MULTIDIMENSIONAL BORG-LEVINSON THEOREM

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Abstract. *We consider the inverse problem of the reconstruction of a Schrödinger operator on a unknown Riemannian manifold or a domain of Euclidean space. The data used is a part of the boundary Γ and the eigenvalues corresponding to a set of impedances in the Robin boundary condition which vary on Γ. The proof is based on the analysis of the behaviour of the eigenfunctions on the boundary as well as in perturbation theory of eigenvalues. This reduces the problem to an inverse boundary spectral problem solved by the boundary control method.*

Key words: Inverse spectral problems, analysis on manifolds, Schrödinger operator.

1. Introduction

In 1929 Ambartsumyan [2] considered the Sturm-Liouville problem

$$-\psi'' + q(x)\,\psi = \lambda\,\psi, \quad x \in (0,1), \quad \psi'(0) = \psi'(1) = 0, \tag{1}$$

where the potential q is continuous and real valued. Let $\{\lambda_k\}_{k=0}^{\infty}$ be the eigenvalues for this Sturm-Liouville problem. Ambartsumyan proved that if $\lambda_k = k^2$ for $k = 0, 1, \dots,$ then $q \equiv 0$.

The next important contribution was due to Borg [6] who assumed that q is integrable and real valued. His result can be stated as follows. He proved that one spectrum in general does not uniquely determine the corresponding Sturm-Liouville operator and that the result of Ambartsumyan is a special case.

Let $\{\lambda_k\}_{k=0}^{\infty}$ be the eigenvalues for (1) with the boundary conditions

$$\psi'(0) + h_1 \cdot \psi(0) = 0, \quad \psi'(1) + h_3 \cdot \psi(1) = 0,$$

and let $\{\mu_k\}_{k=0}^{\infty}$ be the eigenvalues with the boundary condition

$$\psi'(0) + h_2 \cdot \psi(0) = 0, \quad \psi'(1) + h_3 \cdot \psi(1) = 0,$$

where $h_1 \neq h_2, h_3$ are real numbers. Then, the two sets $\{\lambda_k\}_{k=0}^{\infty}$ and $\{\mu_k\}_{k=0}^{\infty}$ uniquely determine h_1, h_2, h_3 and q. Levinson [20] obtained simpler proofs of some of the results of Borg.

Borg [7] and Marchenko [22] generalized the Borg-Levinson theorem to Sturm-Liouville operators on the half line with a boundary condition at the origin when there is no continuous spectrum. They independently proved that the discrete spectra corresponding to two different boundary conditions at $x = 0$ (with a fixed boundary condition, if required, at $x = +\infty$) uniquely determine the potential and the boundary conditions at the origin.

Borg-Marchenko's result was generalized to the case where there is also a continuous spectrum in [1] where it was proven that the potential and boundary conditions are uniquely determined by an appropriate data set containing the discrete eigenvalues and continuous part of the spectral measure corresponding to one boundary condition at the origin and a subset of the discrete eigenvalues for a different boundary condition. Another extension of the Borg-Marchenko theorem to the case with a continuous spectrum is given by Gesztesy and Simon [9]. The uniqueness result is proven there in the case when Krein's spectral shift function is known.

The Borg-Levinson inverse two spectra problem can be reduced to the inverse boundary spectral problem with data of the form

$$\{\lambda_k, c_k\}_{k=0}^{\infty} \tag{2}$$

where c_k are the norming constants,

$$c_k := \|\psi_k\|_{L^2(0,1)},$$

and ψ_k is the eigenfunction corresponding to λ_k with $\psi_k(0) = 1, \psi_k'(0) = -h_1$. See for example [21], [8]. Clearly, data (2) is equivalent to the following inverse boundary spectral data,

$$\{\lambda_k, \phi_k(0)\}_{k=0}^{\infty}, \tag{3}$$

where now ϕ_k are the unit-norm eigenfunctions.

A multidimensional analog of boundary spectral data is the set

$$\{\lambda_k, \phi_k|_{\partial\Omega}\}_{k=0}^{\infty},$$

in the case of the Neumann or third-type boundary conditions (cf. (3), and the set

$$\{\lambda_k, \partial_n\phi_k|_{\partial\Omega}\}_{k=0}^{\infty},$$

in the case of the Dirichlet boundary condition. Here $\Omega \subset \mathbb{R}^n$ is a (smooth) bounded domain and ∂_n is the interior unit normal derivative to $\partial\Omega$. In comparison with the $1-$dimensional case, not all second-order elliptic operators, even isotropic ones, can be reduced to a Schrödinger operator in Ω. For different classes of isotropic elliptic operators, e.g. for an acoustic operator, or a Schrödinger operator , or a more general second-order operator, namely,

$$Au = -c^{-2}(x)\Delta u, \text{ or } Au = -\Delta u + q(x)u, \text{ or } Au = -\text{div}\,(\varepsilon(x)\nabla u) + q(x)u, \tag{4}$$

where c, ε are positive functions and q is a real-valued function in Ω, the uniqueness of determination of c, or q, or ε and q was proven, correspondingly in [3], [23] and

[24]. It should be noted that the methods used in these papers differed significantly, with [3] introducing the boundary control (BC) method while [23] being based on the complex geometric optics method of [25] and [24] using the ideas of $\overline{\partial}$-problem.

The inverse boundary spectral problem for the anisotropic case was considered in [4], where it was shown that boundary spectral data determine a compact Riemannian manifold and in [17], [18] and [19] where it was shown that boundary spectral data determine, up to a natural group of gauge transformations, a general second-order self-adjoint elliptic operator and a wide class of second-order non-self-adjoint elliptic operators on a compact manifold. It should be noted that, the boundary $\partial\Omega$ of the manifold being given, the manifold itself was not a priori known and was to be recovered from the boundary spectral data which, in this case, is the set

$$(\partial\Omega,\ \{\lambda_k,\ \phi_k|_{\partial\Omega}\}_{k=1}^{\infty}) \tag{5}$$

where λ_k and ϕ_k are the Neumann-eigenvalues and normalized eigenfunctions of the Laplace-Beltrami operator.

In this paper we use invariant formulation of inverse problems, i.e., formulate the problem in terms of manifolds. For clarity, we also apply the obtained results in the Euclidean setting. Unless otherwise specified, (Ω, g) is a smooth connected compact Riemannian manifold with non-empty boundary. On (Ω, g) we study the Schrödinger operator

$$A = -\Delta + q$$

where $\Delta = \Delta_g$ is the Laplace-Beltrami operator. By A^ω we denote the operator A defined in the set of $H^2(M)$ functions that satisfy the third-type boundary condition on $\partial\Omega$,

$$(\partial_\nu u + \omega u)\,|_{\partial\Omega} = 0,$$

with ∂_ν being the interior normal derivative on $\partial\Omega$ in the corresponding metric. Following physical literature, we refer to the real valued function $\omega \in C^\infty(\partial\Omega)$ as the impedance. The proofs in [17], [18], [19] were based on a geometric approach to the BC-method, see [14] for a detailed exposition. It is, however, clear from the considerations above that the mentioned papers on multidimensional inverse problems did not consider a multidimensional analog of the Borg-Levinson inverse problem , but the inverse boundary spectral problem. A multidimensional analog of the Borg-Levinson inverse problem may be formulated as follows:

Definition 1.1. *Let (Ω, g) be a compact connected Riemannian manifold with non-empty boundary $\partial\Omega$, $\Sigma \subset \partial\Omega$ be an open connected non-empty subset and q be a real-valued function in $C^\infty(\Omega)$. Let $\omega_0 \in C^\infty(\partial\Omega)$ be a real valued function. Consider the Schrödinger operators in $L^2(\Omega)$ of the form,*

$$A^\omega u = -\Delta u + qu, \quad D(A^\omega) = \{u \in H^2(\Omega) : (\partial_\nu u + \omega u)\,|_{\partial\Omega} = 0\}, \tag{6}$$

where ω is real valued and $\widetilde{\omega} = \omega - \omega_0 \in C_0^\infty(\Sigma)$. Denote by $\lambda_k(\omega)$, $k = 1, 2, \dots$ the corresponding eigenvalues counting multiplicity. The local spectral data is

(7) Σ *and the map* $\omega \mapsto \{\lambda_k(\omega)\}_{k=1}^\infty$ *defined for* $\omega \in C^\infty(\partial\Omega)$, $\omega - \omega_0 \in C_0^\infty(\Sigma)$.

Note that here Ω is compact manifold so that $C^\infty(\Omega)$ consists of functions that are smooth upto the boundary.

Problem 1.2. *Do local spectral data of form (7) determine (Ω, g), q and ω_0 uniquely?*

Note, that by determination of a Riemannian manifold (Ω, g) we mean determination of its isometry type.

We denote the Gateaux derivatives of $\omega \mapsto \lambda_k(\omega)$ at ω_0 in the direction $\widetilde{\omega}$ by $\lambda_{k,\omega_0}(\widetilde{\omega}) = d\lambda_k|_{\omega_0}(\widetilde{\omega})$. Clearly, local spectral data make it possible to find the $\lambda_{k,\omega_0}(\widetilde{\omega})$ for any $k = 1, 2, \dots$ and $\widetilde{\omega} \in C_0^\infty(\Sigma)$.

In following, we use notation

$$B_\varepsilon^\infty(\omega_0) = \{\omega \in C^\infty(\partial\Omega) :\ ||\omega - \omega_0||_{L^\infty(\partial\Omega)} < \varepsilon,\ \omega - \omega_0 \in C_0^\infty(\Sigma)\}. \tag{8}$$

Depending on degeneracy/non-degeneracy of the spectrum of A^{ω_0}, we prove the following result.

Theorem 1.3. *Let (Ω, g) be a smooth, compact, connected Riemannian manifold with boundary and $\Sigma \subset \partial\Omega$ be an open, connected, non-empty subset and A^{ω_0} be a Schrödinger operator of form (6). Then*

a. If the spectrum of A^{ω_0} is simple, then Σ, the eigenvalues $\lambda_k(\omega_0)$, and their Gateaux derivatives, $\lambda_{k,\omega_0}(\omega)$; $\omega \in C_0^\infty(\Sigma)$ uniquely determine (Ω, g), q and ω_0.

b. For arbitrary A^{ω_0}, given Σ and $\{\lambda_k(\omega)\}_{k=1}^\infty$ for all real-valued $\omega \in B_\epsilon^\infty(\omega_0)$ with some $\epsilon > 0$, one can uniquely determine (Ω, g), q and ω_0.

Note that, in Theorem 1.3, we do not assume an a priori knowledge of either Ω or $\partial\Omega$. We only have to know Σ. Theorem 1.3 has the following corollary in Euclidean setting.

Corollary 1.4. *Let $\Omega \subset \mathbb{R}^n$, $g_{ij}(x) = c(x)\delta_{ij}$ be a conformally isotropic metric on Ω, and $\Sigma \subset \partial\Omega$ be open and non-empty. Let A^{ω_0} be a Schrödinger operator of form (6). Then Σ and $\{\lambda_k(\omega)\}_{k=1}^\infty$ for all real valued $\omega \in B_\epsilon^\infty(\omega_0)$, with some $\epsilon > 0$, determine Ω as a subset of $\mathbb{R}^n$, $c(x)$, q, and ω_0 uniquely.*

2. Boundary behavior of eigenfunctions

In this section we consider the eigenvalues and eigenfunctions of an operator A^ω for a fixed ω. In this connection we skip using ω throughout this section, writing λ_k instead of $\lambda_k(\omega)$ and ϕ_k instead of $\phi_k(\omega)$.

To describe behavior of eigenfunctions near $\partial\Omega$ we employ the boundary normal coordinates $x = (z, \tau)$, where $\tau = \mathrm{dist}(x, \partial\Omega)$ and z is the unique point on $\partial\Omega$ nearest to x with local coordinates $z = (z^1, \dots, z^{n-1})$.

Lemma 2.1. *Let ϕ be an eigenfunction for an eigenvalue λ of an operator A^ω (with some fixed ω). Then, for any $z_0 \in \partial\Omega$, there is a multi-index $\alpha_0 \in \mathbb{Z}_+^{n-1}$ such that*

$$\partial^{\alpha_0}\phi(z_0) \neq 0. \tag{9}$$

Here $\phi(z) = \phi(z, 0)$ and equation (9) is valid in proper local coordinates on $\partial\Omega$, $z = (z^1, \dots, z^{n-1})$ where, without loss of generality, $z_0 = 0$.

Proof. If $\omega \neq 0$ we introduce a gauge transformation [14]

$$u \longrightarrow v = \kappa u, \quad \kappa \in C^\infty(\overline{\Omega}), \quad \kappa(x) > 0 \text{ for } x \in \overline{\Omega},\ \partial_\tau \kappa|_{\tau=0} = -\omega.$$

Then $\psi = \kappa\phi$ is a smooth solution to the equation

$$-\partial_\tau^2\psi - g^{ij}\partial_i\partial_j\psi + a^n\partial_\tau\psi + a^i\partial_i\psi + a^0\psi = \lambda\psi,\ \tau > 0,\ i, j = 1, \dots, n-1, \tag{10}$$

where a^0, a^i, a^n and g^{ij} are functions of (z, τ), and

$$\partial_\tau\psi|_{\tau=0} = 0. \tag{11}$$

Assume now that, for any $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{Z}_+^{n-1}$, $\partial^\alpha\phi(0) = \partial_{z_1}^{\alpha_1} \dots \partial_{z_{n-1}}^{\alpha_{n-1}}\phi(0) = 0$ and, therefore, $\partial^\alpha\psi(0) = 0$. Using (10), (11), this implies that for any $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$,

$$\partial_{z_1}^{\beta_1} \dots \partial_{z_{n-1}}^{\beta_{n-1}}\partial_\tau^{\beta_n}\psi(0) = 0. \tag{12}$$

Let $\widehat{\psi}, \widehat{a}^0, \widehat{a}^i, \widehat{g}^{ij}$ be even continuations of these functions across the boundary $\tau = 0$ and $\widehat{a}^n$ be an odd continuation of a_n. Then, in an open set $U \subset \mathbb{R}^n, 0 \in U$, the function $\widehat{\psi}$ is a $C^2(U)$ solution of the equation

$$-\partial_\tau^2\widehat{\psi} - \widehat{g}^{ij}\partial_i\partial_j\widehat{\psi} + \widehat{a}^n\partial_\tau\widehat{\psi} + \widehat{a}^i\partial_i\widehat{\psi} + \widehat{a}^0\widehat{\psi} = \lambda\widehat{\psi}, \tag{13}$$

with $\widehat{g}^{ij} \in C^{0,1}(U)$ and $\widehat{a}^p \in L^\infty(U)$, $p = 0, \dots, n$. Moreover, by (12), for any $N > 0$ there is C_N so that

$$|\widehat{\psi}(z, \tau)| \leq C_N|x|^N, \quad |x|^2 = \sum_{i=1}^{n-1}|z^i|^2 + \tau^2.$$

This, together with equation (13) imply, due to the Hörmander strong uniqueness principle, [10], that $\psi = \phi = 0$. □

It will be shown in the next section that, under some additional assumptions, local spectral data determine $|\phi_k^{\omega_0}(x)|$, $x \in \Sigma$, $k = 1, 2, \dots$. Moreover, the following result holds:

Theorem 2.2. *Given $\xi \in C^\infty(\Sigma)$ such that $\xi(z) = |\phi(z)|$, $z \in \Sigma$, where ϕ is an eigenfunction of an operator A^{ω_0}, it is possible to find $\phi|_\Sigma$ up to multiplication by ± 1.*

Proof. To fix the sign of ϕ, choose a point $z_0 \in \Sigma$ where $\xi(z_0) > 0$ and take $\phi(z_0) = \xi(z_0) > 0$. Let $(z^1, \dots, z^{n-1}) \in B_r \subset \mathbb{R}^{n-1}$ be Riemannian normal coordinates in the metric ball $B_r(z_0) \subset \Sigma$, where $(\partial\Omega, g)$ is endowed with the metric induced by (Ω, g). Note that we can choose

$$r = \min(\text{inj}(\partial\Omega),\, d_{\partial\Omega}(z_0, \partial\Sigma)). \tag{14}$$

We first show that ξ determines ϕ everywhere in B_r. By continuity of ϕ, it is clear that ξ determines ϕ in ball $B_{\widetilde{r}}$ for sufficiently small $\widetilde{r}$. Let ρ be the largest possible value $\rho \le r$ such that ξ determines ϕ in ball B_ρ. We want to show that $\rho = r$. Assuming the contrary, we note that ϕ is defined on the closure $\overline{B_\rho}$. Let $y \in \partial B_\rho \subset \Sigma$. If $\phi(y) \neq 0$, then, by continuity, ξ determines ϕ in a vicinity of y. If $\phi(y) = 0$, by Lemma 2.1 there is $m > 0$ such that

$$\phi(z) = \sum_{|\alpha|=m} b_\alpha (z-y)^\alpha + O\left(|z-y|^{m+1}\right), \quad b_{\alpha_0} \neq 0 \text{ for some } \alpha_0,\ |\alpha_0| = m. \tag{15}$$

It follows from (15) that there is an open dense set $W \subset S^{n-2}$ such that, for $e = (e^1, \dots, e^{n-1}) \in W$,

$$\partial_e^m \phi(y) = (e^j \partial_j)^m \phi(y) \neq 0.$$

Choosing e transversal to ∂B_ρ at y and assuming that, without loss of generality, $\partial_e = \partial_{n-1}$, we obtain, using Malgrange Preparation Theorem, e.g. [11, Th. 7.5.5], that, in a vicinity of y,

$$\phi(\widehat{z}, z^{n-1}) = c(\widehat{z}, z^{n-1}) \sum_{l=0}^m a_l(\widehat{z})(z^{n-1} - y^{n-1})^l. \tag{16}$$

Here $\widehat{z} = (z^1, \dots, z^{n-2})$, the function $c(z) = c(\widehat{z}, z^{n-1})$ is a $C^\infty-$function near $z = y$ with $c(y) \neq 0$ and $a_m(\widehat{z}) = 1$. Therefore, $\phi(\widehat{z},\, z^{n-1})$, for a fixed $\widehat{z}$, has only a finite number of real roots, $r_j(\widehat{z})$. The function $\phi(\widehat{z}, z^{n-1})$, considered as a function of z^{n-1}, changes its sign at $r_j(\widehat{z})$ when this root is of an odd order and does not change the sign when the root is of an even order. As the lines $\widehat{z} = \text{const}$ are transversal to ∂B_ρ near y, we obtain the continuation of ϕ into a vicinity of y. As $y \in \partial B_\rho$ is arbitrary, we obtain the continuation of ϕ into an open neighborhood of $\overline{B_\rho}$. Thus, $\rho = r$, i.e. ϕ can be uniquely determined everywhere in the ball B_r. It also follows from the above arguments that $\{z : \phi(z) \neq 0\} \cap B_r(z_0)$ is an open set of full measure.

To proceed further, let $\widetilde{z} \in \Sigma$ and L be a curve in Σ connecting z_0 with $\widetilde{z}$. We cover L by a finite number of balls $B_{r_j/2}(z_j)$, $j = 0, 1, \dots, J$, $z_J = \widetilde{z}$, such that $B_{r_j/2}(z_j) \cap B_{r_{j+1}/2}(z_{j+1}) \neq \emptyset$. In particular, there is a point $\widetilde{z}_1 \in B_{r_0/2}(z_0) \cap B_{r_1/2}(z_1)$

with $\phi(\widetilde{z}_1) \neq 0$. By the previous construction we find ϕ in $B_{\widetilde{r}_1}(\widetilde{z}_1)$ which contains $B_{r_1/2}(z_1)$. Continuing this process, we find $\phi(\widetilde{z})$. □

3. Generic behavior of eigenvalues

Consider the quadratic form Q^ω related to the operator A^ω,

$$Q^\omega(u) = \int_\Omega \left(|\nabla u|^2 + q|u|^2\right)\, dV + \int_{\partial\Omega} \omega|u|^2\, dS, \tag{17}$$

where dV, dS are the volume and area forms generated by the metric g in Ω and $\partial\Omega$.

Let $A(t)$ be an analytic, for $|t| < \varepsilon$, one-parameter family of Schrödinger operators of the form (6), where the impedance $\omega(t)$ of the form

$$\omega(t) = \omega_0 + t\widetilde{\omega}, \quad \text{with real } \widetilde{\omega} \in C_0^\infty(\Sigma). \tag{18}$$

Then $A(t)$ is a self-adjoint homomorphic operator family of type (B), in the sense of Kato [16, Section 7.4], so that the eigenvalues $\lambda_k(\omega(t))$ and eigenfunctions $\phi_k^{\omega(t)}$ may be chosen to be analytic with respect to t. In this case we can find the Gateaux derivative of λ_k with respect to t. A bit more generally, the following result holds:

Lemma 3.1. *Let $\lambda_k(t)$, $\phi_k(t)$ be an eigenvalue and a corresponding normalized eigenfunction of $A(t)$ which are differentiable with respect to t. Then*

$$\dot{\lambda}_k(t) = -\int_{\partial\Omega} |\phi_k(t)|^2 \widetilde{\omega}\, dS, \tag{19}$$

where $\dot{\lambda}$ stands for the $t-$differentiation of λ.

Proof. Differentiating with respect to t the equation for $\phi_k(t)$, we get

$$(-\Delta + q - \lambda_k(t))\, \dot{\phi}_k(t) = \dot{\lambda}_k(t)\phi_k(t).$$

Thus, due to $||\phi_k(t)|| = 1$,

$$\begin{aligned} \dot{\lambda}_k(t) &= \int_\Omega \Big((-\Delta + q - \lambda_k(t))\dot{\phi}_k(t)\Big)\, \overline{\phi_k(t)}\, dV \\ &= \int_{\partial\Omega} \Big(\partial_\nu \dot{\phi}_k(t)\, \overline{\phi_k(t)} - \dot{\phi}_k(t)\, \partial_\nu \overline{\phi_k(t)}\Big)\, dS. \end{aligned} \tag{20}$$

By the boundary condition in (6),

$$\partial_\nu \dot{\phi}_k(z,t) = -\Big(\omega(z,t)\dot{\phi}_k(z,t) + \dot{\omega}(z,t)\phi_k(z,t)\Big), \quad z \in \partial\Omega.$$

This together with (20) imply equation (19) due to (18). □

Denote by $\mu_k(\omega)$ the multiplicity of λ_k^ω and assume that $\mu_k(\omega)$ is constant near ω_0.

Corollary 3.2. *Assume that for some $\varepsilon > 0$, $\lambda_{k-j-1}(\omega) < \lambda_{k-j}(\omega) = \dots \lambda_k(\omega) = \dots = \lambda_{k+p-1}(\omega) < \lambda_{k+p}(\omega)$, $p + j = \mu_k(\omega_0)$, for all*

$$\omega \in B_\varepsilon^\infty(\omega_0). \tag{21}$$

Then for any $\widetilde{\omega} \in C_0^\infty(\Sigma)$ and a normalized eigenfunction ϕ of A^{ω_0} corresponding to the eigenvalue $\lambda_k(\omega_0)$ there is an eigenvalue $\lambda(t)$ and a normalized eigenfunction $\phi(t)$ of $A^{\omega(t)}$, $\omega(t) = \omega_0 + t\widetilde{\omega}$ such that $\phi(0) = \phi$ and that the equation (19) is valid.

This result is standard for the perturbation theory for quadratic forms, e.g. [16], [5]. We repeat its proof for the convenience of the reader.

Proof. By the perturbation theory for quadratic forms, e.g. [16], [5], a sufficiently small disk centered in $\lambda_k = \lambda_k(\omega_0)$ does not contain eigenvalues of A^ω, except for $\lambda_{k-j}(\omega), \dots, \lambda_{k+p-1}(\omega)$, when ω satisfies (21) with sufficiently small ε. Consider the Riesz projectors, P_k^ω, to the eigenspace corresponding to $\lambda_k(\omega)$,

$$P_k^\omega = \frac{1}{2\pi i} \int_\Gamma R_z^\omega \, dz, \tag{22}$$

where R_z^ω is the resolvent for A^ω and Γ is a sufficiently small circle around $\lambda_k(\omega_0)$. When $\omega = \omega(t)$ is of form (18), $R_z(t)$ is an analytic, with respect to t, operator-valued function in $L^2(\Omega)$. Therefore, P_k^ω are also analytic with respect to t. Moreover, for sufficiently small ε and real t, $\widetilde{\phi}(t) = P_k^{\omega(t)}\phi \neq 0$ so that $\phi(t) = \widetilde{\phi}(t)/||\widetilde{\phi}(t)||$ is a desired normalized eigenfunction for $A(t)$ which smoothly depends on t. This implies also that $\lambda_k(t)$ is smooth with respect to t and the considerations of Lemma 3.1 are valid. □

Combining Corollary 3.2 with Theorem 2.2 we obtain the following result.

Corollary 3.3. *Assume that $\lambda_k(\omega)$ has a constant multiplicity, $\mu_k(\omega) = \mu_k(\omega_0)$ for all ω satisfying equation (21). Then $\mu_k(\omega) = 1$.*

Proof. By corollary 3.2, any $\phi \in P_k^{\omega_0} L^2(\Omega)$, $||\phi|| = 1$ satisfies equation (19). Thus, for any two different normalized eigenfunctions ϕ, $\widetilde{\phi}$ for λ_k,

$$\int_{\partial\Omega} |\phi|^2 \widetilde{\omega} \, dS = \int_{\partial\Omega} |\widetilde{\phi}|^2 \widetilde{\omega} \, dS,$$

with arbitrary $\widetilde{\omega} \in C_0^\infty(\Sigma)$.This implies that $|\phi| = |\widetilde{\phi}|$ on Σ, so that $\phi|_\Sigma = \pm\widetilde{\phi}|_\Sigma$. This, together with the boundary condition in (6), yield that also $\partial_\nu\phi|_\Sigma = \pm\partial_\nu\widetilde{\phi}|_\Sigma$. Using the similar arguments as in proof of Lemma 2.1 and applying the Hörmander unique continuation theorem [10], $\phi = \pm\widetilde{\phi}$ on Ω. □

We now investigate the multiplicity of eigenvalues under small perturbations of the impedance.

Lemma 3.4. *For any $k \in \mathbb{Z}_+$, $\varepsilon > 0$ there is $\omega \in C^\infty(\partial\Omega)$ satisfying equation (21) such that $\lambda_i(\omega)$ are simple for $i = 1, \dots, k$.*

Proof. By the perturbation theory for quadratic forms e.g. [16], [5], for any $\omega_0 \in C^\infty(\partial\Omega)$ and $i \in \mathbb{Z}_+$, there are ϵ, $\delta > 0$ such that

$$\dim P^\omega(\delta) L^2(\Omega) = \dim P_i^{\omega_0} L^2(\Omega),$$

for all $\omega \in B_\epsilon^\infty(\omega_0)$, where $P^\omega(\delta)$ is the projector onto the sum of eigenspaces of A^ω corresponding to the eigenvalues from the interval $(\lambda_i(\omega_0)-\delta,\ \lambda_i(\omega_0)+\delta)$. Therefore, $\mu_i(\omega_0)$ is an upper-semicontinuous function of $\omega_0 \in L^\infty(\partial\Omega)$.

Let

$$\underline{\mu}_i(\omega_0) = \liminf_{\omega\to\omega_0,\ \omega-\omega_0\in C_0^\infty(\Sigma)} \mu_i(\omega),\ \ \overline{\mu}_i(\omega_0) = \limsup_{\omega\to\omega_0,\ \omega-\omega_0\in C_0^\infty(\Sigma)} \mu_i(\omega).$$

As $\mu_i(\omega) \in \mathbb{Z}_+$, there is $\delta_i = \delta_i(\omega_0) > 0$, such that

$$\min \mu_i(\omega) = \underline{\mu}_i(\omega_0),\ \ \max \mu_i(\omega) = \overline{\mu}_i(\omega_0),$$

where minimum and maximum are taken over the set $\omega \in B^\infty_{\delta_i(\omega_0)}(\omega_0)$. Choose ω_1 with $\mu_1(\omega_1) = \underline{\mu}_1(\omega_0)$ such that

$$||\omega_1 - \omega_0||_{L^\infty(\partial\Omega)} < \min(\varepsilon/k,\ \delta_1(\omega_0)), \quad \omega_1 - \omega_0 \in C_0^\infty(\Sigma).$$

Then, due to the mentioned upper-semicontinuity of μ_1, there is $\widetilde{\delta_1} > 0$ so that

$$\mu_1(\omega) = \mu_1(\omega_1) \quad \text{for } \omega \in B^\infty_{\widetilde{\delta_1}}(\omega_1). \tag{23}$$

By Corollary 3.3,

$$\underline{\mu}_1(\omega_0) = \mu_1(\omega) = 1,$$

for $\omega \in B^\infty_{\widetilde{\delta_1}}(\omega_1)$.

Next we find $\delta_2 < \widetilde{\delta_1}$ such that

$$\min \mu_2(\omega) = \underline{\mu}_2(\omega_1)$$

where minimum is taken over the set

$$\omega \in B^\infty_{\delta_2}(\omega_1). \tag{24}$$

This makes it possible to choose ω_2 satisfying (24) and also

$$\mu_2(\omega_2) = \underline{\mu}_2(\omega_1), \quad ||\omega_2 - \omega_1||_{L^\infty(\partial\Omega)} < \min(\varepsilon/k,\ \delta_2).$$

Repeating the same arguments as for μ_1, there is $\widetilde{\delta_2} < \min(\varepsilon/k,\ \delta_2)$ such that

$$\underline{\mu}_2(\omega_1) = \mu_2(\omega) = 1, \tag{25}$$

for $\omega \in B^\infty_{\widetilde{\delta_2}}(\omega_2)$, and the ball $B^\infty_{\widetilde{\delta_2}}(\omega_2)$ lies inside the ball $B^\infty_{\widetilde{\delta_1}}(\omega_1)$ so that also

$$\mu_1(\omega) = 1.$$

Continuing this procedure, we find $\omega_k \in C^\infty(\partial\Omega), \omega_k - \omega_0 \in C_0^\infty(\Sigma)$ with

$$\mu_1(\omega_k) = \cdots = \mu_k(\omega_k) = 1. \tag{26}$$

Moreover, it is seen easily from the above construction that

$$||\omega_k - \omega_0||_{L^\infty(\partial\Omega)} < \varepsilon.$$

$\square$

Remark 3.5. *A slight modification of the previous arguments shows that, in any $C_0^\infty(\Sigma)-$neighborhood of ω_0 there is an impedance ω such that the spectrum of A^ω is simple. Indeed, we can easily generalize Lemma 3.4 to show that, for any $k \in \mathbb{Z}_+$, $\varepsilon > 0$ and ω there is ω_k satisfying (26) such that*

$$||\omega_k - \omega||_{C^k(\partial\Omega)} < \frac{\varepsilon}{2^k}. \tag{27}$$

To construct ω with simple spectrum, we first find ω_1 with $\mu_1(\omega_1) = 1$ satisfying (27) with $k = 1$ and ω_0 instead of ω. Then we find ω_2 with $\mu_1(\omega_2) = \mu_2(\omega_2) = 1$ and (27) with $k = 2$ and ω_1 instead of ω. By taking, if necessary, ω_2 to be $L^\infty-$closer to ω_1, we obtain that

$$|\lambda_1(\omega_2) - \lambda_2(\omega_2)| > (1/2 - 1/2^2)|\lambda_1(\omega_1) - \lambda_2(\omega_1)|. \tag{28}$$

Next we find ω_3 with $\mu_1(\omega_3) = \mu_2(\omega_3) = \mu_3(\omega_3) = 1$ and (27) with $k = 3$ and ω_2 instead of ω. By taking, if necessary, ω_3 to be $L^\infty-$closer to ω_2, we obtain that

$$\begin{aligned} |\lambda_1(\omega_3) - \lambda_2(\omega_3)| &> (1/2 - 1/2^3)|\lambda_1(\omega_1) - \lambda_2(\omega_1)|, \\ |\lambda_2(\omega_3) - \lambda_3(\omega_3)| &> (1/2 - 1/2^3)|\lambda_2(\omega_2) - \lambda_3(\omega_2)|. \end{aligned} \tag{29}$$

Continuing the above procedure, we construct a converging, in $C^\infty(\partial\Omega)$, sequence ω_k. Denote by ω its limit, $\omega = \lim \omega_k$. By (27), for any $p \in \mathbb{Z}_+$,

$$||\omega_0 - \omega||_{C^p(\partial\Omega)} < \varepsilon. \tag{30}$$

As $\lambda_i(\omega)$ depends continuously on ω, equations (28), (29), and analogous equations for further ω_k show that

$$|\lambda_k(\omega) - \lambda_{k+1}(\omega)| \geq \frac{1}{2}|\lambda_k(\omega_k) - \lambda_{k+1}(\omega_k)| > 0,$$

so that A^ω has simple spectrum. It is clear from the above construction that the set of impedances ω with degenerate spectrum is of the first Baire category.

We note that the above result can be also obtained using [26], however, the method of [26] is different from the one in Remark 3.5 being based on the ideas of [27] rather than the quadratic forms perturbation theory and unique continuation for elliptic equation.

4. From local spectral data to boundary spectral data . Proof of main results.

We are now in the position to prove our main results. We start with the following technical theorem:

Theorem 4.1. *For any real $\omega_0 \in C^\infty(\partial\Omega)$ and any open, non-empty connected $\Sigma \subset \partial\Omega$, the local spectral data determine the traces $\phi_k|_\Sigma$, $k = 1, \dots,$ up to a sign, of the eigenfunctions of the Schrödinger operator A^{ω_0}.*

Proof. If $\mu_i(\omega_0) = 1$, Corollary 3.2 makes possible to find, for an arbitrary $\widetilde{\omega} \in C_0^\infty(\Sigma)$,

$$\int_{\partial\Omega} \widetilde{\omega}|\phi_i|^2 \, dS, \tag{31}$$

where ϕ_i is the normalized eigenfunction of A^{ω_0} corresponding to $\lambda_i(\omega_0)$.

Let now $\mu_i(\omega_0) = p > 1$, say $\lambda_l = \dots = \lambda_i = \dots = \lambda_m$, $l \le i \le m$, $m - l = p - 1$. By Lemma 3.4, there are smooth impedances ω_n, $n = 1, 2, \dots,$ which converge to ω_0 while their eigenvalues $\lambda_j(\omega_n)$, $1 \le j \le m$, remain simple. By Corollary 3.2 it is possible to find $\int_{\partial\Omega} \widetilde{\omega}|\phi_j^n|^2 \, dS$, where ϕ_j^n, for $l \le j \le m$, are the orthonormalized eigenfunctions of A^{ω_n} corresponding to $\lambda_j(\omega_n)$. As $||\phi_j^n||_{H^1(\Omega)}$ are uniformly bounded, there is a subsequence $n(k)$, which we assume to be the whole sequence, such that

$$\lim_{n\to\infty} \phi_j^n = \phi_j, \quad 1 \le j \le m. \tag{32}$$

The convergence in (32) is weak in $H^1(\Omega)$ and strong in $H^s(\Omega)$ for any $s < 1$. As

$$\lim_{n\to\infty} \lambda_j(\omega_n) = \lambda_j(\omega_0), \quad 1 \le j \le m,$$

ϕ_j satisfy the equation

$$(-\Delta + q)\phi_j = \lambda_j(\omega_0)\phi_j.$$

Moreover, as

$$\lim_{n\to\infty} \phi_j^n|_{\partial\Omega} = \phi_j|_{\partial\Omega} \quad \text{in } L^2(\partial\Omega),$$

we see that ϕ_j are normalized eigenfunctions of A^{ω_0} for λ_j, $1 \le i \le m$. In addition, for multiple eigenvalues of A^{ω_0}, the corresponding eigenfunctions remain orthogonal because the eigenfunctions ϕ_j^n, ϕ_k^n, $j, k \le m$ are orthogonal for any n and $j \ne k$. Thus, ϕ_j are the first m orthonormal eigenfunctions of A^{ω_0}.

Also,

$$\lim_{n\to\infty} \int_{\partial\Omega} |\phi_j^n|^2 \widetilde{\omega} \, dS = \int_{\partial\Omega} |\phi_j|^2 \widetilde{\omega} \, dS, \tag{33}$$

for any $\widetilde{\omega} \in C_0^\infty(\Sigma)$, so that we know all integrals (31) when $i \le m$. Since $m \in \mathbb{Z}_+$ is arbitrary, we determine the integrals (31) for any $i \in \mathbb{Z}_+$ and $\widetilde{\omega} \in C_0^\infty(\Sigma)$. In turn, this determines all functions $|\phi_i|_\Sigma$. Applying Theorem 2.2 we find ϕ_i, $i = 1, 2, \dots,$ on Σ up to a sign. □

Proof of Theorem 1.3.

a. If all eigenvalues of A^{ω_0} are simple, then, by upper semicontinuity of μ_k, it follows from Corollary 3.2 that the Gateaux derivatives of $\lambda_k(\omega_0)$ determine the integrals

(31) for any $\widetilde{\omega} \in C_0^\infty(\Sigma)$. It then follows from the proof of Theorem 4.1 that the Gateaux derivatives of $\lambda_k(\omega_0)$ determine $\phi_i(\omega_0)$, $i = 1, 2, \ldots$, on Σ.

b. In general, Theorem 4.1 shows that $\lambda_i(\omega)$ for ω satisfying (21) with any $\varepsilon > 0$, determine $\phi_i(\omega_0)$, $i = 1, 2, \ldots$, on Σ. By [13, Thm. 7.3], this data determines uniquely the isometry type of (Ω, g) and the gauge-equivalence class $\{\kappa^{-1} A^{\omega_0} \kappa : \kappa \in C^\infty(\Omega),\ \kappa(x) > 0\}$ of the operator A^{ω_0}. By [14, Lemma 2.29] this equivalence class contains a unique Schrödinger operator of the form (6). Thus we can find q and ω_0. This completes the proof of Theorem 1.3. $\square$

Corollary 1.4 is a direct consequence of Theorem 1.3 and the fact that by Liouville Theorem [12], an isometric embedding of a conformally Euclidean n-manifold to $\mathbb{R}^n$ is unique.

Remark 4.2. *In the case where some of the eigenvalues of A^{ω_0} are simple and some are degenerate, our proof gives a result that is slightly more general than (b) in Theorem 1.3.*

We have actually proven that $(\Omega, g), q$ and ω_0 are uniquely determined by the data consisting of Σ, the simple eigenvalues $\lambda_k(\omega_0)$ and their Gateaux derivatives, $\lambda_{k,\omega_0}(\omega)$; $\omega \in C_0^\infty(\Sigma)$, and moreover, for each degenerate eigenvalue, $\lambda_k(\omega_0)$, with multiplicity $\mu_k(\omega_0)$, the local spectral data, $\{\lambda_l(\omega)\}_{l=k-j}^{k+p-1}$ for all $\omega \in B_\epsilon^\infty(\omega_0)$ for some $\epsilon > 0$, and where p, j are the only integers such that, $p + j = \mu_k(\omega_0)$, and

$$\lambda_{k-j-1}(\omega_0) < \lambda_{k-j}(\omega_0) = \cdots = \lambda_k(\omega_0)) = \cdots = \lambda_{k+p-1}(\omega_0) < \lambda_{k+p}(\omega_0).$$

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