On analytic continuation of solutions of linear partial differential equations*

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Abstract

We study analytic continuation properties of solutions of linear partial differential equations of Cauchy-Kowalevskaya type, with respect to one complex variable in sectors of infinite radius. Growth estimates are also investigated.

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1 Introduction

The Cauchy-Kowalevskaya theorem provides the local existence of holomorphic solutions to the noncharacteristic Cauchy problem for analytic partial differential equations. In general, the global existence or the possibility of analytic extendability of a local solution in the whole plane cannot be expected even if the data are all entire functions. On the other hand, it is known that if one wants to get global solutions in the above sense, one has to impose some additional conditions of polynomial growth for the coefficients of the equation (see [10]).

In this paper, we study linear partial differential equations of the form,

$$(E) \qquad \partial_z^{\beta} V(t,z) = P(t^K, z, \partial_t^K, \partial_z) V(t,z),$$

where β, K are integers larger than one, and P is a polynomial in t^K, ∂_t^K, ∂_z with holomorphic coefficients in the variable z in some neighborhood of the origin in $\mathbb{C}$.

Our purpose is to construct local holomorphic solutions of (E) that can be analytically continued with respect to the variable t in sectors of infinite radius. Moreover, we show growth estimates for these solutions (Theorem 1).

For the proof, we use a previous result which provides sufficient conditions for summability of formal series solutions of linear partial differential equations with non Kowalevskaya type in the complex domain (see [9]).

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2 Summability of formal solutions of linear partial differential equations

In this section we recall a result obtained in a previous paper [9], which gives sufficient conditions for the existence of κ-summable formal series solutions of linear partial differential equations.

Definition 1 *A formal series*

$$(\hat{X}) \qquad \hat{X}(t,z) = \sum_{j=0}^{\infty} \frac{t^j}{j!} X_j(z)$$

where X_j are holomorphic functions on a common neighborhood Ω of the origin in $\mathbb{C}$, is said to be $k-$summable in the direction $d \in \mathbb{R}$ if

i) *There exist $\rho \in \mathbb{R}_+$ and $R(\rho) > 0$, such that the following formal series, called formal Borel transform with respect to the variable t of order k,*

$$W(t,z) = \sum_{j=0}^{\infty} \frac{X_j(z)t^j}{j!\Gamma(1+j/k)},$$

is absolutely convergent for $z \in D(0,\rho) \subset \Omega$ and $|t| < R(\rho)$,

ii) *There exists $\delta > 0$ such that for all $z \in D(0,\rho) \subset \Omega$, the series $W(t,z)$ can be analytically continued with respect to t in a sector $S_{d,\delta} = \{t \in \mathbb{C} : |d - \arg(t)| < \delta\}$. Moreover, there exist $\delta_1 < \delta$, $C > 0$, and $M > 0$ such that*

$$\sup_{z \in \bar{D}(0,\rho)} |W(t,z)| \leq C e^{M|t|^k}$$

for all $t \in S_{d,\delta_1}$.

If this is so, the Laplace transform of order k of $W(t,z)$,

$$(X) \qquad X(t,z) = t^{-k} \int_0^{\infty(\gamma)} W(\tau,z) e^{-(\tau/t)^k} d\tau^k,$$

where the integration is made along the ray $\arg(\tau) = \gamma$ with $|d - \gamma| < \delta$, is called the k-sum of the formal series $\hat{X}(t,z)$.

Rewrite the formal series $(\hat{X})$ in a formal series with respect to the variable z,

$$\hat{X}(t,z) = \sum_{n \geq 0} \frac{\hat{X}_n(t)}{n!} z^n,$$

where $\hat{X}_n(t)$, $n \geq 0$, are now formal series. We have the following result (see [9]),

Proposition 1 *Consider the linear partial differential equation,*

$$(E_{PV}) \qquad \partial_z^S \hat{X}(t,z) = \sum_{(k_0,k_1) \in \mathcal{S}_1} A_{(k_0,k_1)}(t,z) \partial_t^{k_0} \partial_z^{k_1} \hat{X}(t,z),$$

where $\mathcal{S}_1$ is a finite subset of $\mathbb{N}^2$ and $A_{(k_0,k_1)}(t,z)$ are holomorphic functions in a neighborhood of the origin in $\mathbb{C}^2$. Let $(\hat{X})$ be a formal solution of (E_{PV}). Assume that there exists an integer $K>1$ such that

$$\mathcal{S}_1 = \{(k_0,k_1)\in\mathbb{N}^2 : k_0K+k_1\le S, (k_0,k_1)\neq(0,S)\}.$$

We assume that the formal series $\hat{X}_\mu(t)$, for $0\le\mu\le S-1$, are $1/(K-1)$-summable with respect to t in a direction $d\in\mathbb{R}$.

Moreover, we assume that their sums $X_\mu(t)$ are holomorphic in a sectorial region $G\subset D(0,R(\rho))$, independent of μ, which has bisecting direction d and opening greater than $(K-1)\pi$ and satisfy the following estimates.
For all closed sector $\bar{S}\subset G$, there exist $C,\tilde{K}>0$ such that

$$\sup_{t\in\bar{S}}|\partial_t^{n_0}X_{m_1}(t)|\le C\tilde{K}^{n_0+m_1}(Kn_0)!m_1!,$$

for all $0\le m_1\le S-1$ and all $n_0\in\mathbb{N}$.

Then the formal series $\hat{X}(t,z)$, solution of (E_{PV}), is $1/(K-1)$-summable with respect to t in the direction $d\in\mathbb{R}$.

3 Analytic continuation of solutions of linear partial differential equations

Let $K\ge1$ be an integer. We denote by $\mathcal{F}_K$ the vector space of formal series of the form

$$(V)\qquad V(t,z)=\sum_{j\ge0}\frac{t^{Kj}}{(Kj)!}u_j(z),$$

where u_j, $j\ge0$, are holomorphic functions on a common neighborhood Ω of the origin in $\mathbb{C}$. It is easy to see that $\mathcal{F}_K$ is stable by multiplication by t^K and by derivation with respect to t of order K. That is, if $V\in\mathcal{F}_K$, then $t^KV(t,z)\in\mathcal{F}_K$ and $\partial_t^KV(t,z)\in\mathcal{F}_K$.
Consider the linear application $\mathbf{P}_K$, which is a bijection from $\mathcal{F}_K$ into $\mathcal{F}_1$, defined by

$$\mathbf{P}_K(V(t,z))=\sum_{j\ge0}\frac{t^j}{j!}u_j(z).$$

In the next proposition, we give important commutations formulas for the applications $\mathbf{P}_K$ with the operations t^K, ∂_t^K and $a(z)\partial_z^l$ that will be useful in the sequel.

Proposition 2 a) *Let $V\in\mathcal{F}_K$, $K\ge1$. Then,*

$$\mathbf{P}_K(a(z)\partial_z^lV(t,z))=a(z)\partial_z^l\mathbf{P}_K(V(t,z)),$$

where $a(z)$ is an holomorphic function on Ω.
b) *Let $V\in\mathcal{F}_K$, $K\ge1$. Then,*

$$\mathbf{P}_K(\partial_t^KV(t,z))=\partial_t\mathbf{P}_K(V(t,z)).$$

c) *Let $K\ge1$. Then, there exist $a_q\in\mathbb{C}$, $0\le q\le K-1$, which depend on K, such that the following holds. Let $V\in\mathcal{F}_K$, $K\ge1$.*

$$\mathbf{P}_K(t^KV(t,z))=\left(\sum_{q=0}^{K-1}a_qt(t\partial_t)^q\right)\mathbf{P}_K(V(t,z)),$$

where $(t\partial_t)^q$ is the q-th composition of the operator $t\partial_t$.

Proof The part **a)** follows directly from the definition of $\mathbf{P}_K$. For the part **b)**, let

$$V(t,z) = \sum_{j\geq 0} \frac{t^{Kj}}{(Kj)!} u_j(z).$$

Then

$$\partial_t^K V(t,z) = \sum_{j\geq 0} \frac{x^{K(j-1)}}{(K(j-1))!} u_j(z) = \sum_{j\geq 0} \frac{t^{Kj}}{(Kj)!} u_{j+1}(z).$$

On the other hand, let

$$U(t,z) = \sum_{j\geq 0} \frac{t^j}{j!} u_j(z),$$

and as before,

$$\partial_t U(t,z) = \sum_{j\geq 0} \frac{t^j}{j!} u_{j+1}(z).$$

Part **c)**. We have that

$$t^K V(t,z) = \sum_{j\geq 0} \frac{t^{K(j+1)}}{(Kj)!} u_j(z) = \sum_{j\geq 0} \frac{t^{K(j+1)}}{(K(j+1))!} M_K(j) u_j(z) = \sum_{j\geq 0} \frac{t^{Kj}}{(Kj)!} P_K(j) u_{j-1}(z),$$

where $P_K(j) = (Kj)(Kj-1)\cdots(Kj-K+1)$ is a polynomial of order K in the variable j. Note that $P_K(0) = 0$.

On the other hand, we have that

$$t\partial_t U(t,z) = \sum_{j\geq 0} \frac{t^{j+1}}{j!} u_{j+1}(z) = \sum_{j\geq 0} \frac{t^j}{j!} j u_j(z),$$

so that, for all $q \geq 0$,

$$(t\partial_t)^q U(t,z) = \sum_{j\geq 0} \frac{t^j}{j!} j^q u_j(z).$$

Moreover, as before

$$tU(t,z) = \sum_{j\geq 0} \frac{t^j}{j!} j u_{j-1}(z).$$

From the latter formulas, we deduce that for all $q \geq 0$,

$$t(t\partial_t)^q U(t,z) = \sum_{j\geq 0} \frac{t^j}{j!} \mu_q(j) u_{j-1}(z),$$

where $\mu_q(j) = j(j-1)^q$ is a polynomial of order $q+1$ in the variable j. Note that $\mu_q(0) = 0$.

Now, the polynomial P_K can be written as a linear combination with coefficients in $\mathbb{C}$ of the polynomials μ_q, $0 \leq q \leq K-1$,

$$P_K(j) = \sum_{q=0}^{K-1} a_q \mu_q(j),$$

where $a_q \in \mathbb{C}$, for all $j \geq 0$. From this we deduce that

$$\left(\sum_{q=0}^{K-1} a_q t(t\partial_t)^q\right) U(t,z) = \sum_{j\geq 0} \frac{t^j}{(j)!} P_K(j) u_{j-1}(z).$$

$\square$

In the next proposition, we study the linear partial differential equations satisfied by $\mathbf{P}_K(V)$ when V satisfies itself a linear partial differential equation.

Proposition 3 *Assume that a formal series $V(t,z) \in \mathcal{F}_K$ satisfies a linear partial differential equation*

$$(E_V) \qquad \partial_z^S V(t,z) = \sum_{(l_0,l_1)\in\mathcal{S}} B_{(l_0,l_1)}(t^K,z)\partial_t^{Kl_0}\partial_z^{l_1} V(t,z),$$

where $\mathcal{S}$ is a finite subset of $\mathbb{N}^2$ and $B_{(l_0,l_1)}(t^K,z)$ are polynomials with respect to t^K of order $d_{(l_0,l_1)}$ with holomorphic coefficients in Ω.

Then, the formal series $\mathbf{P}_K(V(t,z))$ satisfies a linear partial differential equation of the form

$$(E_{PV}) \qquad \partial_z^S \mathbf{P}_K(V(t,z)) = \sum_{(k_0,k_1)\in\mathcal{S}_1} A_{(k_0,k_1)}(t,z)\partial_t^{k_0}\partial_z^{k_1} \mathbf{P}_K(V(t,z)),$$

where $\mathcal{S}_1$ is a finite subset of $\mathbb{N}^2$ and $A_{(k_0,k_1)}(t,z)$ are polynomials with respect to t with holomorphic coefficients in Ω. Moreover, the highest power $M_0(l_1)$ of ∂_t which appears in monomials of the form $\partial_t^{k_0}\partial_z^{l_1}$ for fixed l_1 in the right handside of (E_{PV}) is of the form

$$M_0(l_1) = \max_{(l_0,l_1)\in\mathcal{S}} \{d_{(l_0,l_1)}(K-1) + l_0\}.$$

Proof Apply the linear operator $\mathbf{P}_K$ the equation (E_V). The result follows from the proposition 2. $\square$

The next result gives sufficient conditions for the existence of analytic solutions $V(t,z)$ of partial differential equations of Cauchy-Kowalevskaya type in a neighborhood of the origin in $\mathbb{C}^2$ that can be analytically continued with respect to the variable t in sectors of infinite radius. Growth estimates for these solutions are also obtained.

Theorem 1 *Let $K > 1$ an integer. Consider a formal series $V(t,z) \in \mathcal{F}_K$ solution of the linear partial differential equation,*

$$(E_V) \qquad \partial_z^S V(t,z) = \sum_{(l_0,l_1)\in\mathcal{S}} B_{(l_0,l_1)}(t^K,z)\partial_t^{Kl_0}\partial_z^{l_1} V(t,z),$$

where $B_{(l_0,l_1)}(t^K,z)$ are polynomials with respect to t^K of order $d_{(l_0,l_1)}$ with holomorphic coefficients in Ω and $\mathcal{S}$ is a finite subset of $\mathbb{N}^2$ of the form

$$\mathcal{S} = \{(l_0,l_1) \in \mathbb{N}^2 : d_{l_0,l_1}K(K-1) + Kl_0 + l_1 \leq S, l_1 < S\}.$$

Let

$$\mathbf{P}_K(V(t,z)) = \hat{X}(t,z) = \sum_{n\geq 0} \frac{\hat{X}_n(t)}{n!} z^n.$$

We assume that the formal series $\hat{X}_\mu(t)$, for $0 \le \mu \le S-1$, are $1/(K-1)$-summable with respect to t in a direction $d \in \mathbb{R}$.

Moreover, we assume that their sums $X_\mu(t)$ are holomorphic in a sectorial region $G \subset D(0, R(\rho))$, independent of μ, which has bisecting direction d and opening greater than $(K-1)\pi$ and satisfy the following estimates.
For all closed sector $\bar{S} \subset G$, there exist $C, \tilde{K} > 0$ such that

$$\sup_{t \in \bar{S}} |\partial_t^{n_0} X_{m_1}(t)| \le C \tilde{K}^{n_0+m_1} (Kn_0)! m_1!,$$

for all $0 \le m_1 \le S-1$ and all $n_0 \in \mathbb{N}$.

Then, the formal solution $V(t,z)$ of (E_V) converges in a neighborhood of the origin in $\mathbb{C}^2$. Moreover, $V(t,z)$ can be analytically continued with respect to t in a sector $S_{d/K,\delta} = \{t \in \mathbb{C} : |d/K - \arg(t)| < \delta\}$. Moreover, there exist $\delta_1 < \delta$, $C > 0$, and $M > 0$ such that

$$\sup_{z \in \bar{D}(0,\rho)} |V(t,z)| \le C e^{M|t|^{\frac{K}{(K-1)}}}$$

for all $t \in S_{d/K,\delta_1}$.

Proof From the proposition 3, we deduce that the formal series $\mathbf{P}_K(V(t,z)) = \hat{X}(t,z)$ satisfies a partial differential equation of the form

$$(E_{PV}) \qquad \partial_z^S \hat{X}(t,z) = \sum_{(k_0,k_1) \in \mathcal{S}_1} A_{(k_0,k_1)}(t,z) \partial_t^{k_0} \partial_z^{k_1} \hat{X}(t,z),$$

where $A_{(k_0,k_1)}(t,z)$ are holomorphic functions on a neighborhood of the origin in $\mathbb{C}^2$ and $\mathcal{S}_1$ is a finite subset of the form

$$\mathcal{S}_1 = \{(k_0,k_1) \in \mathbb{N}^2 : k_0 K + k_1 \le S, k_1 < S\}.$$

From the assumptions on the formal series $\hat{X}(t,z)$, we then deduce, that the assumptions of Proposition 1 are fulfilled. So that the formal series $\hat{X}(t,z)$ is $1/(K-1)$-summable with respect to t in the direction d. As a result, the Borel transform of order $1/(K-1)$ of $\hat{X}$,

$$W(t,z) = \sum_{j=0}^{\infty} \frac{X_j(z) t^j}{j! \Gamma(1+(K-1)j)},$$

converges in a neighborhood of the origin in $\mathbb{C}^2$ and can be analytically continued in a sector $S_{d,\delta}$, with exponential growth estimates of the form

$$\sup_{z \in \bar{D}(0,\rho)} |W(t,z)| \le C e^{M|t|^{\frac{1}{K-1}}}$$

for all $t \in S_{d,\delta_1}$, $\delta_1 < \delta$.

From the general theory of moment summability (see [1]), we get that the formal series

$$\tilde{W}(t,z) = \sum_{j \ge 0} \frac{t^j}{(Kj)!} X_j(z)$$

also converges in a neighborhood of the origin in $\mathbb{C}^2$ and can be analytically continued in the sector $S_{d,\delta}$ with the same exponential growth estimates as W.

By contruction, we have that

$$V(t,z) = \mathbf{P}_K^{-1}(\hat{X}(t,z)) = \tilde{W}(t^K, z).$$

We conclude that the series $V(t,z)$, solution of (E_V), converges in a neighborhood of the origin in $\mathbb{C}^2$ and can be analytically continued in a sector $S_{d/K,\delta'}$. Moreover, there exist $\delta_1' < \delta'$, $C > 0$, and $M > 0$ such that

$$\sup_{z \in \bar{D}(0,\rho)} |V(t,z)| \leq Ce^{M|t|^{\frac{K}{K-1}}}$$

for all $t \in S_{d/K,\delta_1'}$. $\square$

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