# On the Polyharmonic Operator with a Periodic Potential

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**Abstract**

In this paper we obtain asymptotic formulas for eigenvalues and Bloch functions of the polyharmonic operator $L(r,q(x))=-\Delta^r+q(x)$, of arbitrary dimension $d$ with periodic, with respect to arbitrary lattice, potential $q(x)$, where $r\geq 1$. Then we prove that the number of gaps in the spectrum of the operator $L(r,q(x))$ is finite which is the generalisation of the Bethe -Sommerfeld conjecture for this operator. In particular, taking $r=1$ we get the proof of the Bethe -Sommerfeld conjecture for arbitrary dimension and arbitrary lattice.

*INTRODUCTION*

In this paper we consider the operator

$$L(r,q(x))=-\Delta^r+q(x),\ x\in R^d,\ d\geq 2 \tag{1}$$

with a periodic (relative to a lattice $\Omega$) potential $q(x)\in W_2^s(F)$, where $s\geq s_0=\frac{3d-1}{2}(3^d+d+2)+\frac{1}{4}d3^d+d+6,\ F\equiv R^d/\Omega$ is a fundamental domain of $\Omega$, and, without loss of generality, we assume that the measure $\mu(F)$ of $F$ is 1. Let $L_t(r,q(x))$ be the operator generated in $F$ by (1) and the quasiperiodic conditions:

$$u(x+\omega)=e^{i(t,\omega)}u(x),\ \forall\omega\in\Omega, \tag{2}$$

where $t \in F^* \equiv R^d/\Gamma$ and $\Gamma$ is the lattice dual to $\Omega$, that is, $\Gamma$ is the set of all vectors $\gamma \in R^d$ satisfying $(\gamma, \omega) \in 2\pi Z$ for all $\omega \in \Omega$. It is well-known that the operator $L_t(r, q(x))$ has a discrete spectrum denoted by $\Lambda_1(t) \leq \Lambda_2(t) \leq$ ....These functions are called band functions of the operator $L$ and they form the spectrum $Spec(L)$ of the operator $r$, that is, $Spec(L) = \{\Lambda_n(t) : t \in F^*, n = 1, 2, ...\}$. The corresponding eigenfunctions $\Psi_{1,t}(x),\ \Psi_{2,t}(x), ...,$ of $r_t$ are known as Block functions. In the case $q(x) = 0$ these eigenvalues and eigenfunctions are $\mid \gamma + t \mid^{2r}$ and $e^{i(\gamma+t,x)}$ respectively, for $\gamma \in \Gamma$.

The paper consists of three sections.

In section 1 the eigenvalues $|\gamma + t|^{2r}$, for big $\gamma \in \Gamma$, are divided into two groups: non-resonance ones and resonance ones and for the perturbations of each group various asymptotic formulae are obtained. To describe the non-resonance and resonance eigenvalues $|\gamma + t|^{2r}$ of the order of $\rho^{2r}$ ( written as $|\gamma + t|^{2r} \sim \rho^{2r}$ which means that $c_1\rho < |\gamma + t| < c_2\rho$ ,where by $c_i$ for $i = 1, 2, ...$ we denote the positive, independent of $\rho$, constants, whose exact values are inessential) for big paramater $\rho$ we write the potential $q(x) \in W_2^s$ in the form

$$q(x) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1} e^{i(\gamma_1, x)} + O(\rho^{-p\alpha}), \tag{3}$$

where $\Gamma(\rho^\alpha) = \{\gamma \in \Gamma : 0 <\mid \gamma \mid< \rho^\alpha)\}$, $p = s - d$, $q_0 = 0$, $\alpha = \frac{1}{q}$, $q = 3^d + d + 2$. Note that $q(x) \in W_2^s$ means that $\sum_\gamma \mid q_\gamma \mid^2 (1+ \mid \gamma \mid^{2s}) < \infty$. If $s \geq d$ then

$$\sum_\gamma \mid q_\gamma \mid< c_3,\ \ \sup \mid \sum_{\gamma \notin \Gamma(\rho^\alpha)} q_\gamma e^{i(\gamma,x)} \mid \leq \sum_{|\gamma| \geq \rho^\alpha} \mid q_\gamma \mid= O(\rho^{-p\alpha}), \tag{4}$$

i.e., (3) holds. Define $\alpha_k = 3^k\alpha$, for $k = 1, 2, ..., d-1$, and introduce the sets $V^r_{\gamma_1}(\rho^{\alpha_1}) \equiv \{x \in R^d :\mid\mid x \mid^{2r} - \mid x + \gamma_1 \mid^{2r}\mid< \rho^{\alpha_1}\}$,

$$E_1^r(\rho^{\alpha_1}, p) \equiv \bigcup_{\gamma_1 \in \Gamma(p\rho^\alpha)} V^r_{\gamma_1}(\rho^{\alpha_1}),\ \ U^r(\rho^{\alpha_1}, p) \equiv R^d \backslash E_1^r(\rho^{\alpha_1}, p)$$

$$E_k^r(\rho^{\alpha_k}, p) \equiv \bigcup_{\gamma_1, \gamma_2, ..., \gamma_k \in \Gamma(p\rho^\alpha)} (\cap_{i=1}^k V^r_{\gamma_i}(\rho^{\alpha_k})),$$

where the intersection $\cap_{i=1}^k V^r_{\gamma_i}$ in the definition of $E_k^r$ is taken over $\gamma_1, \gamma_2, ..., \gamma_k$ which are linearly independent and the length of $\gamma_i$ is not greater than the length of the other vectors in $\Gamma \cap \gamma_i R$. The set $U^r(\rho^{\alpha_1}, p)$ is said to be a non-resonance domain and the eigenvalue $\mid \gamma+t \mid^{2r}$ is called a non-resonance eigenvalue if $\gamma + t \in U^r(\rho^{\alpha_1}, p)$. The domains $V^r_{\gamma_1}(\rho^{\alpha_1})$ for $\gamma_1 \in \Gamma(p\rho^{\alpha})$ are called resonance domains and the eigenvalue $\mid \gamma + t \mid^{2r}$ is called a resonance eigenvalue if $\gamma + t \in V^r_{\gamma_1}(\rho^{\alpha_1})$. In Section 1 we prove that for every $\gamma + t \in U^r(\rho^{\alpha_1}, p)$ there exists an eigenvalue $\Lambda_N$ of the operator $L_t(r, q(x))$ satisfying the following formulae

$$\Lambda_N =\mid \gamma + t \mid^{2r} + F_{k-1}(\gamma + t) + O(\rho^{-3k\alpha}), \tag{5}$$

for $k = 1, 2, ..., [\frac{1}{3}(p - \frac{1}{2}q(d-1))]$, where $F_{k-1}$ explicitly expresed by the potential and eigenvalues of $L_t(r, 0)$ and $[a]$ denotes the integer part of $a$. Moreover we renumber the eigenvalues $\Lambda_N(t)$ by $\Lambda_{N(\gamma+t)}(t)$ or by $\Lambda(\gamma+t)$ if

$$\mid b(N,\gamma) \mid= \max_{\gamma\in\Gamma} \mid b(N,\gamma) \mid, \tag{6}$$

where the max is taken under condition (15) ( in the case when the max is attained for several $\gamma$, then the selection from them is arbitrary), $b(N,\gamma) = (\Psi_{N,t}, e^{i(\gamma+t,x)})$, $\Psi_{N,t}(x)$ is a normalized eigenfunction of $L_t(r, q(x))$ corresponding to $\Lambda_N(t)$, and prove:

(a) if $\gamma + t$ is in non-resonance domain then $\Lambda(\gamma + t)$ satisfies (5).

(b) if $\gamma + t \in E^r_s \backslash E^r_{s+1}$, where $s = 1, 2, ..., d-1$, then

$$\Lambda(\gamma + t) = \lambda_j(\gamma + t) + O(\rho^{-k\alpha}), \tag{7}$$

where $\lambda_j$ is an eigenvalue of the matrix $C(\gamma + t)$ (more precisely about this see Theorem 1(c)). Every big eigenvalue of the operator $L_t(r, q(x))$ for all values of quasimomenta $t$ satisfies one of these formulae.

In section 2 we construct the subset $B$ of $U^r(\rho^{\alpha_1}, p)$ with the following properties:

Pr.1. If $\gamma + t \in B$ then there exists a unique eigenvalue $\Lambda_{N(\gamma+t)}(t) \equiv \Lambda(\gamma + t)$ of operator $L_t(r, q(x))$ satisfying (5). This is a simple eigenvalue of $L_t(r, q(x))$. Therefore we call the set $B$ the simple set of quasimomenta.

Pr.2. The eigenfunction $\Psi_{N(\gamma+t)}(x) \equiv \Psi_{\gamma+t}(x)$ corresponding to the eigenvalue $\Lambda(\ \gamma + t)$, for $\gamma + t \in B$, is close to $e^{i\langle\gamma+t,x\rangle}$, namely

$$\Psi_{\gamma+t}(x) = e^{i(\gamma+t,x)} + \Phi_{k-1}(x) + O(\rho^{-k\alpha_1}), k = 1, 2, ... \ , \tag{8}$$

where $\Phi_0(x) = 0$, and $\Phi_{k-1}(x)$, for $k > 1$, is a linear conbination of $e^{i(\gamma+t+\gamma',x)}$ for $\gamma' \in \Gamma((k-1)\rho^{\alpha}) \cup \{0\}$ with explicitly expressed coefficients.

In section 3 we prove that the set $B$ has asymptotically ful measure in $R^d$ and contains the intervals $\{a+sb : s \in [-1,1]\}$ such that $\Lambda(a-b) < \rho^{2r}$, $\Lambda(a+b) > \rho^{2r}$ and $\Lambda(\gamma+t)$ is continuous on this intervals. Hence there exists $\gamma+t$ for which $\Lambda(\gamma+t) = \rho^{2r}$. It implies the validity of Bethe-Sommerfeld conjecture for $L(r, q(x))$. Construction of the set $B$ consists of two steps.

Step 1. We prove that all eigenvalues $\Lambda(\gamma+t)$, for $\mid \gamma \mid \sim \rho$, of the operator $r_t(r, q(x))$, lie in the $\varepsilon_1 = \rho^{-d-2\alpha}$, neighborhood of a numbers $F(\gamma+t) = \mid \gamma+t \mid^{2r} + F_{k_1-1}(\gamma+t)$, $\lambda_j(\gamma+t)$ ( see (5), (7)), where $k_1 = [\frac{d}{3\alpha}] + 2$. We call these number as the known part of the eigenvalues.

Step 2. By eliminating the set of quasimomenta $\gamma+t$, for which the known parts $F(\gamma+t)$ of $\Lambda(\gamma+t)$ are situated from the known parts $F(\gamma^{'}+t)$, $\lambda_j(\gamma^{'}+t)$ $(\gamma^{'} \neq \gamma)$ of other eigenvalues at a distance less than $2\varepsilon_1$ we constructed the set $B$ such that if $\gamma+t \in B$ then the following conditions (called simplicity conditions for eigenvalue $\Lambda(\gamma+t) \equiv \Lambda_{N(\gamma+t)}(t)$ ) hold

$$\mid F(\gamma+t) - F(\gamma^{'}+t) \mid \geq 2\varepsilon_1, \tag{9}$$

for $\gamma^{'} \in K \backslash \{\gamma\}$, $\gamma^{'}+t \in U^1(\rho^{\alpha_1}, p)$ and

$$\mid F(\gamma+t) - \lambda_j(\gamma^{'}+t) \mid \geq 2\varepsilon_1, \tag{10}$$

for $\gamma^{'} \in K, \gamma^{'}+t \in E_k^1 \backslash E_{k+1}^1, j = 1, 2, ...,$ where $K$ is the set of $\gamma^{'} \in \Gamma$ satisfying $\mid F(\gamma+t) - \mid \gamma^{'}+t \mid^{2r} \mid < \frac{1}{3}\rho^{\alpha_1}$. Thus $B$ is the set of quasimomenta $\gamma+t \in U^1(\rho^{\alpha_1}, p)$ satisfying the simplicity conditions (9), (10). By these conditions the eigenvalue $\Lambda(\gamma+t)$ can not coincide with other eigenvalues.

The listed all results ( formulas (5), (7), construction and investigations of the simply set $B$, the proof of asymptotic formulas (8) for Bloch function and implication the proof of the Bethe-Sommerfeld conjecture for arbitraty dimension and arbitrary lattices from this formulas ) for the first time were proved in papers [12-15] for the Schrodinger operator $L(1, q(x))$. The main difficults and the crucial point of papers [12-14] were the construction and investigations of the simple set $B$ of quasimomenta in neighborhood of the surface $\{\gamma+t \in U^1(\rho^{\alpha_1}, p) : F(\gamma+t) = \rho^2\}$. If $d = 2, 3$ then $F(\gamma+t) = \mid \gamma+t \mid^2$, and the matrix $C(\gamma+t)$ corresponds to the Schrodinger operator with directional potential $q_{\gamma_1}(x) = \sum_{n \in Z} q_{n\gamma_1} e^{i(n\gamma_1, x)}$ ( see [13]). So for construction of the simple set $B$ of quasimomenta we eliminate the vicinities of the diffraction planes ant the the sets connected with directional potential ( see (9), (10)). The simple sets $B$ of quasimomenta for the first time is constructed and investigated in [13] for $d = 3$ and in [12] for $d \geq 2$ ( if $d = 2$ then for nonsmooth potential $q(x) \in L_2(F)$, if $d > 2$ for smooth potential). Then Yu.E. Karpeshina proved ( see [6],[7],[8]) the convergense of pertubation series for a set, that is similar to $B$, of quasimomenta in the cases 1. $2r > d$; 2. $4r > d+1$, $(2r \leq d)$; 3. $d = 3, r = 1$, and using it she proved the validity of the Bethe-Sommerfeld conjecture in these cases. In papers [2,3] asymptotic formulas for eigenvalues and Bloch function of the two and three dimensional operator $L_t(1, q(x))$ were obtained. In [4] asymptotic formulae for nonresonance eigenvalues of $L_0(1, q(x))$ were obtained.

For the first time M.M. Skriganov [10,11] proved the validity of the Bethe-Sommerfeld conjecture for the Scrodinger operator for dimension $d = 2, 3$ for arbitrary lattice, for dimension $d > 3$ for rational lattice and for the operator $L(r, q(x))$, for $2r > d$. The Skriganov's method is based on the detail investigation of the arifmetical and geometrical properties of the lattice. B.E.J.Dahlberg and E.Trubowits [1] using an asymptotic of Bessel function, gave the simpler proof of this conjecture for the two dimensional Scrodinger operator. B. Helffer and A. Mohamed [5] by investigations the integrated density of states proved the validity of this conjecture for the Scrodinger operator for $d \leq 4$, for arbitrary lattice. Recently Parnovski and Sobelev [9] proved this conjecture for the operator $L(r, q(x))$, for $8r > d+3$.

The method of this paper and papers [12-15] is a first and uniqie for the present by which the validity of the Bethe-Sommerfeld conjecture for arbitrary lattice, and for arbitrary dimension is proved. For the operator $L(r, q)$, in order to avoid eclipsing the essence by technical details, we assume that $r \geq 1$. It can be replaced by $r > n_{s,d}$, where $n_{s,d} < 1$ and depends on the smoothness $s$ of potential $q(x) \in W_2^s(R^d/\Omega)$ and the dimension $d$ of $R^d$ .

In this paper for the different types of the measures of the subset $A$ of $R^d$ we use the same notation $\mu(A)$, because it will be clear which measure we mean. By $\mid A \mid$ we denote the number of elements of the set $A \subset \Gamma$ and use the following obvious fact. If $a \sim \rho$ then the number of elements of the set $\{\gamma + t : \gamma \in \Gamma\}$ satisfying $\mid\mid \gamma + t \mid -a \mid < 1$ is less than $c_\Gamma \rho^{d-1}$, where $c_\Gamma$ is a constant depending on $\Gamma$. Therefore the number of eigenvalues of $L_t(r, q)$ lying in $(a^{2r} - \rho^{2r-1}, a^{2r} + \rho^{2r-1})$ is less than $c_\Gamma \rho^{d-1}$. Besides we use the inequalities:

1. $\alpha_1 + d\alpha < 1 - \alpha$ ; 2. $d\alpha < \frac{1}{2}\alpha_d$; 3. $k_1 \leq \frac{1}{3}(p - \frac{1}{2}(q(d-1))$; 4. $p_1\alpha_1 \geq p\alpha$
5. $3k_1\alpha > d + 2\alpha$; 6. $\alpha_k + (k-1)\alpha < 1$; 7. $\alpha_{k+1} > 2(\alpha_k + (k-1))\alpha$,

for $k = 1, 2, ..., d$, that can be easily verified by using the definitions $p = s - d$, $\alpha_k = 3^k\alpha$, $\alpha = \frac{1}{q}$, $q = 3^d + d + 2$, $k_1 = [\frac{d}{3\alpha}] + 2$, $p_1 \equiv [\frac{p}{3}] + 1$ of the numbers $p, q, \alpha_k, \alpha, k_1, p_1$.

# 1 Asymptotic Formulae for Eigenvalues

In this section we obtain the asymptotic formulae for nonresonance eigenvalues by iteration of the well-known formula

$$(\Lambda_N - \mid \gamma + t \mid^{2r}) b(N,\gamma) = (\Psi_{N,t}(x) q(x), e^{i(a+t,x)}). \tag{11}$$

Putting the decomposition (3) of $q(x)$ into (11), we get

$$(\Lambda_N - \mid \gamma + t \mid^{2r}) b(N,\gamma) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1} b(N, \gamma - \gamma_1) + O(\rho^{-p\alpha}). \tag{12}$$

From the relations (11), (12) it follows that

$$b(N,\gamma^{'}) = \frac{(\Psi_{N,t} q(x), e^{i(\gamma^{'}+t,x)})}{\Lambda_N - \mid \gamma^{'} + t \mid^{2r}} = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \frac{q_{\gamma_1} b(N, \gamma^{'} - \gamma_1)}{\Lambda_N - \mid \gamma^{'} + t \mid^{2r}} + O(\rho^{-p\alpha}) \tag{13}$$

for every vector $\gamma^{'} \in \Gamma$ satisfying the inequality

$$\mid \Lambda_N - \mid \gamma^{'} + t \mid^{2r} \mid > \frac{1}{2} \rho^{\alpha_1}, \tag{14}$$

which is called the iterability condition. If

$$\mid \Lambda_N - \mid \gamma + t \mid^{2r} \mid < \frac{1}{2} \rho^{\alpha_1}, \tag{15}$$

and $\mid \gamma + t \mid^{2r}$ is a non-resonance eigenvalue, i.e., $\gamma + t \in U^r(\rho^{\alpha_1}, p)$ then

$$\mid\mid \gamma + t \mid^{2r} - \mid \gamma - \gamma_1 + t \mid^{2r} \mid > \rho^{\alpha_1}, \mid \Lambda_N - \mid \gamma - \gamma_1 + t \mid^{2r} \mid > \frac{1}{2} \rho^{\alpha_1}, \tag{16}$$

for all $\gamma_1 \in \Gamma(p\rho^\alpha)$. Hence the vector $\gamma - \gamma_1$ satisfies the iterability condition (14). Therefore, in (13) one can replace $\gamma^{'}$ by $\gamma - \gamma_1$ and write

$$b(N, \gamma - \gamma_1) = \sum_{\gamma_2 \in \Gamma(\rho^\alpha)} \frac{q_{\gamma_2} b(N, \gamma - \gamma_1 - \gamma_2)}{\Lambda_N - \mid \gamma - \gamma_1 + t \mid^{2r}} + O(\rho^{-p\alpha}).$$

Putting it into right side of (12) and isolating the terms with the coefficient $b(N,\gamma)$, i.e., isolating the case $\gamma_1 + \gamma_2 = 0$ we get

$$(\Lambda_N - \mid \gamma + t \mid^{2r}) b(N,\gamma) = \sum_{\gamma_1, \gamma_2 \in \Gamma(\rho^\alpha)} \frac{q_{\gamma_1} q_{\gamma_2} b(N, \gamma - \gamma_1 - \gamma_2)}{\Lambda_N - \mid \gamma - \gamma_1 + t \mid^{2r}} + O(\rho^{-p\alpha}) =$$

$$\sum_{\gamma_1\in\Gamma(\rho^\alpha)}\frac{|\ q_{\gamma_1}\ |^2\ b(N,\gamma)}{\Lambda_N-\mid\gamma-\gamma_1+t\mid^{2r}}+\sum_{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha)}\frac{q_{\gamma_1}q_{\gamma_2}b(N,\gamma-\gamma_1-\gamma_2)}{\Lambda_N-\mid\gamma-\gamma_1+t\mid^{2r}}+O(\rho^{-p\alpha}).$$

The last sum is taken under the additional condition $\gamma_1+\gamma_2\neq 0$. Repeating this process $p_1\equiv[\frac{p}{3}]+1$ times, i.e., in the last formula replacing $b(N,\gamma^{'})$ for $\gamma^{'}=\gamma-\gamma_1-\gamma_2$, and then for $\gamma^{'}=\gamma-\gamma_1-\gamma_2-\gamma_3, ...,$ by its expression from (13) and isolating each time the terms with coefficient $b(N,\gamma)$ we obtain

$$(\Lambda_N-\mid\gamma+t\mid^{2r})b(N,\gamma)=A_{p_1}(\Lambda_N,\gamma+t)b(N,\gamma)+C_{p_1}+O(\rho^{-p\alpha}),\quad (17)$$

where $A_{p_1}(\Lambda_N,\gamma+t)=\sum_{k=1}^{p_1}S_k(\Lambda_N,\gamma+t)$ ,

$$S_k(\Lambda_N,\gamma+t)=\sum_{\gamma_1,...,\gamma_k\in\Gamma(\rho^\alpha)}\frac{q_{\gamma_1}q_{\gamma_2}...q_{\gamma_k}q_{-\gamma_1-\gamma_2-...-\gamma_k}}{\prod_{j=1}^{k}(\Lambda_N-\mid\gamma+t-\sum_{i=1}^{j}\gamma_i\mid^{2r})},$$

$$C_{p_1}=\sum_{\gamma_1,...,\gamma_{p_1+1}\in\Gamma(\rho^\alpha)}\frac{q_{\gamma_1}q_{\gamma_2}...q_{\gamma_{p_1+1}}b(N,\gamma-\gamma_1-\gamma_2-...-\gamma_{p_1+1})}{\prod_{j=1}^{p_1}(\Lambda_N-\mid\gamma+t-\sum_{i=1}^{j}\gamma_i\mid^{2r})}.$$

Here the sums in the expressions of $S_k$ and $C_{p_1}$ are taken under the additional conditions $\gamma_1+\gamma_2+...+\gamma_s\neq 0$ for $s=1,2,...,k$ and for $s=1,2,...,p_1$ respectively. These conditions and the inclusion $\gamma_i\in\Gamma(\rho^\alpha)$, for $i=1,2,...,p_1$ imply the relation $\sum_{i=1}^{j}\gamma_i\in\Gamma(p\rho^\alpha)$ for the sum in the denominators of the fractions in expressions of $S_k$ and $C_{p_1}$. Therefore from the relation (16) it follows that the absolute values of the denominators of the fractions in expressions of $S_k$ and $C_{p_1}$ more than $(\frac{1}{2}\rho^{\alpha_1})^k$ and $(\frac{1}{2}\rho^{\alpha_1})^{p_1}$ respectively. Hence the first inequality in (4) and $p_1\alpha_1\geq p\alpha$ ( see the inequality 4. in the end of introduction) give

$$C_{p_1}=O(\rho^{-p_1\alpha_1})=O(\rho^{-p\alpha}),\ \ S_k(\Lambda_N,\gamma+t)=O(\rho^{-k\alpha_1}),\forall k=1,2,...,p_1.\quad (18)$$

Taking into account that for $\Lambda_N$ we only used condition (15) we obtain

$$S_k(a,\gamma+t)=O(\rho^{-k\alpha_1}),\forall k=1,2,...,p_1\quad (19)$$

for all $a$ satisfying $\mid a-\mid\gamma+t\mid^{2r}\mid<\frac{1}{2}\rho^{\alpha_1}$. Thus finding the number $N$ such that $\Lambda_N$ is close to $\mid\gamma+t\mid^{2r}$ and $b(N,\gamma)$ is not very small then dividing both sides of (17) by $b(N,\gamma)$ we can get the asymptotic formulas for $\Lambda_N$ (see Theorem 1(a)). These formulas show that in the nonresonance case the eigenvalue of the perturbed operator $L_t(r,q(x))$ is close to the eigenvalue of the unperturbed operator $L_t(r,0)$. However in Theorem 1(c) we prove that if $\mid \gamma+t \mid^{2r}$ is resonance eigenvalue, say, if $\gamma+t \in (\cap_{i=1}^{k} V_{\gamma_i}^{r}(\rho^{\alpha_k}))$ for $k \geq 1$, where $\gamma_1, \gamma_2, ..., \gamma_k$ are linearly independent vectors of $\Gamma(p\rho^{\alpha})$ and the length of $\gamma_i$ is not more than the length of other vectors in $\Gamma \cap \gamma_i R$, then the corresponding eigenvalue of $L_t(r, q(x))$ is close to the eigenvalue of the matrix constructed as follows. Introduse the sets

$$B_k = B_k(\gamma_1, \gamma_2, ..., \gamma_k) = \{b : b = \sum_{i=1}^{k} n_i \gamma_i, n_i \in Z, \mid b \mid < \frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}\},$$

$$B_k(\gamma+t) = \gamma + t + B_k \equiv \{\gamma + t + b : b \in B_k, \},$$

$$B_k(\gamma+t, p_1) = B_k(\gamma+t) + \Gamma(p_1\rho^{\alpha}).$$

Denote by $h_i + t$ for $i = 1, 2, ..., b_k$ the vectors of $B_k(\gamma + t, p_1)$, where

$b_k \equiv b_k(\gamma_1, \gamma_2, ..., \gamma_k)$ is the number of the vectors of $B_k(\gamma+t, p_1)$. Define the matrix $C(\gamma+t, \gamma_1, \gamma_2, ..., \gamma_k) = (c_{i,j})$ by the formula

$c_{i,j} = q_{h_i - h_j}, \forall i \neq j, c_{i,i} = \mid h_i + t \mid^{2r}$, $i, j = 1, 2, ..., b_k$.

We use essentially the following obvious statement:

if $x \in R^d$, $\mid x \mid \sim \rho$, $\gamma_1 \in \Gamma$, $\mid x + \gamma_1 \mid \sim \rho$ then

$$\mid x \mid^{2r} - \mid x + \gamma_1 \mid^{2r} = a^{2(r-1)}(\mid x \mid^2 - \mid x + \gamma_1 \mid^2) \tag{20}$$

where $a \sim \rho$. Therefore $V_{\gamma_1}^{r}(\rho^{\alpha_1}) \subset V_{\gamma_1}^{1}(\rho^{\alpha_1})$,

$$(\cap_{i=1}^{k} V_{\gamma_i}^{r}(\rho^{\alpha_k})) \subset \cap_{i=1}^{k} V_{\gamma_i}^{1}(\rho^{\alpha_k})), \tag{21}$$

$$U^1(\rho^{\alpha_1}, p) \subset U^r(\rho^{\alpha_1}, p) \tag{22}$$

for $r \geq 1$ and $k = 1, 2, ....$Taking into account these inclusions we consider the resonance eigenvalue $\mid \gamma + t \mid^{2r}$ for $\gamma + t \in (\cap_{i=1}^{k} V_{\gamma_i}^{1}(\rho^{\alpha_k}))$.

**Theorem 1** *(a) Suppose $\gamma + t \in U^r(\rho^{\alpha_1}, p)$, $\mid \gamma \mid \sim \rho$. If (15) and*

$$\mid b(N, \gamma) \mid > c_4 \rho^{-c\alpha} \tag{23}$$

*hold then $\Lambda_N$ satisfies formulas (5), for $k = 1, 2, ..., [\frac{1}{3}(p-c)]$, where*

$$F_s = O(\rho^{-\alpha_1}), \forall s = 0, 1, ..., \tag{24}$$

*and $F_0 = 0$, $F_s = A_s(\mid \gamma + t \mid^{2r} + F_{s-1}, \gamma + t)$, for $s = 1, 2, ....$*

($b$) *For* $\gamma+t\in U^{r}(\rho^{\alpha_{1}},p)$, $\mid\gamma\mid\sim\rho$ *there exists an eigenvalue* $\Lambda_{N}$ *of* $L_{t}(r,q(x))$ *satisfying (5).*

($c$)*Suppose* $\mid\gamma\mid\sim\rho$, $\gamma+t\in(\cap_{i=1}^{k}V_{\gamma_{i}}^{1}(\rho^{\alpha_{k}}))\backslash E_{k+1}^{1}$, *where* $1\leq k\leq d-1$. *If (15) and (23) hold then there is an indeks* $j$ *such that*

$$\Lambda_{N}=\lambda_{j}(\gamma+t)+O(\rho^{-(p-c-\frac{1}{4}d3^{d})\alpha}),\tag{25}$$

*where* $\lambda_{1}(\gamma+t)\leq\lambda_{2}(\gamma+t)\leq...\leq\lambda_{b_{k}}(\gamma+t)$ *are the eigenvalues of the matrix* $C(\gamma+t,\gamma_{1},\gamma_{2},...,\gamma_{k})$.

($d$) *Every eigenvalue* $\Lambda(\gamma+t)\sim\rho^{2r}$ *( see (6) for the definition of* $\Lambda(\gamma+t)$*) of the operator* $L_{t}(r,q(x))$ *satisfies either (5) or (25) for* $c=\frac{q(d-1)}{2}$, *namely* $\Lambda(\gamma+t)$ *satisfies (5) for* $\gamma+t\in U^{r}(\rho^{\alpha_{1}},p)$ *and satisfies (25) for* $\gamma+t\in E_{k}^{1}\backslash E_{k+1}^{1}$, *where* $k=1,2,...,d-1$.

($a$) To prove (5) in case $k=1$ we divide both side of (17) by $b(N,\gamma)$ and use the estimations (18). Then we obtain

$$\Lambda_{N}-\mid\gamma+t\mid^{2r}=\mid\gamma+t\mid^{2r}+O(\rho^{-\alpha_{1}})\tag{26}$$

This and $\alpha_{1}=3\alpha$ ( see the end of the introduction) imply that formula (5) for $k=1$ holds and $F_{0}=0$. Hence (24) for $s=0$ is also proved. Moreover by (19) we have

$S_{k}(\mid\gamma+t\mid^{2r}+O(\rho^{-\alpha_{1}}),\gamma+t)=O(\rho^{-\alpha_{1}})$ for $k=1,2,....$ Therefore (24) for arbitrary $s$ follows from the definition of $F_{s}$ by induction. Now we prove (5) for all $k$ by induction. If (5) is true for $k=j$, then putting the expression from (5) (for $k=j$), for $\Lambda_{N}$ into $A_{p_{1}}(\Lambda_{N},\gamma+t)$ in (17) and dividing both sides of (17) by $b(N,\gamma)$ we get

$$\begin{gathered}\Lambda_{N}=\mid\gamma+t\mid^{2r}+A_{p_{1}}(\mid\gamma+t\mid^{2r}+F_{j-1}+O(\rho^{-j\alpha_{1}}),\gamma+t)+O(\rho^{-(p-c)\alpha})=\\ \mid\gamma+t\mid^{2r}+\{A_{p_{1}}(\mid\gamma+t\mid^{2r}+F_{j-1}+O(\rho^{-j\alpha_{1}}),\gamma+t)-\\ A_{p_{1}}(\mid\gamma+t\mid^{2r}+F_{j-1},\gamma+t)\}+A_{p_{1}}(\mid\gamma+t\mid^{2r}+F_{j-1},\gamma+t)+O(\rho^{-(p-c)\alpha}).\end{gathered}$$

To prove ($a$) for $k=j+1$ we need to show that the expression in curly brackets is equal to $O(\rho^{-2(j+1)\alpha_{1}})$. It can be cheked by using (4), (16), (24) and the obvious relation

$$\frac{1}{\prod_{j=1}^{s}(\mid \gamma+t\mid^{2r}+F_{j-1}+O(\rho^{-j\alpha_1})-\mid \gamma+t-\sum_{i=1}^{s}\gamma_i\mid^{2r})}-$$
$$\frac{1}{\prod_{j=1}^{s}(\mid \gamma+t\mid^{2r}+F_{j-1}-\mid \gamma+t-\sum_{i=1}^{s}\gamma_i\mid^{2r})}$$
$$=\frac{1}{\prod_{j=1}^{s}(\mid \gamma+t\mid^{2r}+F_{j-1}-\mid \gamma+t-\sum_{i=1}^{s}\gamma_i\mid^{2r})}(\frac{1}{1-O(\rho^{-(j+1)\alpha_1})}-1)$$

$=O(\rho^{-(j+1)\alpha_1}),\ \forall s=1,2,...,p_1.$

$(b)$ Let $A$ be the set of indices $N$ satisfying (15). Using (11) and Bessel inequality, we obtain

$$\sum_{N\notin A}\mid b(N,\gamma)\mid^2=\sum_{N\notin A}\mid\frac{(\Psi_N(x),(q(x)e^{i(\gamma+t,x)})}{\Lambda_N-\mid\gamma+t\mid^{2r}}\mid^2=O(\rho^{-2\alpha_1})$$

Hence by the Parseval equality we have $\sum_{N\in A}\mid b(N,\gamma)\mid^2=1-O(\rho^{-2\alpha_1})$. This together with $\mid A\mid< c_\Gamma\rho^{d-1}=c_\Gamma\rho^{(d-1)q\alpha}$ ( see the end of the introduction) implies that there exsists a number $N$ satisfying $\mid b(N,\gamma)\mid>\frac{1}{2}(c_\Gamma)^{-1}\rho^{-\frac{(d-1)q}{2}\alpha}$ , that is, (23) holds, for $c=\frac{(d-1)q}{2}$ . Thus $\Lambda_N$ satisfies (5) according to $(a)$.

$(c)$Writing the equation (12) for all $h_i+t\in B_k(\gamma+t,p_1)$, we obtain

$$(\Lambda_N-\mid h_i+t\mid^{2r})b(N,h_i)=\sum_{\gamma'\in\Gamma(\rho^\alpha)}q_{\gamma'}b(N,h_i-\gamma')+O(\rho^{-p\alpha}),\tag{27}$$

for $i=1,2,...,b_k$. First we show that if $(h_i-\gamma'+t)\notin B_k(\gamma+t,p_1)$ then

$$b(N,h_i-\gamma')=$$
$$\sum_{\gamma_1,...,\gamma_{p_1-1}\in\Gamma(\rho^\alpha)}\frac{q_{\gamma_1}q_{\gamma_2}...q_{\gamma_{p_1}}b(N,h_i-\gamma'-\sum_{i=1}^{p_1}\gamma_i)}{\prod_{j=0}^{p_1-1}(\Lambda_N-\mid h_i-\gamma'+t-\sum_{i=1}^{j}\gamma_i\mid^{2r})}+\tag{28}$$
$$+O(\rho^{-p\alpha})=O(\rho^{-p\alpha}).$$

For this, we apply the formula (13) $p_1$ times and use the inequality

$$\mid\Lambda_N-\mid h_i-\gamma'-\gamma_1-\gamma_2-...-\gamma_s+t\mid^{2r}\mid>\frac{1}{6}\rho^{\alpha_{k+1}},\tag{29}$$

for $h_i+t\in B_k(\gamma+t,p_1)$, $(h_i-\gamma^{'}+t)\notin B_k(\gamma+t,p_1)$, and $\gamma_j\in\Gamma(\rho^{\alpha})$, where $j=1,2,...,s$ and $s=0,1,...,p_1-1$. So now we prove (29). The relations $p>2p_1$ ( see the end of the introduction), $h_i+t\in B_k(\gamma+t,p_1)$, $(h_i-\gamma^{'}+t)\notin B_k(\gamma+t,p_1)$ and

$\mid\gamma^{'}\mid,\mid\gamma_1\mid,...,\mid\gamma_{p_1-1}\mid<\rho^{\alpha}$ imply that

$a_s\equiv h_i-\gamma^{'}-\gamma_1-\gamma_2-...-\gamma_s+t\in B_k(\gamma+t,p)\backslash B_k(\gamma+t)$ for all $s=0,1,...,p_1-1$. Since $B_k(\gamma+t,p)=\gamma+t+B_k+\Gamma(p\rho^{\alpha})$ we have a decomposition $a_s=\gamma+t+b+a$, where $b\in B_k, a\in\Gamma(p\rho^{\alpha})$, that is,

$$\mid b\mid<\frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}},\mid a\mid<p\rho^{\alpha}. \tag{30}$$

Using this notation we see that (29) follows from (15) and

$$\mid\mid\gamma+t+a+b\mid^{2r}-\mid\gamma+t\mid^{2r}\mid>\frac{1}{5}\rho^{\alpha_{k+1}} \tag{31}$$

It follows from (20) that, to verify (31) it is enough to prove it for $r=1$. To prove (31) for $r=1$ we consider two cases:

Case 1. $a\in P\equiv Span\{\gamma_1,\gamma_2,...,\gamma_k\}$. Then the relations $a+b\in P$, $\gamma+t+a+b\notin\gamma+t+B_k$ imply that $a+b\in P\backslash B_k$ ,i.e., $\mid a+b\mid\geq\frac{1}{2}\ \rho^{\frac{1}{2}\alpha_{k+1}}$. Consider orthogonal decomposition $\gamma+t=x+v$ of $\gamma+t$, where $v\in P$ and $x\perp P$. It is not hard to show that (see equality (40) in Remark 1)

$$\mid v\mid=O(\rho^{(k-1)\alpha+\alpha_k}). \tag{32}$$

It together with above inequality for $\mid a+b\mid$ and the relations

$\alpha_{k+1}>2(\alpha_k+(k-1)\alpha)$ ( see the inequality 7. in the end of the introduction), $\gamma+t=x+v$, $(x,v)=(x,a)=(x,b)=0$,

$$\mid\gamma+t+a+b\mid^2-\mid\gamma+t\mid^2=\mid a+b+v\mid^2-\mid v\mid^2 \tag{33}$$

give us the estimation (31) for $r=1$.

Case 2. $a\notin P$. First we show that

$$\mid\mid\gamma+t+a\mid^2-\mid\gamma+t\mid^2\mid\geq\rho^{\alpha_{k+1}}. \tag{34}$$

Suppose that this inequality does not hold. Then $\gamma+t\in V_a^1(\rho^{\alpha_{k+1}})$. On the other hand $\gamma+t\in\cap_{i=1}^{k}V_{\gamma_i}^1(\rho^{\alpha_{k+1}})$. Therefore we have $\gamma+t\in E_{k+1}^1$ which contradicts to the conditions of the Theorem 1(c). The difference $\mid\gamma+t+a+b\mid^2-\mid\gamma+t\mid^2$ can be written as the sum of the differences

$d_1 \equiv\mid \gamma+t+a+b\mid^2 - \mid \gamma+t+b\mid^2$ and $d_2 \equiv\mid \gamma+t+b\mid^2 - \mid \gamma+t\mid^2$ . Since $d_1 =\mid \gamma+t+a\mid^2 - \mid \gamma+t\mid^2 +2\langle a,b\rangle$, it follows from the inequalities (34), (30) that $\mid d_1\mid > \frac{2}{3}\ \rho^{\alpha_{k+1}}$. Taking $a=0$ in (33) we have $d_2 =\mid b+v\mid^2 - \mid v\mid^2$ . Therefore (32), (30) and the inequality 7. in the end of the introduction imply that

$\mid d_2\mid < \frac{1}{3}\ \rho^{\alpha_{k+1}}$, $\mid d_1\mid - \mid d_2\mid > \frac{1}{3}\rho^{\alpha_{k+1}}$, that is, (31) holds.

Thus in all cases (31) and hence (29) is proved. Therefore relation (28) follows from (4) and (29), since $p_1\alpha_{k+1} > p_1\alpha_1 \geq p\alpha$ ( see the inequality 4. in the end of introduction). Hence (27) has form

$$(\Lambda_N - \mid h_i+t\mid^2)b(N,h_i) = \sum_{\gamma'} q_{\gamma'}b(N,h_i-\gamma') + O(\rho^{-p\alpha}), i=1,2,...,b_k,$$

where the sum is taken under the conditions $\gamma' \in \Gamma(\rho^\alpha)$ and

$h_i-\gamma'+t \in B_k(\gamma+t,p_2)$. It can be written as

$$(C-\Lambda_N I)(b(N,h_1),b(N,h_2),...b(N,h_{b_k})) = O(\rho^{-p\alpha}),$$

where the rigth side of this system is a vector having the norm

$\parallel O(\rho^{-p\alpha})\parallel = O(\sqrt{b_k}\rho^{-p\alpha})$. Now by (23) we have

$$c_4\rho^{-c\alpha} < (\sum_{h_i}\mid b(N,h_i)\mid^2)^{\frac{1}{2}} \leq \parallel (C-\Lambda_N I)^{-1}\parallel \sqrt{b_k}c_5\rho^{-p\alpha}, \tag{35}$$

$$\max_{i=1,2,...,b_k}\mid \Lambda_N-\lambda_i\mid^{-1} = \parallel (C-\Lambda_N I)^{-1}\parallel > c_4c_5^{-1}b_k^{-\frac{1}{2}}\rho^{-c\alpha+p\alpha}. \tag{36}$$

Using that $\mid B_k\mid = O(\rho^{\frac{k}{2}\alpha_{k+1}})$, $\mid \Gamma(p_1\rho^\alpha)\mid = O(\rho^{d\alpha})$ and $d\alpha < \frac{1}{2}\alpha_d$ ( see the end of introduction) we get

$$b_k = O(\rho^{d\alpha+\frac{k}{2}\alpha_{k+1}}) = O(\rho^{\frac{d}{2}\alpha_d}) = O(\rho^{\frac{d}{2}3^d\alpha}), \forall k=1,2,...,d-1 \tag{37}$$

Thus the formula (25) follows from (36) and (37).

$(d)$ Let $\Lambda_N \equiv \Lambda(\gamma+t)$ be any eigenvalue of order $\rho^{2r}$ of the operator $L_t(r,q(x))$. Denote by $D$ the set of all vectors $\gamma\in\Gamma$ satisfying (15). From (11), arguing as in the proof of $(b)$, we obtain $\sum_{\gamma\in D}\mid b(N,\gamma)\mid^2 = 1 - O(\rho^{-2\alpha_1})$. Since $\mid D\mid = O(\rho^{d-1})$ ( see the end of the introduction), from definition of $\Lambda(\gamma+t)$ we get

$\mid b(N,\gamma)\mid > c_6\rho^{-\frac{(d-1)}{2}} = c_6\rho^{-\frac{(d-1)q}{2}\alpha}$, that is, condition (23) for $c=\frac{(d-1)q}{2}$ holds. Now the proof follows from $(a)$ and $(c)$ since either $\gamma+t\in U^1(\rho^{\alpha_1},p)$

or $\gamma+t\in E_k^1\backslash E_{k+1}^1$ for $k=1,2,...,d-1$ ( see (41) in Remark 1). The Theorem is proved.$\diamondsuit$

**Remark 1** *Here we note that the nonresonance domain $U^r(c_7\rho^{\alpha_1},p)$ has an asymptotically full measure on $R^d$ in the sence that $\frac{\mu(U\cap B(\rho))}{\mu(B(\rho))}$ tends to 1 as $\rho$ tends to infinity, where $B(\rho)=\{x\in R^d :\mid x\mid=\rho\}$. By (22) it is enough to prove this for $r=1$. It is clear and well known that $B(\rho)\cap V_b^1(c_7\rho^{\alpha_1})$ is the part of sphere $B(\rho)$ which is contained between two parallel hyperplanes*

*$\{x:\mid x\mid^2-\mid x+b\mid^2=-c_7\rho^{\alpha_1}\}$and $\{x:\mid x\mid^2-\mid x+b\mid^2=c_7\rho^{\alpha_1}\}$. The distance of these hyperplanes from origin are $\frac{c_7\rho^{\alpha_1}}{|b|}$. This, the relation $\mid\Gamma(p\rho^\alpha)\mid=O(\rho^{d\alpha})$, and $\alpha_1+d\alpha<1-\alpha$ ( see the inequality 1. in the end of the introduction) imply*

$$\mu(B(\rho)\cap V_b^1(c_7\rho^{\alpha_1}))\sim\frac{\rho^{\alpha_1+d-2}}{\mid b\mid},\ \mu(E_1^1\cap B(\rho))=O(\rho^{d-1-\alpha}) \tag{38}$$

$$\mu(U^1(c_7\rho^{\alpha_1},p)\cap B(\rho))=(1+O(\rho^{-\alpha}))\mu(B(\rho)). \tag{39}$$

*Now we consider $E\equiv(\cap_{i=1}^k V_{\gamma_i}^1(\rho^{\alpha_k}))\cap B(\rho)$. We turn the coordinate axis so that $Span\{\gamma_1,\gamma_2,...,\gamma_k\}$ coincides with the span of the vectors*

*$e_1=(1,0,0,...,0)$, $e_2=(0,1,0,...,0)$,..., $e_k$. Then $\gamma_s=\sum_{i=1}^k\gamma_{s,i}e_i$ for $s=1,2,...,k$ . Therefore the relation $x\in\cap_{i=1}^k V_{\gamma_i}^1(\rho^{\alpha_k})$ implies that*

$$\sum_{i=1}^k\gamma_{s,i}x_i=O(\rho^{\alpha_k}), s=1,2,...,k;\ x_n=\frac{\det(b_{j,i}^n)}{\det(\gamma_{j,i})}\ ,n=1,2,...,k,$$

*where $x=(x_1,x_2,...,x_d)$, $\gamma_j=(\gamma_{j,1},\gamma_{j,2},...,\gamma_{j,k},0,0,...,0)$, $b_{j,i}^n=\gamma_{j,i}$ for $n\neq j$ and $b_{j,i}^n=O(\rho^{\alpha_k})$ for $n=j$. Taking into account that the determinant $\det(\gamma_{j,i})$ of the system is a volume of the parallelepiped*

*$\{\sum_{i=1}^k b_i\gamma_i : b_i\in[0,1], i=1,2,...,k\}$ , hence greater than some constant $c$ and using that $\mid\gamma_{j,i}\mid<p\rho^\alpha$ ( since $\gamma_j\in\Gamma(p\rho^\alpha)$ ) we get the estimations*

$$x_n=O(\rho^{\alpha_k+(k-1)\alpha})\ ,\forall n=1,2,...,k \tag{40}$$

*From this for projection $v$ of $x\in\cap_{i=1}^k V_{\gamma_i}(\rho^{\alpha_k})$ onto $Span\{\gamma_1,\gamma_2,...,\gamma_k\}$ we obtain estimation (32). Moreover, note that if $k=d$, then $v=x$ and by (40) we have $\mid x\mid=O(\rho^{\alpha_d+(d-1)\alpha})$ which is imposible, becouse $\alpha_d+(d-1)\alpha<1$ ( see the inequality 6. in the end of the introduction) and $x\in B(\rho)$. It means* *that the intersection of d resonance domains lying in* $\{| x |> \rho_0\}$ *for* $\rho_0 \gg 1$ *is empty set. Therefore we have*

$$R^d \cap \{| x |> \rho_0\} = (U^1 \cup (\cup_{s=1}^{d-1}(E_s^1 \backslash E_{s+1}^1))) \cap \{| x |> \rho_0\}. \qquad (41)$$

**Remark 2** *Here we note some properties of the known part* $| \gamma+t |^{2r} + F_k(\gamma+t)$ *( for* $k = 1, 2, ...$ *see Theorem 1) of the nonresonance eigenvalues of* $L_t(r, q(x))$*. Denoting* $\gamma + t$ *by* $x$ *, where* $| \gamma + t |\sim \rho$*,* $\gamma + t \in U^1(\rho^{\alpha_1}, p)$*, we prove*

$$\frac{\partial F_k(x)}{\partial x_i} = O(\rho^{2-2r-2\alpha_1+\alpha}), \forall i = 1, 2, ..., d; \forall k = 1, 2, ... \qquad (42)$$

*We prove (42) by induction with respect to k. Using (20) one can easily verify that*

$$| x |^{2r} - | x - \gamma_1 |^{2r} \sim \rho^{2r-2}(| x |^2 - | x - \gamma_1 |^2),$$
$$| x |^{2r-2} - | x - \gamma_1 |^{2r-2} \sim \rho^{2r-4}(| x |^2 - | x - \gamma_1 |^2),$$

*where* $|| x |^2 - | x - \gamma_1 |^2|> \rho^{\alpha_1}$*, since* $x \notin V_{\gamma_1}^1(\rho^{\alpha_1})$*. Therefore*

$$\frac{\partial}{\partial x_i}(\frac{1}{| x |^{2r} - | x - \gamma_1 |^{2r}}) =$$
$$\frac{2rx_1(| x |^{2r-2} - | x - \gamma_1 |^{2r-2})}{(| x |^{2r} - | x - \gamma_1 |^{2r})^2})+ \qquad (43)$$
$$\frac{-2\gamma_1(i) | x - \gamma_1 |^{2r-2}}{(| x |^{2r} - | x - \gamma_1 |^{2r})^2} = O(\rho^{2-2r-2\alpha_1+\alpha}),$$

*where* $\gamma_1(i)$ *is the i-th component of the vector* $\gamma_1 \in \Gamma(p\rho^\alpha)$ *hence is equal to* $O(\rho^\alpha)$*. Now (42) for* $k = 1$ *follows from (4) and (43). Suppose that (42) holds for* $k = s$*. Using this and (24), replacing* $| x |^{2r}$ *by* $| x |^{2r} + F_s(x)$ *in (43) and evaluating as above we obtain*

$$\frac{\partial}{\partial x_i}(\frac{1}{| x |^{2r} + F_s(x) - | x - \gamma_1 |^{2r}}) =$$
$$O(\rho^{2r-2-2\alpha_1+\alpha}) + \frac{\frac{\partial F_s(x)}{\partial x_i}}{(| x |^{2r} + F_s - | x - \gamma_1 |^{2r})^2} =$$
$$O(\rho^{2-2r-2\alpha_1+\alpha}) + O(\rho^{2-2r-4\alpha_1+\alpha}) = O(\rho^{2-2r-2\alpha_1+\alpha})$$

*This formula with the definition of* $F_k$ *implies (42) for* $k = s + 1$*.*

# 2 Asymptotic Formulas for Bloch Functions

In this section using the asymptotic formulas for eigenvalues ( Theorem 1) and simplisity conditions (9), (10) we prove asymptotic formulas for Bloch functions with a quasimomenta of the simple set $B$. To use the simplicity conditions we write the asymptotic formulas in the following form. By Theorem 1(b) for $\gamma+t\in B\subset U^1(\rho^{\alpha_1},p)$ there exists an eigenvalue $\Lambda_N$ satisfying (5). Taking $k=k_1$ in (5), where $k_1=[\frac{d}{3\alpha}]+2$, ( we can take $k=k_1$, since $k_1\leq\frac{1}{3}(p-\frac{1}{2}q(d-1))$ (see the inequality 3. in the end of introduction)) using the relation $3k_1\alpha>d+2\alpha$ ( see the inequality 5. in the end of the introduction) and notations $F(\gamma+t)=\mid\gamma+t\mid^{2r}+F_{k_1-1}(\gamma+t)$, $\varepsilon_1=\rho^{-d-2\alpha}$ ( see introduction) we obtain

$$\Lambda_N-F(\gamma+t)=o(\varepsilon_1) \tag{44}$$

**Theorem 2** *If $\gamma+t\in B$ and $\mid\gamma+t\mid\sim\rho$ then there exists a unique eigenvalue $\Lambda_N$ satisfying (5) for $k=1,2,...,n=[\frac{p}{3}]$. This is a simple eigenvalue and the corresponding eigenfunction $\Psi_N(x)$ satisfies*

$$\Psi_N(x)=e^{i(\gamma+t,x)}+O(\rho^{-\alpha_1}), \tag{45}$$

*if $q(x)\in W_2^{s_0}(F)$, where $s_0=\frac{3d-1}{2}(3^d+d+2)+\frac{1}{4}d3^d+d+6$. This eigenvalue is the $\Lambda(\gamma+t)$ ( see (6) for the definition of $\Lambda(\gamma+t)$ ).*

We noted that ( see above) there exists an eigenvalue $\Lambda_N$ satisfying (44). Let $\Psi_N$ be any normalized eigenfunction corresponding to $\Lambda_N$. Since the normalized eigenfunction is defined up to constant of modulas 1 we can suppose that $\arg b(N,\gamma)=0$, where $b(N,\gamma)=(\Psi_N,e^{i(\gamma+t,x)})$. Therefore to prove (45) it suffices to verify the equality

$$\sum_{\gamma'\in\Gamma\backslash\{\gamma\}}\mid b(N,\gamma')\mid^2=O(\rho^{-2\alpha_1}). \tag{46}$$

Thus we need to show that (46) holds. First consider the case $\gamma'\notin K$. Using (44), definition of $K$ (see introduction), and then (11), we get

$$\mid\Lambda_N-\mid\gamma'+t\mid^{2r}\mid>\frac{1}{4}\rho^{\alpha_1},\gamma'\notin K, \tag{47}$$

$$\sum_{\gamma'\notin K}\mid b(N,\gamma')\mid^2=\parallel\Psi_N(x)q(x)\parallel^2 O(\rho^{-2\alpha_1})=O(\rho^{-2\alpha_1}).$$

If $\gamma^{'} \in K$ , then clearly

$$\mid \Lambda_N - \mid \gamma^{'} + t \mid^{2r} \mid < \frac{1}{2}\rho^{\alpha_1}, \tag{48}$$

Now we prove that the simplicity conditions (9), (10) imply

$$\mid b(N,\gamma^{'}) \mid \leq c_4\rho^{-c\alpha}, \forall \gamma^{'} \in K\backslash\{\gamma\}, \tag{49}$$

where $c = p - dq - \frac{1}{4}d3^d - 3$. If for $\gamma^{'} + t \in U^1(\rho^{\alpha_1}, p)$ and $\gamma^{'} \in K\backslash\{\gamma\}$ the inequality in (49) is not true then by Theorem 1(a) ( see Theorem 1(a), and inequalities (48), (49)) we have

$$\Lambda_N = \mid \gamma^{'} + t \mid^{2r} + F_{k-1}(\gamma^{'} + t) + O(\rho^{-3k\alpha}), \tag{50}$$

for $k = 1, 2, ..., [\frac{1}{3}(p - c)] = [\frac{1}{3}(dq + \frac{1}{4}d3^d + 3)]$. Now taking $k = k_1 = [\frac{d}{3\alpha}] + 2$ ( we can take $k = k_1$, since one can easily verify that $\frac{d}{3\alpha} + 2 \leq \frac{1}{3}(dq + \frac{1}{4}d3^d + 3)$ ), and arguing as in the prove of (44) we get

$$\Lambda_N - F(\gamma^{'} + t) = o(\varepsilon_1) \ ,$$

because $3k_1\alpha > d + 2\alpha$ (see the inequality 5. in the end of introduction). This with (44) contradicts the simplicity condition (9). Similarly if, for $\gamma^{'} + t \in (E_k^1 \backslash E_{k+1}^1)$ and $\gamma^{'} \in K$, the inequality in (49) does not hold then by Theorem 1(c)

$$\Lambda_N = \lambda_j(\gamma^{'} + t) + O(\rho^{-(p-c-\frac{1}{4}d3^d)\alpha}). \tag{51}$$

where $(p - c - \frac{1}{4}d3^d)\alpha = (dq + 3)\alpha > d + 2\alpha$ . Therefore we have

$$\Lambda_N - \lambda_j(\gamma^{'} + t) = o(\varepsilon_1)$$

This with (44) contradicts the simplicity condition (10). Hence the inequality in (49) holds. Therefore using $\mid K \mid = O(\rho^{q(d-1)\alpha})$ we have

$$\sum_{\gamma^{'} \in K\backslash\{\gamma\}} \mid b(N,\gamma^{'}) \mid^2 = O(\rho^{-(2c-q(d-1))\alpha}) = O(\rho^{-(2p-(3d-1)q-\frac{1}{2}d3^d-6)\alpha}). \tag{52}$$

If $s = s_0$, that is, $p = s_0 - d$ then $2p - (3d - 1)q - \frac{1}{2}d3^d - 6 = 6$. Since $\alpha_1 = 3\alpha$ the equality (52) and the equality in (47) imply (46). Thus we proved that for any normalized eigenfunction $\Psi_N$ corresponding to any eigenvalue $\Lambda_N$ satisfying (5) the equality (45) holds. If there exist two different eigenvalues or multiple eigenvalue satisfying (5) then there exist two orthogonal normalized eigenfunction satisfying (45) which is imposible. Therefore $\Lambda_N$ is a simple eigenvalue. It follows from Theorem 1(a) that $\Lambda_N$ satisfies (5) for $k = 1, 2, ..., n = [\frac{p}{3}]$, because the inequality (23) holds for $c = 0$ ( see (45)). Since (45) holds it follows from the definition of $\Lambda(\gamma + t)$ ( see (6)) that if $\gamma + t \in B$ then the eigenvalue $\Lambda_N$ satisfying (5) is $\Lambda(\gamma + t)$. The theorem is proved■

Now we prove the asymptotic formulas of arbitrary order for Bloch functions.

**Theorem 3** *If $\gamma + t \in B$ and $\mid \gamma + t \mid \sim \rho$ then the eigenfunction $\Psi_{\gamma+t}(x) \equiv \Psi_{N(\gamma+t)}(x)$ corresponding to the eigenvalue $\Lambda_N \equiv \Lambda(\gamma + t)$ satisfies formulas (8), for $k = 1, 2, ..., n = [\frac{1}{6}(2p - (3d - 1)q - \frac{1}{2}d3^d - 6)]$, where,*

$$\Phi_0(x) = 0,\ \Phi_1(x) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \frac{q_{\gamma_1} e^{i(\gamma+t+\gamma_1,x)}}{(\mid \gamma + t \mid^{2r} - \mid \gamma + \gamma_1 + t \mid^{2r})},$$

*and $\Phi_{k-1}(x)$ is a linear conbination of $e^{i(\gamma+t+\gamma',x)}$ for $\gamma' \in \Gamma((k-1)\rho^\alpha) \cup \{0\}$ with explicitly expressed coefficients (58), (59).*

By Theorem 2 formula (8) for $k = 1$ is proved. To prove formula (8) for arbitrary $k \leq n$ we prove the following equivalent relations

$$\sum_{\gamma' \in \Gamma^c(k-1)} \mid b(N, \gamma + \gamma') \mid^2 = O(\rho^{-2k\alpha_1}), \tag{53}$$

$$\Psi_N = b(N, \gamma)e^{i(\gamma+t,x)} + \sum_{\gamma' \in \Gamma((k-1)\rho^\alpha)} b(N, \gamma + \gamma')e^{i(\gamma+t+\gamma',x)} + H_k(x), \tag{54}$$

where $\Gamma^c(m) \equiv \Gamma \backslash (\Gamma(m\rho^\alpha) \cup \{0\})$ and $\parallel H_k(x) \parallel = O(\rho^{-k\alpha_1})$. The case $k = 1$ is proved accoding to (46). Assume that (53) is true for $k = m$ . Then by (54) for $k = m$, and by (3) we have $\Psi_N(x)(q(x)) = H(x) + O(\rho^{-m\alpha_1})$, where $H(x)$ is linear conbination of $e^{i(\gamma+t+\gamma',x)}$ for $\gamma' \in \Gamma(m\rho^\alpha) \cup \{0\}$. Hence $(H(x), e^{i(\gamma+t+\gamma',x)}) = 0$ for $\gamma' \in \Gamma^c(m)$. Therefore using (11) and the inequality in (47), we get

$$\sum_{\gamma'} \mid b(N, \gamma + \gamma') \mid^2 = \sum_{\gamma'} \mid \frac{(O(\rho^{-m\alpha_1}), e^{i(\gamma+t+\gamma',x)})}{\Lambda_N - \mid \gamma + \gamma' + t \mid^{2r}} \mid^2 = O(\rho^{-2(m+1)\alpha_1}), \tag{55}$$

where the sum is taken under conditions $\gamma^{'} \in \Gamma^c(m)$ and $\gamma+\gamma^{'} \notin K$. On the other hand, using $\alpha_1 = 3\alpha$ and definition of $n$, from (52) we obtain

$$\sum_{\gamma^{'}\in K\backslash\{\gamma\}} \mid b(N,\gamma^{'}) \mid^2 = O(\rho^{-2n\alpha_1}),$$

This with (55) implies (53) for $k = m+1$. Thus formula (54) is also proved. Here $b(N,\gamma)$ and $b(N,\gamma+\gamma^{'})$ for $\gamma^{'} \in \Gamma((n-1)\rho^{\alpha})$ can be calculated as follows. In (12) replase $\gamma$ by $\gamma+\gamma^{'}$. Iterate it $n$ times and every times isolate the terms with maltiplicant $b(N,\gamma)$. In other word apply (13) for $b(N,\gamma+\gamma^{'})$, where $\gamma^{'} \in \Gamma((n-1)\rho^{\alpha})$, and hence $\gamma^{'} \neq 0$. Then apply (13) for $b(N,\gamma+\gamma^{'}-\sum_{i=1}^{j}\gamma_i)$ when $\gamma^{'}-\sum_{i=1}^{j}\gamma_i \neq 0$ where $\gamma_i \in \Gamma(\rho^{\alpha})$, $j = 1,2,...,n-1$. Then using (4) and the relation

$\mid \Lambda_N - \mid \gamma+t+\gamma^{'}-\sum_{i=1}^{j}\gamma_i \mid^{2r} \mid > \frac{1}{2}\rho^{\alpha_1}$ ( see (16) and use the fact that $\gamma^{'}-\sum_{i=1}^{j}\gamma_i \in \Gamma(p\rho^{\alpha})$ since $p > 2n$),

$\Lambda_N = P(\gamma+t) + O(\rho^{-n\alpha_1})$, where $P(\gamma+t) = \mid \gamma+t \mid^{2r} + F_{[\frac{p}{3}]}(\gamma+t)$ ( see Theorem 2), we obtain

$$b(N,\gamma+\gamma^{'}) = \sum_{k=1}^{n-1} A_k(\gamma^{'})b(N,\gamma) + O(\rho^{-n\alpha_1}), \tag{56}$$

where

$$A_1(\gamma^{'}) \equiv \frac{q_{\gamma^{'}}}{P(\gamma+t) - \mid \gamma+\gamma^{'}+t \mid^{2r}} = \frac{q_{\gamma^{'}}}{\mid \gamma+t \mid^{2r} - \mid \gamma+\gamma^{'}+t \mid^{2r}} + O(\rho^{-3\alpha_1})$$

$$A_k(\gamma^{'}) = \sum_{\gamma_1,\dots,\gamma_{k-1}} \frac{q_{\gamma_1}q_{\gamma_2}\dots q_{\gamma_{k-1}}q_{\gamma^{'}-\gamma_1-\gamma_2-\dots-\gamma_{k-1}}}{\prod_{j=0}^{k-1}(P(\gamma+t) - \mid \gamma+t+\gamma^{'}-\sum_{i=1}^{j}\gamma_i \mid^{2r})} = O(\rho^{-k\alpha_1}),$$

$$\sum_{\gamma^*\in\Gamma((n-1)\rho^{\alpha})} \mid A_1(\gamma^*) \mid^2 = O(\rho^{-2\alpha_1}), \quad \sum_{\gamma^*\in\Gamma((n-1)\rho^{\alpha})} \mid A_k(\gamma^*) \mid = O(\rho^{-k\alpha_1}), \tag{57}$$

for $k > 1$. Now from (54) for $k = n$ and (56), we obtain

$$\Psi_N = b(N,\gamma)e^{i(\gamma+t,x)} + \sum_{\gamma^*\in\Gamma((n-1)\rho^{\alpha})} \sum_{k=1}^{n-1} (A_k(\gamma^*)b(N,\gamma) + O(\rho^{-n\alpha_1}))e^{i(\gamma+t+\gamma^*,x)}) + H_n$$

Since $\parallel \Psi_N \parallel = 1, \arg b(N,\gamma) = 0$, the functions $e^{i(\gamma+t,x)}$, $H_n(x)$, $e^{i(\gamma+t+\gamma^*,x)}$, $(\gamma^* \in \Gamma((n-1)\rho^{\alpha}))$ are orthogonal and $\parallel H_n \parallel = O(\rho^{-n\alpha_1})$ we have

$$1 =\mid b(N,\gamma)\mid^2 + \sum_{k=1}^{n-1}(\sum_{\gamma^*\in\Gamma((n-1)\rho^\alpha)} \mid A_k(\gamma^*)b(N,\gamma)\mid^2 + O(\rho^{-n\alpha_1})),$$

$$b(N,\gamma) = (1+\sum_{k=1}^{n-1}(\sum_{\gamma^*\in\Gamma((n-1)\rho^\alpha)} \mid A_k(\gamma^*)\mid^2))^{-\frac{1}{2}} + O(\rho^{-n\alpha_1})). \qquad (58)$$

(See the second equality in (57).) Now from (56) we obtain

$$b(N,\gamma+\gamma') = (\sum_{k=1}^{n-1} A_k(\gamma'))(1+\sum_{k=1}^{n-1}\sum_{\gamma^*} \mid A_k(\gamma^*)\mid^2)^{-\frac{1}{2}} + O(\rho^{-n\alpha_1}) \qquad (59)$$

Consider the case $n = 2$. By (58), (57), (59) we have $b(N,\gamma) = 1 + O(\rho^{-2\alpha_1})$,

$$b(N,\gamma+\gamma') = A_1(\gamma') + O(\rho^{-2\alpha_1}) = \frac{q_{\gamma'}}{\mid\gamma+t\mid^{2r} - \mid\gamma+\gamma'+t\mid^{2r}} + O(\rho^{-2\alpha_1})$$

for all $\gamma' \in \Gamma(\rho^\alpha)$. These and (54) for $k = 2$ imply the formula for $\Phi_1\diamondsuit$

# 3 Simple Sets and Bethe-Sommerfeld conjecture

In this section we construct a part of the simple set in neighbourhood of the surface $S_\rho = \{x \in U^1(2\rho^{\alpha_1}, p) : F(x) = \rho^{2r}\}$. Due to (5) ( replace $\gamma + t$ by $x$ in $F(\gamma + t)$ ) it is natural to call $S_\rho$ the approximated isoenergetic surfaces in the nonresonance domain. As we noted in introduction in order that the nonresonance eigenvalue $\Lambda(\gamma + t)$ can not coincide with any other nonresonance eigenvalue $\Lambda(\gamma + t + b)$ we choose the $\gamma + t$ such that

$\mid F(\gamma + t) - F(\gamma + t + b)\mid > 2\varepsilon_1$ for $\gamma + t + b \in U^1(\rho^{\alpha_1}, p)$ and $b \in \Gamma\backslash\{0\}$ (see (9)). Therefore we eliminate

$$P_b = \{x : x, x + b \in U^1(\rho^{\alpha_1}, p), \mid F(x) - F(x + b)\mid < 3\varepsilon_1\} \qquad (60)$$

for $b \in \Gamma\backslash\{0\}$ from $S_\rho$. Denote the remaining part of $S_\rho$ by $S'_\rho$. Then we consider the $\varepsilon$ neighbourhood $U_\varepsilon(S'_\rho) = \cup_{a\in S'_\rho} U_\varepsilon(a)\}$ of $S'_\rho$, where $\varepsilon = \frac{\varepsilon_1}{7r\rho^{2r-1}}$, $U_\varepsilon(a) = \{x \in R^d :\mid x - a\mid < \varepsilon\}$. In this set the first simplicity condition (9) holds (see Lemma 4(a)). Denote by

$Tr(E) = \{\gamma + x \in U_\varepsilon(S'_\rho) : \gamma \in \Gamma, x \in E\}$ and

$Tr_{F^\star}(E) \equiv \{\gamma + x \in F^\star : \gamma \in \Gamma, x \in E\}$ the translations of $E \subset R^d$ into $U_\varepsilon(S'_\rho)$ and $F^\star$ respectively. In order that the second simplicity condition (10) holds, we discart from $U_\varepsilon(S'_\rho)$ the translation $Tr(A(\rho))$ of

$$A(\rho) = \cup_{k=1}^{d-1}(\cup_{\gamma_1,\gamma_2,\ldots,\gamma_k\in\Gamma(p\rho^\alpha)}(\cup_{i=1}^{b_k} A_{k,i}(\gamma_1, \gamma_2, \ldots, \gamma_k))), \qquad (61)$$

where

$A_{k,i}(\gamma_1, ..., \gamma_k) = \{x \in (\cap_{i=1}^{k} V_{\gamma_i}^1(\rho^{\alpha_k}) \backslash E_{k+1}^1) \cap K_\rho : \lambda_i(x) \in (\rho^{2r} - 3\varepsilon_1, \rho^{2r} + 3\varepsilon_1)\}$, and

$$K_\rho = \{x \in R^d :\mid\mid x \mid^{2r} - \rho^{2r} \mid < \rho^{\alpha_1}\}. \tag{62}$$

As a result we construct the part $U_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho))$ of the simple set $B$ (see Theorem 5(a)) which contains the intervals $\{a + sb : s \in [-1, 1]\}$ such that $\Lambda(a - b) < \rho^{2r}$, $\Lambda(a + b) > \rho^{2r}$ and $\Lambda(\gamma + t)$ is continuous on this intervals. Hence there exists $\gamma + t$ for which $\Lambda(\gamma + t) = \rho^{2r}$. It implies the validity of Bethe-Sommerfeld conjecture for $L(r, q)$. For this we need the following lemma.

**Lemma 4** *(a) If $x \in U_\varepsilon(S_\rho^{'})$ and $x + b \in U^1(\rho^{\alpha_1}, p)$, where $b \in \Gamma$, then*

$$\mid F(x) - F(x + b) \mid > 2\varepsilon_1 \tag{63}$$

*Hence for $\gamma + t \in U_\varepsilon(S_\rho^{'})$ the first simplicity condition (9) holds.*

*(b) If $x \in U_\varepsilon(S_\rho^{'})$ then $x + b \notin U_\varepsilon(S_\rho^{'})$ for all $b \in \Gamma$ .*

*(c)If $E \subset R^d$ is bounded set then $\mu(Tr(E)) \leq \mu(E)$.*

*(d) If $E \subset U_\varepsilon(S_\rho^{'})$ then $\mu(Tr_{F^\star}(E)) = \mu(E)$*

$(a)$ If $x \in U_\varepsilon(S_\rho^{'})$ then there is a point $a$ in $S_\rho^{'}$ such that $x \in U_\varepsilon(a)$. Since $S_\rho^{'}$ does not contain the set $P_b$ ( see (60)) we have

$$\mid F(a) - F(a + b) \mid \geq 3\varepsilon_1 \tag{64}$$

On the other hand using (42) and the relations

$\mid x \mid < \rho + 1$, $\mid x - a \mid < \varepsilon$, $\mid x + b - a - b \mid < \varepsilon$ we obtain

$$\begin{aligned} &\mid F(x) - F(a) \mid < 3r\rho^{2r-1}\varepsilon, \\ &\mid F(x + b) - F(a + b) \mid < 3r\rho^{2r-1}\varepsilon. \end{aligned} \tag{65}$$

These inequalities together with (64) give (63), since $6r\rho^{2r-1}\varepsilon < \varepsilon_1$.

$(b)$ This is a simple consequence of $(a)$. Indeed if $x$ and $x+b$ lie in $U_\varepsilon(S_\rho^{'})$ then there are points $a$ and $c$ in $S_\rho^{'}$ such that $x \in U_\varepsilon(a)$ and $x + b \in U_\varepsilon(c)$. Repeating the proof of (65) we get $\mid F(c) - F(x + b) \mid < 3r\rho^{2r-1}\varepsilon$. This, the first ineguality in (65), and the relations $F(a) = \rho^2, F(c) = \rho^2$ (see the definition of $S_\rho$) give $\mid F(x) - F(x + b) \mid < \varepsilon_1$ which contradicts (63).

$(c)$ Clearly for any bounded set $E$ there are only finite number of the vectors $\gamma_1, \gamma_2, ..., \gamma_s$ such that $E(k) \equiv (E + \gamma_k) \cap U_\varepsilon(S_\rho^{'}) \neq \emptyset, k = 1, 2, ..., s$

and $Tr(E)$ is the union of the sets $E(k)$. For $E(k)-\gamma_k$ we have the relations $\mu(E(k)-\gamma_k)=\mu(E(k)), E(k)-\gamma_k\subset E$. Moreover by $(b)$

$(E(k)-\gamma_k)\cap(E(j)-\gamma_j)=\emptyset$ for $k\neq j$. Therefore $(c)$ is true.

$(d)$ Now let $E\subset U_\varepsilon(S_\rho^{'})$. Then by $(b)$ the set $E$ can by devided into finite number of the pairwise disjoint sets $E_1,E_2,...,E_n$ such that there are the vectors $\gamma_1,\gamma_2,...,\gamma_n$ satisfying $(E_k+\gamma_k)\subset F^\star,(E_k+\gamma_k)\cap(E_j+\gamma_j)\neq\emptyset$,for $k\neq j$ and $k,j=1,2,...,n$. Using $\mu(E_k+\gamma_k)=\mu(E_k)$ we get the proof of $(d)$, becouse $Tr_{F^\star}(E)$ and $E$ are union of piarwise disjoint sets $E_k+\gamma_k$ and $E_k$ for $k=1,2,...,n$ respectively$\diamondsuit$

**Theorem 5** $(a)$ *The set $U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho))$ is a subset of $B$. Hence if $\gamma+t$ lies in this subset then Theorem 2 and Theorem 3 hold. For every connected open subset $E$ of $U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho)$ there exists a unique index $N$ such that*

*$\Lambda(\gamma+t)=\Lambda_N(t)$ for all $\gamma+t\in E$.*

$(b)$ *For the part $V_\rho\equiv S_\rho^{'}\backslash U_\varepsilon(Tr(A(\rho)))$ of the approximated isoenergetic surface the following holds*

$$\mu(V_\rho)>(1-c_8\rho^{-\alpha}))\mu(B(\rho)). \tag{66}$$

*Moreover $U_\varepsilon(V_\rho)$ lies in the subset $U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho))$ of the simple set $B$.*

$(c)$ *The number $\rho^{2r}$ for $\rho\gg1$ lies in the spectrum of $L(r,q(x))$, that is, the number of the gaps in the spectrum of $L(r,q(x))$ is finite, where $r\geq1$, $q(x)\in W_2^{s_0}(R^d/\Omega)$, $d\geq2$, and $\Omega$ is an arbitrary lattice.*

$(a)$ Suppose $\gamma+t\equiv x\in U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho))$, that is , there is $a\in S_\rho^{'}$ such that $x\in U_\varepsilon(a)$. Then, by lemma 4(a), the simplisity condition (9) holds. Now we prove that (10) holds too. The first inequality in (65) and the relation $F(a)=\rho^{2r}$ give

$$F(\gamma+t)\in(\rho^{2r}-3r\rho^{2(r-1)}\varepsilon,\rho^{2r}+3r\rho^{2(r-1)}\varepsilon)\subset(\rho^{2r}-\varepsilon_1,\rho^{2r}+\varepsilon_1) \tag{67}$$

for $\gamma+t\in U_\varepsilon(S_\rho^{'})$. On the other hand $\gamma+t\notin Tr(A(\rho))$. It means that for any $\gamma^{'}\in\Gamma$, we have $\gamma^{'}+t\notin A(\rho)$. If $\gamma^{'}\in K$ and $\gamma^{'}+t\in E_k^1\backslash E_{k+1}^1$ then by definition of $K$ ( see introduction) the inequality $\mid F(\gamma+t)-\mid\gamma^{'}+t\mid^{2r}\mid<\frac{1}{3}\rho^{\alpha_1}$ holds. This (24) and (67) imply that $\gamma^{'}+t\in(E_k^1\backslash E_{k+1}^1)\cap K_\rho$ ( see (62) for the definition of $K_\rho$). Sinse $\gamma^{'}+t\notin A(\rho)$ we have $\lambda_i(\gamma^{'}+t)\notin(\rho^{2r}-3\varepsilon_1,\rho^{2r}+3\varepsilon_1)$ for $\gamma^{'}\in K$ and $\gamma^{'}+t\in E_k\backslash E_{k+1}$. Therefore (10) follows from (67). Moreover it is clear that the inclusion $S_\rho^{'}\subset U(2\rho^{\alpha_1},p)$ ( see definition of $S_\rho$ and $S_\rho^{'}$) implies that $U_\varepsilon(S_\rho^{'})\subset U(\rho^{\alpha_1},p)$. Thus $U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho))\subset B$.

Now let $E$ be a connected open subset of $U_\varepsilon(S'_\rho)\backslash Tr(A(\rho) \subset B$. By Theorem 2 for $a \in E \subset U_\varepsilon(S'_\rho)\backslash Tr(A(\rho)$ the eigenvalue $\Lambda(a)$ is simple and the corresponding eigenfunction $\Psi_a(x)$ satisfies $\mid (\Psi_a(x), e^{i(a,x)}) \mid^2 > \frac{1}{2}$. So for $y \in E$ there exists a unique $N(y)$ such that $\Lambda(y) = \Lambda_{N(y)}(y)$,$\Psi_y(x) = \Psi_{N(y),y}(x)$,

$$\mid (\Psi_{N(y),y}(x), e^{i(y,x)}) \mid^2 > \frac{1}{2}. \tag{68}$$

On the other hand for fixed $N$ the functions $\Lambda_N(t)$ and $(\Psi_{N,t}(x), e^{i(t,x)})$ are continuous in neighborhood of $t$ if $\Lambda_N(t)$ is a simple eigenvalue. Therefore for $a \in E$ there exists a neighborhood $U(a) \subset E$ of $a$ such that $\mid (\Psi_{N(a),y}(x), e^{i(y,x)}) \mid^2 > \frac{1}{2}$, for every $y \in U(a)$. This and (68) yield that $N(y) = N(a)$ for $y \in U(a)$, since there is a unique integer $N$ satisfying (68). Hence we proved the Proposition 1 which is

$\forall a \in E, \exists U(a) \subset E : N(y) = N(a), \forall y \in U(a)$.

Now let $a_1$ and $a_2$ be two points of $E$ and $C \subset E$ be the arc which joins these points. (Note that the open connected subset of $R^d$ is arcwise connected). Let $U(y_1), U(y_2), ..., U(y_k)$ be finite subcover of the open cover $\cup_{a \in C} U(a)$ of the compact $C$, where $U(a)$ is a neighborhood of $a$ satisfying Proposition 1. By Proposition 1 $N(y) = N(y_i) = N_i$ for $y \in U(y_i)$. Clearly if $U(y_i) \cap U(y_j) \neq \emptyset$ then $N_i = N_j$ since for $a \in U(y_i) \cap U(y_j)$ there is a unique $N(a)$. Thus $N_1 = N_2 = ... = N_k$ and $N(a_1) = N(a_2)$.

$(b)$ First we note that if $a \in V_\rho$ then $U_\varepsilon(a) \in U_\varepsilon(S'_\rho)\backslash Tr(A(\rho))$, because the relation $a \in S'_\rho$ implies that $U_\varepsilon(a) \in U_\varepsilon(S'_\rho)$ and the relation $a \notin U_\varepsilon(Tr(A(\rho)))$ implies that $U_\varepsilon(a) \cap Tr(A(\rho)) = \emptyset$. To prove (66) first we estimate the measure of $S_\rho, S'_\rho, U_{2\varepsilon}(A(\rho))$, namely we prove

$$\mu(S_\rho) > (1 - c_9\rho^{-\alpha})\mu(B(\rho)), \tag{69}$$

$$\mu(S'_\rho) > (1 - c_{10}\rho^{-\alpha})\mu(B(\rho)), \tag{70}$$

$$\mu(U_{2\varepsilon}(A(\rho))) = O(\rho^{-\alpha})\mu(B(\rho))\varepsilon. \tag{71}$$

( See below, Estimations 1, 2, 3). The estimation (64) of the measure of set $V_\rho$ is done in Estimation 4 by using Estimations 1, 2, 3.

$(c)$ Since $F(a) = \rho^{2r}$ for $a = (a_1, a_2, ..., a_d) = \sum_{i=1}^d a_i e_i \in V_\rho \subset S_\rho$ it follows from (24) that $\rho - 1 <\mid a \mid< \rho + 1$, and there is an indeks $i$ such that $\mid a_i \mid> \frac{1}{d}\rho$ . Without loss of generality, assume that $a_i > 0$. Then (42) and (44) imply that $F(a - \varepsilon e_i) < \rho^{2r} - c_{11}\varepsilon_1$, $F(a + \varepsilon e_i) > \rho^{2r} + c_{11}\varepsilon_1$ and $\Lambda(a - \varepsilon e_i) < \rho^{2r}$, $\Lambda(a + \varepsilon e_i) > \rho^{2r}$. Since $\Lambda(\gamma + t)$ is continuous there exists a point $y(a) \in [a-\varepsilon e_i, a+\varepsilon e_i]$ such that $\Lambda(y(a,i)) = \rho^{2r}$. The Theorem is proved $\diamondsuit$

In Estimations 1-4 we use the notations: $G(+i,a) = \{x \in G, x_i > a\}$, $G(-i,a) = \{x \in G, x_i < -a\}$, where $x = (x_1, x_2, ..., x_d), a > 0$. It is not hard to verify that for any subset $G$ of $U_\varepsilon(S_\rho^{'}) \cup U_{2\varepsilon}(A(\rho))$ , hence for all considered sets $G$ in these estimations, and for any $x \in G$ the followings hold

$$\rho - 1 <\mid x \mid< \rho + 1, G \subset (\cup_{i=1}^d (G(+i, \rho d^{-1}) \cup G(-i, \rho d^{-1})) \tag{72}$$

Indeed if $x \in S_\rho^{'}$ then $F(x) = \rho^{2r}$ and by (24) we have $\mid x \mid = \rho + O(\rho^{-1-\alpha_1})$. Hence the inequalities in (72) hold for $x \in U_\varepsilon(S_\rho^{'})$. If $x \in A(\rho)$ then by definition of $A(\rho)$ we have $x \in K_\rho$, and hence $\mid x \mid = \rho + O(\rho^{-1+\alpha_1})$. Therefore the inequalities in (72) hold for $x \in U_{2\varepsilon}(A(\rho))$ too. The inclusion in (72) follows from these inequalities.

If $G \subset S_\rho$ then for $x \in G(+k, \rho^{-\alpha})$ by (42) we have $\frac{\partial F(x)}{\partial x_k} > 0,$ . Therefore to calculate the measure of $G(+k,a)$ for $a \geq \rho^{-\alpha}$ we use the formula

$$\mu(G(+k,a)) = \int\limits_{\Pr_k(G(+k,a))} (\frac{\partial F}{\partial x_k})^{-1} \mid grad(F) \mid dx_1...dx_{k-1}dx_{k+1}...dx_d, \tag{73}$$

where $\Pr_k(G) \equiv \{(x_1, x_2, ..., x_{k-1}, x_{k+1}, x_{k+2}, ..., x_d) : x \in G\}$ is the projection of $G$ on the hyperplane $x_k = 0$. Instead of $\Pr_k(G)$ we write $\Pr(G)$ if $k$ is unambiguous. If $D$ is $m-$dimensional subset of $R^m$ then we use

$$\mu(D) = \int\limits_{\Pr_k(D)} \mu(D(x_1, ...x_{k-1}, x_{k+1}, ..., x_m))dx_1...dx_{k-1}dx_{k+1}...dx_m, \tag{74}$$

where $D(x_1, ...x_{k-1}, x_{k+1}, ..., x_m) = \{x_k : (x_1, x_2, ..., x_m) \in D\}$.

ESTIMATION 1. Here we prove (69) by using (73). During this estimation the set $S_\rho$ is redenoted by $G$. If $x \in G$ then $x \notin V_b^1(\rho^{\alpha_1}))$, for all $b \in \Gamma(p\rho^\alpha)$. Since the rotation does not change the measure we choose the coordinate axis so that the direction of some $b \in \Gamma(p\rho^\alpha)$ coinsides with the direction of $(1,0,0,...,0)$, that is, $b = (b_1, 0, 0, ..., 0), b_1 > 0$. Then the relation $x \notin V_b^1(\rho^{\alpha_1}))$ implies that $\mid x_1 \mid> a$, where $a = (\rho^{\alpha_1} - b_1^2)(2b_1)^{-1} > \rho^\alpha$ because $\mid b \mid< p\rho^\alpha, \alpha_1 = 3\alpha$. Therefore $G = G(+1,a) \cup G(-1,a)$. Now we estimate $\mu(G(+1,a))$ by using (73) and following relations

$$\frac{\partial F(x)}{\partial x_1} = r \mid x \mid^{2(r-1)} (2x_1 + O(\rho^{-2\alpha})) \tag{75}$$

$$\frac{\partial F}{\partial x_1} > \rho^{\alpha},\ (\frac{\partial F}{\partial x_1})^{-1} \mid grad(F) \mid = \frac{\mid x \mid}{x_1} + O(\rho^{-2\alpha}), \tag{76}$$

$$\Pr(G(+1,a)) \supset \Pr(A(+1,2a)), \tag{77}$$

where $x \in G(+1,a)$, $A = B(\rho) \cap U^1(3\rho^{\alpha_1}, p)$. Here (75) follows from (42) and (76) follows from (75). Now we prove the inclusion in (77). If $(x_2, ..., x_d) \in \Pr(A(+1,2a))$ then by definition of $A(+1,2a)$ there exists $x_1$ such that

$$x_1 > 2a > 2\rho^{\alpha},\ x_1^2 + x_2^2 + ... + x_d^2 = \rho^2, \mid \sum_{i \geq 1} (2x_i b_i - b_i^2) \mid \geq 3\rho^{\alpha_1} \tag{78}$$

for all $(b_1, b_2, ..., b_d) \in \Gamma(p\rho^{\alpha})$. Therefore for $h = \rho^{-\alpha}$ we have

$(x_1+h)^2 + x_2^2 + ... + x_q^2 > \rho^2 + \rho^{-\alpha}$, $(x_1-h)^2 + x_2^2 + ... + x_q^2 < \rho^2 - \rho^{-\alpha}$. This and (24) give $F(x_1+h, x_2, ..., x_d) > \rho^{2r}$, $F(x_1-h, x_2, ..., x_d) < \rho^{2r}$. Since $F$ is a continuous function there is $y_1 \in (x_1 - h, x_1 + h)$ such that (see (78))

$$y_1 > a, F(y_1, x_2, ..., x_d) = \rho^2, \mid 2y_1 b_1 - b_1^2 + \sum_{i \geq 2} (2x_i b_i - b_i^2) \mid > 2\rho^{\alpha_1}, \tag{79}$$

because the expression under the absolute value in (79) differ from the expression under the absolute value in (78) by $2(y_1 - x_1)b_1$, whose absolute value is less than $\rho^{\alpha_1}$. The relations in (79) means that $(x_2, ..., x_d) \in \Pr G(+1,a)$. Hence the inclusion in (77) is proved. Now it follows from (73), (76) and the obvious relation $\mu(\Pr G(+1,a)) = O(\rho^{d-1})$ ( see the inequality in (72))

$$\mu(G(+1,a)) = \int\limits_{\Pr(G(+1,a))} \frac{\mid x \mid}{x_1} dx_2 dx_3 ... dx_d + O(\rho^{-\alpha})\mu(B(\rho)) \geq$$

$$\int\limits_{\Pr(A(+1,2a))} \frac{\mid x \mid}{x_1} dx_2 dx_3 ... dx_d - c_{11}\rho^{-\alpha}\mu(B(\rho)) =$$

$$\mu(A(+1,2a)) - c_{12}\rho^{-\alpha}\mu(B(\rho)).$$

Similarly $\mu(G(+1,a)) \geq \mu(A(-1,2a)) - c_{13}\rho^{-\alpha}\mu(B(\rho))$. Hence $\mu(G) > \mu(A) - c_{14}\rho^{-\alpha}\mu(B(\rho))$. This inequality implies (69) since $\mu(A)) = (1 + O(\rho^{-\alpha}))\mu(B(\rho))$ (see (39) ).

ESTIMATION 2 Here we prove (70). For this we estimate the measure of the set $S_\rho \cap P_b$ ( see (60)) by using (73). During this estimation the set $S_\rho \cap P_b$ is redenoted by $G$. We choose the coordinate axis so that the direction of $b$ coincides with the direction of $(1,0,0,...,0)$, i.e., $b=(b_1,0,0,...,0)$ and $b_1>0$. Clearly if $(x_1,x_2,...,x_d)\in G$ then ( see definition of $S_\rho, G$ and $F(x)$)

$$(x_1^2+x_2^2+...+x_d^2)^r+F_{k_1-1}(x)=\rho^{2r}, \tag{80}$$

$$((x_1-b_1)^2+x_2^2+x_3^2+...+x_d^2)^r+F_{k_1-1}(x+b)=\rho^{2r}+h \tag{81}$$

where $h\in(-3\varepsilon_1,3\varepsilon_1)$. It follows from (80), (81), and (24) that

$$(x_1^2+x_2^2+x_3^2+...+x_d^2)=\rho^2+O(\rho^{-\alpha_1}) \tag{82}$$

$$((x_1-b_1)^2+x_2^2+x_3^2+...+x_d^2)=\rho^2+O(\rho^{-\alpha_1}) \tag{83}$$

Subtrracting (82) from (83) we get

$$(2x_1-b_1)b_1=O(\rho^{-\alpha_1}), \tag{84}$$

This with inequalities in (72) implies

$$\mid b_1\mid<2\rho+3,\ x_1=\frac{b_1}{2}+O(\rho^{-\alpha_1}b_1^{-1}),\mid x_1^2-(\frac{b_1}{2})^2\mid=O(\rho^{-\alpha_1}) \tag{85}$$

Conside two cases. Case 1: $b\in\Gamma_1\equiv\{b\in\Gamma:\mid\rho^2-(\frac{b_1}{2})^2\mid<3d\rho^{-2\alpha}\}$. In this case from (85), and (82) we obtain

$$x_1^2=\rho^2+O(\rho^{-2\alpha}),\mid x_1\mid=\rho+O(\rho^{-2\alpha-1}),\ x_2^2+x_3^2+...+x_d^2=O(\rho^{-2\alpha}). \tag{86}$$

Therefore $G\subset G(+1,a)\cup G(-1,a)$, where $a=\rho-\rho^{-1}$. Using (73), the obvious relation $\mu(\Pr_1(G(+1,a))=O(\rho^{-(d-1)\alpha})$ (see the lsat equality in (86)) and the fact that the expression under the integral (73) for $k=1$ is equal to $1+O(\rho^{-\alpha})$ (see (76) and (86)) and doing the same for $G(-1,a)$ we get $\mu(G)=O(\rho^{-(d-1)\alpha})$. Since $\mid\Gamma_1\mid=O(\rho^{d-1})$ we have

$$\mu(\cup_{b\in\Gamma_1}(S_\rho\cap P_b)=O(\rho^{-(d-1)\alpha+d-1})=O(\rho^{-\alpha})\mu(B(\rho)). \tag{87}$$

Case 2: $b\notin\Gamma_1$, i.e., $\mid\rho^2-(\frac{b_1}{2})^2\mid\geq3d\rho^{-2\alpha}$. By (85) and (82) we have

$$\mid x_1^2-\rho^2\mid>2d\rho^{-2\alpha},x_2^2+x_3^2+...+x_d^2>d\rho^{-2\alpha},\mid x_k\mid>\rho^{-\alpha}, \tag{88}$$

for some $k=2,3,...,d$. Therefore $G\subset\cup_{k\geq2}(G(+k,\rho^{-\alpha})\cup G(-k,\rho^{-\alpha}))$. We calculate $\mu(G(+d,\rho^{-\alpha}))$ by (73). Redenote by $D$ the set $\Pr_d G(+d,\rho^{-\alpha})$. If

$x \in G(+d, \rho^{-\alpha})$ then by (82) and (42) the under integral expression in (73) for $k = d$ is $O(\rho^{1+\alpha})$. Therefore the first equality in

$$\mu(D) = O(\varepsilon_1 \mid b \mid^{-1} \rho^{d-2}),\ \mu(G(+d, \rho^{-\alpha})) = O(\rho^{d-1+\alpha}\varepsilon_1 \mid b \mid^{-1}) \tag{89}$$

implies the second equality in (89). To prove the first equality of (89) we use (74) for $m = d-1$; $k = 1$ and the relations $\mu(\Pr_1 D) = O(\rho^{d-2})$,

$$\mu(D(x_2, x_3, ..., x_{d-1})) < 6\varepsilon_1 \mid b \mid^{-1},\ \forall (x_2, x_3, ..., x_{d-1}) \in \Pr D. \tag{90}$$

First relation follows from the fact that $\mid x \mid < \rho + 1$ for $x \in G(+d, \rho^{-\alpha})$ (see (72)). So we need to prove (90). If $x_1 \in D(x_2, x_3, ..., x_{d-1})$ then (80) and (81) holds. Subtructing (80) from (81), we get

$$((x_1 - b_1)^2 + x_2^2 + x_3^2 + ... + x_d^2)^r - ((x_1^2 + x_2^2 + x_3^2 + ... + x_d^2)^r +$$

$$F_{k_1-1}(x - b) - F_{k_1-1}(x) = h \tag{91}$$

where $x_2, x_3, ..., x_{d-1}$ are fixed . Hence we have two equations (80) and (91) with respect two unknown $x_1$ and $x_d$. Using (42), the implicit function theorem, and the inequalities $\mid x_d \mid > \rho^{-\alpha}$, $\alpha_1 > 2\alpha$ from (80), we obtain

$$x_d = f(x_1), \frac{df}{dx_1} = \frac{2x_1 + O(\rho^{-2\alpha_1+\alpha})}{2x_d + O(\rho^{-2\alpha_1+\alpha})} = \frac{x_1}{x_d} + O(\rho^{-\alpha_1}) \tag{92}$$

Substituting this in (91), we get

$$((x_1 - b_1)^2 + x_2^2 + x_3^2 + ... + f^2(x_1))^r - ((x_1^2 + x_2^2 + x_3^2 + ... + f^2(x_1))^r +$$

$$F_{k_1-1}(x_1 - b_1, x_2, ..., x_{d-1}, f(x_1)) - F_{k_1-1}(x_1, ..., x_{d-1}, f) = h \tag{93}$$

Using (42), (92), the first equality in (85) and $\mid x_d \mid > \rho^{-\alpha}$ we see the derivative (w.r.t. $x_1$) of the rigth side $a_r(x)$ of (93) is

$$\frac{\partial a_r(x)}{\partial x_1} = r \mid x - b \mid^{2(r-1)} (2(x_1 - b_1) + 2f(x_1)f'(x_1)) \tag{94}$$

$$-r \mid x \mid^{2(r-1)} (2x_1 + 2f(x_1)f'(x_1)) + O(\rho^{-2\alpha_1+\alpha})(1 + \tfrac{x_1}{x_d}) + O(\rho^{-3\alpha_1+\alpha}).$$

If $r = 1$ then using the first equality in (85) and $\mid x_d \mid > \rho^{-\alpha}$ we have

$$\mid \frac{\partial a_r(x)}{\partial x_1} \mid > b_1. \tag{95}$$

If $r>1$ then it follows from (80), (81), and (24) that

$$\mid x\mid^{2r}=\rho^{2r}(1+O(\rho^{-2r-\alpha_1})),\quad \mid x-b\mid^{2r}=\rho^{2r}(1+O(\rho^{-2r-\alpha_1}))$$

$$\mid x\mid^{2(r-1)}=\rho^{2(r-1)}(1+O(\rho^{-2r-\alpha_1}))^{\frac{r-1}{r}}=\rho^{2(r-1)}+O(\rho^{-2-\alpha_1})$$
$$\mid x-b\mid^{2(r-1)}=\rho^{2(r-1)}(1+O(\rho^{-2r-\alpha_1}))^{\frac{r-1}{r}}=\rho^{2(r-1)}+O(\rho^{-2-\alpha_1})$$

Using these in (94) and arguing as in proof of (95), for $r=1$, we get the proof of (95) for $r>1$. Thus in any case (95) holds. Therefore from (93) by implicit function theorem, we get $\mid \frac{dx_1}{dh}\mid<\frac{1}{|b|}$. This inequality and relation $h\in(-3\varepsilon_1,3\varepsilon_1)$ imply (90). Hence (89) is proved. In the same way we get the same estimation for $G(+k,\rho^{-\alpha})$ and $G(-k,\rho^{-\alpha})$ for $k\geq 2$. Thus $\mu(S_\rho\cap P_b)=O(\rho^{d-1+\alpha}\varepsilon_1\mid b\mid^{-1})$, for $b\notin\Gamma_1$. Since $\mid b\mid<2\rho+3$ ( see (85)) and $\varepsilon_1=\rho^{-d-2\alpha}$, we infer $\mu(\cup_{b\notin\Gamma_1}(S_\rho\cap P_b))=O(\rho^{2d-1+\alpha}\varepsilon_1)=O(\rho^{-\alpha})\mu(B(\rho))$. This, (87) and (69) give the proof of (70).

ESTIMATION 3. Here we prove (71). Let $G\equiv U_{2\varepsilon}(A_{k,j}(\gamma_1,\gamma_2,...,\gamma_k))$, where $\gamma_1,\gamma_2,...,\gamma_k\in\Gamma(p\rho^\alpha), k=\leq d-1$. We turn the coordinate axis so that $Span\{\gamma_1,\gamma_2,...,\gamma_k\}=\{x=(x_1,x_2,...,x_k,0,0,...,0):x_1,x_2,...,x_k\in R\}$. Then by (40) we have $x_n=O(\rho^{\alpha_k+(k-1)\alpha})$, for $n\leq k$, $x\in G$. This, (72), and $\alpha_k+(k-1)\alpha<1$ ( see the inequality 6. in the end of the introduction) give $G\subset(\cup_{i>k}(G(+i,\rho d^{-1})\cup G(-i,\rho d^{-1}))$, $\mu(\Pr_i(G(+i,\rho d^{-1})))=O(\rho^{k(\alpha_k+(k-1)\alpha)+(d-1-k)})$, for $i>k$. Now using this and (74) for $m=d$ we prove

$$\mu(G(+i,\rho d^{-1}))=O(\varepsilon\rho^{k(\alpha_k+(k-1)\alpha)+(d-1-k)}),\forall i>k. \tag{96}$$

For this we redenote by $D$ the set $G(+i,\rho d^{-1})$ and prove

$$\mu((D(x_1,x_2,...x_{i-1},x_{i+1},...x_d))\leq(42d^2+4)\varepsilon,\forall i>k, \tag{97}$$

for all $(x_1,x_2,...x_{i-1},x_{i+1},...x_d)\in\Pr_i(D)$. To prove (97) it is sufficient to show that if both $x=(x_1,x_2,...,x_i,...x_d)$ and $x^{'}=(x_1,x_2,...,x_i^{'},...,x_d)$ are in $D$ then $\mid x_i-x_i^{'}\mid\leq(42d^2+4)\varepsilon$. Assume the converse. Then $\mid x_i-x_i^{'}\mid>(42d^2+4)\varepsilon$. We may assume that $\rho d^{-1}<x_i<x_i^{'}$ ( see definition of $D$). Since $x$ and $x^{'}$ lie in the $2\varepsilon$ neighborhood of $A_{k,j}$ there exist points $a$ and $a^{'}$ in $A_{k,j}$ such that $\mid x-a\mid<2\varepsilon$ and $\mid x^{'}-a^{'}\mid<2\varepsilon$. Then we have the inequalities $\rho d^{-1}-2\varepsilon<a_i<a_i^{'}$, $a_i^{'}-a_i>42d^2\varepsilon$, $(a_i^{'})^2-(a_i)^2>2(\rho d^{-1}-2\varepsilon)(a_i^{'}-a_i)$, $\mid\mid a_s\mid-\mid a_s^{'}\mid\mid<4\varepsilon$, $\mid\mid a_s\mid^2-\mid a_s^{'}\mid^2\mid<12\rho\varepsilon$ for $s\neq i$, and hence

$\sum_{s\neq i} || a_s |^2 - | a_s^{'} |^2| < 12d\rho\varepsilon < \frac{2}{7}\rho d^{-1}(a_i - a_i^{'})$,

$$|| a |^2 - | a^{'} |^2| > \frac{3}{2}\rho d^{-1} | a_i^{'} - a_i |, | a_i^{'} - a_i | > 42d^2\varepsilon \tag{98}$$

Moreover using mean value theorem and the relations $| a |= \rho + O(1), | a^{'} |= \rho + O(1)$, we get

$$| a |^{2r} - | a^{'} |^{2r} = r(\rho + O(1))^{2(r-1)}(| a |^2 - | a^{'} |^2) \tag{99}$$

Let $r_i(x) = \lambda_i(x) - | x |^{2r}$ . Hence $r_1(x) \leq r_2(x) \leq ... \leq r_{b_k}(x)$ be the eigenvalue of the matrix $C(x) - | x |^{2r} I \equiv C^{'}(x)$, where $C(x)$ is defined in Section 1 (see Theorem 1(c)). By definition of $C^{'}(x)$ only diagonal elements of the matrix $C^{'}(x)$ depend on $x$ and they are $| x - d_i |^{2r} - | x |^{2r}$, where $d_i = h_i + t - \gamma - t \in B_k + \Gamma(p_1\rho^{\alpha})$. Clearly

$$| d_i | < \rho^{\frac{1}{2}\alpha_d}, \quad | r_j(a^{'}) - r_j(a) | \leq \| C^{'}(a^{'}) - C^{'}(a) \| = \max_i | a_{i,i} | . \tag{100}$$

where $C^{'}(a^{'}) - C^{'}(a) = (a_{i,j})$, $a_{i,i} = | a |^{2r} - | a - d_i |^{2r} - | a^{'} |^{2r} + | a^{'} - d_i |^{2r}$, $a_{i,j} = 0$ for $i \neq j$, that is $C^{'}(x) - C^{'}(x^{'})$ is diagonal matrix. To estimate $| a_{i,i} |$ using mean value theorem and the relations $| a |= \rho + O(1), | a^{'} |= \rho + O(1)$, $| a - d_i |= \rho + O(\rho^{\frac{1}{2}\alpha_d}), | a^{'} - d_i |= \rho + O(\rho^{\frac{1}{2}\alpha_d})$, we obtain

$$a_{i,i} = r(\rho + O(\rho^{\frac{1}{2}\alpha_d}))^{2(r-1)}(| a |^2 - | a^{'} |^2) -$$

$$r(\rho + O(\rho^{\frac{1}{2}\alpha_d}))^{2(r-1)}(| a - d_i |^2 - | a^{'} - d_i |^2) =$$

$$r(\rho + O(\rho^{\frac{1}{2}\alpha_d}))^{2(r-1)}(| a |^2 - | a - d_i |^2 - | a^{'} |^2 + | a^{'} - d_i |^2) +$$

$$O(\rho^{\frac{1}{2}\alpha_d + 2r - 3})(| a |^2 - | a^{'} |^2).$$

Since $| a |^2 - | a - d_i |^2 - | a^{'} |^2 + | a^{'} - d_i |^2 = 2(a - a^{'}, d_i)$ we have ( see (100)) $| r_j(a) - r_j(a^{'}) | \leq 3r\rho^{\frac{1}{2}\alpha_d + 2r - 2} | a - a^{'} | - c_{15}\rho^{\frac{1}{2}\alpha_d + 2r - 3} || a |^2 - | a^{'} |^2|$ . Therefore using $\frac{1}{2}\alpha_d < 1$, (99), (98) and $\lambda_i(x) = r_i(x) + | x |^{2r}$ we obtain

$| \lambda_j(a) - \lambda_j(a^{'}) | \geq || a |^{2r} - | a^{'} |^{2r}| - | r_j(a) - r_j(a^{'}) | > r\rho^{2r-1}d^{-1} | a_i^{'} - a_i | >$

$r\rho^{2r-1}42d\varepsilon > 6\varepsilon_1$ which controdicts the fact that both $\lambda_j(a)$ and $\lambda_j(a^{'})$ lie in $(\rho^2 - 3\varepsilon_1, \rho^2 + 3\varepsilon_1)$, becouse $a$ and $a^{'}$ lie in $A_{k,j}$ ( see definition $A_{k,j}$). Thus (97), hence (96) is proved. In the same way we get the same formula for $G(-i, \frac{\rho}{d})$. So $\mu(U_{2\varepsilon}(A_{k,j}(\gamma_1, \gamma_2, ..., \gamma_k))) = O(\varepsilon\rho^{k(\alpha_k + (k-1)\alpha) + d - 1 - k})$, where $j = 1, 2, ..., b_k(\gamma_1, \gamma_2, ..., \gamma_k)$, and $\gamma_1, \gamma_2, ..., \gamma_k \in \Gamma(p\rho^{\alpha})$. From this using that

$b_k = O(\rho^{d\alpha+\frac{k}{2}\alpha_{k+1}})$ ( see (37)) and the number of the vectors $(\gamma_1, \gamma_2, ..., \gamma_k)$ for $\gamma_1, \gamma_2, ..., \gamma_k \in \Gamma(p\rho^\alpha)$ is $O(\rho^{dk\alpha})$, we obtain (71) if
$d\alpha + \frac{k}{2}\alpha_{k+1} + dk\alpha + k(\alpha_k + (k-1)\alpha) + d - 1 - k \le qd - 1 - \alpha$ or

$$(d+1)\alpha + \frac{k}{2}\alpha_{k+1} + dk\alpha + k(\alpha_k + (k-1)\alpha) \le k \qquad (101)$$

for $1 \le k \le d-1$. Dividing both side of (101) by $k\alpha$ and using $\alpha_k = 3^k\alpha$, $\alpha = \frac{1}{q}$, $q = 3^d + d + 2$ we see that (101) is equivalent to
$\frac{d+1}{k} + \frac{3^{k+1}}{2} + 3^k + k - 1 \le 3^d + 2$

The right-hand side of this inequality gets its maximum value at $k = d-1$. Therefore we need to show that $\frac{d+1}{d-1} + \frac{5}{6}3^d + d \le 3^d + 4$ which follows from $\frac{d+1}{d-1} \le 3, d < \frac{1}{6}3^d + 1$ for $d \ge 2$.

ESTIMATION 4. Here we prove (66). During this estimation we denote by $G$ the set $S_\rho^{'} \cap U_\varepsilon(Tr(A(\rho))$. Since $V_\rho \equiv S_\rho^{'} \backslash G$, by (70) and (72), it is enough to prove $\mu(G(+i, \rho d^{-1})) = O(\rho^{-\alpha})\mu(B(\rho))$, for $i = 1, 2, ..., d$. (The same estimation for $G(-i, \rho d^{-1})$ can be proved in the same way.) We prove this by using (73). By (42) for $x \in G(+i, \rho d^{-1})$ the under integral expression in (73), for $k = i; a = \rho d^{-1}$, is less than $d+1$. Therefore it is sufficient to prove

$$\mu(\Pr(G(+i, \rho d^{-1})) = O(\rho^{-\alpha})\mu(B(\rho)) \qquad (102)$$

Clearly, if $(x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_d) \in \Pr_i(G(+i, \rho d^{-1}))$ then
$\mu(U_\varepsilon(G)(x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_d)) \ge 2\varepsilon$ and by (74),

$$\mu(U_\varepsilon(G)) \ge 2\varepsilon\mu(\Pr(G(+i, \rho d^{-1})). \qquad (103)$$

Hence to prove (102) we need to estimate $\mu(U_\varepsilon(G))$. For this we prove that

$$U_\varepsilon(G) \subset U_\varepsilon(S_\rho^{'}), U_\varepsilon(G) \subset U_{2\varepsilon}(Tr(A(\rho))), U_\varepsilon(G) \subset Tr(U_{2\varepsilon}(A(\rho))). \qquad (104)$$

The first and second inclusions follow from $G \subset S_\rho^{'}$ and $G \subset U_\varepsilon(Tr(A(\rho)))$ respectively (see definition of $G$ ). Now we prove the third inclusion in (104). If $x \in U_\varepsilon(G)$ then by the second inclusion of (104) there exists $b$ such that $b \in Tr(A(\rho))$, $\mid x - b \mid < 2\varepsilon$. Then by the definition of $Tr(A(\rho))$ there are $\gamma \in \Gamma$ and $c \in A(\rho)$ such that $b = \gamma + c$. Therefore $\mid x - \gamma - c \mid = \mid x - b \mid < 2\varepsilon$,
$x - \gamma \in U_{2\varepsilon}(c) \subset U_{2\varepsilon}(A(\rho))$. This together with $x \in U_\varepsilon(G) \subset U_\varepsilon(S_\rho^{'})$ (see the first inclusion of (104)) give $x \in Tr(U_{2\varepsilon}(A(\rho)))$ , i.e., the third inclusion in (104) is proved. The third inclusion, Lemma 4 (c) and (71) imply that $\mu(U_\varepsilon(G)) = O(\rho^{-\alpha})\mu(B(\rho))\varepsilon$. This and (103) give the proof of (102)$\diamondsuit$

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