Large Deviations in Quantum Lattice Systems: One-phase Region

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Abstract

We give large deviation upper bounds, and discuss lower bounds, for the Gibbs-KMS state of a system of quantum spins or an interacting Fermi gas on the lattice. We cover general interactions and general observables, both in the high temperature regime and in dimension one.

1 Introduction and statement of the results

The study of large deviations of macroscopic observables plays a fundamental role in classical statistical physics, for example in the study of the equivalence of ensembles and in hydrodynamical limits. The large deviation principle for Gibbs measures in classical mechanics, initiated in the seminal papers [31, 20] is now a classical subject, see e.g. [13, 29, 9, 14, 16, 12, 23, 24, 30]. It is maybe surprising that, in comparison, very little is known about large deviations in quantum statistical mechanics. To our knowledge, results on large deviations have been obtained only for the fluctuations of the particle density in [21], for ideal fermionic and bosonic quantum gases, and in [15], for dilute fermionic and bosonic gases in (using cluster expansion techniques). As we were completing this paper a preprint [27] appeared where large deviation results for observables that depend only on one site are established in the high temperature regime (using again cluster expansions). In this work, however, we consider general observables. Also, from a technical point of view, we do not use cluster expansions, but a simple matrix inequality, combined with analyticity estimates on the dynamics and subadditivity arguments.

Other large deviation results in quantum mechanical models can be found in [4, 5, 28, 34, 35]. In the larger context of probabilistic results for quantum lattice systems we want to mention [17, 18, 25, 26], about the Central Limit Theorem and the algebra of normal fluctuations, and [6, 7], on Shannon–McMillan type of theorems.

For simplicity we will consider quantum systems of spins or fermions on a lattice. The statistical mechanics of spins and fermions is naturally expressed in the formalism of C^*-algebras, which we will use throughout the paper (refer to [8, 33] for spin systems and to the recent [3] for fermions). For classical Gibbs systems, the DLR condition is a crucial ingredient in establishing the large deviations principle. For quantum systems, Araki introduced an analogous condition, the so-called Gibbs condition, which will play an important role in our proofs. Our results presumably extend to some bosonic systems or lattices of oscillators, but such systems present more technical difficulties as they are most naturally expressed in the W^*-algebraic formalism.

In order to illustrate the scope—and limitations—of this work, let us consider an example of a system of (fermionic) particles of spin 1/2. Our results, however, are more general and will be detailed in Section 3.

Let $c_{x,\sigma}$ and $c^*_{x,\sigma}$ denote the annihilation and creation operators for fermions of spin $\sigma \in \{\uparrow, \downarrow\}$ at site $x \in \mathbf{Z}^d$. These operators satisfy the Canonical Anticommutation Relations. We denote by $n_{x,\sigma} = c^*_{x,\sigma}c_{x,\sigma}$ the operator for the number of particles in x with spin σ, so that $n_x = n_{x,\uparrow} + n_{x,\downarrow}$ indicates the operator for the total number of particles in x. The finite-volume Hamiltonian (with free boundary conditions) for a finite subset Λ of the lattice is taken to be

$$\begin{aligned} H_\Lambda &= -\sum_{\{x,y\}\subset\Lambda} T_{x-y} \sum_\sigma \left(c^*_{y,\sigma}c_{x,\sigma} + c^*_{x,\sigma}c_{y,\sigma} \right) + \sum_{x\in\Lambda} U n_{x,\uparrow} n_{x,\downarrow} \\ &\quad + \sum_{\{x,y\}\subset\Lambda} J_{x-y}\, n_x n_y \,. \end{aligned} \tag{1.1}$$

Special cases of this Hamiltonian are the Hubbard models and the tJ-models, which are widely used in applications. We define the number operator for a finite subset Λ of the lattice by

$$N_\Lambda = \sum_{x\in\Lambda} n_x \,. \tag{1.2}$$

We work in the grand canonical ensemble, whose finite-volume Gibbs states are given by

$$\omega_\Lambda^{(\beta,\mu)}(\,\cdot\,) \;=\; \frac{\mathrm{tr}\big(\,\cdot\; e^{-\beta(H_\Lambda-\mu N_\Lambda)}\big)}{\mathrm{tr}\big(e^{-\beta(H_\Lambda-\mu N_\Lambda)}\big)}\,, \tag{1.3}$$

where β is the inverse temperature and is the chemical potential μ. We denote by $\omega^{(\beta,\mu)}$ the Gibbs states of the infinite systems, which, roughly speaking, are constructed as limit points of ω_Λ as $\Lambda \nearrow \mathbf{Z}^d$. Mathematically the Gibbs states of the infinite system are characterized, equivalently, by either the variational principle, the KMS condition, or the Gibbs condition.

Let us consider a local (microscopic) observable A, i.e., a self-adjoint operator depending only on even products of creation and annihilation operators (this is a natural limitation for Fermi systems; see Section 2). For example, say that A is an even polynomial of the creation and annihilation operators. Assume that A depends only on the sites of $X \subset \mathbf{Z}^d$, and denote by $\upsilon_x A$ the translate of the operator A by $x \in \mathbf{Z}^d$. For a cube Λ we define the macroscopic observable K_Λ by

$$K_\Lambda \;=\; \sum_{X+x\subset\Lambda} \upsilon_x A\,, \tag{1.4}$$

In this way, $|\Lambda|^{-1}K_\Lambda$ represents the average of A in Λ ($|\Lambda|$ denotes the cardinality of the set Λ). If we denote by $\mathbf{1}_B$ the indicator function of a Borel set $B \subset \mathbf{R}$, then, for a Gibbs state $\omega^{(\beta,\mu)}$,

$$\rho_\Lambda(B) \;=\; \omega^{(\beta,\mu)}\left(\mathbf{1}_B\left(|\Lambda|^{-1}K_\Lambda\right)\right) \tag{1.5}$$

defines a probability measure on $\mathbf{R}$. Namely, (1.5) is the probability that, in the state $\omega^{(\beta,\mu)}$, the observable $|\Lambda|^{-1}K_\Lambda$ takes values in B. Large deviation theory studies the asymptotic behavior of the family of measures ρ_Λ on an exponential scale in $|\Lambda|$. This asymptotic behavior is expressed in terms of a rate function $I(x)$, which is a lower continuous function with compact level sets.

Let C be a closed set. We say that we have a large deviation upper bound for C if

$$\limsup_{\Lambda\nearrow\mathbf{Z}^d} \frac{1}{|\Lambda|}\log\omega^{(\beta,\mu)}\left(\mathbf{1}_C\left(|\Lambda|^{-1}K_\Lambda\right)\right) \le -\inf_{x\in C} I(x)\,. \tag{1.6}$$

Similarly, if O is open, we have a large deviation lower bound for O if

$$\liminf_{\Lambda\nearrow\mathbf{Z}^d} \frac{1}{|\Lambda|}\log\omega^{(\beta,\mu)}\left(\mathbf{1}_O\left(|\Lambda|^{-1}K_\Lambda\right)\right) \ge -\inf_{x\in O} I(x)\,. \tag{1.7}$$

One says that $\{\rho_\Lambda\}$ satisfies the large deviation principle if we have upper and lower bound for all closed and open sets, respectively.

In order to study the large deviations for ρ_Λ as $\Lambda \nearrow \mathbf{Z}^d$ (along sequences of cubes), one considers the corresponding logarithmic moment generating function, defined as

$$f(\alpha) \;=\; \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \log \omega^{(\beta,\mu)}(e^{\alpha K_\Lambda})\,. \tag{1.8}$$

The Gärtner–Ellis Theorem (see, e.g., [11]), shows that the existence of $f(\alpha)$ implies large deviation upper bounds with a rate function $I(x)$ that is the Legendre transform of $f(\alpha)$. One obtains lower bounds if, in addition, the moment generating function is smooth, at least a C^1. If the moment generating function is not smooth, one has a weaker result: in (1.7), the infimum over O is replaced by the infimum over $O \cap E$, where E is the set of the so-called exposed points (see [11] for details).

Our results apply both in one dimension and at high temperature. In both cases the parameters β and μ are such that there is a unique Gibbs-KMS state $\omega^{(\beta,\mu)}$.

Dimension one. Let us assume that the lattice is one-dimensional and that the interaction has finite range, i.e., there exists an $R > 0$ such that T_{x-y} and J_{x-y} vanish whenever $|x-y| > R$. Our core result is that, for any macroscopic observable K_Λ and all values of β and μ, the moment generating function $f(\alpha)$ exists and is finite for all $\alpha \in \mathbf{R}$. Furthermore $f(\alpha)$ is given by the formula

$$f(\alpha) \;=\; \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \log \frac{\mathrm{tr}(e^{\alpha K_\Lambda}\, e^{-\beta(H_\Lambda - \mu N_\Lambda)})}{\mathrm{tr}(e^{-\beta(H_\Lambda - \mu N_\Lambda)})}\,, \tag{1.9}$$

which involves only finite-dimensional objects.

As recalled above, if $I(x)$ is the Legendre transform of $f(\alpha)$, the Gärtner–Ellis Theorem entails the large deviation upper bounds with I as the rate function. As for the lower bounds, it is tempting to conjecture that the function $f(\alpha)$ is smooth, in one dimension. We have not proved it so far.

It is instructive, at this point, to compare our results on quantum systems with their classical analogs. In the classical case, using the DLR equations, one shows that a formula similar to Eq. (1.9) holds, with the trace replaced by the expectation with respect to the counting measure. In that case, one sees that $f(\alpha)$ is simply the translated pressure corresponding to the Hamiltonians $\beta H_\Lambda - \beta\mu N_\Lambda - \alpha K_\Lambda$. Therefore, classically, the smoothness of $f(\alpha)$ follows immediately from the lack of phase transitions in one dimension, together with the identification of Gibbs states with functionals tangent to the pressure [19, 33]. In the quantum case, K_Λ does not commute with $H_\Lambda - \mu N_\lambda$, in general, so the thermodynamic interpretation of the moment generating function is not obvious. This is the main difference.

Such interpretation is possible, however, when K_Λ commutes with $H_\Lambda - \mu N_\Lambda$. In our example (1.1), H_Λ does commute with N_Λ (this is not hard to verify, using the CAR; see (2.1)), so that we can fully treat the physically important large deviations in the energy and in the density. More in detail, if the pressure function for our system is defined as

$$P(\beta,\mu) \;=\; \lim_{\Lambda\nearrow\mathbf{Z}^d} \frac{1}{|\Lambda|} \log \operatorname{tr}\left(e^{-\beta(H_\Lambda - \mu N_\Lambda)}\right)\,, \tag{1.10}$$

then (1.6)-(1.7) hold for the energy ($K_\Lambda = H_\Lambda$), with $I(x)$ being the Legendre transform of

$$\alpha \;\mapsto\; P\left(\beta-\alpha, \frac{\beta}{\beta-\alpha}\mu\right) - P(\beta,\mu)\,. \tag{1.11}$$

They also hold for the number of particles ($K_\Lambda = N_\Lambda$), and in that case $I(x)$ is the Legendre transform of

$$\alpha \;\mapsto\; P\left(\beta, \mu + \frac{\alpha}{\beta}\right) - P(\beta,\mu)\,. \tag{1.12}$$

High temperature. For arbitrary space dimension we assume that that the interaction is summable: $\sum_{x\in\mathbf{Z}^d} |T_x| + |J_x| < \infty$. Our main result is that there exist two constants, β_0 (which depends only on the Hamiltonians H_Λ) and α_0 (which depends only on the observable K_Λ), such that the function $f(\alpha)$ exists for $|\alpha| < \alpha_0$, $|\beta| < \beta_0$, and arbitrary $\mu \in \mathbf{R}$. Furthermore, in the special case in which the macroscopic observable is a sum of terms depending only on one site, α_0 can be taken to be infinity. Again, the function $f(\alpha)$ is also given by (1.9).

This yields large deviation upper bounds for closed sets which are contained in a neighborhood of the average $\overline{K} = \lim_{\Lambda\nearrow\mathbf{Z}^d} |\Lambda|^{-1}\omega^{(\beta,\mu)}(K_\Lambda)$. At high temperature, one expects $f(\alpha)$ to be smooth, in fact analytic, and this can be proved using a cluster expansion [22].

For the case of commuting observables we show that the moment generating function exists for any α, provided $|\beta| < \beta_0$. It is known that, for sufficiently high temperature and any value of the chemical potential, there is a unique Gibbs state (see Theorem 6.2.46 of [8]). Using this, we obtain a full large deviation principle for the particle number (or density). As for the energy, we expect $f(\alpha)$ to have a singularity at some $\alpha \neq 0$; at any rate, we have upper bounds for all closed sets and lower bounds for sets that are contained in a neighborhood of the mean energy. For both the energy and the particle density, the rate functions are again the Legendre transforms of (1.11)-(1.12).

Once again, the precise statements for a general quantum lattice system will be presented—and proved—in Section 3.

2 Quantum lattice systems

We consider a quantum mechanical system on the d-dimensional lattice $\mathbf{Z}^d$, as seen, e.g., in [32, 19, 8, 33] for spin systems, and in [3] for fermions.

2.1 Observable algebras

We first describe quantum spin systems. Let $\mathcal{H}$ be a finite-dimensional Hilbert space. One associates with each lattice site $x \in \mathbf{Z}^d$ a Hilbert space $\mathcal{H}_x$ isomorphic to $\mathcal{H}$ and with each finite subset $X \subset \mathbf{Z}^d$ the tensor product space $\mathcal{H}_X = \bigotimes_{x\in X} \mathcal{H}_x$. The local algebra of observables is given by $\mathcal{O}_X = \mathcal{B}(\mathcal{H}_X)$, the set of all bounded operators on $\mathcal{H}_X$. If $X \subset Y$, there is a natural inclusion of $\mathcal{O}_X$ into $\mathcal{O}_Y$, and the algebras $\{\mathcal{O}_X\}$ form a partially ordered family of matrix algebras. The norm-completion of the union of the local algebras is a C^*-algebra denoted by $\mathcal{O}$ which correspond to the physical observables of the system. In particular we have that $[\mathcal{O}_X\,,\,\mathcal{O}_Y] = 0$ whenever $X \cap Y = \emptyset$. A state ω is a positive normalized linear functional on $\mathcal{O}$, i.e., $\omega : \mathcal{O} \longrightarrow \mathbf{C}$, $\omega(\mathbf{1}) = 1$ and $\omega(A) \geq 0$, whenever $A \geq 0$. The group $\mathbf{Z}^d$ acts as a *-automorphic group on $\mathcal{O}$: For $x \in \mathbf{Z}^d$, $\upsilon_x(\mathcal{O}_X) = \mathcal{O}_{X+x}$. A state is called translation invariant if $\omega \circ \upsilon_x = \omega$ for all $x \in \mathbf{Z}^d$ and we denote by Ω_I the set of all translation invariant states. The action of υ is asymptotically abelian: therefore Ω_I is a Choquet simplex and one can decompose a state into ergodic components (see [33]).

The structure of the algebra of observables for fermionic lattices gases is a little more involved, due to the anticommutativity properties of creation and annihilation operators (see [3, 10]). We construct it as follows.

Let $\mathcal{I}$ be the finite set that is supposed to describe the spin states of a particle. For X a finite subset of $\mathbf{Z}^d$, $\mathcal{F}_X$ is defined formally as the C^*-algebra generated by the elements $\{c^*_{x,\sigma},\, c_{x,\sigma}\}_{x\in X,\sigma\in\mathcal{I}}$ together with the relations

$$\begin{aligned} \{c^*_{x,\sigma}\,,\, c_{y,\sigma'}\} &= \delta_{x,y}\delta_{\sigma,\sigma'}\mathbf{1} \\ \{c^*_{x,\sigma}\,,\, c^*_{y,\sigma'}\} &= \{c_{x,\sigma}\,,\, c_{y,\sigma'}\} = 0\,. \end{aligned} \tag{2.1}$$

The above are referred to as CAR (Canonical Anticommutation Relations). $c^*_{x,\sigma}$ and $c_{x,\sigma}$ are called the annihilation and creation operators and are taken to be mutually adjoint by definition. It is easy to realize that, as a vector space,

$$\mathcal{F}_X \;=\; \mathrm{span}\left\{c^{\sharp_1}_{x_1,\sigma_1}\, c^{\sharp_2}_{x_2,\sigma_2} \cdots c^{\sharp_m}_{x_m,\sigma_m}\right\}, \tag{2.2}$$

where the span is taken over all (finite) sequences $\{(x_j, \sigma_j, \sharp_j)\}_{j=1}^m$ in $X \times \mathcal{I} \times \{\cdot, *\}$ that are strictly increasing w.r.t. a predetermined order. If $X \subset Y$, there is a natural inclusion $\mathcal{F}_X \subset \mathcal{F}_Y$, and we define the fermionic C^*-algebra $\mathcal{F}$ to be the norm-completion of $\bigcup_{X \subset \mathbf{Z}^d} \mathcal{F}_X$.

Elements of $\mathcal{F}$ localized on disjoint parts of the lattice do not necessarily commute (they might either commute or anticommute) and so $\mathcal{F}$ is not asymptotically abelian. We have to restrict the class of allowed observables to a smaller algebra. Let us denote by Θ the automorphism of $\mathcal{F}$ determined by $\Theta(c^\sharp_{x,\sigma}) = -c^\sharp_{x,\sigma}$. The observable algebra of a fermionic lattice gas $\mathcal{O}$ is defined to be the even part of $\mathcal{F}$, i.e.,

$$\mathcal{O} \;=\; \{A \in \mathcal{F} \,|\, \Theta(A) = A\}\,. \tag{2.3}$$

Clearly, $\mathcal{O}_X = \mathcal{O} \cap \mathcal{F}_X$ is given by the same r.h.s. of (2.2), restricted to m even. Hence $[\mathcal{O}_X\,,\mathcal{F}_Y] = 0$ whenever $X \cap Y = \emptyset$, which is the commutativity property we need. The algebra $\mathcal{O}$ is thus quasilocal and similar considerations as for quantum spins systems apply.

Example 2.1 For quantum spin systems with spin 1/2, the Hilbert spaces $\mathcal{H}_x$, $x \in \mathbf{Z}^d$ is isomorphic to $\mathbf{C}^2$.

Example 2.2 For fermionic systems of particles with spin 1/2, for each x, the algebra generated by $c^*_{x,\sigma}$ and $c_{x,\sigma}$ is isomorphic to $\mathcal{B}(\mathbf{C}^4)$.

2.2 Interactions and macroscopic observables

An interaction $\Phi = \{\phi_X\}$ is a map from the finite subsets X of $\mathbf{Z}^d$ (denoted $\mathcal{P}_f(\mathbf{Z}^d)$) into the self-adjoint elements of the observable algebras $\mathcal{O}_X$ (denoted $\mathcal{O}_X^{(sa)}$). We will always assume the interaction to be translation invariant, i.e., $\upsilon_x \phi_X = \phi_{X+x}$ for all $x \in \mathbf{Z}^d$ and all $X \in \mathcal{P}_f(\mathbf{Z}^d)$. An interaction is said to have finite range if there exists an $R > 0$ such that $\phi_X = 0$ whenever $\mathrm{diam}(X)$, the diameter of X, exceeds R. (One usually says that the range is R if R is the smallest positive number that verifies the previous condition.) We denote by $\mathcal{B}^{(f)}$ the set of all finite range interactions. The set of interactions can be made into a Banach space by completing $\mathcal{B}^{(f)}$ with respect to various norms. In this paper we use the norm

$$\|\Phi\|_\lambda = \sum_{X \ni 0} \|\phi_X\| \, e^{\lambda |X|} \,, \tag{2.4}$$

where $\lambda > 0$ and $|X|$ denotes the cardinality of X. We call $\mathcal{B}_\lambda$ the corresponding Banach space of interactions. To a given Φ one associates a family of Hamiltonians (or energy operators) $\{H_\Lambda\}_{\Lambda \in \mathcal{P}_f(\mathbf{Z}^d)}$ via

$$H_\Lambda = H_\Lambda(\Phi) = \sum_{X \subset \Lambda} \phi_X . \tag{2.5}$$

As in [20], we define a finite-range macroscopic observable K of range R to be a mapping $K : \mathcal{P}_f(\mathbf{Z}^d) \longrightarrow \mathcal{O}^{(sa)}$ such that

1. $K_{\Lambda+x} = \upsilon_x K_\Lambda$ for all $x \in \mathbf{Z}^d$ and for all $\Lambda \in \mathcal{P}_f(\mathbf{Z}^d)$.

2. $K_{\Lambda \cup \Lambda'} = K_\Lambda + K_{\Lambda'}$ if Λ and Λ are at distance greater than R.

The kind of example that we have in mind, and that covers most applications, is $K_\Lambda = \sum_{X+x \in \Lambda} \upsilon_x A$, for a given self-adjoint $A \in \mathcal{O}_X$ (which could be, say, the magnetization or the occupation operator at the origin, or the energy in a finite region, or so).

Given a finite-range observable K, we can recursively define a finite-range interaction $\Psi \in \mathcal{B}^{(f)}$ by means of the equalities $K_\Lambda = \sum_{X \subset \Lambda} \psi_X$. We have a one-to-one correspondence between finite-range macroscopic observables and finite range interactions. We can and will consider more general macroscopic observables by replacing condition 2 with the condition that the interaction Ψ, corresponding to K, belongs to some Banach space.

2.3 Gibbs-KMS states

There are several equivalent ways to characterize the equilibrium states corresponding to an interaction Φ. These equivalences certainly hold if $\Phi \in \mathcal{B}_\lambda$, for some $\lambda > 0$ [33, 8]. A more general result of this type has been proved recently in [3], both for spin and fermion systems, for a nearly optimal class of interactions.

In this paper, the notation $\Lambda \nearrow \mathbf{Z}^d$ will always mean that we take the limit along an increasing sequence of hypercubes Λ. All our results can presumably also be proved for more general sequences (Van-Hove limits), but, for simplicity, we will refrain from doing so.

We denote by $P(\Phi)$ the pressure for the interaction Φ, given by the limit $P(\Phi) = \lim_{\Lambda \nearrow \mathbf{Z}^d} |\Lambda|^{-1} \mathrm{tr}(e^{-H_\Lambda})$. Here tr is the normalized trace in $\mathcal{H}_\Lambda$ and H_Λ is specified by (2.5). Let ω be a translation invariant state. The mean energy relative to ω is defined as $e^{(\Phi)}(\omega) = \overline{H} = \lim_{\Lambda \nearrow \mathbf{Z}^d} |\Lambda|^{-1} \omega(H_\Lambda)$. Denoting by ω_Λ the restriction of ω to $\mathcal{O}_\Lambda$, we define the mean entropy in the state ω by $s(\omega) = \lim_{\Lambda \nearrow \mathbf{Z}^d} |\Lambda|^{-1} S(\omega_\Lambda)$, where $S(\omega_\Lambda) = \omega_\Lambda(\log \rho_\Lambda) = \mathrm{tr}(\rho_\Lambda \log \rho_\Lambda)$ and ρ_Λ is the density matrix of ω_Λ. The existence of the limits for the pressure, mean energy and entropy is a standard result.

The variational principle states that

$$P(\Phi) = \sup_{\omega \in \Omega_I} \left(s(\omega) - e^{(\Phi)}(\omega) \right) . \tag{2.6}$$

We denote by $\Omega_I^{(\Phi)}$ the set of states for which the supremum in Eq. (2.6) is attained, and we call such states the equilibrium states for the interaction Φ. The set $\Omega_I^{(\Phi)}$ is a simplex and each of its states has a unique decomposition into ergodic states.

The second characterization of equilibrium states is via the KMS condition. Let us consider τ_t, a strongly continuous unitary action of $\mathbf{R}$ on $\mathcal{O}$. It is known that, on a norm-dense subalgebra of $\mathcal{O}$, τ_t can be extend to a (pointwise analytic) action of $\mathbf{C}$ [8]. So, a state ω is said to be τ-KMS if

$$\omega(A\tau_i(B)) = \omega(BA) \tag{2.7}$$

for all A, B in a norm-dense τ-invariant subalgebra of $\mathcal{O}$. For a given interaction Φ, one constructs the dynamics $\tau_t^{(\Phi)}$ as the limit of finite volume dynamics defined, on a local observable A, by $e^{iH_\Lambda t} A e^{-iH_\Lambda t}$. Then one can speak of a KMS state for the interaction Φ.

The third characterization is through the Gibbs condition. This condition is analog to the DLR equations for classical spin systems. Stating it properly would require considerable machinery, including the Tomita-Takesaki theory. Detailed expositions can be found in [8, 33] and we will be brief here. Given an element $P \in \mathcal{O}^{(sa)}$ and a state ω, one can define a perturbed state ω^P in the following way: Using the Tomita-Takesaki theory one constructs (in the GNS representation) a dynamics τ_t that makes ω a τ-KMS state. One then perturbs the dynamics τ_t by formally adding the term $i[P\,,\,\cdot]$ to its generator (this would correspond to adding P to the Hamiltonian). Finally, one defines ω^P as the KMS state for the perturbed dynamics (Araki's perturbation theory).

For an interaction Φ, let us consider the perturbation

$$W_\Lambda \;=\; \sum_{\substack{X\cap\Lambda\neq\emptyset\\ X\cap\Lambda^c\neq\emptyset}} \phi_X\,, \tag{2.8}$$

which is well-defined under our assumptions. The state ω satisfies the Gibbs condition if, for every finite subset Λ, there exists a state ω' on $\mathcal{O}_{\Lambda^c}$ such that

$$\omega^{-W_\Lambda} \;=\; \omega_\Lambda^{(\Phi)} \otimes \omega'\,. \tag{2.9}$$

Here $\mathcal{O}_{\Lambda^c}$ is the subalgebra of observables that "do not depend on Λ" (we omit the formal definition; suffices to say that $\mathcal{O} = \mathcal{O}_\Lambda \otimes \mathcal{O}_{\Lambda^c}$). Also, which is crucial, $\omega_\Lambda^{(\Phi)}$ is the finite-volume Gibbs state on $\mathcal{O}_\Lambda$ given by

$$\omega_\Lambda^{(\Phi)}(A) \;=\; \frac{\mathrm{tr}(Ae^{-H_\Lambda})}{\mathrm{tr}(e^{-H_\Lambda})}\,, \tag{2.10}$$

The Gibbs condition is very similar to the DLR equations in classical lattice systems, and it is not difficult to check that the DLR equations and the Gibbs condition are indeed equivalent for classical spin systems.

Nor is it hard to verify that finite-volume Gibbs states satisfy all the previous three conditions. A fundamental result of quantum statistical mechanics, due to Lanford, Robinson, Ruelle and Araki, asserts that the three characterizations are indeed equivalent for infinite-volume translation invariant states of spins or fermions. The key to the proof is the Gibbs condition, introduced by Araki. In the very recent [3], equivalence has been proved for a very large class of interactions, much larger than the one considered in this paper.

3 Moment generating function

Given an interaction Φ with a corresponding Gibbs-KMS state $\omega \in \Omega_I^{(\Phi)}$, and a macroscopic observable $\{K_\Lambda\}$, uniquely determined by the interaction Ψ, we introduce the moment generating function

$$f^{(\Psi,\Phi)}(\alpha) \;=\; \lim_{\Lambda\nearrow\mathbf{Z}^d} \frac{1}{|\Lambda|}\log\omega(e^{\alpha K_\Lambda})\,; \tag{3.1}$$

that is, when the limit exists. A priori it is not obvious that $f^{(\Psi,\Phi)}(\alpha)$ depends only on Φ and not the choice of $\omega \in \Omega_I^{(\Phi)}$. In this paper, however, we will always work in the one-phase regime, see Remark 3.3. Furthermore one expects that, as in the classical case, $f^{(\Psi,\Phi)}(\alpha)$ would depend only on Φ.

We will make one of the following assumptions.

H1: High temperature. Both Φ and Ψ belong to some $\mathcal{B}_\lambda$ and

$$\frac{\lambda}{4}\|\Phi\|_\lambda < 1\,. \tag{3.2}$$

H2: High temperature improved. Φ is the sum of two interactions, $\Phi = \Phi' + \Phi''$, where $\Phi'' = \{\phi''_x\}_{x\in\mathbf{Z}^d}$ involves only observables depending on one site, and, for all $\Lambda \subset \mathbf{Z}^d$, we have $[H'_\Lambda\,,\, H''_\Lambda] = 0$. Also, we assume that Φ' and Ψ belong to some $\mathcal{B}_\lambda$ with

$$\frac{\lambda}{4}\|\Phi'\|_\lambda < 1\,. \tag{3.3}$$

No smallness assumption on Φ'' is made.

H3: Dimension one. The lattice has dimension one and both Φ and Ψ have finite range R.

Remark 3.1 Condition **H2** is important in physical applications where Φ'' is a chemical potential or an external magnetic field. It allows us to prove our results at high temperature for *any* value of the chemical potential/magnetic field (see the example in the introduction).

Our main result is

Theorem 3.2 *Let ω, Φ, Ψ be as above.*

1. **(High temperature)** *If* **H1** *or* **H2** *is satisfied, then the moment generating function $f^{(\Psi,\Phi)}(\alpha)$ exists and is finite for all real α such that*
$$|\alpha| < \frac{4}{\lambda\|\Psi\|_\lambda}\,. \tag{3.4}$$
If the macroscopic observable is the sum of observables depending only on one site, i.e., $K_\Lambda = \sum_{x\in\Lambda}\psi_x$, with $\psi_x \in \mathcal{O}_{\{x\}}$, then $f^{(\Psi,\Phi)}(\alpha)$ exists and is finite for all $\alpha \in \mathbf{R}$.

2. **(Dimension one)** *If* **H3** *is satisfied, then $f^{(\Psi,\Phi)}(\alpha)$ exists and is finite for all $\alpha \in \mathbf{R}$.*

The moment generating function $f^{(\Psi,\Phi)}(\alpha)$ is convex and Lipschitz continuous; more precisely,

$$\left| f^{(\Psi,\Phi)}(\alpha_1) - f^{(\Psi,\Phi)}(\alpha_2) \right| \leq \|\Psi\|_0 |\alpha_1 - \alpha_2| \,, \tag{3.5}$$

where $\|\Psi\|_0 = \sum_{X \ni 0} \|\psi_X\|$). Moreover

$$f^{(\Psi,\Phi)}(\alpha) = \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \log \left(\frac{\operatorname{tr}(e^{\alpha K_\Lambda} e^{-H_\Lambda})}{\operatorname{tr}(e^{-H_\Lambda})} \right) . \tag{3.6}$$

Remark 3.3 Although our proof does not directly use this fact, the assumptions of Theorem 3.2 imply that there is a unique KMS state (in [8], for instance, check Theorem 6.2.45 for **H1**, Theorem 6.2.46 for **H2**, and Theorem 6.2.47 for **H3**).

Remark 3.4 The equality of the two limits (3.1) and (3.6) implies that—using the terminology of [21]—semi-local large deviations are the same as global large deviations. In other words, $\omega(\mathbf{1}_B(|\Lambda|^{-1} K_\Lambda))$ decreases at the same exponential rate as $\omega_\Lambda(\mathbf{1}_B(|\Lambda|^{-1} K_\Lambda))$. Global large deviations are so named because they gauge the probability of deviation from the expected value when a microscopic observable is averaged over *all* the available volume.

For particular, physically important observables, the results of Theorem 3.2 can be improved.

Corollary 3.5 *Suppose that, for all $\Lambda \in \mathcal{P}_f(\mathbf{Z}^d)$, the observable K_Λ commutes with the energy H_Λ.*

1. *If* **H1** *or* **H2** *holds, then $f^{(\Psi,\Phi)}(\alpha)$ exists and is finite for all $\alpha \in \mathbf{R}$, and is C^1 in a neighborhood of 0. If K_Λ is the sum of observables depending only on one site, then $f^{(\Psi,\Phi)}(\alpha)$ is C^1 for all α.*

2. *If* **H3** *holds, then $f^{(\Psi,\Phi)}(\alpha)$ exists, is finite, and is C^1 for all $\alpha \in \mathbf{R}$.*

Proof: If $[H_\Lambda \,, K_\Lambda] = 0$ then, by Theorem 3.2 and Eq. (3.6),

$$\begin{aligned} f^{(\Psi,\Phi)}(\alpha) &= \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \log \left(\frac{\operatorname{tr}(e^{\alpha K_\Lambda - H_\Lambda})}{\operatorname{tr}(e^{-H_\Lambda})} \right) \\ &= P(\Phi - \alpha\Psi) - P(\Phi) \,, \end{aligned} \tag{3.7}$$

so that, as in the classical case, $f^{(\Psi,\Phi)}(\alpha)$ is the translated pressure. There is a unique Gibbs-KMS state for the interaction $\Phi - \alpha\Psi$, provided $\|\Phi - \alpha\Psi\|_\lambda$ is sufficiently small ([8], Theorem 6.2.45), so, by the equivalence between Gibbs-KMS states and functionals tangent to the pressure [19, 33], $f^{(\Psi,\Phi)}(\alpha)$ is differentiable if α is sufficiently small. If the interaction Ψ consists only of observables depending on one site, and H_Λ commutes with K_Λ, then there is a unique Gibbs-KMS state for $\Phi - \alpha\Psi$, for all α, provided $\|\Phi\|_\lambda$ is small ([8], Theorem 6.2.46). If condition **H2** is satisfied, similar considerations apply (see [8], Theorem 6.2.46). If condition **H3** is satisfied there is a unique Gibbs-KMS state for $\Phi - \alpha\Psi$ ([8], Theorem 6.2.47). ∎

The proof of Theorem 3.2 is in two steps. In the first step, instead of $f^{(\Psi,\Phi)}(\alpha)$, we consider

$$g^{(\Psi,\Phi)}(\alpha) \;=\; \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \log \operatorname{tr}(e^{\alpha K_\Lambda} e^{-H_\Lambda})\,. \tag{3.8}$$

In the second step we show that

$$f^{(\Psi,\Phi)}(\alpha) \;=\; g^{(\Psi,\Phi)}(\alpha) - P(\Phi)\,. \tag{3.9}$$

The function $g^{(\Psi,\Phi)}(\alpha)$ is defined via finite-dimensional objects. We will prove the existence of the limit using a subadditivity argument, as in the proof of the existence of the pressure. The equality (3.9) is proved using perturbation theory for KMS states.

3.1 Perturbation of KMS states

A basic ingredient in the proof of the existence of the pressure is the following matrix inequality:

$$\left|\log \operatorname{tr}\left(e^{H+P}\right) - \log \operatorname{tr}\left(e^{H}\right)\right| \;\le\; \|P\|\,, \tag{3.10}$$

where H and P are symmetric $n \times n$ matrices. In order to study the function $g^{(\Psi,\Phi)}(\alpha)$, where we have two (generally non-commuting) exponentials under the trace, one needs to estimate quantities like

$$\left|\log \operatorname{tr}\left(Ce^{H+P}\right) - \log \operatorname{tr}\left(Ce^{H}\right)\right|\,, \tag{3.11}$$

where C is a positive-definite $n \times n$ matrix. A little thinking convinces one that an estimate of (3.11) by a constant times $\|P\|$ cannot possibly hold true, if the constant is required not to depend on C or n.

The following lemma gives an upper bound for (3.11) which is independent of C and n, although it has a different form than Eq. (3.10).

Lemma 3.6 *Let $H, P \in \mathbf{C}^{n\times n}$, with $H^* = H$ and $P^* = P$.*

1. *We have*
$$\left|\log \operatorname{tr}\left(e^{H+P}\right) - \log \operatorname{tr}\left(e^{H}\right)\right| \;\leq\; \|P\|\,. \tag{3.12}$$

2. *Also, if $C \in \mathbf{C}^{n\times n}$ with $C > 0$,*
$$\left|\log \operatorname{tr}\left(Ce^{H+P}\right) - \log \operatorname{tr}\left(Ce^{H}\right)\right| \;\leq\; \sup_{0\leq t\leq 1}\;\sup_{-\frac{1}{2}\leq s\leq \frac{1}{2}} \left\|U^{-s}(t)\,P\,U^{s}(t)\right\|\;, \tag{3.13}$$
where
$$U^{s}(t) = e^{s(H+tP)}\,. \tag{3.14}$$

Proof: The proof of part 1 is standard. One writes

$$\begin{aligned}\left|\log \operatorname{tr}\left(e^{H+P}\right) - \log \operatorname{tr}\left(e^{H}\right)\right| \;&=\; \left|\int_0^1 dt\,\frac{d}{dt}\log \operatorname{tr}\left(e^{H+tP}\right)\right| \\ \leq\;\; \int_0^1 ds\left|\frac{\operatorname{tr}(Pe^{H+tP})}{\operatorname{tr}(e^{H+tP})}\right| \;&\leq\; \|P\|\,,\end{aligned} \tag{3.15}$$

having used the fact that, for $E \geq 0$,
$$\left|\frac{\operatorname{tr}(AE)}{\operatorname{tr}(E)}\right| \leq \|A\|\,. \tag{3.16}$$

To prove part 2, we recall DuHamel's identity for the derivative of $e^{F(t)}$, when $F(t)$ is a bounded operator:
$$\frac{d}{dt}e^{F(t)} \;=\; \int_0^1 du\, e^{uF(t)}\,F'(t)\,e^{(1-u)F(t)}\,. \tag{3.17}$$

We write
$$\log \operatorname{tr}\left(Ce^{H+P}\right) - \log \operatorname{tr}\left(Ce^{H}\right) \;=\; \int_0^1 dt\,\frac{d}{dt}\log \operatorname{tr}\left(Ce^{H+tP}\right) \tag{3.18}$$

and

$$\begin{aligned}&\frac{d}{dt}\log \operatorname{tr}\left(C\,e^{H+tP}\right) \\ &=\; \frac{\operatorname{tr}\left(\int_0^1 du\, C\, e^{u(H+tP)}\,P\,e^{(1-u)(H+tP)}\right)}{\operatorname{tr}\left(C\,e^{H+tP}\right)} \\ &=\; \frac{\operatorname{tr}\left(e^{(H+tP)/2}\,C\,e^{(H+tP)/2}\int_0^1 du\, e^{(u-1/2)(H+tP)}\,P\,e^{(1/2-u)(H+tP)}\right)}{\operatorname{tr}\left(e^{(H+tP)/2}\,C\,e^{(H+tP)/2}\right)} \\ &\leq\; \left\|\int_{-1/2}^{1/2} ds\, e^{-s(H+tP)}\,C\,e^{s(H+tP)}\right\|\;,\end{aligned} \tag{3.19}$$

where we have used the bound (3.16) with $E = e^{(H+tP)/2}Ce^{(H+tP)/2}$. This concludes the proof of Lemma 3.6. ∎

Lemma 3.6 involves the quantity $U^{-s}(t)PU^{s}(t)$, which is the time evolution (in imaginary time) of the observable P, relative to the dynamics generated by $H+tP$. One needs to estimate the dynamics for imaginary times between $-i/2$ and $i/2$. The connection with the KMS boundary conditions is evident.

If we define a (finite-volume) state ω and a perturbed state ω^P by

$$\omega(A) \;=\; \frac{\operatorname{tr}(Ae^{H})}{\operatorname{tr}(e^{H})}\,, \qquad \omega^P(A) \;=\; \frac{\operatorname{tr}(Ae^{H+P})}{\operatorname{tr}(e^{H+P})}\,, \tag{3.20}$$

then Lemma 3.6 immediately implies that, for $C>0$,

$$\Big|\log\omega^P(C) - \log\omega(C)\Big| \;\leq\; \|P\| + \sup_{0\leq t\leq 1}\ \sup_{-\frac{1}{2}\leq s\leq\frac{1}{2}} \|U^{-s}(t)\,P\,U^{s}(t)\|\,. \tag{3.21}$$

We will generalize this bound for Gibbs-KMS states of the infinite system, using results from the perturbation theory of KMS states (see, e.g., Chapter 5.4 of [8] or Chapter IV.5 of [33]). For a τ-KMS state ω, we denote by $(\mathcal{G}_\omega, \pi_\omega, O_\omega) = (\mathcal{G}, \pi, O)$ its GNS representation. The scalar product on $\mathcal{G}$ is indicated with $\langle\,\cdot\,,\,\cdot\,\rangle$. For any $A\in\mathcal{O}$ we have

$$\omega(A) \;=\; \langle O, \pi(A)O\rangle \tag{3.22}$$

and the dynamics τ is implemented by some self-adjoint operator H on $\mathcal{G}$:

$$\pi(\tau_s(A)) \;=\; e^{isH}\pi(A)\,e^{-isH}\,. \tag{3.23}$$

From now on we will identify an element A with its representative $\pi(A)$. This is possible since the two-sided ideal $\{A\in\mathcal{O}\,|\,\omega(A^*A)=0\}$ is trivial ([33], Theorem IV.4.10), therefore π is the left multiplication on $\mathcal{O}$ ([33], Theorem I.7.5).

For $P\in\mathcal{O}^{(sa)}$, $\tau^P(A)$ given by

$$\begin{aligned}&\tau^P_s(A) = \tau_s(A)\\ &\quad + \sum_{n=1}^{\infty} i^n \int_0^s ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n\, [\tau_{s_n}(P), [\cdots[\tau_{s_1}(P), \tau_s(A)]]\cdots]\,.\end{aligned} \tag{3.24}$$

defines a strongly continuous semigroup of automorphims of $\mathcal{O}$ implemented by $H+P$:

$$\tau^P_s(A) \;=\; e^{is(H+P)}A\,e^{-is(H+P)}\,. \tag{3.25}$$

Moreover we have

$$\tau_s^P(A) \;=\; \Gamma_s^P \tau_s(A)(\Gamma_s^P)^* \;=\; \Gamma_s^P \tau_s(A)(\Gamma_s^P)^{-1}\,, \tag{3.26}$$

where the unitary operator

$$\Gamma_s^P \;=\; e^{is(H+P)}e^{-isH} \tag{3.27}$$

has the following representation as norm-convergent series:

$$\Gamma_s^P = \mathbf{1} + \sum_{n=1}^{\infty} i^n \int_0^s ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \, \tau_{s_n}(P) \cdots \tau_{s_1}(P)\,. \tag{3.28}$$

Furthermore, $f(s,P) = \Gamma_s^P O$, defined on $\mathbf{R}$, extends to a holomorphic function $f(z,P)$ on $\{z \in \mathbf{C} \,|\, 0 \leq \mathrm{Im} z \leq 1/2\}$ (i.e., the function is continuous and bounded on the close strip, and analytic on its interior). In particular, O belongs to the (maximal) domain of $\Gamma_{i/2}^P$, so that one can set

$$O^P \;=\; \Gamma_{i/2}^P O \;=\; e^{-(H+P)/2}\, e^{H/2}\, O\,. \tag{3.29}$$

Araki's perturbation theory asserts that the state ω^P given by

$$\omega^P(A) \;=\; \frac{\langle O^P\,,\, A\,O^P\rangle}{\langle O^P\,,\, O^P\rangle} \;=\; \frac{\Big\langle O\,,\, (\Gamma_{i/2}^P)^* A\,(\Gamma_{i/2}^P)\,O\Big\rangle}{\Big\langle O\,,\, (\Gamma_{i/2}^P)^*(\Gamma_{i/2}^P)\,O\Big\rangle} \tag{3.30}$$

is a τ^P-KMS state.

The bound in Lemma 3.6 involves the norm of the imaginary-time evolution of the perturbation P. Therefore, for infinite systems, we will assume that P is an analytic element for the dynamics in the strip $\{|\mathrm{Im} z| \leq 1/2\}$: by this mean that $\tau_z(P)$ extends to a holomorphic function in the strip, in the sense specified above. This is clearly a strong assumption and the main limitation of our approach.

Theorem 3.7 *Let ω be a τ-KMS state and let $P \in \mathcal{O}$ be a self-adjoint analytic element in the strip $\{|\mathrm{Im} z| \leq 1/2\}$. Then, for all positive $C \in \mathcal{O}$ we have*

$$\Big|\log \omega^P(C) - \log \omega(C)\Big| \;\leq\; \|P\| + \sup_{0\leq t\leq 1} \sup_{-\frac{1}{2}\leq s\leq \frac{1}{2}} \|\tau_{is}^{tP}(P)\| \tag{3.31}$$

Proof: The proof of Theorem 3.7 follows closely the proof of Lemma 3.6. We first assume that $C^{1/2}$ is an analytic element for the dynamics τ—such elements form a dense subalgebra of $\mathcal{O}$ ([33], Proposition IV.4.6). Rewriting Eq. (3.28) as

$$\Gamma_s^P = \mathbf{1} + \sum_{n=1}^{\infty} (is)^n \int_0^1 du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{n-1}} du_n \, \tau_{su_n}(P) \cdots \tau_{su_1}(P) \quad (3.32)$$

and recalling the hypothesis on P, it is easy to extend Γ_s^P to a holomorphic function on $\{|\mathrm{Im} s| \le 1/2\}$. In light of Eq. (3.26), then, we conclude that $C^{1/2}$ is an analytic element for τ^{tP} in that same strip, for all $0 \le t \le 1$.

Using Eq. (3.30) we have

$$\begin{aligned} \log \omega^P(C) - \log \omega(C) &= \int_0^1 dt \, \frac{d}{dt} \log \omega^{tP}(C) \\ &= \int_0^1 dt \, \frac{d}{dt} \left[\log \left\langle O \,,\, (\Gamma_{i/2}^{tP})^* \, C \, \Gamma_{i/2}^{tP} \, O \right\rangle - \log \left\langle O^{tP} \,,\, O^{tP} \right\rangle \right] . \end{aligned} \quad (3.33)$$

We now claim that

$$\frac{d}{dt} \Gamma_{i/2}^{tP} = - \int_0^{1/2} ds \, \tau_{is}^{tP}(P) \, \Gamma_{i/2}^{tP} . \quad (3.34)$$

Verifying (3.34) would amount to a simple application of DuHamel's formula (3.17), if H were a bounded operator. In the case at hand we need to work a little harder, even though we use the same idea. For $e > 0$, let Π_e be the projection on the invariant space of H defined by values of its spectral measure in $[-e, e]$. Then $\Pi'_e = \mathbf{1} - \Pi_e$ is the projection on the orthogonal space. Set

$$H_e = \Pi_e \, H \, \Pi_e \, , \qquad H'_e = \Pi'_e \, H \, \Pi'_e \, , \qquad P_e = \Pi_e \, P \, \Pi_e \, . \quad (3.35)$$

Clearly, H_e and P_e are bounded operators and $[H'_e \,,\, H_e] = [H'_e \,,\, P_e] = 0$. By means of (3.17), and after a change of variable, we verify that

$$\frac{d}{dt} e^{-(H_e + tP_e)/2} e^{H_e/2} = - \int_0^{1/2} ds \, e^{-s(H_e + tP_e)} P_e e^{s(H_e + tP_e)} e^{-(H_e + tP_e)/2} \, e^{H_e/2} . \quad (3.36)$$

Now we multiply each factor above by the corresponding term $e^{uH'_e}$ ($u = \pm 1/2, \pm s$); these terms commute with everything. We obtain

$$\frac{d}{dt} \Gamma_{i/2}^{tP_e} = - \int_0^{1/2} ds \, \tau_{is}^{tP_e}(P_e) \, \Gamma_{i/2}^{tP_e} . \quad (3.37)$$

That (3.37) becomes (3.34), as $e \to +\infty$, follows from (3.24)—or rather its analytic continuation—and (3.32), since P_e is entire analytic for τ_s, and $\|P_e - P\| \to 0$.

Once (3.34) is settled, we can write

$$\begin{aligned} & \frac{d}{dt} \log \left\langle O\,,\, (\Gamma^{tP}_{i/2})^* \, C \, \Gamma^{tP}_{i/2} \, O \right\rangle \\ = & \; - \frac{\omega^{tP}\left(\int_{-1/2}^{0} ds\, \tau^{tP}_{is}(P) C + \int_{0}^{1/2} ds\, C \tau^{tP}_{is}(P) \right)}{\omega^{tP}(C)} \,. \end{aligned} \tag{3.38}$$

The symmetric form of the KMS condition for ω^{tP} is easily derived from (2.7): for A, B analytic in the strip,

$$\omega^{tP}\left(\tau^{tP}_{-i/2}(A)\, \tau^{tP}_{i/2}(B) \right) = \omega^{tP}\left(BA \right) . \tag{3.39}$$

Applying the above twice,

$$\begin{aligned} \omega^{tP}\left(\tau^{tP}_{is}(P) C \right) &= \omega^{tP}\left(\tau^{tP}_{-i/2}(C^{1/2})\, \tau^{tP}_{i(s+1/2)}(P)\, \tau^{tP}_{i/2}(C^{1/2}) \right) ; \\ \omega^{tP}\left(C \tau^{tP}_{is}(P) \right) &= \omega^{tP}\left(\tau^{tP}_{-i/2}(C^{1/2})\, \tau^{tP}_{i(s-1/2)}(P)\, \tau^{tP}_{i/2}(C^{1/2}) \right) . \end{aligned} \tag{3.40}$$

We thus turn (3.38) into

$$\begin{aligned} & \frac{d}{dt} \log \left\langle O^{tP}\,,\, C\, O^{tP} \right\rangle \\ = & \; - \frac{\omega^{tP}\left(\tau^{tP}_{-i/2}(C^{1/2}) \int_{-1/2}^{1/2} ds\, \tau^{tP}_{is}(P)\, \tau^{tP}_{i/2}(C^{1/2}) \right)}{\omega^{tP}\left(\tau^{tP}_{-i/2}(C^{1/2}) \tau^{tP}_{i/2}(C^{1/2}) \right)} \,, \end{aligned} \tag{3.41}$$

and therefore

$$\left| \frac{d}{dt} \log \left\langle O^{tP}\,,\, C\, O^{tP} \right\rangle \right| \le \sup_{-\frac{1}{2} \le s \le \frac{1}{2}} \left\| \tau^{tP}_{is}(P) \right\| . \tag{3.42}$$

Here we have used the fact that

$$A \mapsto \frac{\omega(B^* A B)}{\omega(B^* B)} \tag{3.43}$$

defines a state on $\mathcal{O}$ if $\omega(B^* B) \neq 0$.

As for the second term in (3.33), we plug $C = \mathbf{1}$ in (3.41), use the invariance of ω^{tP} with respect to τ^{tP}_z, and conclude that

$$\left| \frac{d}{dt} \log \left\langle O^{tP}\,,\, O^{tP} \right\rangle \right| \le \|P\| . \tag{3.44}$$

This gives the desired bound when $C^{1/2}$ is analytic. The general statement follows by density, see Corollary IV.4.4 in [33]. ∎

3.2 Analyticity estimates

As is apparent from the previous section, we need estimates on the evolution of observables (in imaginary time). We will use two results, one valid at high temperature and one valid in dimension one.

The first is due to Ruelle, has no restriction on the dimension, and is a standard.

Proposition 3.8 *Let $\Phi \in \mathcal{B}_\lambda$, for some $\lambda > 0$ (see Section 2.2). For any $\Lambda \in \mathcal{P}_f(\mathbf{Z}^d)$ and any collection of numbers $\{u_X\}_{X\subset\Lambda}$, with $u_X = u_X(\Lambda) \in [0,1]$, set*

$$H_\Lambda^{(u)} = \sum_{X\subset\Lambda} u_X \phi_X \tag{3.45}$$

(of course, $H_\Lambda^{(u)} = H_\Lambda$, if $u_X = 1$ for all X). If $A \in \bigcup_X \mathcal{O}_X$ is a local observable and z belongs to the strip $\{|\mathrm{Im}z| \leq 2/(\lambda\|\Phi\|_\lambda)\}$, then

$$\left\| e^{izH_\Lambda^{(u)}} A e^{-izH_\Lambda^{(u)}} \right\| \leq \frac{1}{1 - |\mathrm{Im}z|\, \frac{\lambda}{2}\, \|\Phi\|_\lambda} \|A\|\, e^{\lambda|X|}\,. \tag{3.46}$$

This estimate is uniform in Λ (and $\{u_X\}$) and thus holds in the limit $\Lambda \nearrow \mathbf{Z}^d$, when this limit exists. In particular it holds for the infinite-volume dynamics τ_z.

Proof: Follows trivially from the estimates of Theorem 6.2.4 in [8]. ∎

Theorem 3.8 implies that, in the high-temperature regime

$$\frac{\lambda}{4}\|\Phi\|_\lambda < 1\,, \tag{3.47}$$

local observables are analytic elements for the dynamics at least in the strip $\{|\mathrm{Im}z| \leq 1/2\}$, which is what we need.

The second estimate is due to Araki [1] and applies only in dimension one. It was used recently in [25] to prove a central limit theorem in one-dimensional spin systems.

In order to state it we introduce the concept of exponentially localized observables. Denote $\mathcal{O}_n = \mathcal{O}_{[-n,n]}$. Given $A \in \mathcal{O}$, we set $\|A\|^{[0]} = \|A\|$ and

$$\|A\|^{[n]} = \inf_{A_n\in\mathcal{O}_n} \|A - A_n\|\,. \tag{3.48}$$

This allows us to define, for $0 < \theta < 1$, the norm

$$\|A\|_{(\theta)} = \sum_{n\geq 0} \theta^{-n}\, \|A\|^{[n]}\,. \tag{3.49}$$

An element A of $\mathcal{O}$ is said to be exponentially localized with rate θ if, and only if, $\|A\|_{(\theta)} < \infty$. The symbol $\mathcal{O}^{(\theta)}$ will denote the space of all such observables.

We consider an interaction Φ of finite range R, and set

$$S(\Phi) \;=\; \left\| \sum_{X \ni 0} \frac{\phi_X}{\mathrm{diam}(X)} \right\| . \tag{3.50}$$

Also, for $s > 0$, we define

$$F_R(s) \;=\; \exp\left[(-R+1)s + 2 \sum_{k=1}^{R} \frac{e^{kR}-1}{k} \right] . \tag{3.51}$$

We have

Proposition 3.9 *Let $\Phi \in \mathcal{B}^{(f)}$, with range R. If $\theta \in (0,1)$ and $h > 0$ verify $\theta e^{4hS(\Phi)} = \theta' < 1$, then there exists a constant $M = M(R, \theta, h)$ (independent of Φ) such that, for $A \in \mathcal{O}^{(\theta)}$ and $|\mathrm{Im} z| \leq h$,*

$$\left\| e^{izH_\Lambda^{(u)}} A e^{-izH_\Lambda^{(u)}} \right\|_{(\theta')} \;\leq\; M\, F_R(2S(\Phi))\, \|A\|_{(\theta)} . \tag{3.52}$$

Here $H_\Lambda^{(u)}$ is defined as in (3.45). This estimate is uniform in Λ (and $\{u_X\}$) and thus holds in the limit $\Lambda \nearrow \mathbf{Z}^d$, when this limit exists. In particular it holds for the infinite-volume dynamics τ_z.

Proof: Follows from the results of [1]; see also [26]. ∎

We will use this result in the particular case in which the macroscopic observable $\{K_\Lambda\}$ has finite range. Hence notice that, if A is a local observable, then $A \in \bigcap_\theta \mathcal{O}^{(\theta)}$. Furthermore, for every $\theta \in (0,1)$, there exists a constant $D = D(\theta, R')$ such that $\|A\|_{(\theta)} \leq D\|A\|$, for all $A \in \mathcal{O}_X$ with $\mathrm{diam}(X) \leq R'$. The reverse bound, $\|A\| \leq \|A\|_{(\theta)}$, is of course valid for every $A \in \mathcal{O}$. These considerations and Proposition 3.9 imply that, for any such A, there exists a constant $G = G(R, R', S(\Phi))$ such that, for $|\mathrm{Im} z| \leq 1/2$,

$$\left\| e^{izH_\Lambda} A e^{-izH_\Lambda} \right\| \;\leq\; G\, \|A\| . \tag{3.53}$$

Once again, this is what we need to apply Lemma 3.6 and Theorem 3.7.

3.3 Subadditivity

We now give sufficient conditions for the limit

$$g^{(\Psi,\Phi)}(\alpha) \;=\; \lim_{\Lambda\nearrow\mathbf{Z}^d} \frac{1}{|\Lambda|}\log\mathrm{tr}(e^{\alpha K_\Lambda}e^{-H_\Lambda}) \tag{3.54}$$

to exist.

Theorem 3.10 *The following holds true:*

1. **(High temperature)** *If condition* **H1** *or* **H2** *applies, then the function $g^{(\Psi,\Phi)}(\alpha)$ defined by Eq. (3.54) exists and is finite for α real, with*

$$|\alpha| < \frac{4}{\lambda\|\Psi\|_\lambda}\,. \tag{3.55}$$

   Furthermore, if $\Psi = \{\psi_x\}_{x\in\mathbf{Z}^d}$ with $\psi_x \in \mathcal{O}_{\{x\}}$, (observables depending only on one site), then $g^{(\Psi,\Phi)}(\alpha)$ exists and is finite for all $\alpha\in\mathbf{R}$.

2. **(Dimension one)** *If condition* **H3** *applies, then $g^{(\Psi,\Phi)}(\alpha)$ exists and is finite for all $\alpha\in\mathbf{R}$.*

Proof: We start with item 1 under the condition **H1**. The proof combines Lemma 3.6, the analyticity estimates of Section 3.2, and a subadditivity argument as in the proof of the existence of the pressure. Let Λ to be an hypercube of side length L. We choose $a>0$ and write $L = na+b$, with $0\le b<a$. We divide the L-cube into disjoint adjacent n^d a-cubes, $\Delta_1,\Delta_2,\cdots,\Delta_{n^d}$ and a "rest" region Δ_0 which contains $L^d-(na)^d$ lattice points. We write

$$H_\Lambda \;=\; \sum_{j=1}^{n^d} H_{\Delta_j} + H_{\Delta_0} + W\,, \qquad K_\Lambda \;=\; \sum_{j=1}^{n^d} K_{\Delta_j} + K_{\Delta_0} + U\,. \tag{3.56}$$

where

$$W \;=\; \sum_X{}' \phi_X\,, \qquad U \;=\; \sum_X{}' \psi_X \tag{3.57}$$

and $\sum'$ indicates a sum over all $X\subset\Lambda$ such that, for some $j=0,1,\ldots,n^d$, $X\cap\Delta_j\neq\emptyset$ and $X\cap\Delta_j^c\neq\emptyset$. We denote by

$$g_\Lambda^{(\Psi,\Phi)}(\alpha) \;=\; \frac{1}{|\Lambda|}\log\mathrm{tr}\left(e^{\alpha K_\Lambda}e^{-\beta H_\Lambda}\right) \tag{3.58}$$

the function whose limit we are set to take. By the commutativity property of local observables and the translation invariance,

$$\begin{aligned}\log \mathrm{tr}\left(e^{\alpha\sum_{j=1}^{n^d} K_{\Delta_j}-\sum_{j=1}^{n^d} H_{\Delta_j}}\right) &= \log\prod_{j=1}^{n^d}\mathrm{tr}\left(e^{\alpha K_{\Delta_j}}e^{-\beta H_{\Delta_j}}\right)\\ &= (na)^d g_{\Delta_1}^{(\Psi,\Phi)}(\alpha)\,. \end{aligned}\tag{3.59}$$

Set now

$$P = H_{\Delta_0}+W\,, \qquad Q = K_{\Delta_0}+U\,. \tag{3.60}$$

Using Eq. (3.59), the triangle inequality, Lemma 3.6 and Proposition 3.8, we are able to estimate

$$\begin{aligned}&\left|g_\Lambda^{(\Psi,\Phi)}(\alpha)-\frac{(na)^d}{|\Lambda|}g_{\Delta_1}^{(\Psi,\Phi)}(\alpha)\right|\\ &\le \frac{1}{|\Lambda|}\sup_{0\le t\le 1}\sup_{-\frac{1}{2}\le s\le\frac{1}{2}}\|e^{-s(H_\Lambda-tP)}\,P\,e^{s(H_\Lambda-tP)}\|\\ &\quad+\frac{1}{|\Lambda|}\sup_{0\le t\le 1}\sup_{-\frac{1}{2}\le s\le\frac{1}{2}}\|e^{-s\alpha(K_\Lambda-tQ)}\,\alpha T\,e^{s\alpha(K_\Lambda-tQ)}\|\\ &\le \frac{1}{1-\frac{\lambda}{4}\|\Phi\|_\lambda}\frac{1}{|\Lambda|}\left(\sum_{X\subset\Delta_0}+{\sum_X}'\right)\|\phi_X\|\,e^{\lambda|X|}\\ &\quad+\frac{|\alpha|}{1-|\alpha|\frac{\lambda}{4}\|\Phi\|_\lambda}\frac{1}{|\Lambda|}\left(\sum_{X\subset\Delta_0}+{\sum_X}'\right)\|\psi_X\|\,e^{\lambda|X|}\,. \end{aligned}\tag{3.61}$$

We take the limit $\Lambda\nearrow\mathbf{Z}^d$ of the various parts of Ineq. (3.61). First,

$$\begin{aligned}\frac{1}{|\Lambda|}\sum_{X\in\Delta_0}\|\phi_X\|\,e^{\lambda|X|} &\le \frac{1}{L^d}\sum_{x\in\Delta_0}\sum_{X\ni x}\|\phi_X\|\,e^{\lambda|X|}\\ &\le \frac{L^d-(na)^d}{L^d}\|\Phi\|_\lambda \longrightarrow 0 \end{aligned}\tag{3.62}$$

as $L\to\infty$; similarly for $\sum_{X\in\Delta_0}\|\psi_X\|\,e^{\lambda|X|}$. Also, in the same limit,

$$\begin{aligned}\frac{1}{|\Lambda|}{\sum_X}'\|\phi_X\|\,e^{\lambda|X|} &\le \frac{1}{L^d}\sum_{j=1}^{n^d}\sum_{\substack{X\cap\Delta_j\neq\emptyset\\ X\cap\Delta_j^c\neq\emptyset}}\|\phi_X\|\,e^{\lambda|X|}\\ &\le \frac{n^d}{L^d}\sum_{\substack{X\cap\Delta_1\neq\emptyset\\ X\cap\Delta_1^c\neq\emptyset}}\|\phi_X\|\,e^{\lambda|X|}\end{aligned}$$

$$\longrightarrow \quad \frac{1}{|\Delta_1|} \sum_{\substack{X\cap\Delta_1\neq\emptyset \\ X\cap\Delta_1^c\neq\emptyset}} \|\phi_X\| \, e^{\lambda|X|} \,. \tag{3.63}$$

Once again, a similar estimate holds for $\sum_X' \|\psi_X\| \, e^{\lambda|X|}$. In the remainder, for the sake of the notation, we rename $\Delta_1 = \Delta$. From (3.61)-(3.63) we obtain

$$\begin{aligned}
&\left| \limsup_{\Lambda\nearrow\mathbf{Z}^d} g_\Lambda^{(\Psi,\Phi)}(\alpha) - g_\Delta^{(\Psi,\Phi)}(\alpha) \right| \\
&\quad \leq \; \frac{1}{1-\frac{\lambda}{4}\|\Phi\|_\lambda} \frac{1}{|\Delta|} \sum_{\substack{X\cap\Delta\neq\emptyset \\ X\cap\Delta^c\neq\emptyset}} \|\phi_X\| e^{\lambda|X|} \\
&\qquad + \frac{|\alpha|}{1-|\alpha|\frac{\lambda}{4}\|\Psi\|_\lambda} \frac{1}{|\Delta|} \sum_{\substack{X\cap\Delta\neq\emptyset \\ X\cap\Delta^c\neq\emptyset}} \|\psi_x\| e^{\lambda|X|} \,.
\end{aligned} \tag{3.64}$$

It is now time to take the limit $\Delta \nearrow \mathbf{Z}^d$. Denote by Δ' the cube of side length $a - a^{1/2}$ and concentric to Δ. We have

$$\begin{aligned}
&\frac{1}{|\Delta|} \sum_{\substack{X\cap\Delta\neq\emptyset \\ X\cap\Delta^c\neq\emptyset}} \|\phi_X\| \, e^{\lambda|X|} \\
&\quad \leq \; \frac{1}{|\Delta|} \sum_{x\in\Delta'} \sum_{\substack{X\ni x \\ X\cap\Delta^c\neq\emptyset}} \|\phi_X\| \, e^{\lambda|X|} + \frac{1}{|\Delta|} \sum_{x\in\Delta\setminus\Delta'} \sum_{\substack{X\ni x \\ X\cap\Delta^c\neq\emptyset}} \|\phi_X\| \, e^{\lambda|X|} \\
&\quad \leq \; \frac{|\Delta'|}{|\Delta|} \sum_{\mathrm{diam}(X)\geq a^{1/2}} \|\phi_X\| \, e^{\lambda|X|} + \frac{|\Delta\setminus\Delta'|}{|\Delta|} \|\Phi\|_\lambda \longrightarrow 0 \,,
\end{aligned} \tag{3.65}$$

as $a \to \infty$. The same holds for the second term of (3.64). Finally, then,

$$\left| \limsup_{\Lambda\nearrow\mathbf{Z}^d} g_\Lambda^{(\Psi,\Phi)}(\alpha) - \liminf_{\Lambda\nearrow\mathbf{Z}^d} g_\Lambda^{(\Psi,\Phi)}(\alpha) \right| \; = \; 0 \,. \tag{3.66}$$

which proves the existence and finiteness of the limit $g^{(\Psi,\Phi)}(\alpha)$, in the high temperature regime.

In the special case in which Ψ consists only of one-body interactions, we have

$$K_\Lambda \; = \; \sum_{j=1}^{n^d} K_{\Delta_j} + K_{\Delta_0} \,, \tag{3.67}$$

i.e., $U = 0$ and all the observables involved commute. Thus, in the first inequality of (3.61), the second term simplifies to

$$\|e^{-s\alpha(K_\Lambda - tK_{\Delta_0})}\alpha K_{\Delta_0} e^{s\alpha(K_\Lambda - tK_{\Delta_0})}\| = \|\alpha K_{\tilde{\Delta}}\| \leq (L^d - (na)^d)|\alpha| \, \|\Psi\|_0 \, . \tag{3.68}$$

Proceeding as above, one proves the existence of $g^{(\Psi,\Phi)}(\alpha)$ for all $\alpha \in \mathbf{R}$.

If condition **H2** holds instead of **H1** we have to modify the argument a little: using the same notation as above and because Φ'' only involves one-site interactions, we have

$$H'_\Lambda \;=\; \sum_{j=1}^{n^d} H'_{\Delta_j} + H'_{\Delta_0} + W' \, , \qquad H''_\Lambda \;=\; \sum_{j=1}^{n^d} H''_{\Delta_j} + H''_{\Delta_0} \, , \tag{3.69}$$

We note that since $[H'_V \, , \, H''_V] = 0$ for all $V \in \mathcal{P}_f(\mathbf{Z}^d)$, then the decomposition (3.69) implies that

$$\left[H''_V \, , \, W'\right] \;=\; 0 \, . \tag{3.70}$$

In order to estimate

$$\log \operatorname{tr}\left(C\, e^{-H_\Lambda}\right) - \log \operatorname{tr}\left(C\, e^{-\sum_{j=1}^{n^d} H_{\Delta_j}}\right) \tag{3.71}$$

for positive C, we proceed in two steps, using Lemma 3.6. We have, using (3.70),

$$\begin{aligned}
&\left|\log \operatorname{tr}\left(C\, e^{-H_\Lambda}\right) - \log \operatorname{tr}\left(C\, e^{-H_\Lambda - W'}\right)\right| \\
&\quad\leq\; \sup_{0\leq t\leq 1} \; \sup_{-\frac{1}{2}\leq s\leq \frac{1}{2}} \left\|e^{-s(H_\Lambda - tW')}\, W'\, e^{s(H_\Lambda - tW')}\right\| \\
&\quad=\; \sup_{0\leq t\leq 1} \; \sup_{-\frac{1}{2}\leq s\leq \frac{1}{2}} \left\|e^{-s(H'_\Lambda - tW')}\, e^{-sH''_\Lambda}\, W'\, e^{sH''_\Lambda}\, e^{s(H'_\Lambda - tW')}\right\| \\
&\quad=\; \sup_{0\leq t\leq 1} \; \sup_{-\frac{1}{2}\leq s\leq \frac{1}{2}} \left\|e^{-s(H'_\Lambda - tW')}\, W'\, e^{s(H'_\Lambda - tW')}\right\| \, .
\end{aligned} \tag{3.72}$$

This term does not involve Φ'' anymore and is estimated as under condition **H1**. On the other hand, since $H_\Lambda - W' \;=\; \sum_j H_{\Delta_j} + H_{\Delta_0}$ is a sum of commuting terms, we have

$$\begin{aligned}
&\left|\log \operatorname{tr}\left(C\, e^{-(H_\Lambda - W')}\right) - \log \operatorname{tr}\left(C\, e^{-\sum_{j=1}^{n^d} H_{\Delta_j}}\right)\right| \\
&\quad\leq\; \sup_{0\leq t\leq 1} \; \sup_{-\frac{1}{2}\leq s\leq \frac{1}{2}} \left\|e^{-s\left(\sum_j H_{\Delta_j} - tH_{\Delta_0}\right)}\, H_{\tilde{\Delta}}\, e^{s\left(\sum_j H_{\Delta_j} - tH_{\Delta_0}\right)}\right\| \\
&\quad\leq\; \|H_{\tilde{\Delta}}\| \, ,
\end{aligned} \tag{3.73}$$

which is estimated as in (3.68).

If one works under condition **H3** the proof is similar, using estimate (3.53). This concludes the proof of Theorem 3.10. ∎

Theorem 3.11 *If any of the conditions* **H1**, **H2**, *or* **H3** *hold, and* ω *is a Gibbs-KMS state for* Φ, *then*

$$\lim_{\Lambda\nearrow\mathbf{Z}^d}\left|\frac{1}{|\Lambda|}\log\omega\left(e^{\alpha K_\Lambda}\right)-\frac{1}{|\Lambda|}\log\frac{\mathrm{tr}(e^{\alpha K_\Lambda}\,e^{-H_\Lambda})}{(e^{-H_\Lambda})}\right| \;=\; 0\,. \tag{3.74}$$

Proof: We will give two different proofs of Theorem 3.11. The first uses the Gibbs condition and not, a priori, the fact that we are in a one-phase region.

Defining W_Λ as in (2.8), we apply the Gibbs condition (2.9) for ω to the observable $e^{\alpha K_\Lambda}$:

$$\omega^{-W_\Lambda}\left(e^{\alpha K_\Lambda}\right) \;=\; \omega_\Lambda^{(\Phi)}\left(e^{\alpha K_\Lambda}\right)\omega'(\mathbf{1}) \;=\; \frac{\mathrm{tr}\left(e^{\alpha K_\Lambda}e^{-H_\Lambda}\right)}{\mathrm{tr}\left(e^{-H_\Lambda}\right)}\,. \tag{3.75}$$

On the other hand, Theorem 3.7 asserts that

$$\left|\log\omega^{-W_\Lambda}\left(e^{\alpha K_\Lambda}\right)-\log\omega\left(e^{\alpha K_\Lambda}\right)\right| \;\leq\; \|W_\Lambda\|+\sup_{0\leq t\leq 1}\;\sup_{-\frac12\leq s\leq\frac12}\left\|\tau_{is}^{-tW_\Lambda}(W_\Lambda)\right\|\,. \tag{3.76}$$

By proposition 3.8, then, if Φ is in the the high temperature regime and $|s|<1/2$,

$$\frac{1}{|\Lambda|}\left\|\tau_{is}^{-tW_\Lambda}(W_\Lambda)\right\| \;\leq\; \frac{1}{1-\frac{\lambda}{4}\|\Phi\|_\lambda}\frac{1}{|\Lambda|}\sum_{\substack{X\cap\Lambda\neq\emptyset\\ X\cap\Lambda^c\neq\emptyset}}\|\phi_X\|\,e^{\lambda|X|}\,, \tag{3.77}$$

which vanishes when $\Lambda\nearrow\mathbf{Z}^d$, as we have checked in (3.65). The same, of course, happens to $|\Lambda|^{-1}\|W_\Lambda\|$. Putting together (3.75), (3.76) and the last two estimates proves the theorem in the case **H1**.

If **H2** applies, we have only two relations that have to do with the specific case at hand: the rest of the proof are algebraic manipulations for KMS states. The first relation is

$$\tau_s \;=\; \tau_s^{(\Phi'+\Phi'')} \;=\; \tau_s^{(\Phi')}\circ\tau_s^{(\Phi'')} \tag{3.78}$$

(the notation should be clear), and the second is

$$\tau_s^{(\Phi'')}(W_\Lambda) = W_\Lambda\,. \tag{3.79}$$

Eq. (3.78) comes from the fact that $[H'_V\,,\, H''_V] = 0$, for all finite sets $V \subset \mathbf{Z}^d$, and that τ_s is the limit of finite-volume dynamics. As concerns Eq. (3.79), we define

$$W_\Lambda(V) \;=\; \sum_{\substack{X\subset V\\ X\cap\Lambda\neq\emptyset\,;\,X\cap\Lambda^c\neq\emptyset}} \phi_X\,. \tag{3.80}$$

As in the proof of Theorem 3.10, $[H''_V\,,\, W_\Lambda(V)] = 0$, so, taking again the limit $V \nearrow \mathbf{Z}^d$, and noting that $\|W_\Lambda(V) - W_\Lambda\| \to 0$, we derive (3.79).

Now, using (3.78) and (3.79) in (3.24), we get that, for the perturbed dynamics,

$$\tau_s^{-tW_\Lambda}(A) \;=\; \tau_s^{(\Phi'),\,-tW_\Lambda}\left(\tau_s^{(\Phi'')}(A)\right)\,. \tag{3.81}$$

We plug $A = W_\Lambda$ in the above, exploit (3.79) again, and take the analytic continuation of the result: for $|s| \le 1/2$,

$$\tau_{is}^{-tW_\Lambda}(W_\Lambda) \;=\; \tau_{is}^{(\Phi'),\,-tW_\Lambda}(W_\Lambda) \tag{3.82}$$

which is estimated as in case **H1**.

One proceeds similarly when **H3** holds. This concludes the first proof of Theorem 3.11.

The second proof is based on the fact that—as we have thoroughly recalled earlier—the Gibbs-KMS state is unique, under our assumptions. Therefore we can write ω as limit of finite-volume Gibbs states with free boundary conditions:

$$\omega(A) \;=\; \lim_{V\nearrow\mathbf{Z}^d} \frac{\mathrm{tr}(A\,e^{-H_V})}{\mathrm{tr}(e^{-H_V})}\,, \tag{3.83}$$

for $A \in \bigcup_X \mathcal{O}_X$. Let us write $H_V = H_\Lambda + H_{\Lambda^c} + W_\Lambda(V)$, where $W_\Lambda(V)$ was defined in Eq. (3.80). If $A \in \mathcal{O}_\Lambda$, with $\Lambda \subset V$,

$$\frac{\mathrm{tr}(Ae^{-H_V})}{\mathrm{tr}(e^{-H_V})} = \frac{\mathrm{tr}(Ae^{-H_\Lambda})}{\mathrm{tr}(e^{-H_\Lambda})}\,\frac{\mathrm{tr}(Ae^{-H_\Lambda-H_{\Lambda^c}-W_\Lambda(V)})}{\mathrm{tr}(Ae^{-H_\Lambda-H_{\Lambda^c}})}\,\frac{\mathrm{tr}(e^{-H_\Lambda-H_{\Lambda^c}})}{\mathrm{tr}(e^{-H_\Lambda-H_{\Lambda^c}-W_\Lambda(V)})}\,, \tag{3.84}$$

because the trace factorizes, when evaluating the product of two observables with disjoint support. Now, via Lemma 3.6, a couple of estimates of the type seen in Theorem 3.10 yield

$$\begin{aligned}
\lim_{\Lambda\nearrow\mathbf{Z}^d} \frac{1}{|\Lambda|}\left|\log \frac{\mathrm{tr}(e^{-H_\Lambda-H_{\Lambda^c}})}{\mathrm{tr}(e^{-H_\Lambda-H_{\Lambda^c}-W_\Lambda(V)})}\right| &= 0\,,\\
\lim_{\Lambda\nearrow\mathbf{Z}^d} \frac{1}{|\Lambda|}\left|\log \frac{\mathrm{tr}(Ae^{-H_\Lambda-H_{\Lambda^c}-W_\Lambda(V)})}{\mathrm{tr}(Ae^{-H_\Lambda-H_{\Lambda^c}})}\right| &= 0\,,
\end{aligned} \tag{3.85}$$

uniformly in $A \in \mathcal{O}_\Lambda$, $A > 0$, and in $V \supset \Lambda$. Thanks to this uniformity, one obtains the assertion of Theorem 3.11 from (3.84). ∎

We conclude by proving what we have called our main result.

Proof of Theorem 3.2: Combining Theorems 3.10 and 3.11 we have that

$$\begin{aligned} f^{(\Psi,\Phi)}(\alpha) &= \lim_{\Lambda\nearrow\mathbf{Z}^d} \frac{1}{|\Lambda|} \log \omega(e^{\alpha K_\Lambda}) \\ &= \lim_{\Lambda\nearrow\mathbf{Z}^d} \frac{1}{|\Lambda|} \log \mathrm{tr}(e^{\alpha K_\Lambda} e^{-H_\Lambda}) - \lim_{\Lambda\nearrow\mathbf{Z}^d} \frac{1}{|\Lambda|} \log \mathrm{tr}(e^{-H_\Lambda}) \\ &= g^{(\Psi,\Phi)}(\alpha) - P(\Phi)\,. \end{aligned} \tag{3.86}$$

The existence of the pressure is of course a standard result not harder than Theorem 3.10.

The convexity of $f^{(\Psi,\Phi)}$ follows from the convexity of $\alpha \mapsto \log \omega(e^{\alpha K_\Lambda})$, which is verified with a standard application of Hölder's inequality, noting that $\omega(e^{\alpha K_\Lambda}) = \int d\nu(x)\, e^{\alpha x}$, for some Borel measure ν (coming from the spectral measure of K_Λ in the GNS representation).

To obtain the Lipschitz continuity, we apply Lemma 3.6 with $H = \alpha_2 K_\Lambda$, $P = (\alpha_1 - \alpha_2) K_\Lambda$, and $C = e^{-H_\Lambda}$. Since H and P commute,

$$\begin{aligned} &\frac{1}{|\Lambda|} \left| \log \mathrm{tr}\left(e^{\alpha_1 K_\Lambda} e^{-H_\Lambda}\right) - \log \mathrm{tr}\left(e^{\alpha_2 K_\Lambda} e^{-H_\Lambda}\right) \right| \\ &\quad \leq \frac{1}{|\Lambda|} \|(\alpha_1 - \alpha_2) K_\Lambda\| \leq |\alpha_1 - \alpha_2| \sum_{X \ni 0} \|\psi_X\|\,, \end{aligned} \tag{3.87}$$

which easily leads to (3.5). ∎

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