respectively implies that

$$\nu > \frac{\alpha d}{\alpha - d + \alpha d} , \tag{5.107}$$

and

$$\kappa \geq \nu + \alpha(1-\nu) . \tag{5.108}$$

Note that (5.107) yields, in view of (5.101), that

$$\alpha > d . \tag{5.109}$$

On the other hand, if we combine the rightmost inequality in (5.102) with (5.108) and (5.103) we obtain

$$\eta < \nu[\alpha - d + \alpha d] - \alpha d < \alpha - d \tag{5.110}$$

[the last inequality is due to (5.101)]. Finally, we use this last bound, in combination with (5.104) and (5.105) to obtain a lower bound for the parameter a:

$$a > \frac{\alpha d(\alpha+1)}{\alpha - d} \qquad \text{with } \alpha > d . \tag{5.111}$$

Small values of a compatible with this restriction are obtained when

$$\alpha = d + \sqrt{d^2 + d} \tag{5.112}$$

and satisfy the inequality

$$a \;>\; 2d^2\Big(1+\sqrt{1+\frac{1}{d}}+\frac{1}{2d}\Big)\;. \tag{5.113}$$

Reversing the preceding analysis, we see that the best strategy is to take α and a acording to (5.112) and (5.113) and to choose ν and p satisfying (5.43) and (5.44). This yields the smallest possible choices of a that allow choices of η, b and κ respecting the constraints (5.100)–(5.105). This explains the definition adopted for the parameters in the statement of the change-of-scale Theorem 5.42.

This is also a good opportunity to comment on choices for time scales. They are determined by the demands impossed throughout the proofs of Sublemma 5.63 and Lemma 5.92. There are two major constraints: First, the scales $\widehat{\delta}(l)$ and $\widehat{\tau}(l)$ must be truly intermediate in the sense that

$$l \;\prec\; \widehat{\delta}(l) \;\prec\; L=l^\alpha \qquad \text{and} \qquad T(l) \;\prec\; \widehat{\tau}(l) \;\prec\; T(L)=T(l^\alpha)\;, \tag{5.114}$$

where $f(l) \prec g(l)$ means that $f(l)/g(l)$ tends to zero as l tends to infinity. Second, the exponent $h(l)$ defined in (5.77)/(5.78) must grow with l. This implies that

$$\widehat{\tau}(l) \;\prec\; e^{\widehat{\delta}(l)}\;. \tag{5.115}$$

The combination of the two preceding displays yields the relations

$$T(l) \;\prec\; \widehat{\tau}(l) \;\prec\; e^{\widehat{\delta}(l)} \;\prec\; e^{l^\alpha} \tag{5.116}$$

which shows that at best the time scale can grow as a stretched exponential.

5.4.5 Conclusion of the proof of Theorem 5.42

We prove (5.28) by induction in k. The case $k=0$ follows from hypothesis (5.45) because (m_0,L)-regularity gives (m_∞,L)-regularity for $m_\infty < m_0$. Suppose now (5.28) is true for some $k>0$. Lemmas 5.87 and 5.92 —with the replacements $(m,l)\to(m_k,L_k)$ and $L\to L_{k+1}$— and the preceding discussion on optimal choices imply that if

$$\mathbb{P}\Big\{x \text{ is } (m_k,L_k)-\text{regular}\Big\} \;\geq\; 1-\frac{1}{L_k^p}\;, \tag{5.117}$$

then

$$\mathbb{P}\{\mathcal{A}\cap\mathcal{B}_\Delta\} \;\geq\; 1-\frac{1}{L_{k+1}^p} \tag{5.118}$$

for l, and hence L_k, sufficiently large. On the other hand, Lemma 5.49 —with $(m,l) \to (m_k, L_k)$ and $(M,L) \to (m_{k+1}, L_{k+1})$— implies that if $L_k^{-\theta} < m_k < m_0$ for $0 < \theta < \theta_0 = \alpha(1-\nu)$, then

$$\mathcal{A} \cap \mathcal{B}_\Delta \ \subseteq\ \Big\{ x \text{ is } (m_{k+1}, L_{k+1}) - \text{regular} \Big\}\ , \tag{5.119}$$

with

$$m_{k+1} \ \geq\ m_k - \frac{a_0}{L_k^{\theta_0}} \ \geq\ \frac{1}{L_{k+1}^{\theta}} \tag{5.120}$$

for l, and hence L_k, sufficiently large. ¿From (5.118) and (5.119) we get that

$$\mathbb{P}\Big\{ x \text{ is } (m_k, L_k) - \text{regular} \Big\} \ \geq\ 1 - \frac{1}{L_k^p} \tag{5.121}$$

for all natural k, for l suffciently large.

To conclude we must check that

$$m_\infty \ \leq\ m_k \quad , \quad \text{for } k = 1, 2, \ldots \tag{5.122}$$

¿From (5.120) we have

$$m_k \ \geq\ m_0 - a_0 \sum_{j=0}^{k-1} \frac{1}{L_j^{\theta_0}} \ =\ m_0 - a_0 \sum_{j=0}^{k-1} \Big(\frac{1}{L_0^{\theta_0}} \Big)^{\alpha^j} . \tag{5.123}$$

Therefore, (5.122) is verified if L_0 is chosen so large that

$$a_0 \sum_{j=0}^{\infty} \Big(\frac{1}{L_0^{\theta_0}} \Big)^{\alpha^j} \ <\ m_0 - m_\infty\ . \quad \square \tag{5.124}$$

5.5 A last probabilistic estimate: The choice of initial scale

To finish the proof of the key Lemma 3.18 we show that for every real $\widetilde{L}$ (though we are interested in $\widetilde{L}$ large) there exists $L_0 \geq \widetilde{L}$ such that (5.45) holds. It is only here that hypothesis (2.18) is needed. Let us suppose that a function $S : \mathbf{G} \to [1, \infty)$ has been selected and consider the resulting quantity $\Psi^{\mathbf{J}}$ defined in (2.23).

Lemma 5.125 *Given $\widetilde{L}, p > 0$, for each $\rho \in [0,1)$ there exist $m(\rho), \varepsilon(\rho) > 0$ and $L(\rho) \geq \widetilde{L}$ such that*

$$\mathbb{P}\Big\{ \Psi^{\mathbf{J}} > \rho \Big\} \ <\ \varepsilon(\rho) \quad \Longrightarrow \quad \mathbb{P}\Big\{ x \textit{ is } (m(\rho), L(\rho)) - \textit{regular} \Big\} \ \geq\ 1 - \frac{1}{L(\rho)^p}\ . \tag{5.126}$$

Proof. By Theorem 5.1 in Fernández, Ferrari and Garcia (2001), the condition $\Psi^{\mathbf{J}} \leq \rho$ implies that there exist $c(\rho), \widetilde{m}(\rho) > 0$ such that

$$G^{\mathbf{J}}_{B_{L+\delta,T}(X)}(X,Y) \;\leq\; c(\rho)\, \exp\Big\{-\widetilde{m}(\rho)\, \max(\|x-y\|\;, -t_Y)\Big\}\,, \tag{5.127}$$

for all $Y = (y, t_Y)$ with $t_Y \leq 0$. Taking, for instance, $m(\rho) = \widetilde{m}(\rho)/2$ we conclude that there exists $\widetilde{L}(\rho)$ such that

$$\Big\{x \text{ is } (m(\rho), L) - \text{regular}\Big\} \;\supset\; \Big\{\Psi^{\mathbf{J}} \leq \rho\Big\} \tag{5.128}$$

for all $L \geq \widetilde{L}(\rho)$. Implication (5.126) follows by choosing $L(\rho) = \max[\widetilde{L}\,, \widetilde{L}(\rho)]$ and $\varepsilon(\rho) = L(\rho)^{-p}$. $\square$

Chebyshev inequality shows that (2.18) is a sufficient condition for the validity of the left-hand side of (5.126), as long as $\varepsilon \leq \rho\,\varepsilon(\rho)$, for any given choice of ρ. This observation, the preceding Lemma 5.125 and Theorems 5.42 and 5.29 constitute the proof of the main Lemma 3.18. We observe that, in our proof, $q = \frac{1}{\nu}$ and $q_0 = \frac{a[\alpha - d + \alpha d]}{\alpha d[\alpha + a + 1]}$ with $\alpha = d + \sqrt{d^2 + d}$.

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References

[1] A. J. Baddeley, W. S. Kendall, and M. N. M. Van Lieshout. Quermass-interaction porcesses, 1996. Preprint.

[2] A. J. Baddeley and M. N. M. van Lieshout. Area-interaction point processes. *Ann. Inst. Statist. Math.*, 47(4):601–619, 1995.

[3] C. Bezuidenhout and G. Grimmett. Exponential decay for subcritical contact and percolation processes. *Ann. Probab.*, 19(3):984–1009, 1991.

[4] J. Bricmont and A. Kupiainen. High temperature expansions and dynamical systems. *Comm. Math. Phys.*, 178(3):703–732, 1996.

[5] M. Campanino and A. Klein. Decay of two-point functions for $(d+1)$-dimensional percolation, Ising and Potts models with d-dimensional disorder. *Comm. Math. Phys.*, 135(3):483–497, 1991.

[6] M. Campanino, A. Klein, and J. F. Perez. Localization in the ground state of the Ising model with a random transverse field. *Comm. Math. Phys.*, 135(3):499–515, 1991.

[7] R. L. Dobrushin. Perturbation methods of the theory of Gibbsian fields. In *Lectures on probability theory and statistics (Saint-Flour, 1994)*, pages 1–66. Springer, Berlin, 1996.

[8] R. Fernández, P. A. Ferrari, and N. L. Garcia. Measures on contour, polymer or animal models. A probabilistic approach. *Markov Process. Related Fields*, 4(4):479–497, 1998. I Brazilian School in Probability (Rio de Janeiro, 1997).

[9] R. Fernández, P. A. Ferrari, and N. L. Garcia. Loss network representation of Peierls contours. *Ann. Probab.*, 29:902–937, 2001.

[10] R. Fernández, P. A. Ferrari, and N. L. Garcia. Perfect simulation for interacting point processes, loss networks and ising models. *Stoch. Proc. Appl.*, 102:63–88, 2002.

[11] P. A. Ferrari and N. L. Garcia. One-dimensional loss networks and conditioned $M/G/\infty$ queues. *J. Appl. Probab.*, 35(4):963–975, 1998.

[12] G. Gielis and C. Maes. Percolation techniques in disordered spin flip dynamics: relaxation to the unique invariant measure. *Comm. Math. Phys.*, 177(1):83–101, 1996.

[13] G. Grimmett. The stochastic random-cluster process and the uniqueness of random-cluster measures. *Ann. Probab.*, pages 1461–1450, 1995.

[14] F. P. Kelly. Loss networks. *Ann. Appl. Probab.*, 1(3):319–378, 1991.

[15] W. S. Kendall. Perfect simulation for the area-interaction point process. In L. Accardi and C. C. Heyde, editors, *Probability Towards 2000*, pages 218–234. Springer, 1998.

[16] A. Klein. Extinction of contact and percolation processes in a random environment. *Ann. Probab.*, 22, 1994.

[17] A. Klein. Multiscale analysis in disordered systems: Percolation and contact process in a random environment. Preprint, 1995.

[18] R. Meester and R. Roy. *Continuum percolation*, volume 119 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[19] J. Møller. A review of perfect simulation in stochastic geometry. In *Selected Proceedings of the Symposium on Inference for Stochastic Processes*, pages 333–55. IMS Lecture Notes & Monographs Series, Volume 37, 2001.

[20] J. Møller and R. P. Waagepetersen. *Statistical inference and simulation for spatial point processes*, volume 100 of *Monographs on Statistics and Applied Probability*. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[21] D. J. Strauss. A model for clustering. *Biometrika*, 62(2):467–475, 1975.

[22] J. van den Berg. A note on disjoint-occurrence inequalities for marked Poisson point processes. *J. Appl. Probab.*, 33(2):420–426, 1996.

[23] H. von Dreifus. On the effects of randomness in ferromagnetic models and Schrödinger operators. Phd. Thesis, New York University, 1987.

[24] B. Widow and J. S. Rowlinson. New model for the study of liquid-vapor phase transitions. *J. Chem. Phys.*, 52:1670–1684, 1970.