Kneading determinants and spectra of transfer operators in higher dimensions, the isotropic case

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Abstract. Transfer operators $\mathcal{M}_k$ are associated to C^r transversal local diffeomorphisms ψ_ω of $\mathbb{R}^n$, and C^r compactly supported functions g_ω. A formal trace $\operatorname{tr}^{\#}\mathcal{M}$, yields a formal Ruelle-Lefschetz determinant $\operatorname{Det}^{\#}(\mathrm{Id} - z\mathcal{M})$. We use the Milnor-Thurston-Kitaev equality recently proved by Baillif [Bai] to relate zeroes and poles of $\operatorname{Det}^{\#}(\mathrm{Id} - z\mathcal{M})$ with spectra of the transfer operators $\mathcal{M}_k$, under additional assumptions. As an application, we get a new proof of a result of Ruelle [Ru3] on the spectral interpretation of zeroes and poles of the dynamical zeta function $\exp \sum_{m\ge 1}(z^m/m)\sum_{f^m(x)=x} |\det Df(x)|^{-1}$ for smooth expanding endomorphisms f.

1. Introduction

A weighted (Ruelle) dynamical zeta function may be associated to a transformation f (on a compact space M, say) and a "weight" function $g : M \to \mathbb{C}$ by setting

$$\zeta_{f,g}(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x\in \operatorname{Fix}(f^m)} \prod_{j=0}^{m-1} g(f^j(x)) \,, \tag{1.1}$$

assuming, e.g., that each set $\operatorname{Fix} f^m = \{x \in M \mid f^m(x) = x\}$ is finite. If f is a one-sided subshift of finite type associated to a Markov matrix A, and g is constant, it is an easy

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algebraic exercise to see that $\zeta_{f,g}(z) = 1/\det(\mathrm{Id} - zA)$. If g is not constant, but has exponentially decaying variations on small cylinders, the transfer operator

$$\mathcal{L}\varphi(x) = \sum_{f(y)=x} g(y)\varphi(y) \tag{1.2}$$

has essential spectral radius bounded above by $\theta \exp(P \log(|g|))$ (with P the topological pressure) when acting on the Banach space Lip of Lipschitz functions on the one-sided shift space ([Po]). (Throughout this introduction, we refer to the survey [Ba1] for more information and references.) Haydn then proved (following crucial steps by Ruelle and Pollicott) that $\zeta_{f,g}(z)$ is meromorphic in the disc of radius $\theta^{-1}\exp(-P\log(|g|))$, where its poles are the inverse eigenvalues of $\mathcal{L}$ on Lip.

Using Markov methods (symbolic dynamics via Markov partitions, Markov covers, tower-like Markov extensions, etc.), similar results were obtained for weighted dynamical zeta functions associated to smooth data on manifolds: Axiom A diffeomorphisms (Ruelle [Ru2], Pollicott [Po], Haydn [Ha]), one-dimensional piecewise monotone maps (Hofbauer-Keller, Baladi-Keller [BaKe], Keller-Nowicki [KN]), and higher-dimensional piecewise smooth and injective maps (Buzzi-Keller [BuKe]). There, one assumes not only (piecewise) smoothness of the dynamics f and the weight g but also some "hyperbolicity" (or expansion) in f.

Note also that, for smooth expanding maps on manifolds, Ruelle [Ru1, Ru3] and Fried [Fr0, Fr1] studied dynamical Fredholm-type determinants. These determinants will only be mentioned in passing in the present article (see (4.2) below). We just point out here that, in the analytic case, the transfer operator is nuclear on a suitable Banach space, and this dynamical determinant can further be interpreted as an honest Grothendieck-Fredholm determinant: see Mayer [Ma], Ruelle [Ru1], Keller [Ke], and more recently Rugh [Rug], Fried [Fr1]. The underlying topological mechanism in these works is still the Markov partition.

In the early nineties, a new approach was launched [BaRu and references therein], initially for piecewise monotone interval maps. The original motivation was to understand the links between, on the one hand the Milnor-Thurston [MT] identity relating the kneading matrix and an unweighted dynamical zeta function, and on the other hand the spectral interpretation of the zeroes and poles of (weighted) dynamical zeta functions. This kneading approach is now quite well understood in one real dimension ([Ru4, BaRu, BKRS, Ba2, Go]), We recall its key ingredients next, very briefly.

The initial data is a finite set of local homeomorphisms $\psi_\omega : U_\omega \to \psi_\omega(U_\omega)$, where each U_ω is an open interval of $\mathbb{R}$, and of associated weight functions g_ω which are continuous, of bounded variation, and have support inside U_ω. In particular, the ψ_ω can be the inverse branches of a single piecewise monotone interval map f, and g_ω can be $g \circ \psi_\omega$ for a single g. For a more specific example, if f is unimodal on $[-1, 1]$ with $\max f = f(0) = b < 1$ and $\min f = f(+1) = f(-1) = -1$ then we may take $U_- = U_+ = (b + \epsilon, -1 - \epsilon)$, and $\psi_\pm$ (somewhat arbitrary extensions of) the two inverse branches of f, with images in $(-1 - \delta, \delta)$ and $(-\delta, 1 + \delta)$. We may also take g a continuous function of bounded variation on $(-1-\delta, 1+\delta)$ supported in $[-1,+1]$. Beware however that this setup does *not* allow to take for g the characteristic function of $[-1,1]$. Gouëzel [Go] recently obtained a significant weakening of the continuity assumption (while avoiding boundary terms in integration by parts), which allows e.g. the choice $g = 1_{[-1,+1]}$ for our unimodal example *if* 0 is not preperiodic, i.e., if there is no finite Markov partition.

We emphasize that no contraction assumption is required on the ψ_ω: their graph can even coincide with the diagonal on a segment. The transfer operator is now

$$\mathcal{M}\varphi = \sum_\omega g_\omega \cdot (\varphi \circ \psi_\omega)\,. \tag{1.3}$$

Ruelle [Ru4] obtained an estimate, noted $\widehat{R}$, for the essential spectral radius of $\mathcal{M}$ acting on the Banach space BV of functions of bounded variation. The main result of [BaRu] links the eigenvalues of $\mathcal{M}$ (outside of the disc of radius $\widehat{R}$), acting on BV, with the zeroes of the "sharp determinant"

$$\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M}) = \exp - \sum_{m=1}^\infty \frac{z^m}{m} \mathrm{tr}^{\#} \mathcal{M}^m\,, \tag{1.4}$$

where (with the understanding that $y/|y| = 0$ if $y = 0$)

$$\mathrm{tr}^{\#} \mathcal{M} = \sum_\omega \int \frac{1}{2} \frac{\psi_\omega(x) - x}{|\psi_\omega(x) - x|}\, dg_\omega(x)\,. \tag{1.5}$$

If the ψ_ω are strict contractions which form the set of inverse branches of a piecewise monotone interval map f, and $g_\omega = g \circ \psi_\omega$ then integration by parts together with the key property that

$$d\frac{x}{2|x|} = \delta\,, \text{ the Dirac delta at the origin of } \mathbb{R}\,,$$

show that $\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M}) = 1/\zeta_{f,g}(z)$. If one only assumes that the graph of each admissible composition $\psi^m_{\vec\omega}$ of m successive ψ_ωs (with $m \ge 1$) intersects the diagonal transversally, then

$$\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M}) = \exp - \sum_{m=1}^\infty \frac{z^m}{m} \sum_{\text{admissible } \psi^m_{\vec\omega}} \sum_{x \in \mathrm{Fix}\, \psi^m_{\vec\omega}} L(x, \psi^m_{\vec\omega}) \prod_{j=0}^{m-1} g_{\omega_j}(\psi^j_{\vec\omega}(x))\,, \tag{1.6}$$

where $L(x, \psi) \in \{-1, 1\}$ is the Lefschetz number of a transversal fixed point $x = \psi(x)$ (if ψ is C^1 this is just $\mathrm{sgn}\,(1 - \psi'(x))$). This is why we call the sharp determinant $\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M})$ a Ruelle-Lefschetz (dynamical) determinant. It is not difficult to see, for non Markov unimodal f and $g = 1_{[-1,1]}$ as above, that the expression (1.6) with

Lefschetz numbers, coming from our additional transversality assumption, gives that $\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M})$ is just $1/\zeta^-(z)$, where the "negative zeta function"

$$\zeta^-(z) = \exp + \sum_{m=1}^{\infty} \frac{z^m}{m} \big(2\#\mathrm{Fix}^-(f^m) - 1\big)$$

is defined by counting the sets

$$\mathrm{Fix}^-(f^m) = \{x \in (-1,1] \mid f^m(x) = x\,, f \text{ strictly decreasing in a nbhd of } x\}$$

of "negative fixed points." This zeta function was studied by Milnor and Thurston [MT], who proved the remarkable identity

$$(\zeta^-(z))^{-1} = \det(1 + \widehat{D}(z))$$

where $\widehat{D}(z)$ is a 1×1 "matrix" which is a power series in z with coefficients in $\{-1, 0, +1\}$ given by the signed itinerary of $f(0)$ (i.e., the kneading data). Milnor and Thurston's equation also applies to unimodal maps, but $\widehat{D}(z)$ is here a finite $(N-1) \times (N-1)$ matrix (with N the number of laps of f) constructed from the signed kneading series of the $N-1$ turning points.

Let us return now to our general setup ψ_ω, g_ω. The crucial step in the proof of the spectral interpretation of the zeroes of this Ruelle-Lefschetz determinant consists in establishing the following continuous version of the Milnor-Thurston identity:

$$\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M}) = \mathrm{Det}^{*}(\mathrm{Id} + \widehat{\mathcal{D}}(z))\,, \tag{1.7}$$

where the "kneading operator" $\widehat{\mathcal{D}}(z)$ replaces (formally) the finite kneading matrix of Milnor and Thurston. (In fact, approaching the g_ω by locally constant functions, $\mathrm{Det}^{*}(\mathrm{Id} + \widehat{\mathcal{D}}(z))$ can be approached by the determinants of a sequence of finite weighted Milnor Thurston matrices, see the appendix of [BaRu].)

If $|z| < 1/R$ then $\widehat{\mathcal{D}}(z)$ is a Hilbert-Schmidt operator on a suitable L^2 space (its kernel is in fact bounded and compactly supported), thus allowing the use of regularised determinants of order two. In fact, $\mathrm{Det}^{*}(\mathrm{Id} + \widehat{\mathcal{D}}(z))$ is the product of the regularised determinant with the exponential of the average of the kernel of $\widehat{\mathcal{D}}(z)$ along the diagonal, which is well-defined. Another kneading operator, $\mathcal{D}(z)$, is essential. If $1/z$ is not in the spectrum of $\mathcal{M}$ (on BV) then $\mathcal{D}(z)$ is also Hilbert-Schmidt, and we have $\mathrm{Det}^{*}(\mathrm{Id} + \widehat{\mathcal{D}}(z)) = \mathrm{Det}^{*}(\mathrm{Id} + \mathcal{D}(z))^{-1}$.

The kneading operator approach to study Ruelle-Lefschetz dynamical determinants in one complex dimension is presented in [BKRS]. There, a more conceptual definition of the $\mathcal{D}(z)$ was suggested, which was later implemented in one real dimension [Ba2]:

$$\mathcal{D}(z) = \mathcal{N}(\mathrm{Id} - z\mathcal{M})^{-1}\mathcal{S}\,, \tag{1.8}$$

where $\mathcal{N}$ is an auxiliary transfer operator and $\mathcal{S}$ is the convolution (μ being an auxiliary nonnegative finite measure)

$$\mathcal{S}\varphi(x) = \int \frac{1}{2}\frac{x-y}{|x-y|}\varphi(y)d\mu\,. \tag{1.9}$$

It is indeed clear from (1.8) that the kneading operator is a regularised (through the convolution $\mathcal{S}$) object which describes the inverse spectrum of the transfer operator: the resolvent $(\mathrm{Id} - z\mathcal{M})^{-1}$ in (1.8) means that poles can only appear if $1/z$ is an eigenvalue, and since $\mathrm{Det}^*(\mathrm{Id} + \widehat{\mathcal{D}}(z)) = \mathrm{Det}^*(\mathrm{Id} + \mathcal{D}(z))^{-1}$, this can be translated into a statement for zeroes of $\mathrm{Det}^*(\mathrm{Id} + \widehat{\mathcal{D}}(z))$. The Milnor-Thurston identity (1.7)) then implies that any zero of $\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M})$ (recall $|z| < 1/R$) is an inverse eigenvalue of $\mathcal{M}$.

In our opinion, the kneading approach we just described is not only interesting because it extends previous results (see e.g. [BaKe]) on piecewise monotone interval maps, but also because of its conceptual simplicity. We believe it sheds a new light on weighted dynamical zeta functions associated to smooth (non analytic) maps: The mechanism relating the zeroes of the zeta function with eigenvalues is literally visible in (1.7–1.8), as we just explained. As a side-effect, the sometimes cumbersome Markov partition tool is bypassed. (The unimodal example discussed above shows that in fact this one-dimensional approach works best for piecewise monotone maps with no finite Markov partition.)

Implementing this strategy in higher dimensions is a natural goal: Assume that the U_ω are (finitely many, say) open subsets of $\mathbb{R}^n$ and that the $\psi_\omega : U_\omega \to \psi_\omega(U_\omega)$ are local C^r homeomorphisms or diffeomorphisms, while the $g_\omega : U_\omega \to \mathbb{C}$ are compactly supported C^r functions, for $r \geq 1$. We assume that $\cup_\omega U_\omega$ is contained in a compact set. We consider the n-tuple $\mathcal{M}$ of operators $\mathcal{M}_k$ acting on k-forms by

$$\mathcal{M}_k\varphi = \sum_\omega g_\omega \cdot \psi_\omega^*\varphi\,. \tag{1.10}$$

As in dimension one, constant (or discontinuous) weights g_ω are not allowed. Since we may reduce to the model above from a smooth dynamical system on a manifold with the help of charts and partitions of unity, this causes in fact less problems than in dimension one. For example the perhaps most natural transfer operator acting on functions,

$$\mathcal{L}_f\varphi(x) = \sum_{f(y)} \varphi(y)/|\det D(f(y))|\,,$$

where f is the expanding linear map on the two torus defined by $(y_1, y_2) \mapsto (2y_1, 2y_2)$ (modulo 1), and $\det Df \equiv 4$, can easily be modelled by (1.10) with the help of local inverse branches and partitions of unity. We refer to Section 4 for details. (If f is an Anosov toral map, a partition of unity can also be used to obtain a representation (1.10), without the need of Markov partitions.)

In 1995, A. Kitaev wrote a two-page sketch of the (statement and) proof of a higher-dimensional Milnor-Thurston formula. He made an additional transversality assumption, which guarantees that the set of fixed points of each fixed period m is finite, allowing him to define a Ruelle-Lefschetz determinant $\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M})$ through formula (1.6). (Note that the Lefschetz numbers $L(x, \psi_{\vec{\omega}}^m) = \mathrm{sgn}\,(\det(\mathrm{Id} - D_x\psi_{\vec{\omega}}^m))$ in (1.6) are nonzero by the transversality assumption, and that if we assume that the ψ_ω are contractions, they are all $+1$.) Baillif turned Kitaev's unpublished note into a mathematical proof [Bai, see also Ba2] and proved the following "Milnor–Thurston" formula:

$$\mathrm{Det}^{\#}(\mathrm{Id} - z\mathcal{M}) = \prod_{k=0}^{n-1} \mathrm{Det}^{\flat}(\mathrm{Id} + \mathcal{D}_k(z))^{(-1)^{k+1}} . \tag{1.11}$$

We refer the reader to Section 2 for the definition of the kneading operators $\mathcal{D}_k(z)$ and their flat traces and determinants. We just mention here that they are kernel operators acting on $k+1$ forms, constructed with the resolvent $(\mathrm{Id} - z\mathcal{M}_k)^{-1}$ (where the transfer operator $\mathcal{M}_k$ acts on k forms via the pull-back ψ_ω^*), together with a convolution operator $\mathcal{S}_k$, mapping $k+1$-forms to k-forms and which satisfies the homotopy equation $d\mathcal{S} + \mathcal{S}d = 1$. The kernel $\sigma_k(x, y)$ of $\mathcal{S}_k$ has singularities of the form $(x - y)/\|x - y\|^n$. The transversality assumption allows Baillif to interpret the determinant obtained by integrating the kernels along the diagonal as a flat determinant in the sense of Atiyah and Bott [AB1, AB2], whence the notation $\mathrm{Det}^{\flat}$ in the right-hand-side of (1.11).

Although Baillif did not give a spectral interpretation of zeroes or poles of the sharp determinant (1.11), he did notice that for $|z|$ very small, suitably high iterates of the $\mathcal{D}_k(z)$ are trace-class on $L^2(\mathbb{R}^n)$, showing that the corresponding regularised determinant has a nonzero radius of convergence under rather weak assumptions.

In the present work, we carry out the spectral interpretation of the sharp determinant (1.6) in arbitrary dimension, under the transversality assumptions of Baillif and Kitaev (see Section 3). We concentrate here on the isotropic setting where the forward dynamics is (uniformly) expanding in all space directions. We decided however to formulate our main result in a way suitable for generalisations to anisotropic (in particular, Anosov) cases. This result says (Theorems 13–14 in Section 3) that *if* (Axiom 1 in Section 3) there are suitable Banach spaces $\mathcal{B}_k$ of coefficients for k-forms on which (Axiom 2 in Section 3) the essential spectral radii of all transfer operators $\mathcal{M}_k$ are smaller than some $\widetilde{R} < 1$, while the flat traces of the kernels $\mathcal{D}_k(z)$ are meromorphic in the disc of radius $1/R$ (Axiom 3), then *taking* $R = \widetilde{R}^{1/n}$*:*

Main Theorem. *Under Axioms 1, 2, 3 from Section 3, Det*$^{\#}$*(Id* $- z\mathcal{M}$*) is meromorphic in* $\{|z| < 1/R\}$*. The order of* z *as a zero/pole of Det*$^{\#}$*(Id* $- z\mathcal{M}$*) is*

- *the sum over* $0 \le 2k \le n$ *of the algebraic multiplicity of* $1/z$ *as an eigenvalue of* $\mathcal{M}_{2k}$ *acting on* $2k$ *forms with coefficients in* $\mathcal{B}_{2k}$

minus

- *the sum over* $1 \le 2k+1 \le n$ *of the algebraic multiplicity of* $1/z$ *as an eigenvalue of* $\mathcal{M}_{2k+1}$ *acting on* $2k+1$*-forms with coefficients in* $\mathcal{B}_{2k+1}$*.*

When the dynamics ψ_ω corresponds to the inverse branches of a differentiable uniformly expanding map on a compact manifold and the g_ω are differentiable, Axioms 1 and 2–3 are satisfied by taking the $\mathcal{B}_k$ to be Sobolev spaces. We then recover a (case of a) result of Ruelle [Ru3] (Section 4).

Choosing Banach spaces

In dimension one, the kernel of the convolution operator $\mathcal{S}$ is a function of bounded variation. This is very handy, because the essential spectral radius of the transfer operator on BV can be estimated, and in many cases shown to be strictly smaller than the spectral radius ("quasicompactness"). To adapt the strategy of [BaRu] to our higher-dimensional setting, it is desirable to find Banach spaces which, on the one hand, contain the kernels $\sigma_k(\cdot, y)$ (or some "primitive" of the kernels, see below) and, on the other, allow us to obtain quasicompactness of the transfer operators.

The kernels $\sigma_k(\cdot, y)$ belong to a fractional Sobolev space $W^{\alpha,p}$ for $0 < \alpha < 1$ and small enough $p > 1$. Instead of working with this limited "Hölder"-type fractional regularity, we adapt a technique used by Ruelle [Ru4] in dimension one: he constructed modified kneading operators $\mathcal{D}^{(r)}(z)$ with smoother kernels obtained by taking a "primitive" of the σ_k, which have the same traces as the original operators. This step is carried out in the abstract Lemma 6. In the concrete setting of Section 4, it allows us to let the transfer operators act on forms whose coefficients are (locally supported and) in a Sobolev space $L^t_m(\mathbb{R}^n) = W^{m,t}(\mathbb{R}^n)$, i.e., have their derivatives in $L^t(\mathbb{R}^n)$, up to some finite order $m \geq 0$. To achieve this goal, we need an analogue of the one-dimensional concept of "primitive." An obvious tool [St] is the inverse square root of the Laplacian (Riesz potential), or its better behaved version at infinity, the Bessel potential. This step is emblematic of the most important new feature of this higher-dimensional version of [BaRu]: the *intervention of harmonic analysis.* Indeed, very classical Calderón-Zygmund results are behind the fact (Lemma 3) that the operators $d\mathcal{S}_k$ are bounded on $L^p(\mathbb{R}^n)$. The second new ingredient is the (re)regularisation (3.9) of the kneading operators $\mathcal{D}_k(z)$ (in fact; of their smoothened version $\mathcal{D}^{(r)}_k(z)$): In particular, the resolvent $(1 - z\mathcal{M}_k)^{-1}$ is replaced by (see (3.6)) $(1 - \mathcal{M}_{k,L}(z))^{-1}$, where $\mathcal{M}_{k,L}(z) = \mathrm{Id} - (\mathrm{Id} - z\mathcal{M}_k)\exp\left(\sum_{j=1}^{L-1} \frac{z^j \mathcal{M}_k^j}{j}\right)$, for large enough L. (This allows us to "kill" constants and essentially consider the kneading operators as compact, see Corollary of Lemma 5.) Thirdly, one of the difficulties of the higher-dimensional kneading approach is that d and $\mathcal{M}_k$ do not commute. The spectra of operators $d\mathcal{S}_{k-1}\mathcal{M}_k$ (note that $d\mathcal{S}$ is a projection to $\mathrm{Ker}\, d$) arise naturally from the definitions of the kneading operators (see Lemma 11). In order to show that, outside of the disc of radius R, the eigenvalues λ of $d\mathcal{S}_{k-1}\mathcal{M}_k$ which are not eigenvalues of $\mathcal{M}_k$ do not affect the zeroes and poles of the sharp determinant (in the corresponding disc), we show that they are not intrinsic (see Lemma 12). An indirect argument (Lemma 11, Theorem 13) then proves that the contributions of such eigenvalues must cancel in the alternated product of kneading determinants. These two other new tricks are simple-minded, but they seem not to have been used before in dynamical contexts. We hope they will be helpful elsewhere, the last one may be relevant in dynamical cohomological settings.

Further extensions – beyond the smooth isotropic case

A natural extension is to apply kneading operators to uniformly hyperbolic dynamics, in order to improve on the results of Pollicott, Ruelle, and Haydn mentioned above and to contribute to a spectral interpretation of the work of Kitaev [Ki], see Fried [Fr2] and Blank-Keller-Liverani [BKL]. In a forthcoming work by the second named author, we shall see that replacing the isotropic Sobolev spaces of Section 4 by Banach spaces of anisotropic distributions, with leafwise smoothness, is the key to this extension. We also hope that the methods introduced in the present work will eventually be used to tackle the nonuniformly (expanding or) hyperbolic case (such as Hénon-like maps). A less ambitious but interesting problem is to adapt these methods to higher-dimensional (non-Markov) piecewise expanding maps, in order to give a different proof (and perhaps improve on) the results of Buzzi and Keller [BuKe]. Ultimately, one would also like to get rid of the transversality assumption.

Contents of the paper

In Section 2 we introduce notations, recall the Kitaev-Milnor-Thurston formula and prove some facts about the operators $\mathcal{S}_k$. Section 3 contains a statement of our assumptions (Axioms 1 and 2–3) and our main result (Theorem 13). (The reader will notice that the axioms below are more general than the ones stated in [Ba3], but we have to take $R = \widetilde{R}^{1/n}$. We have also corrected Axioms 2 and 3 there.) Assuming only Axioms 1 and 2, we get a weaker result (Theorem 14). In Section 4 we apply Theorem 13 to the smooth expanding setting, recovering a result of Ruelle [Ru3].

2. Preliminaries – the Milnor and Thurston formula

Notations and basic data - ψ_ω and g_ω

Let us fix once and for all an integer order of differentiability $r \geq 1$ and a dimension $n \geq 2$. The case $n = 1$ is dealt with in [BaRu, Ru4] and [Go] (see also [Ba1, Ba2]).

The data we consider is a system (indexed by a finite set Ω) of

(1) local C^r diffeomorphisms $\psi_\omega : U_\omega \to \psi_\omega(U_\omega)$, where $r \geq 1$ is an integer, and each U_ω is a nonempty bounded open subset of $\mathbb{R}^n$;
(2) local C^r "weight functions" $g_\omega : \mathbb{R}^n \to \mathbb{C}$, such that the support of g_ω is contained in U_ω.

We need more hypotheses and notation. Let $K = \{\|x\| \leq T\}$ in $\mathbb{R}^n$ be so that $\cup_\omega U_\omega \subset K$ and $\cup_\omega \psi_\omega(U_\omega) \subset K$. We make the following *transversality* assumption: for each $x \in \mathbb{R}^n$ and $m \geq 1$ such that there are $\omega_1, \ldots, \omega_m$ for which the composition

$$\psi_{\vec{\omega}}^m = \psi_{\omega_m} \circ \cdots \circ \psi_{\omega_1} \tag{2.1}$$

is well defined in a neighbourhood of x and fixes x, the derivative $D_x(\psi_{\omega_m} \circ \cdots \circ \psi_{\omega_1})$ does not have 1 as an eigenvalue. We often consider products of weights along orbits

$$g_{\vec{\omega}}^{(\ell)}(x) := g_{\omega_\ell}(\psi_{\omega_{\ell-1}} \circ \cdots \circ \psi_{\omega_1}(x)) \cdots g_{\omega_2}(\psi_{\omega_1}(x)) \cdot g_{\omega_1}(x) \,. \tag{2.2}$$

Since the mth factor $g_{\omega_m}(\psi_{\omega_{m-1}}\cdots\psi_{\omega_1}(x))$ in the above product vanishes whenever the composition (2.1) is not well defined (assuming the previous compositions, for $j<m$ make sense), we shall not worry much about domains of definition. Note also that the transversality hypothesis, combined with our compactness assumption, implies that for each ℓ, the set $\mathrm{Fix}\,\psi^\ell_{\vec\omega}$ contains only finitely many points x for which (2.2) isn't zero.

The transfer operators $\mathcal{M}_k$ – Forms $\mathcal{A}_k$ – Sharp determinant

We write $\mathcal{A}_{k,C^m}$, $\mathcal{A}_{k,C^m(K)}$ for the vector spaces of k-forms on $\mathbb{R}^n$ with C^m coefficients ($m\ge 0$ integer), respectively C^m coefficients supported in the compact set K. We shall also work with k-forms with coefficients in $L^q = L^q(\mathrm{Lebesgue}\,,\mathbb{R}^n)$ or in $L^q(U) = L^q(\mathrm{Lebesgue}\,,U)$, with U a bounded domain in $\mathbb{R}$ and $1\le q<\infty$, writing $\mathcal{A}_{k,L^q}$, $\mathcal{A}_{k,L^q(U)}$ for the corresponding spaces. More generally, if $\mathcal{B}$ is a Banach space of functions or distributions on $\mathbb{R}^n$, then $\mathcal{A}_{k,\mathcal{B}}$ denotes the space of k-forms with coefficients in $\mathcal{B}$. Sometimes we also consider $\mathcal{A}^C_{k,\mathcal{B}}$ the space of compactly supported k-forms with coefficients in $\mathcal{B}$. If $\mathcal{B}$ is a Banach space, we use the Banach norm $\|\phi\|_{\mathcal{B}} = \|\phi\|_{\mathcal{A}_{k,\mathcal{B}}} = \max_{j\in I(k)}\|\phi_j\|_{\mathcal{B}}$ for $\phi\in\mathcal{A}_{k,\mathcal{B}}$, where $I(k)$ denotes the set of ordered k-tuples in $\{1,\dots,n\}$ and

$$\phi(x) = \sum_{j\in I(k)} \phi_j(x)\, dx_{j_1}\wedge\cdots\wedge dx_{j_k}\,,\ \text{with } \phi_j\in\mathcal{B}\,.$$

The map ψ_ω induces a transformation ψ^*_ω on k-forms by the usual pullback operation (which involves differentiating the ψ_ω if $k\ne 0$). Our object of interest is the transfer operator associated to the system $\{\psi_\omega, g_\omega\}$ by setting, for $k=0,\dots,n$,

$$\mathcal{M}_k : \mathcal{A}_{k,C^{r-1}} \to \mathcal{A}_{k,C^{r-1}(K)}\,,\qquad \mathcal{M}_k\phi(x) = \sum_\omega g_\omega(x)\,(\psi^*_\omega\phi)(x)\,. \tag{2.3}$$

(In fact, $\mathcal{M}_0$ maps $\mathcal{A}_{0,C^r}$ to $\mathcal{A}_{0,C^r(K)}$.) We write $\mathcal{M}$ for the $(n{+}1)$-tuple $(\mathcal{M}_0,\dots,\mathcal{M}_n)$. We want to relate the spectra of the $\mathcal{M}_k$ on suitable Banach spaces $\mathcal{A}_{k,\mathcal{B}_k}$ to the zeroes and poles of the sharp determinant [Bai] associated to the data $\{\psi_\omega, g_\omega\}$ by the (a priori) formal power series

$$\mathrm{Det}^{\#}(\mathrm{Id}\,-z\mathcal{M}) = \exp - \sum_{m=1}^\infty \frac{z^m}{m}\sum_{\vec\omega\in\Omega^m}\sum_{x\in\mathrm{Fix}\,\psi^m_{\vec\omega}} g^{(m)}_{\vec\omega}(x)\, L(x,\psi^m_{\vec\omega})\,, \tag{2.4}$$

where $L(x,\psi)\in\{1,-1\}$ (the value 0 is excluded by our transversality assumption) is the Lefschetz number of the diffeomorphism ψ at x, i.e. $L(x,\psi) = \mathrm{sgn}\,\det(1-D_x\psi)$.

In other words $\mathrm{Det}^{\#}(\mathrm{Id}-z\mathcal{M}) = \exp-\sum_{m=1}^\infty \frac{z^m}{m}\mathrm{tr}^{\#}\mathcal{M}^m$, the (formal) sharp trace being defined by $\mathrm{tr}^{\#}\mathcal{M} = \sum_{\omega\in\Omega}\sum_{x\in\mathrm{Fix}\,\psi_\omega} g_\omega(x)\,L(x,\psi_\omega)$.

Kneading operators $\mathcal{D}_k(z)$ – Homotopy operators $\mathcal{S}_k$

We shall introduce next the kneading operators $\mathcal{D}_k(z)$, for $k=0,\dots,n-1$. Each $\mathcal{D}_k(z)$ is a power series in z with coefficients operators $\mathcal{A}_{k+1,C^{r-1}(K)}\to\mathcal{A}_{k+1,C^{r-1}(K)}$

(other Banach spaces of coefficients will be specified later). For some values of z, these operators will have a kernel and, for a suitable iterate $\mathcal{D}_k^{m(n)}(z)$, this kernel will be in L^2, so that Hilbert-Schmidt theory will furnish a regularised determinant of order $2m(n)$. Note that we use the sign conventions of [Bai] when integrating forms depending on two variables, for instance when considering kernel operators. For example if $\phi(x,y)$ is a n-form in y, this definition implies $d^x \int_y \phi(x,y) = (-1)^n \int d^x \phi(x,y)$, where d^x denotes derivative with respect to x (see [Bai (5)–(7)])

The operators $\mathcal{D}_k(z)$ were first defined in [Bai], following an idea of Kitaev. They involve the auxiliary transfer operators $\mathcal{N}_k : \mathcal{A}_{k,C^{r-1}} \to \mathcal{A}_{k+1,C^{r-1}(K)}$:

$$\mathcal{N}_k = d_k \mathcal{M}_k - \mathcal{M}_{k+1} d_k \, , \tag{2.5}$$

where $d = d_k : \mathcal{A}_{k,C^{r-1}} \to \mathcal{A}_{k+1,C^{r-2}}$ (or $\mathcal{A}^C_{k,C^{r-1}} \to \mathcal{A}^C_{k+1,C^{r-2}}$) denotes the exterior derivative. The fact that $\mathcal{N}_k$ is well-defined for $r=1$ and does not lower the regularity comes, e.g., from the equivalent expression $\mathcal{N}_k\phi(x) = \sum_\omega dg_\omega(x) \wedge \psi_\omega^* \phi(x)$.

The good properties of the kernels of the kneading operators $\mathcal{D}_k(z)$ are due to the convolution operators $\mathcal{S}_k : \mathcal{A}^C_{k+1,C^{r-1}} \to \mathcal{A}_{k,C^{r-1}}$. We shall see in the proof of Lemma 4 that $\mathcal{S}_k$ can be written as $d^*_{k+1}\Delta^{-1}_{k+1}$, we recall here the definition suggested by Kitaev and used in [Bai]: Introduce an $n-1$ form solving $d\sigma = \delta$, the Dirac current at $0 \in \mathbb{R}^n$,

$$\sigma(x) = \frac{\Gamma(n/2)}{2(\pi^{n/2})} \sum_{i=1}^n \sigma^{\hat{i}}(x)\, dx_1 \wedge \cdots \wedge d\hat{x}_i \wedge \cdots \wedge dx_n \, , \, \sigma^{\hat{i}}(x) = (-1)^{i+1} \frac{x_i}{|x|^n} \, , \tag{2.6}$$

where Γ is the Euler function and the hat means that the coordinate is deleted.

Lemma 0 (Definition and properties of the $\mathcal{S}_k$ [Bai]). *For each $k \in \{0, \ldots, n-1\}$ let $\mathcal{S}_k$ be the convolution operator*

$$\mathcal{S}_k \phi(x) = (-1)^n \int_y \sigma_k(x,y) \wedge \phi(y) \, , \tag{2.7}$$

where $\sigma_k(x,y)$ is a k-form in $x \in \mathbb{R}^n$ and an $n-k-1$-form in $y \in \mathbb{R}^n$, measurable on $\mathbb{R}^{2n}$, defined by the decomposition ($\sigma(x-y)$ denotes $(F^*\sigma)(x,y)$ where $F(x,y) = x-y$)

$$\sigma(x-y) = \sum_{k=0}^{n-1} (-1)^n \sigma_k(x,y) \, . \tag{2.8}$$

(1) For every bounded open ball $D \subset \mathbb{R}^n$, the form $\sigma_k(\cdot, y)$ belongs to $\mathcal{A}_{k,L^s(D)}$ for each $1 \le s < n/(n-1)$ and $\sup_{y\in D} \|\sigma_k(\cdot,y)\|_{L^s(D)} < \infty$ for all k. Outside of the diagonal $x=y$ in $\mathbb{R}^{2n}$, $\sigma_k(x,y)$ has C^∞ coefficients.
(2) The following homotopy equation holds: $d_{k-1}\mathcal{S}_{k-1} + \mathcal{S}_k d_k = Id \quad$ on $\mathcal{A}^C_{k,C^2}$.

Proof of Lemma 0. By definition, the singular kernel $\sigma_k(x,y)$ of $\mathcal{S}_k$ can be written $\sigma_k(x,y) = \sum_{\vec{i}} \tau_{k,\vec{i}}(x-y) dy^{\vec{i}}$, where the coefficients of each k-form $\tau_{k,\vec{i}}(u)$ are linear combinations of:

$$\frac{u_j}{\|u\|^n} \, , j = 1, \ldots, n \, , \tag{2.9}$$

where $\|u\|$ denotes the Euclidean norm of $u \in \mathbb{R}^n$. The fact that $\sigma_k(x,y)$ is C^∞ outside of the diagonal is obvious.

It was pointed out in [Bai] that the singularity in (2.9) belongs to $L^s(D)$ for each $1 \le s < n/(n-1)$ and every bounded ball D. The integrability claim in Lemma 0 easily follows from this. Claim (2) is not difficult (see [Bai]). $\square$

Finally, the kneading operators are defined (in the sense of formal power series with coefficients linear operators) as

$$\mathcal{D}_k(z) = z\mathcal{N}_k(\mathrm{Id} - z\mathcal{M}_k)^{-1}\mathcal{S}_k = \sum_{\ell=0}^{\infty} z^{\ell+1}\mathcal{N}_k\mathcal{M}_k^\ell\mathcal{S}_k\,, \quad k = 0, \ldots, n-1\,. \tag{2.10}$$

The higher dimensional Milnor-Thurston formula

In order to state the results from [Bai], let us view the kneading operators as kernel operators. By Lemma 0, since both $\mathcal{M}_k$ and $\mathcal{N}_k$ act boundedly on $\mathcal{A}_{k,L^q}$, we may write

$$\mathcal{D}_k(z)\phi(x) = (-1)^n \sum_{\ell=0}^{\infty} z^{\ell+1} \int_y (\mathcal{N}_k\mathcal{M}_k^\ell)_x \sigma_k(x,y) \wedge \phi(y)\,. \tag{2.11}$$

(The subscript x means that the operator acts on the x variable.) Hence, $\mathcal{D}_k(z)$ has a formal power series kernel $\sum_{\ell\ge 0} z^{\ell+1}\mathcal{K}_{k,\ell}(x,y)$ (just set $\mathcal{K}_{k,\ell}(x,y) = (\mathcal{N}_k\mathcal{M}_k^\ell)_x\sigma_k(x,y)$). The kernels of iterates $\mathcal{D}_k^m(z)$ are obtained in a similar way and analogous expressions may be written for $\widehat{\mathcal{D}}_k(z)$ and $\mathcal{D}_k^*(z)$.

Baillif [Bai] proved the following lemma and theorem (see also [Ba2, §3]) using transversality and elementary properties of the $\sigma_k(x,y)$:

Lemma 1 [Bai]. *For each $k = 0, \ldots, n-1$ and $\ell \ge 0$, we have $\mathcal{K}_{k,\ell}(x,x) \in L^1(\mathbb{R}^n)$, so that we may define the formal flat trace of $\mathcal{D}_k(z)$ by integrating the kernel along the diagonal (with the correct sign):*

$$tr^\flat \mathcal{D}_k(z) = \sum_{\ell \ge 0} z^{\ell+1}(-1)^{(k+1)} \int \mathcal{K}_{k,\ell}(x,x)\,. \tag{2.12}$$

The same integrability property holds for the kernels of $\mathcal{D}_k(z)^m$ for all $m \ge 1$. Therefore, the traces $tr^\flat\mathcal{D}_k^m(z)$ are power series with complex coefficients.

For $k = 0, \ldots, n-1$, we may thus define a (formal) flat determinant by

$$\mathrm{Det}^\flat(\mathrm{Id} + \mathcal{D}_k(z)) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \mathrm{tr}^\flat \mathcal{D}_k(z)^m\,.$$

The coefficients of $\mathrm{tr}^\flat\mathcal{D}_k(z)$ coincide with the Atiyah-Bott flat traces of the corresponding kernel operators if the ψ_ω and g_ω are C^∞, see [Bai].

Theorem 2 (Milnor-Thurston identity) [Bai]. *In the sense of formal power series*

$$Det^{\#}(Id - z\mathcal{M}) = \prod_{k=0}^{n-1} Det^{\flat}(Id + \mathcal{D}_k(z))^{(-1)^{k+1}} . \tag{2.13}$$

We shall not need the following result, although we shall exploit and revisit key ideas in its proof (see [Bai, Lemma 6.2]) in Lemma 6 below:

Theorem [Bai]. *There is $\delta > 0$ so that in the disc $\{|z| < \delta\}$, for all $m \geq n/2 + 1$ the kernel of $\mathcal{D}_k^m(z)$ has coefficients in $L^2(dx \times dy)$. In particular, the regularised determinant of order $n+1$ (if n is odd) or $n+2$ (if n is even) of $\mathcal{D}_k(z)$ on L^2 is holomorphic in the disc of radius δ.*

Harmonic analysis and algebra with the $\mathcal{S}_k$

Lemma 3 (Properties of the $\mathcal{S}_k$ and $d_k\mathcal{S}_k$: analysis). *Let $k \in \{0, \dots, n-1\}$.*
(1) If $1 < q < n$ then $\mathcal{S}_k$ is bounded from $\mathcal{A}_{k+1,L^q}$ to $\mathcal{A}_{k,L^{q'}}$ for $q' = qn/(n-q)$. For $1 \leq q \leq n$ and any continuous compactly supported function χ, the operator $\chi\mathcal{S}_k\chi$ is bounded from $\mathcal{A}_{k+1,L^q}$ to $\mathcal{A}_{k,L^{q''}}$ for all $q < q'' < qn/(n-q)$.
(2) $d_k\mathcal{S}_k$ extends to a bounded operator on $\mathcal{A}_{k+1,L^q}$ for $1 < q < \infty$. In fact, for each i, the operator $\partial_i\mathcal{S}_k$ extends to a bounded operator from $\mathcal{A}_{k+1,L^q}$ to $\mathcal{A}_{k,L^q}$.

Note that $q' > q > 1$, in fact $q' > \eta q$ for each $1 \leq \eta < 1 + q/(n-q)$.

Proof of Lemma 3. The second claim of (1) follows from usual properties of the convolution (see e.g. [Sch, p.151]) since $\chi\sigma_k \in L^t$ for all $1 \leq t < n/(n-1)$.

Noting that the kernel σ_k is of weak-type $\frac{n}{n-1}$, we may apply the Hardy-Littlewood-Sobolev fractional integration theorem (see e.g. [St, V.Theorem 1 and Comment 1.4]; the comment explains why Theorem 1 extends from the Riesz potential to more general weak-type kernels), which yields that $\mathcal{S}_k$ is a bounded operator from $\mathcal{A}_{k+1,L^q}$ to $\mathcal{A}_{k,L^{q'}}$ if $1 < q < n$.

It remains to prove (2), i.e., study $\partial_i\mathcal{S}_k$. We first claim that $\partial_i\mathcal{S}_k$ can be written as the sum of, on the one hand a convolution operator with kernel a form having coefficients which are linear combinations of expressions of the type

$$\frac{x_i x_j}{\|x\|^{n+2}} , \frac{\|x\|^2 - n x_i^2}{\|x\|^{n+2}} , 1 \leq i, j \leq n , \tag{2.14}$$

and on the other hand a distribution ν_k which extends to a bounded operator on $\mathcal{A}_{k+1,L^q}$. Indeed, computing the functional partial derivatives $\partial_i = \frac{\partial}{\partial x_i}$ of (2.9) produces (2.14), so that we only need to study the distributional contribution to $\partial_i\mathcal{S}_k$. For this, let us take a kernel singularity $x_j/\|x\|^n$, and for each $t > 0$ let $\chi_t : \mathbb{R} \to \mathbb{R}$ be a C^∞ function, identically zero on $[-t, t]$ and which is identically equal to 1 outside of $[-2t, 2t]$. We may assume that $\sup|\chi'| \leq 2/t$. Let us consider

$$\partial_i \left(\frac{\chi_t(\|x\|)x_j}{\|x\|^n} \right) = \frac{x_j \partial_i(\chi_t(\|x\|))}{\|x\|^n} + \chi_t(\|x\|)\partial_i \left(\frac{x_j}{\|x\|^n} \right) .$$

Since the second term of the above sum converges to the already mentioned functional partial derivative as $t \to 0$, we should check that the distribution with kernel corresponding to the first term, which is just $\chi'_t(\|x\|)x_i x_j/\|x\|^{n+1}$, acts boundedly on $\mathcal{A}_{k+1,L^q}$, uniformly in $t \geq 0$. By density and completeness, it is enough to consider $\phi \in \mathcal{A}_{k+1,L^q} \cap C^\infty$ and take $\varphi = \phi_\ell$ for $\ell \in I(k+1)$. We may formally write for every $w \in \mathbb{R}^n$

$$\begin{aligned}|\nu_k(\varphi)(w)| = \left|\int_{\mathbb{R}^n} \chi'_t(\|x\|)\frac{x_i x_j}{\|x\|^{n+1}}\varphi(w-x)\,dx\right| &\leq \frac{2}{t}\int_{\|x\|\leq t}\left|\frac{x_i x_j}{\|x\|^{n+1}}\varphi(w-x)\,dx\right| \\ &\leq \frac{2}{t}\sup_{\|x\|\leq t}|\varphi(w+x)|\int_{\|x\|\leq t}\left|\frac{x_i x_j}{\|x\|^{n+1}}\right| dx\,.\end{aligned} \tag{2.15}$$

Going to polar coordinates, it is not difficult to see that there is a constant $C > 0$ so that $\int_{\|x\|\leq t} |x_i x_j| \|x\|^{-n-1}\,dx \leq Ct$. We end our analysis of the distributional contribution by noting that $\lim_{t\to 0}\int \sup_{\|x\|\leq t}|\varphi(w+x)|^q\,dw = \int |\varphi(w)|^q\,dw$.

(Since our assumptions imply that $|\varphi(w+x) - \varphi(w)| \leq t\sup\|D\varphi\|$, the above is easily checked.) As pointed out to us by S. Gouëzel, a more careful analysis shows that in fact ν_k is zero if $i \neq j$ and is a scalar multiple of the Dirac mass if $i = j$.

We next observe that the expressions (2.14) in the functional term of the kernel of $\partial_j \mathcal{S}_k$ exhibit the same kind of singularity as the Riesz transform. More precisely, this kernel is a linear combination of $\frac{\Omega_\ell(x)}{\|x\|^n}$ where each Ω_ℓ is homogeneous of degree zero, has vanishing integral on the unit $(n-1)$-dimensional sphere and is C^1 on this sphere. We may thus apply e.g. Theorem 3 in Chapter II of [St], which immediately guarantees that $d_k\mathcal{S}_k$ extends to a bounded operator on $\mathcal{A}_{k+1,L^q}$, for $1 < q < \infty$. $\square$

Lemma 4 (Properties of the $\mathcal{S}_k$: algebra). *Let $1 < q < \infty$.*
(1) $\mathcal{S}_{k-2}\mathcal{S}_{k-1} = 0$ on $\mathcal{A}_{k,L^q}$.
(2) On $\mathcal{A}_{k,L^q}$, and suppressing the indices for simplicity, $d\mathcal{S} = (d\mathcal{S})^2$, and $(\mathcal{S}d)^2 = \mathcal{S}d$, $d\mathcal{S}\mathcal{S}d = 0 = \mathcal{S}dd\mathcal{S}$. In other words, $d\mathcal{S}$ and $\mathcal{S}d$ are two orthogonal bounded projectors onto $Im\,d = Ker\,d$, respectively $Im\,\mathcal{S} = Ker\,\mathcal{S}$, in $\mathcal{A}_{k,L^q} = Im\,d \oplus Ker\,\mathcal{S} = Im\,\mathcal{S} \oplus Ker\,d$.

Proof of Lemma 4. The facts that $\mathcal{S}d$ is onto $\mathrm{Ker}\,\mathcal{S}$ and that $\mathrm{Ker}\,\mathcal{S} \subset \mathrm{Im}\,\mathcal{S}$ do not depend on $\mathcal{S}^2 = 0$: Use $\phi = \mathcal{S}d\phi$ if $\phi \in \mathrm{Ker}\,\mathcal{S}$.

To prove that $\mathcal{S}^2 = 0$, we shall give an equivalent definition of $\mathcal{S}$, which was indicated to us by D. Ruelle. Using Lemmas 0 and 3, it suffices to show the result on C^∞ compactly supported forms (by density). Recall that there is a scalar (or hermitian) product on the space of compactly supported C^∞ k-forms $\mathcal{A}^C_{k,C^\infty}$, defined by:

$$< \phi(x)dx_{i_1}\wedge\cdots\wedge dx_{i_k} \,|\, \varphi(x)dx_{j_1}\wedge\cdots\wedge dx_{j_k} >= \begin{cases}\displaystyle\int \phi\cdot\varphi \ \text{ if } i_1,\ldots,i_k = j_1,\ldots,j_k \\ 0 \ \text{ otherwise.}\end{cases} \tag{2.16}$$

(The scalar product is still defined if only one of ϕ, φ is compactly supported.) Now, $d : \mathcal{A}^C_{k,C^\infty} \to \mathcal{A}_{k+1,C^\infty}$ defines a dual $d^* : \mathcal{A}^C_{k+1,C^\infty} \to \mathcal{A}_{k,C^\infty}$ by $< d^*\phi, \varphi >=< \phi, d\varphi >$.

The Laplace-Beltrami operator is defined by $\Delta = dd^* + d^*d$ where d is the exterior derivative of forms. In $\mathbb{R}^n$, we have (see, e.g., [Li]):

$$\Delta \left[\sum_{j \in I(k)} \phi_j(x) dx_j \right] = - \sum_{j \in I(k)} (\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \phi_j)(x) dx_j \,. \tag{2.17}$$

Define the Green kernel to be $E(x) = -\frac{\Gamma(n/2)}{(n-2)2(\pi)^{n/2}} \frac{1}{||x||^{n-2}} dx_1 \wedge \cdots \wedge dx_n$ if $n \geq 3$, and $E(x) = \frac{1}{2\pi} \log(||x||) dx_1 \wedge dx_2$ if $n = 2$. Then, let $E_k(x,y)$ be the k-form in x and $n-k$-form in y such that $E(x-y) = \sum_{k=0}^n (-1)^n E_k(x,y)$. It is well known (see [Sch]) and has been used in [Bai] that $\Delta E = \delta(x) dx_1 \wedge \cdots \wedge dx_n$ (as a current acting on C^∞ forms vanishing at infinity, or compactly supported C^∞forms). One thus sees easily that the operator $G_k : \phi(x) \mapsto \int_y E_k(x,y) \wedge \phi(y)$ satisfies $\Delta G_k = id$ on $\mathcal{A}^C_{k,C^\infty}$. Using that $\frac{\partial^2}{\partial y_i^2} E(x-y) = \frac{\partial^2}{\partial x_i^2} E(x-y)$, one checks that $G_k \Delta = id$ on the same space.

The operator $\mathcal{S}_k$ defined in [Bai] is in fact $d^* G_{k+1}$. Indeed,

$$d^* \left(\phi(x) dx_1 \wedge \cdots \wedge dx_n \right) = \sum_{j=1}^k (-1)^{j+1} \partial_j \phi(x) dx_1 \wedge \cdots d\hat{x}_j \cdots \wedge dx_n \,. \tag{2.18}$$

Thus, $d^* E(x) = \sigma(x)$, where σ was defined in (2.6). Since $\mathcal{S}_k$ is defined in (2.7) as the convolution with σ_k, the equality is immediate.

By Lemma 3, the composition $\mathcal{S}_{k-1} \mathcal{S}_k = d^* G_k d^* G_{k+1}$ is well defined on L^q forms, and thus on $\mathcal{A}^C_{k,C^\infty}$. We shall show that $d^* G_k = G_{k-1} d^*$, which implies $\mathcal{S}_{k-1} \circ \mathcal{S}_k = 0$ since $d^* d^* = 0$. Since $E(x)$ is an n-form, $dE(x) = 0$. Using $d = d^x + d^y$, we obtain $d^x E_{k-1}(x,y) = -d^y E_k(x,y)$. Integrating by parts, $dG_{k-1} = G_k d$ (recall that $E_k(x,y)$ is an n-form). Notice now that Δ is auto-dual by definition, and thus G_k is also auto-dual. Hence, $d^* G_k = G_{k-1} d^*$. $\square$

3. Spectral interpretation of the zeroes of the sharp determinant

The Axioms

Let r, Ω, ψ_ω, g_ω, $K = \{x \in \mathbb{R}^n \mid |x| \leq T\}$, $\mathcal{M}_k$, $\mathcal{S}_k$, σ_k, $\mathcal{N}_k$, and $\mathcal{D}_k(z)$ be the objects from Section 2. Let 1_K be the characteristic function of K. We fix once and for all

$$K' = \{x \in \mathbb{R}^n \mid |x| < 2T\} \,,$$

and a radial C^∞ function $\chi_K = \chi_{K,K'}$ supported in K' and identically equal to 1 on K. Note that $\mathcal{N}_k$ and $\mathcal{M}_k$ are bounded from k-forms to $k+1$-forms, respectively k-forms, with coefficients in $L^q(K')$. We use the notation $\rho(\mathcal{P})$, $\rho_{ess}(\mathcal{P})$ for the spectral and essential spectral radii of a bounded linear operator $\mathcal{P}$. We are now ready to state our three assumptions:

Axiom 1: For $k = 0, \dots, n$ and $1 < t < \infty$, there are Banach spaces $\mathcal{B}_{k,t}$ of distributions on $\mathbb{R}^n$ so that:

(1) There are real self-adjoint invertible pseudodifferential convolution operators $\widetilde{\mathcal{J}}_k$, $\widetilde{\mathcal{J}}_k^{-1}$ of order (at most) r so that $\mathcal{B}_{k,t}$ is defined by the Banach space isomorphism

$$\widetilde{\mathcal{J}}_k(L^t(\mathbb{R}^n)) = \mathcal{B}_{k,t} \,. \tag{3.1}$$

(2) If ψ is a C^r local diffeomorphism and and g is a C^r function supported in K then $\varphi \mapsto g \cdot \psi^*(\varphi)$ is a bounded operator on each $\mathcal{B}_{k,t}$.

(3) Each operator $d_k : \mathcal{A}_{k,\mathcal{B}_{k,t}} \to \mathcal{A}_{k+1,\mathcal{B}_{k+1,t}}$ is bounded.

Axiom 2: There is $0 \le \widetilde{R} < 1$ so that for $k = 0, \dots, n$:

(1) $\mathcal{M}_k$ is bounded on $\mathcal{A}_{k,\mathcal{B}_{k,t}}$ with $\rho(\mathcal{M}_k|_{\mathcal{A}_{\mathcal{B}_{k,t}}}) \le 1$, $\rho_{ess}(\mathcal{M}_k|_{\mathcal{A}_{\mathcal{B}_{k,t}}}) < \widetilde{R}$, and there are no eigenvalues of modulus $\widetilde{R}$, for each $1 < t < \infty$.

(2) Let $\Pi_{k,t}$ be the (finite-rank) spectral projector on $\mathcal{A}_{k,\mathcal{B}_{k,t}}$ associated to the spectrum of $\mathcal{M}_k$ outside the disc of radius $\widetilde{R}$. Let $\{\varphi_{k,t,s}\}$ be a basis for the corresponding generalised eigenspace. (Our assumptions imply that the dimension does not depend on $1 < t < \infty$.) Then, there is $C > 0$ so that $\max_{k,s} \sup_{1<t<\infty} \|\varphi_{k,t,s}\|_{\mathcal{B}_{k,t}} \le C$, and, for all k, j , we have

$$\sup_{1<t<\infty} \|\mathcal{M}_k^j\|_{\mathcal{B}_{k,t}} \le C \,, \quad \sup_{1<t<\infty} \|\mathcal{M}_k^j(\mathrm{Id} - \Pi_{k,t})\|_{\mathcal{B}_{k,t}} \le C\widetilde{R}^j \,.$$

Axiom 3: There is $\mathcal{E}_0 > 0$ so that, for all $0 < \mathcal{E} < \mathcal{E}_0$, each $j \ge 1$ and each admissible composition $\psi_{\vec{\omega}}^j$, letting $V_{\vec{\omega},\mathcal{E}}$ be the set of x in the domain of definition of $\psi_{\vec{\omega}}^j$ so that $\|\psi_{\vec{\omega}}^{j+\ell}(x) - x\| < \mathcal{E}$, the map $\psi_{\vec{\omega}}^{j+\ell} - \mathrm{Id}$ is injective on $V_{\vec{\omega},\mathcal{E}}$.

Also, for each $\eta > 0$ there is C so that for all k, every sequence $\{\mathcal{E}_j\}$ with $0 < \mathcal{E}_j < \mathcal{E}_0$, and each k-form $\mathbf{1}_k = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

$$\sum_{\vec{\omega}\in\Omega^j} \sup_{x\in V_{\vec{\omega},\mathcal{E}_j}} \frac{|g_{\vec{\omega}}^{(j)}(\psi_{\vec{\omega}}^j x)|}{|\det(D\psi_{\vec{\omega}}^j(x) - \mathrm{Id}\,)|} \big|(\psi_{\vec{\omega}}^j)^*(\mathbf{1}_k)(x)/\mathbf{1}_k(x)\big| \le C\exp(\eta j) \,, \forall j \,.$$

Consequences of Axiom 1. C^∞ functions with compact support are contained in $\mathcal{B}_{k,t}$ and distributions in $\mathcal{B}_{k,t}$ have order at most r. Also, $\mathcal{B}_{k,t} \subset \mathcal{B}_{k,t'}$ boundedly if $t > t'$.

Since $\widetilde{\mathcal{J}}_k$ is a convolution operator it commutes (at most up to sign) with $\mathcal{S}$, $d\mathcal{S}$, $\mathcal{S}d$ and with convolution operators having a smooth kernel $\delta_\epsilon(\|x-y\|)$. Thus, by Lemma 3, for all $1 < t < n$ the operators $\mathcal{S}_k : \mathcal{A}_{k+1,\mathcal{B}_{k,t}} \to \mathcal{A}_{k,\mathcal{B}_{k,tn/(n-t)}}$ are bounded. Also, for all $1 < t < \infty$ the operators $\mathcal{S}_k : \mathcal{A}_{k+1,\mathcal{B}_{k+1,t}} \to \mathcal{A}_{k,\mathcal{B}_{k,t}}$ are bounded, and both operators $d_k\mathcal{S}_k$, $\mathcal{S}_{k+1}d_{k+1}$ are bounded on $\mathcal{A}_{k+1,\mathcal{B}_{k+1,t}}$. Finally, there are C^∞ mollifiers $\delta_\epsilon(\|x-y\|)$ on $K' \times K'$ with $\|\int \delta_\epsilon(\cdot, y)\varphi(y)\, dy - \varphi\|_{\mathcal{B}_{k,t}} \to 0$ as $\epsilon \to 0$ for all φ [Ad, Lemma 2.18].

By Lemma 0 (1), for each fixed $y \in K$, the form $\sigma_k^{(r)}(x, y) = \widetilde{\mathcal{J}}_k(1_K(\cdot)\sigma_k(\cdot, y))$ belongs to $\mathcal{A}_{k,\mathcal{B}_{k,t}}$ for all $1 \le t < n/(n-1)$ and each k, with

$$\max_k \sup_{y\in K} \|\widetilde{\mathcal{J}}_k(1_K(\cdot)\sigma_k(\cdot, y))\|_{\mathcal{A}_{k,\mathcal{B}_{k,t}}} < \infty \,, \forall\, 1 \le t < n/(n-1) \,. \tag{3.2}$$

Write $\mathcal{S}_k^{(r)}$ for the convolution operator corresponding to $\sigma_k^{(r)}(x,y)$. Up to a sign, this is just $1_K \widetilde{K} \mathcal{J}_k \mathcal{S}_k$. Hence, there is $\upsilon_k \in \{+1,-1\}$ so that for all $1<t<\infty$

$$1_K \mathcal{S}_k \widetilde{\mathcal{J}}_k = \upsilon_k \mathcal{S}_k^{(r)}\,, \qquad \text{on } L^t(\mathbb{R}^n)\,. \tag{3.3}$$

Since $\widetilde{\mathcal{J}}^{-1}$ is real and self-adjoint, for all compactly supported $\varphi \in L^\infty(\mathbb{R}^n)$, all $\psi \in \mathcal{B}_{k,t}$

$$\int (\widetilde{\mathcal{J}}_k^{-1}\varphi)(x)\psi(x)dx = \int \varphi(x)(\widetilde{\mathcal{J}}_k^{-1}\psi)(x)dx\,. \tag{3.4}$$

Preliminary Step: The L-regularised kneading determinants $\mathcal{D}_{k,L}(z)$

For each $\ell \geq 1$ the ℓ-regularised version of $\mathcal{M}_k$ is defined to be the following convergent power series with operator (on $\mathcal{A}_{k,\mathcal{B}_{k,t}}$, e.g.) coefficients:

$$\mathcal{M}_{k,\ell}(z) = \mathrm{Id} - (\mathrm{Id} - z\mathcal{M}_k)\exp\Big(\sum_{j=1}^{\ell-1} \frac{z^j \mathcal{M}_k^j}{j}\Big)\,. \tag{3.6}$$

Note that $\mathcal{M}_{k,1}(z) = z\mathcal{M}_k$ and $\mathcal{M}_{k,\ell}(z)$ converges for all z and all ℓ. We shall also use $\mathcal{N}_{k,\ell}(z) = d\mathcal{M}_{k,\ell}(z) - \mathcal{M}_{k+1,\ell}(z)d$. (Note that $\mathcal{N}_{k,1}(z) = z\mathcal{N}_k$.)

Lemma 5. *Assume Axioms 1 and 2. For each $C>1$ and $\xi < 1/\widetilde{R}$ there is $L \geq 1$ so that for each $|z|<\xi$ and all t, k, we have $\|\mathcal{M}_{k,L}(z)|_{Im(\mathrm{Id}-\Pi_{k,t})}\|_{\mathcal{A}_{k,\mathcal{B}_{k,t}}} \leq \frac{|z|^L\widetilde{R}^L}{C}$ and*

$$\|\mathcal{N}_{k,L}(z)|_{Ker(\Pi_{k,t})\cap Ker(\Pi_{k+1,t}\circ d_k)}\|_{\mathcal{A}_{k,\mathcal{B}_{k,t}},\mathcal{A}_{k+1,\mathcal{B}_{k+1,t}}} \leq \frac{|z|^L\widetilde{R}^L}{C}\,.$$

Up to taking a slightly larger value of $\widetilde{R}$, we may assume that L does not depend on ξ.

Proof of Lemma 5. Formally,

$$\mathcal{M}_{k,\ell}(z) = \mathrm{Id} - (\mathrm{Id} - z\mathcal{M}_k)\exp[-\log(\mathrm{Id} - z\mathcal{M}_k) - \sum_{j\geq \ell}\frac{z^j\mathcal{M}_k^j}{j}] = \mathrm{Id} - \exp\sum_{j\geq\ell} -\frac{z^j\mathcal{M}_k^j}{j}\,.$$

Then, we use that for any $0<\theta<1$ and $C>1$ there is L so that for all $\ell \geq L$

$$\sum_{j\geq\ell}\frac{\theta^j}{j} \leq \frac{\theta^\ell}{\ell}\sum_{j\geq 0}\theta^j \leq \frac{\theta^\ell}{\ell(1-\theta)} \leq \frac{\theta^\ell}{2C}\,, \text{ while } 1-\exp(-\frac{\theta^\ell}{2C}) \leq \frac{\theta^\ell}{C}\,.$$

This gives the bound for $\mathcal{M}_{k,\ell}(z)$ by Axiom 2(1). To estimate $\mathcal{N}_{k,\ell}(z)$ note that

$$\mathcal{N}_{k,\ell}(z) = \mathrm{Id} - \exp\sum_{j\geq\ell} -\frac{z^j\mathcal{N}_k^{(j)}}{j}\,, \tag{3.8}$$

with $\mathcal{N}_k^{(j)} = d_k \mathcal{M}_k^j - \mathcal{M}_{k+1}^j d_k$, and use Axiom 1(3). $\square$

Whenever the value of t is clear from the context, we write Π_k instead of $\Pi_{k,t}$ for simplicity. Note that $\Pi_k = \Pi_k \chi_K = \chi_K \Pi_k$. It follows from Lemma 5 that the essential spectral radius of $\mathcal{M}_{k,\ell}(z)$ on any $\mathcal{A}_{k,\mathcal{B}_{k,t}}$ is not larger than $|z|^\ell \widetilde{R}^\ell < 1$ for all ℓ and every $|z| < 1/\widetilde{R}$. It is then an easy algebraic exercise to see that for any $|z| < 1/\widetilde{R}$, the complex number $1/z$ is an eigenvalue of $\mathcal{M}_k$ (on $\mathcal{A}_{k,\mathcal{B}_{k,t}}$) of algebraic multiplicity m if and only 1 is an eigenvalue of multiplicity m for $\mathcal{M}_{k,\ell}(z)$ (on $\mathcal{A}_{k,\mathcal{B}_{k,t}}$), and in particular, for all ℓ, setting $R = \widetilde{R}^{1/n}$,

$$V_{k,t} = \{|z| < 1/R \mid 1/z \notin \operatorname{sp} \mathcal{M}_k\} = \{|z| < 1/R \mid 1 \notin \operatorname{sp} \mathcal{M}_{k,\ell}(z)\} . \tag{3.7}$$

(Just use that $\mathcal{M}_k$ commutes with each $\exp(z^j \mathcal{M}_k^j / j)$.) One can use the same basis of generalised eigenvectors for both eigenspaces, and Π_k commutes with each $\mathcal{M}_{k,\ell}$. (See e.g. [GGK] for analogous results in the case when a power of $\mathcal{M}_k$ is Hilbert-Schmidt.)

We shall work with the regularised kneading operators (for suitably large ℓ)

$$\mathcal{D}_{k,\ell}(z) = \mathcal{N}_{k,\ell}(z)(\mathrm{Id} - \mathcal{M}_{k,\ell}(z))^{-1} \mathcal{S}_k . \tag{3.9}$$

We explain next why Lemma 1 and a modified version of Theorem 2 hold for the $\mathcal{D}_{k,\ell}(z)$. The replacement of $z\mathcal{M}_k$ and $z\mathcal{N}_k$ by $\mathcal{N}_{k,\ell}(z)$, $\mathcal{M}_{k,\ell}(z)$ does not cause any problems in Baillif's [Bai, Ba2] proof of Lemma 1, since $\mathcal{M}_{k,\ell}(z)$ is just an entire series with coefficient transfer operators (acting on $\mathcal{A}_{k,L^q(K')}$). However, since (2.12) implies

$$\mathrm{Det}^\flat(\mathrm{Id} - \mathcal{M}_{k,\ell}(z)) = \mathrm{Det}^\flat(\mathrm{Id} - z\mathcal{M}_k) \exp\Big(\sum_{j=1}^{\ell-1} \frac{z^j}{j} \mathrm{tr}^\flat \mathcal{M}_k^j\Big) ,$$

the first equality in Theorem 2 must be replaced by

$$\mathrm{Det}^\#(\mathrm{Id} - z\mathcal{M}) \exp\Big(\sum_{j=1}^{\ell-1} \frac{z^j}{j} \mathrm{tr}^\# \mathcal{M}^j\Big) = \prod_{k=0}^{n-1} \mathrm{Det}^\flat(\mathrm{Id} + \mathcal{D}_{k,\ell}(z))^{(-1)^{k+1}} . \tag{3.10}$$

The additional factor $\exp \sum_{j=1}^{\ell-1} \frac{z^j}{j} \mathrm{tr}^\# \mathcal{M}^j$ is clearly an entire and non vanishing function of z for each ℓ. The final useful property of the regularised kneading determinants is:

Corollary of Lemma 5. *Assume Axioms 1 and 2. For each $z \in V_{k,t}$ (see (3.7)) and all $0 \le k \le n-1$, $1 < t < \infty$, the essential spectral radius of $\mathcal{D}_{k,L}(z)$ on $\mathcal{A}_{k+1,\mathcal{B}_{k+1,t}}$ goes to zero exponentially fast as $L \to \infty$, in particular it is < 1 for large L.*

Proof of the Corollary. Let $\widetilde{\Pi}_k$ be a projector onto the finite-dimensional space $\{\psi \in (\mathrm{Id} - \Pi_k)(\mathcal{A}_{k,\mathcal{B}_{k,t}}) \mid d_k \psi \in \mathrm{Im}\, \Pi_{k+1}\}$. Use $\Pi_k \mathcal{M}_{k,L}(z) = \mathcal{M}_{k,L}(z) \Pi_k$ to get the decomposition

$$\begin{aligned}\mathcal{D}_{k,L}(z) &= \mathcal{N}_{k,L}(z)(\mathrm{Id} - \widetilde{\Pi}_k)(\mathrm{Id} - \mathcal{M}_{k,L}(z))^{-1}(\mathrm{Id} - \Pi_k)\mathcal{S}_k \\ &\quad + \mathcal{N}_{k,L}(z)\widetilde{\Pi}_k(\mathrm{Id} - \mathcal{M}_{k,L}(z))^{-1}(\mathrm{Id} - \Pi_k)\mathcal{S}_k + \mathcal{N}_{k,L}(z)(\mathrm{Id} - \mathcal{M}_{k,L}(z))^{-1}\Pi_k \mathcal{S}_k\end{aligned}$$

into an operator of arbitrarily small spectral radius (by $|z| < 1/R$ and Lemma 5) and a finite rank operator (since $z \in V_{k,t}$). $\square$

Operators $\mathcal{D}^{(r)}_{k,L}(z)$

We next introduce auxiliary operators $\mathcal{D}^{(r)}_{k,L}(z)$ on $\mathcal{A}_{k+1,L^2(K')}$, extending a one-dimensional construction of Ruelle [Ru4]. Their iterates will be trace-class, and their traces will coincide with the formal flat traces of iterates of the $\mathcal{D}_{k,L}(z)$ in the sense of power series. This will allow us to prove the following crucial lemma:

Lemma 6 (Meromorphic extension of $\mathrm{Det}^\flat(\mathrm{Id} + \mathcal{D}_{k,L}(z))$). *Assume Axioms 1–2 and 3. Set $\mathcal{B}_k = \mathcal{B}_{k,2}$ for $k = 0, \ldots, n$. For $k = 0, \ldots, n-1$ and all large enough L:*
(1) $\mathrm{Det}^\flat(Id + \mathcal{D}_{k,L}(z))$ extends holomorphically to

$$V_k = \{|z| < 1/R \mid 1/z \notin \mathrm{sp}(\mathcal{M}_k|_{\mathcal{A}_{k,\mathcal{B}_k}})\} = \{|z| < 1/R \mid 1 \notin \mathrm{sp}(\mathcal{M}_{k,\ell}(z)|_{\mathcal{A}_{k,\mathcal{B}_k}})\} .$$

(2) If $|z| < 1/R$ and $1/z \in \mathrm{sp}(\mathcal{M}_k|_{\mathcal{A}_{k,\mathcal{B}_k}})$, then $\mathrm{Det}^\flat(Id + \mathcal{D}_{k,L}(z))$ is meromorphic at z with a pole of order at most the algebraic multiplicity of the eigenvalue.

Proof of Lemma 6. Recall $\sigma^{(r)}_k(x, y)$ and $\mathcal{S}^{(r)}_k$ from Axiom 1, and that $\sigma^{(r)}_k(\cdot, y) \in \mathcal{A}_{k,\mathcal{B}_{k,t}}$, for $1 \le t < n/(n-1)$ and $y \in K$. Set:

$$\mathcal{D}^{(r)}_{k,L}(z) = v_k(\widetilde{\mathcal{J}}_k)^{-1}\mathcal{N}_{k,L}(z)(\mathrm{Id} - \mathcal{M}_{k,L}(z))^{-1}\mathcal{S}^{(r)}_k .$$

Axioms 1 and 2 first imply that $z \mapsto \mathcal{D}^{(r)}_{k,L}(z)$ is a well-defined map (taking values in the space of bounded linear operators on $\mathcal{A}_{k+1,L^2(K')}$), holomorphic in V_k, and meromorphic in the disc of radius $1/\widetilde{R}$, with possible poles at the inverse eigenvalues of $\mathcal{M}_k$ on $\mathcal{A}_{k,\mathcal{B}_{k,2}}$ (the order of the pole being at most the algebraic multiplicity of the eigenvalue). We use here that, by Axiom 1(2-3), $\mathcal{N}_{k,L}(z)$ is a z-entire function, bounded from $\mathcal{A}_{k,\mathcal{B}_{k,t}}$ to $\mathcal{A}_{k+1,\mathcal{B}_{k+1,t}}$ for all $1 < t < \infty$.

The kernel of $\mathcal{D}^{(r)}_{k,L}(z)$ is

$$\mathcal{K}^{\mathcal{D}}(x, y) = v_k(\widetilde{\mathcal{J}}_{x,k})^{-1}\mathcal{N}_{x,k,L}(z)(\mathrm{Id} - \mathcal{M}_{x,k,L}(z))^{-1}\sigma^{(r)}_k(x, y) \tag{3.11}$$

(where the transfer operators act on the x-variable). The coefficients of the kernel $\mathcal{K}^{\mathcal{D}}(x, y)$ have no reason to be in $L^2(K \times K)$, but Axioms 1 and 2 imply that the coefficients of $\mathcal{K}^{\mathcal{D}}(\cdot, y)$ are in $L^t(K')$ for all $1 \le t < n/(n-1)$, with the supremum over $y \in K$ of the L^t norm bounded.

The proof of Lemma 6.2 in [Bai] shows that for $z \in V_k$, and all $\ell > n/2$ the operator $(\mathcal{D}^{(r)}_{k,L}(z))^\ell$ is Hilbert-Schmidt on $\mathcal{A}_{k+1,L^2(K')}$. In particular, the regularised determinant of order $[n/2] + 1$,

$$\mathrm{Det}^{\,\mathrm{reg}}_{[n/2]+1}(\mathrm{Id} + \mathcal{D}^{(r)}_{k,L}(z)) = \exp - \sum_{\ell=[n/2]+1}^{\infty} \frac{z^\ell}{\ell}\mathrm{tr}^\flat(\mathcal{D}^{(r)}_{k,L}(z))^\ell ,$$

is holomorphic on V_k. Indeed, $(\mathrm{Id} - \mathcal{M}_{k,L}(z))^{-1}$ is holomorphic in V_k as a bounded operator on k-forms with coefficients in $\mathcal{B}_{k,t}$, all t, and in particular for all values of t between $t_0 < n/(n-1)$ and $t_{[n/2]+1} \geq 2$ which appear along the successive iterations.

By (3.3)

$$\mathcal{D}^{(r)}_{k,L}(z) = (\widetilde{\mathcal{J}}_k)^{-1}\mathcal{N}_{k,L}(z)(\mathrm{Id} - \mathcal{M}_{k,L}(z))^{-1}\mathcal{S}_k\widetilde{\mathcal{J}}_k\,. \tag{3.12}$$

Recall that the wave front set of (the Schwartz kernel of) a pseudo-differential operator is included in $\{(x,x,\xi,-\xi) \mid \xi \neq 0\}$ and that composition with a pseudodifferential operator does not enlarge the wave-front set (see e.g. [AG]). Hence, using the assumptions on $\widetilde{\mathcal{J}}_k$, $(\widetilde{\mathcal{J}}_k)^{-1}$, a convolution with a C^∞ mollifier, the transversality property (2.1) to restrict to the diagonal, and Fubini (see Section 4 of [Bai]), the following equalities between formal power series hold

$$\mathrm{Det}^{\flat}(\mathrm{Id} + \mathcal{D}^{(r)}_{k,L}(z)) = \mathrm{Det}^{\flat}(\mathrm{Id} + \mathcal{D}_{k,L}(z))\,,\ \forall\, k = 0, \dots n-1\,. \tag{3.13}$$

To show that the full flat determinant $\mathrm{Det}^{\flat}(\mathrm{Id} + \mathcal{D}_{k,L}(z))$ is holomorphic in V_k, we shall use Axiom 3 to see that the power series for $\mathrm{tr}^{\flat}(\mathcal{D}_{k,L}(z))^{\ell}$ for each $1 \leq \ell \leq [n/2]$ is holomorphic on V_k (its exponential is thus holomorphic and nonvanishing). We consider the (hardest) case, i.e., $\ell = 1$, leaving higher iterates to the reader. Also, we assume for simplicity that $k = 0$ (dealing with k-forms only introduces notational complications).

We first show that the flat trace of $(\widetilde{\mathcal{J}}_0)^{-1}\mathcal{N}_{0,L}(z)(\mathrm{Id} - \mathcal{M}_{0,L}(z))^{-1}(\mathrm{Id} - \Pi_0)\mathcal{S}_0^{(r)}$ is holomorphic in the disc of radius $1/R$. Let $\mathcal{T}$ denote the restriction to the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$. For each $\mathcal{E} < \mathcal{E}_0$, let $W_{\mathcal{E}}$ be the set of $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $\|x - y\| < \mathcal{E}$ and let $\chi^{\mathcal{E}}(x-y) + (1 - \chi^{\mathcal{E}}(x-y))$ be a C^∞ radial partition of unity of $\mathbb{R}^n \times \mathbb{R}^n$ subordinated to $W_{\mathcal{E}}$ in the sense that $\chi_{\mathcal{E}}(u)$ depends only on $\|u\|$, $\chi^{\mathcal{E}}(u) = 1$ if $\|u\| \leq \mathcal{E}$, and $\chi^{\mathcal{E}}(u) = 0$ if $\|u\| \geq 2\mathcal{E}$. Let δ_ϵ be C^∞ mollifiers (converging to the Dirac mass in $\mathbb{R}^n$ as $\epsilon \to 0$), and write $\sigma^{\epsilon,<\mathcal{E}}$ for the convolution of δ_ϵ with $\chi^{\mathcal{E}}\sigma_0$ and $\sigma^{\epsilon,>\mathcal{E}}$ for the convolution of δ_ϵ with $(1 - \chi^{\mathcal{E}})\sigma_0$

Using the identity $(\mathrm{Id} - \mathcal{M}_{k,L}(z))^{-1} = \exp -[\sum_{i=0}^{L-1} z^i \mathcal{M}_k^i] \cdot (1 - z\mathcal{M}_k)^{-1}$, it suffices to show that there is C so that for all j there is $\mathcal{E}$ so that for each $\epsilon > 0$, all $|z| < 1/R$,

$$\int_x |\mathcal{T}[(\delta_\epsilon \star (\widetilde{\mathcal{J}}_0^{-1}\mathcal{N}_{0,L}(z)(\exp(-\sum_{i=0}^{L-1} z^i \mathcal{M}_0^i)\mathcal{M}_0^j(\mathrm{Id} - \Pi_0)\widetilde{\mathcal{J}}_{x,0}(\chi_K\sigma^{\epsilon,>\mathcal{E}})(x,y))]|dx$$
$$+ \int_x \mathcal{T}|[(\delta_\epsilon \star (\mathcal{N}_{0,L}(z)(\exp(-\sum_{i=0}^{L-1} z^i \mathcal{M}_0^i)\mathcal{M}_0^j(\mathrm{Id} - \Pi_0)\sigma^{\epsilon,<\mathcal{E}})(x,y))]|dx \leq CR^j\,.$$

The entire series $\mathcal{N}_{0,L}(z)\exp(-\sum_{i=0}^{L-1} z^i \mathcal{M}_0^i) = \sum_{q\geq 1} z^q \mathcal{Q}_q$ has coefficients

$$\mathcal{Q}_q = \sum_{q_1+q_2=q} \kappa_{q_1,q_2}\mathcal{N}_0^{(q_1)}\mathcal{M}_0^{q_2} : \mathcal{A}_{0,\mathcal{B}_0} \to \mathcal{A}_{1,\mathcal{B}_1}\,, \quad \kappa_{q_1,q_2} \in \mathbb{R}\,,$$

so that for any $\widehat{R}$ there is C so that $|\kappa_{q_1,q_2}| < C\widehat{R}^{q_1+q_2}$. We may write $\mathcal{N}_0^{(q_1)}\mathcal{M}_0^{q_2}\varphi = \sum_{\vec{\omega}\in\Omega^{q_1+q_2}} h_{q_1,q_2,\vec{\omega}} \wedge (\psi_{\vec{\omega}}^{q_1+q_2})^*\varphi$ where the $h_{q_1,q_2,\vec{\omega}}$ are one-forms with C^{r-1} coefficients supported in the domains of the $\psi_{\vec{\omega}}$.

Write $\sigma^{\epsilon,<,>\mathcal{E}}(x,y) = \sum_m \tau_m^{\epsilon,<,>\mathcal{E}}(x-y)$. Fix j and consider the (hypothetical) case $q_2 = q$. The modulus of the contribution of $\mathcal{M}_0^{q_2}\mathcal{M}_0^j$ may be bounded by the sum over m of

$$\left|\int \mathcal{T}[\delta_\epsilon \star \left(\mathcal{M}_{x,0}^{j+q}(\tau_m^{\epsilon,<\mathcal{E}})(x-y)\right)]\,dx\right| + \left|\int \mathcal{T}[\delta_\epsilon \star \left(\mathcal{M}_{x,0}^{j+q}\Pi_{x,0}(\tau_m^{\epsilon,<\mathcal{E}})(x-y)\right)]\,dx\right|$$
$$+\left|\int \mathcal{T}[\delta_\epsilon \star \left(\widetilde{\mathcal{J}}_0^{-1}\mathcal{M}_{x,0}^{j+q}(\mathrm{Id}-\Pi_{x,0})\widetilde{\mathcal{J}}_0(1_K\tau_m^{\epsilon,>\mathcal{E}})(x-y)\right)]\,dx\right|. \tag{3.15}$$

Recall that for $\mathcal{E} < \mathcal{E}_0$ as in Axiom 3, $V_{\vec{\omega},\mathcal{E}}$ is the set of x so that $\|\psi_{\vec{\omega}}^{j+q}(x) - x\| < \mathcal{E}$. By Axiom 3, writing $\Phi_{\vec{\omega}}$ for the inverse of $\psi_{\vec{\omega}}^{j+q} - \mathrm{Id}$ on $V_{\vec{\omega},\mathcal{E}}$, and performing the corresponding change of variable, the first term in the above sum is bounded by

$$\left|\sum_{\vec{\omega}}\int_{(\psi_{\vec{\omega}}^{j+q}-\mathrm{Id}\,)V_{\vec{\omega},\mathcal{E}}} \frac{g_{\vec{\omega}}^{(j+q)}(u+\Phi_{\vec{\omega}}(u))}{|\det(D\psi_{\vec{\omega}}^{j+q}(\Phi_{\vec{\omega}}(u)) - \mathrm{Id}\,)|}(\tau_m^{\epsilon,<\mathcal{E}})(u)\,du\right|$$
$$\le \sum_{\vec{\omega}} \sup_{V_{\vec{\omega},\mathcal{E}}} \frac{|g_{\vec{\omega}}^{(j+q)} \circ \psi_{\vec{\omega}}^{j+q}|}{|\det(D\psi_{\vec{\omega}}^{j+q}(x) - \mathrm{Id}\,)|} \cdot \int |\tau_m^{\epsilon,<\mathcal{E}}(u)|\,du \le C\exp(\eta(q+j))\mathcal{E}\,.$$

By the consequences of Axiom 1, for all x, y in K, we have $\widetilde{\mathcal{J}}_0((1_K\tau_m^{\epsilon,<\mathcal{E}})(\cdot - y))(x) = \tilde{v}_k\widetilde{\mathcal{J}}_0((1_K\tau_m^{\epsilon,<\mathcal{E}})(x-\cdot))(y)$ for $\tilde{v}_k \in \{+1,-1\}$. Hence, for the second term, use (3.4) to get (in the case where all eigenvalues are semisimple)

$$\left|\int \mathcal{T}[\delta_\epsilon \star \sum_s \lambda_s^{j+q}\varphi_s(x)\nu_s(\widetilde{\mathcal{J}}_{y,0}^{-1}\widetilde{\mathcal{J}}_{0,\cdot}(1_K\tau_m^{\epsilon,<\mathcal{E}})(\cdot - y))]\,dx\right|$$
$$\le \left|\int \mathcal{T}[\delta_\epsilon \star \sum_s \lambda_s^{j+q}\varphi_s(x)\widetilde{\mathcal{J}}_{y,0}^{-1}(\nu_s(\widetilde{\mathcal{J}}_{0,\cdot}(1_K\tau_m^{\epsilon,<\mathcal{E}})(\cdot - y)))]\,dx\right|$$
$$\le \sum_s \Big|\int_K \lambda_s^{j+q}\varphi_s(x)\widetilde{\mathcal{J}}_{x,0}^{-1}(\nu_s(\widetilde{\mathcal{J}}_{0,\cdot}(1_K\tau_m^{\epsilon,<\mathcal{E}})(\cdot - x)))\,dx\Big|$$
$$= \sum_s \Big|\int_K \lambda_s^{j+q}\widetilde{\mathcal{J}}_0^{-1}(\varphi_s)\nu_s(\widetilde{\mathcal{J}}_{0,\cdot}(1_K\chi^{\mathcal{E}}\cdot\tau_m^{\epsilon})(\cdot - x))\,dx\Big|$$
$$\le C\sum_s \sup|\widetilde{\mathcal{J}}_0^{-1}(\varphi_s)| \cdot \sup_x \|\widetilde{\mathcal{J}}_{0,\cdot}(1_K\tau_m^{\epsilon,<\mathcal{E}})(\cdot - x))\|_{\mathcal{B}_{0,t}}$$
$$\le C\sup_{x\in K}\|(\tau_m^{\epsilon,<\mathcal{E}})(\cdot - x)\|_{L^t(K)} \le C\sup_x(\int |\tau_m^{\epsilon,<\mathcal{E}}(u-x)|^t\,du)^{1/t} \le C\mathcal{E}\,,$$

taking $t > 1$ close to 1 and using $|\lambda_s| \le 1$. Nilpotent contributions produce polynomial growth in $q + j$ which gives a bound $C\exp(\eta(q+j))\mathcal{E}$ for arbitrarily small η.

If $a(x,y)$ is continuous then $\int_K |a(x,x)|\,dx \le \mathrm{Vol}(K)\sup_y \sup_x |a(x,y)|$. Recall also that if $\sup_{t\to\infty} \|b(\cdot,y)\|_{L^t(K)} < B(y)$, then $b(\cdot,y)$ is in $L^\infty(K)$ and $\sup_K |b(\cdot,y)| < B(y)$. The third term may thus be estimated by (use both parts of Axiom 2)

$$
\begin{aligned}
&C \sup_{x,y\in K} |\delta_\epsilon \star (\widetilde{\mathcal{J}}_{x,0}^{-1}\mathcal{M}_{x,0}^{j+q}(\mathrm{Id}-\Pi_{x,0})\widetilde{\mathcal{J}}_{x,0}(\tau_m^{\epsilon,>\mathcal{E}})(x-y)))| \\
&\le C \sup_y \sup_{t\to\infty} \|\mathcal{M}_0^{j+q}(\mathrm{Id}-\Pi_0)\widetilde{\mathcal{J}}_0(1_K\tau_m^{\epsilon,>\mathcal{E}})(\cdot - y)\|_{\mathcal{B}_{0,t}} \\
&\le C\widetilde{R}^{j+q}\sup_{y,t}\big(\|\widetilde{\mathcal{J}}_0(1_K\tau_m^{\epsilon,>\mathcal{E}})(\cdot-y)\|_{\mathcal{B}_{0,t}} + \|\Pi_0\widetilde{\mathcal{J}}_0(1_K\tau_m^{\epsilon,>\mathcal{E}})(\cdot-y)\|_{\mathcal{B}_{0,t}}\big) \\
&\le C\widetilde{R}^{j+q}\sup_y\sup_{t\to\infty}\Bigg(\|\tau_m^{\epsilon,>\mathcal{E}}(\cdot-y)\|_{L^t(K)} \\
&\qquad\qquad + \sum_s |\nu_s[\widetilde{\mathcal{J}}_0(1_K\tau_m^{\epsilon,>\mathcal{E}})](\cdot-y)|\cdot\|\widetilde{\mathcal{J}}_0^{-1}\varphi_s\|_{L^t(K)}\Bigg) \\
&\le C\widetilde{R}^{j+q}\big(\sup_{\|u\|\ge\mathcal{E}}|\tau_m^{\epsilon,>\mathcal{E}}(u)| + C\sup_y\|1_K\tau_m^{\epsilon,>\mathcal{E}}(\cdot-y)\|_{L^{t_0}}\big) \\
&\le C\widetilde{R}^{j+q}(\mathcal{E}^{-(n-1)}+C)\,,
\end{aligned}
$$

for some $1 < t_0 < n/(n-1)$. Since $\|\mathcal{N}_0^{(q_1)}\| \le C^{q_1}$, Ω is finite, and $\|h_{q_1,q_2,\vec\omega}\| \le C^{q_1+q_2}$, the case $q_1 \ge 1$ follows from the bounds on κ_{q_1,q_2}. Choosing $\mathcal{E} = \widetilde{R}^{j/n}$ gives claim (1) (because the projection Π_0 has finite rank, see also the proof of Lemma 8 below).

If $|z| < 1/R$ is in the spectrum of $\mathcal{M}_k$ on $\mathcal{B}_k$, the argument with mollifiers described above can be used to invoke the ordinary Plemelj-Smithies formula as in Lemma 4.4.2 of [Go], to see that the determinant $\mathrm{Det}^\flat(\mathrm{Id}+\mathcal{D}_{k,L}^{(r)}(z))$ has at most a pole of order the algebraic multiplicity of the eigenvalue at z, proving claim (2). □

We next relate the zeroes of the analytic continuation of the formal determinant $\mathrm{Det}^\flat(\mathrm{Id}+\mathcal{D}_{k,L}(z))$, to the presence of an eigenvalue -1 for the operators $\mathcal{D}_{k,L}(z)$:

Lemma 7 (Zeroes of Det$^\flat$(Id + $\mathcal{D}_{k,L}(z)$)). *Assume Axioms 1, 2 and 3. For $k = 0,\dots,n-1$ and all large enough L, $\mathcal{D}_{k,L}(z)$ extends holomorphically on V_k to a family of operators on $\mathcal{A}_{k+1,\mathcal{B}_{k+1}}$. The essential spectral radius of each $\mathcal{D}_{k,L}(z)$ is < 1. If $Det^\flat(Id+\mathcal{D}_{k,L}(z)) = 0$ for $z \in V_k$, then -1 is an eigenvalue of $\mathcal{D}_{k,L}(z)$ on $\mathcal{A}_{k+1,\mathcal{B}_{k+1}}$.*

Sublemma. *Let A be a compact subset of $\mathbb{R}^n$. Let $\widetilde{\mathcal{L}}$ be a Hilbert-Schmidt operator on $L^2(A)$ written in kernel form $\widetilde{\mathcal{L}}\varphi(x) = \int \widetilde{\mathcal{K}}_{xy}\varphi(y)\,dy$.*

Let $\mathcal{L}$ be a bounded operator acting on a Banach space $\mathcal{B}$ of distributions over A containing $C^\infty(A)$. Assume that for small $\epsilon > 0$ there are C^∞ kernels $\mathcal{K}_{\epsilon,xy}$ and $\widetilde{\mathcal{K}}_{\epsilon,xy}$ so that the Fredholm determinants of the associated operators coincide

$$Det(Id-\lambda\mathcal{L}_\epsilon) = Det(Id-\lambda\widetilde{\mathcal{L}}_\epsilon)\,,\forall\epsilon\,,$$

and assume also $\lim_{\epsilon\to 0}\|\mathcal{L}_\epsilon-\mathcal{L}\|_{\mathcal{B}\to\mathcal{B}} = 0$, $\lim_{\epsilon\to 0}|\widetilde{\mathcal{K}}_{\epsilon,xy}-\widetilde{\mathcal{K}}_{xy}|_{L^2(K\times K)} = 0$.

Then, for all λ with $1/|\lambda| < \rho_{ess}(\mathcal{L})$, writing

$$Det_2^{reg}(Id - \lambda\widetilde{\mathcal{L}}) = \exp - \sum_{\ell=2}^{\infty} \frac{\lambda^\ell}{\ell} tr\,\widetilde{\mathcal{L}}^\ell \,,$$

we have $Det_2^{reg}(Id - \lambda\widetilde{\mathcal{L}}) = 0 \Longrightarrow \lambda^{-1}$ is an eigenvalue of $\mathcal{L}$ on $\mathcal{B}$.

Proof of the Sublemma. The statements in this proof hold for all $1/|\lambda| < \rho_{ess}(\mathcal{L})$ (uniformly in any compact subset). $\mathrm{Det}\,(\mathrm{Id} - \lambda\mathcal{L}_\epsilon) = \mathrm{Det}\,(\mathrm{Id} - \lambda\widetilde{\mathcal{L}}_\epsilon)$ vanishes if and only if $1/\lambda$ is an eigenvalue of $\mathcal{L}_\epsilon$ (on $L^2(A)$, or equivalently on $C^\infty(A)$, using that the image of an element of $L^2(A)$ by an operator with C^∞ kernel is C^∞), using the l.h.s., if and only if $1/\lambda$ is an eigenvalue of $\widetilde{\mathcal{L}}_\epsilon$ (on $L^2(A)$ or equivalently on $C^\infty(A)$) using the r.h.s. The convergence of the kernels $\widetilde{\mathcal{K}}_{\epsilon,xy}$ implies both that $\mathrm{Det}_2^{\mathrm{reg}}(\mathrm{Id} - \lambda\widetilde{\mathcal{L}}_\epsilon)$ converges to $\mathrm{Det}_2^{\mathrm{reg}}(\mathrm{Id} - \lambda\widetilde{\mathcal{L}})$ (which is entire in λ), and that $\widetilde{\mathcal{L}}_\epsilon$ on $L^2(A)$ converges to the compact operator $\widetilde{\mathcal{L}}$ on $L^2(A)$. Thus every zero λ_0 of $\mathrm{Det}_2^{\mathrm{reg}}(\mathrm{Id} - \lambda\widetilde{\mathcal{L}})$ is a limit of λ_ϵ so that $1/\lambda_\epsilon$ is an eigenvalue of $\widetilde{\mathcal{L}}_\epsilon$ (on $L^2(A)$). (Indeed, such a zero corresponds to $1/\lambda_0$ being an eigenvalue of $\widetilde{\mathcal{L}}$.) By the first observation, $1/\lambda_\epsilon$ is an eigenvalue of $\mathcal{L}_\epsilon$ (on $C^\infty(A)$, so that the eigenfunction is in $\mathcal{B}$). If $|\lambda_0| < 1/\rho_{ess}(\mathcal{L})$, since $\mathcal{L}_\epsilon$ converges to $\mathcal{L}$ in operator norm on $\mathcal{B}$, if $1/\lambda_\epsilon$ is an eigenvalue of $\mathcal{L}_\epsilon$ for all ϵ then λ_0 is an eigenvalue of $\mathcal{L}$ (for an eigenvector which is the $\mathcal{B}$-norm limit of $\varphi_\epsilon \in C^\infty(A)$). $\square$

Proof of Lemma 7. We have shown that the left-hand-side of (3.13) extends holomorphically to V_k. By definition and the Axioms the operator $\mathcal{D}_{k,L}(z)$ extends holomorphically to V_k in the sense of bounded operators on $\mathcal{A}_{k+1,\mathcal{B}_{k+1}}$. By the corollary of Lemma 5, it is enough to show the claim about the zeroes of the determinant.

Taking iterates to get Hilbert-Schmidt operators on $\mathcal{A}_{k+1,L^2(K')}$, one can apply the sublemma to show that if $z \in V_k$ is such that $\mathrm{Det}^\flat(\mathrm{Id} + \mathcal{D}_{k,L}(z)) = \mathrm{Det}^\flat(\mathrm{Id} + \mathcal{D}^{(r)}_{k,L}(z)) = 0$ then -1 is an eigenvalue of $\mathcal{D}_{k,L}(z)$ on $\mathcal{A}_{k+1,\mathcal{B}_{k+1}}$.

More precisely, for fixed $z \in V_k$, set $\mathcal{L} = (\mathcal{D}_{k,L}(z))^m$, $\quad \tilde{\mathcal{L}} = (\mathcal{D}^{(r)}_{k,L}(z))^m$. Then, to construct the smooth kernels required by the assumptions of the sublemma, we (again) smoothen $\mathcal{D}_{k,L}(z)$ by pre and post-convolution with a C^∞ mollifier δ_ϵ, writing $\mathcal{D}^\epsilon_k(z) = \delta_\epsilon \mathcal{D}_{k,L}(z)\delta_\epsilon$ for the new operator. If we mollify $\mathcal{D}^{(r)}_{k,L}(z)$ similarly, the equality (3.13) between determinants remains true, and the kernels of $\mathcal{D}^{\epsilon,(r)}_k(z)$ converge as $\epsilon \to 0$, in the $L^2(K \times K)$ topology. To see that $\mathcal{D}^\epsilon_k(z)$ converges to $\mathcal{D}_{k,L}(z)$, in the sense of operators on $\mathcal{A}_{k+1,\mathcal{B}_{k+1}}$, use that δ_ϵ converges to Id in the $\mathcal{A}_{k+1,\mathcal{B}_{k+1}}$ topology. $\square$

A modification of the homotopy operators

We next explain how to exploit Lemmas 6–7. We first discuss the effect of making a suitable finite-rank perturbation of $\mathcal{S}$ in the definition of the kneading operators:

Lemma 8 (Perturbing the homotopy operators). *Assume Axioms 1 and 2. Let $\mathcal{S}'$ be a finite-rank perturbation of $\mathcal{S}$ (i.e., $\mathcal{S}'_k - \mathcal{S}_k$ is finite-rank from $\mathcal{A}_{k+1,\mathcal{B}_{k+1}}$ to $\mathcal{A}_{k,\mathcal{B}_k}$), so that $d\mathcal{S}' + \mathcal{S}'d = Id$ and $(\mathcal{S}')^2 = 0$. Then the statements of Lemma 1 and* *the modified equality* (3.10) *from Theorem 2 remain true for* $\mathcal{S}'$ *and for the perturbed kneading operators* $\mathcal{D}'_{k,L}(z)$ *defined by*

$$\mathcal{D}'_{k,L}(z) = \mathcal{N}_{k,L}(z)(Id - \mathcal{M}_{k,L}(z))^{-1}\mathcal{S}'_k \, , k = 0, \dots , n-1 \, . \tag{3.16}$$

Proof of Lemma 8. Since $d\mathcal{S}' + \mathcal{S}'d = \mathrm{Id}$, Baillif's proof extends relatively straightforwardly to the modified convolution operators $\mathcal{S}'$. Indeed, we may use the decomposition

$$\mathcal{D}'_{k,L}(z) = \mathcal{D}_{k,L}(z) + (\mathcal{D}'_{k,L}(z) - \mathcal{D}_{k,L}(z)) = \mathcal{D}_{k,L}(z) + \mathcal{F}_{k,L}(z)\, ,$$

into a power series with coefficients operators whose Schwartz kernel is well-behaved along the diagonal (in particular the flat trace exists, and Fubini is allowed, so that all desired commutations hold [Bai]), summed with a power series whose coefficients are finite-rank operators. These finite-rank operators act on $\mathcal{A}_{k+1,\mathcal{B}_{k+1}}$ and each coefficient of $\mathcal{D}_{k,L}(z)$ is bounded on this space. Hence, we may define the formal traces and determinants (as power series only) by setting

$$\mathrm{tr}^\flat(\mathcal{D}_{k,L}(z) + \mathcal{F}_{k,L}(z))^m = \mathrm{tr}^\flat\mathcal{D}_{k,L}(z)^m + \mathrm{tr}^\flat\mathcal{F}_{m,k,L}(z)\, ,$$

where $\mathcal{F}_{m,k,L}(z)$ is a power series with coefficients finite-rank operators on $\mathcal{A}_{k+1,\mathcal{B}_{k+1}}$, for which the flat trace is just defined to be the sum of eigenvalues.

In fact, since $\mathcal{D}'_{k,L}(z) - \mathcal{D}_{k,L}(z)$ starts with a precomposition by $\chi_K(\mathcal{S}'_k - \mathcal{S}_k)$, which is finite-rank from $\mathcal{A}_{k+1,\mathcal{B}_{k+1}}$ to $\mathcal{A}_{k,\mathcal{B}_k}$, the resolvent factor $(\mathrm{Id} - \mathcal{M}_{k,L}(z))^{-1}$ is well defined if $z \in V_k$. Thus, the power series $\mathrm{tr}^\flat\mathcal{F}_{m,k,L}(z)$ defines in fact a holomorphic function in V_k. (This argument will be useful in the proof of Lemma 10 below.) □

Lemma 9 (Adapted homotopy operators $\mathcal{S}'$). *Assume Axioms 1 and 2, and suppose that all eigenvalues of modulus* $> R$ *of the* $\mathcal{M}_k$ *on* $\mathcal{A}_{k,\mathcal{B}_k}$ *(*$k \le n-1$*) are simple, and that the corresponding eigenvectors* φ_k *do not belong to* $Ker d = Im d$ *or to* $Ker\mathcal{N}_k$*. Then there are finite-rank operators*

$$\begin{aligned}&\mathcal{F}_k : \mathcal{A}_{k+1,\mathcal{B}_{k+1}} \to \{\varphi \in \mathcal{A}_k \mid \chi_K\varphi \in \mathcal{A}_{k,\mathcal{B}_k}\}\, , \quad 0 \le k \le n-1\, ,\\ &\textit{so that the perturbed operators} \qquad \mathcal{S}'_k = \mathcal{S}_k + \mathcal{F}_k\, ,\end{aligned}$$

satisfy $\mathcal{S}'\mathcal{S}' = 0$, $\mathcal{S}'d + d\mathcal{S}' = 1$ *and, additionally, if* $0 \le k \le \min(s, n-1)$ *and* $1/z$ *with* $|z| > R$ *is an eigenvalue of* $\mathcal{M}_k$ *on* $\mathcal{A}_{k,\mathcal{B}_k}$ *with corresponding (simple) fixed vector* $\varphi_k(z)$ *for* $\mathcal{M}_{k,\ell}(z)$ *and dual fixed vector* $\nu_k(z)$*, with* $\nu_k(z)(\varphi_k(z)) = 1$*, then for large enough* ℓ

$$|\nu_k(z)(\chi_K\mathcal{S}'_k\mathcal{N}_{k,\ell}\varphi_k(z))| > 0\, .$$

Proof of Lemma 9. For each k we consider the finite set of "bad pairs" (z, φ_k), with $z = z_{k,i} \in \mathbb{C}$ and $\varphi_{k,i}$ an fixed vector such that $\nu_{k,i}(z)(\chi_K\mathcal{S}_k\mathcal{N}_{k,\ell}\varphi_{k,i}(z)) = 0$. We write the argument assuming that this set is either empty or a singleton $\{(z, \varphi_k)\}$ for each

k in order to simplify notation. Our assumptions imply that $\upsilon_k = \nu_k(\mathcal{S}_k d\varphi_k) \neq 0$ and $\mathcal{N}_{k,\ell}(z)\varphi_k(z) \neq 0$ for all ℓ. For each k so that a bad pair (z, φ_k) exists, set $\varphi'_k = \varphi_k$ and let ν'_k, ν'_{k+1} be continuous functionals on $\mathcal{A}_{k,\mathcal{B}_k}$, $\mathcal{A}_{k+1,\mathcal{B}_{k+1}}$, respectively, which satisfy $\nu'_k \circ d = 0$, $\nu'_{k+1} \circ d = 0$, and

$$\alpha_{k,\ell} = \nu'_{k+1}(\mathcal{N}_{k,\ell}\varphi_k) = -\nu'_{k+1}(\mathcal{M}_{k+1,\ell} d\varphi_k) \neq 0\,, \forall \ell\,, \quad \beta_k = \nu'_k(\varphi_k) \neq 0\,.$$

If there is no φ_k we set $\varphi'_k = 0$ and $\nu'_{k+1} = 0$ and allow α_k and $\beta_{k,\ell}$ to vanish. We put, for complex ϵ of small modulus, and all k

$$\mathcal{F}'_k(\varphi) = -\epsilon d\varphi'_{k-1} \cdot \nu'_k(\mathcal{S}_k\varphi) + \epsilon\varphi'_k \cdot \nu'_{k+1}(\varphi)\,.$$

We have $d\mathcal{F}'_{k-1} = -\mathcal{F}'_k d$ so that $d(\mathcal{S} + \mathcal{F}') + (\mathcal{S} + \mathcal{F}')d = \mathrm{Id}$. Thus, setting $\mathcal{F} = \mathcal{F}' - d\mathcal{S}\mathcal{F}' - d\mathcal{F}'\mathcal{S} - d\mathcal{F}'\mathcal{F}'$ we have $d\mathcal{S}' + \mathcal{S}'d = \mathrm{Id}$ and $\mathcal{S}'\mathcal{S}' = 0$. Finally, for each k with a bad pair, we find for uncountably many small values of ϵ and all large enough ℓ

$$\begin{aligned}\nu_k((\mathcal{S}'_k\mathcal{N}_{k,\ell})\varphi_k) = \epsilon\Big(&\nu_k(\mathcal{S}_k d\varphi_k)\cdot\nu'_{k+1}(\mathcal{N}_{k,\ell}(z)\varphi_k)\\ &- \nu_k(d\varphi_{k-1})\cdot\big[\nu'_k(\mathcal{S}_k\mathcal{N}_{k,\ell}(z)\varphi_k) + \epsilon\nu'_k(\varphi_k)\nu'_{k+1}(\mathcal{N}_{k,\ell}(z)\varphi_k)\big]\Big)\\ = \epsilon\big(&\upsilon_k\alpha_{k,\ell} + \nu_k(d\varphi_{k-1})(\nu'_k(\mathcal{S}_k\mathcal{N}_{k,\ell}(z)\varphi_k) + \epsilon\beta_k\alpha_{k,\ell})\big) \neq 0\,.\end{aligned}$$

By taking ϵ small enough we may ensure that no new bad pairs are created. $\square$

Replacing $\mathcal{S}$ by the finite rank perturbation $\mathcal{S}'$, we get a stronger version of Lemmas 6–7:

Lemma 10 (Meromorphic extension – guaranteeing poles). *Under the assumptions of Lemma 9, and up to taking a larger value of L, for each $k = 0, \dots, n-1$, Axiom 3 implies:*
(1) The power series $Det^\flat(Id + \mathcal{D}'_{k,L}(z))$ defines a holomorphic function in V_k.
(2) If $|z_0| < 1/R$ is such that $1/z_0$ is a simple eigenvalue of $\mathcal{M}_k|_{\mathcal{A}_{k,\mathcal{B}_k}}$, then $Det^\flat(Id + \mathcal{D}'_k(z))$ is meromorphic at z_0, with a pole of order exactly one.
(3) $\mathcal{D}'_{k,L}(z)$ extends holomorphically on V_k to a family of operators on $\mathcal{A}_{k+1,\mathcal{B}_{k+1}}$, each such $\mathcal{D}'_{k,L}(z)$ has essential spectral radius strictly less than 1, and $Det^\flat(Id+\mathcal{D}'_{k,L}(z)) = 0$ if and only if $\mathcal{D}'_{k,L}(z)$ has an eigenvalue -1 on $\mathcal{A}_{k+1,\mathcal{B}_{k+1}}$.

Proof of Lemma 10. From the proof of Lemma 6, $\mathcal{D}^{(r)}_k(z)$ has its $[n/2]+1$th iterate trace-class on $\mathcal{A}_{k+1,L^2(K')}$ for all $z \in V_k$. Using Axioms 1–2, for each $z \in V_k$,

$$\mathcal{D}^{(r)'}_{k,L}(z) = \mathcal{D}^{(r)}_{k,L}(z) + (\widetilde{\mathcal{J}}_k)^{-1}\mathcal{N}_{k,L}(z)(\mathrm{Id} - \mathcal{M}_{k,L}(z))^{-1}(\mathcal{S}'_k - \mathcal{S}_k)\widetilde{\mathcal{J}}_k$$

is such that its $[n/2]+1$th iterate is a finite-rank perturbation on $\mathcal{A}_{k+1,L^2(K')}$ of $\mathcal{D}^{(r)}_{k,L}(z)$. The sum of the $L^2(K')$ (or, equivalently, flat) trace of $(\mathcal{D}^{(r)}_{k,L}(z))^{[n/2]+1}$ and the flat trace of a composition (in any order) of j factors of the finite-rank term $[\mathcal{D}^{(r)'}_{k,L}(z) - \mathcal{D}^{(r)}_{k,L}(z)]^{[n/2]+1}$ with $([n/2]+1-j)$ factors $\mathcal{D}^{(r)}_{k,L}(z)$ is holomorphic in V_k. Its power series is equal to the sum of the formal flat trace of $\mathcal{D}^{[n/2]+1}_{k,L}(z)$ with the trace of a finite rank operator which has the same trace as $(\mathcal{D}'_{k,L}(z) - \mathcal{D}_{k,L}(z))^{[n/2]+1}$. (Recall that $\mathcal{D}'_{k,L}(z) - \mathcal{D}_{k,L}(z)$ starts with a precomposition by $\chi_K(\mathcal{S}'_k - \mathcal{S}_k)$ which is finite-rank from $\mathcal{A}_{k+1,\mathcal{B}_{k+1}}$ to $\mathcal{A}_{k,\mathcal{B}_k}$.) Therefore $\mathrm{tr}^\flat(\mathcal{D}^{(r)'}_{k,L}(z))^{[n/2]+1} = \mathrm{tr}^\flat(\mathcal{D}'_{k,L}(z))^{[n/2]+1}$, which yields the first claim of Lemma 10.

Next, we may apply the Sublemma, essentially as in Lemma 7, to see that the zeroes of the determinant $\mathrm{Det}^\flat(\mathrm{Id} + \mathcal{D}'_{k,L}(z))$ correspond to eigenvalues -1, using

$$\mathcal{D}^{\epsilon,(r')}_{k,L}(z) = \mathcal{D}^{\epsilon,(r)}_{k,L}(z) + \delta_\epsilon(\widetilde{\mathcal{J}}_k)^{-1}\mathcal{N}_{k,L}(z)(\mathrm{Id} - \mathcal{M}_{k,L}(z))^{-1}(\mathcal{S}'_k - \mathcal{S}_k)\widetilde{\mathcal{J}}_k\delta_\epsilon$$
$$\mathcal{D}^{\epsilon'}_{k,L}(z) = \mathcal{D}^{\epsilon}_{k,L}(z) + \delta_\epsilon\mathcal{N}_{k,L}(z)(\mathrm{Id} - \mathcal{M}_{k,L}(z))^{-1}(\mathcal{S}'_k - \mathcal{S}_k)\delta_\epsilon\,,$$

which both have a finite-rank (in $L^2(\mathbb{R}^n)$) second term. This gives Lemma 10,(3).

If $1/z_0$ is a simple eigenvalue for $\mathcal{M}_k$ with $z_0 \in V_k$ then one proves, like in Lemma 6, that $\mathrm{Det}^\flat(\mathrm{Id} + \mathcal{D}'_{k,L}(z))$ is meromorphic at z_0. with a pole of order at most one. Finally, we shall prove that $\mathrm{Det}^\flat(\mathrm{Id} + \mathcal{D}'_{k,L}(z))$ does not have a removable singularity at z_0 showing that the pole has order exactly one i.e., claim (2).

For this, we use a spectral decomposition of $\mathcal{M}_{k,\ell}(z)$ for the simple eigenvalue λ_z on $\mathcal{A}_{k,\mathcal{B}_k}$, for z close to z_0 (with $\lambda_{z_0} = 1$):

$$(\mathrm{Id} - \mathcal{M}_{k,\ell}(z))^{-1} = \frac{\varphi_k(z)}{\lambda_z - 1}\cdot\nu_k(z) + \mathcal{R}_k(z)\,, \tag{3.17}$$

with λ_z holomorphic in z; $\mathcal{R}_k(z) : \mathcal{A}_{k,\mathcal{B}_k} \to \mathcal{A}_{k,\mathcal{B}_k}$ depending holomorphically on z at z_0; $\mathcal{R}_k(z)\varphi_k(z) = \nu_k(z)\mathcal{R}_k(z) = 0$; the eigenvector $\varphi_k(z) \in \mathcal{A}_{k,\mathcal{B}_k}$ of unit norm depending holomorphically on z with $\varphi_k(z_0) = \varphi_k \in \mathcal{A}_{k,\mathcal{B}_k}$ so that $\mathcal{M}_k(z_0)\varphi_k = \varphi_k$; and $\nu_k(z)$ unit-norm linear functionals on $\mathcal{A}_{k,\mathcal{B}_k}$, depending holomorphically on z, with $\nu_k(z)(\varphi_k(z)) = 1$.

By Lemma 9, $|\nu_k(z_0)(\chi_K\mathcal{S}'_k\mathcal{N}_{k,\ell}(z_0)\varphi_k(z_0))| > 0$ if ℓ is large enough. It is then easy to see that $(z - z_0)\mathrm{Det}^\flat(\mathrm{Id} + \mathcal{D}'_{k,L}(z))$ does not vanish at z_0. $\square$

Using the new homotopy operators

Making use of the homotopy operators from Lemma 9, we state the final ingredients needed in our main result. The eigenvalues of modulus larger than R of the operators $d\mathcal{S}\mathcal{M}_k$ produce zeroes of the flat kneading determinants (Lemma 11). To show that such eigenvalues do not contribute to the zeroes and poles of the sharp determinant (except when $\mathcal{M}_k\tilde{\varphi}_k = d\mathcal{S}\mathcal{M}_k\tilde{\varphi}_k$, in particular if $k = n$), we shall prove that they are not intrinsic. More precisely, in Lemma 12 we construct perturbed homotopy operators $\mathcal{S}''$ which cause these eigenvalues to vary.

Lemma 11 (Zeroes of the flat kneading determinants). *Under the assumptions of Lemma 9, and up to taking a larger value of L, for all $k \in \{1, \ldots, n\}$ and $|z| < 1/R$,*
(1) If $Det^\flat(Id + \mathcal{D}'_{k-1,L}(z)) = 0$ then $1/z$ is an eigenvalue of $d\mathcal{S}'_{k-1}\mathcal{M}_k$ acting on $\mathcal{A}_{k,\mathcal{B}_k}$ or $k \geq 2$ and $1/z$ is an eigenvalue of $d\mathcal{S}'_{k-2}\mathcal{M}_{k-1}$ acting on $\mathcal{A}_{k-1,\mathcal{B}_{k-1}}$, in either case, the geometric multiplicity of the eigenvalue is at least equal to the order of the zero.
(2) If $1/z$ (with $|z| < 1/R$) is an eigenvalue of $\mathcal{M}_n$ acting on $\mathcal{A}_{n,\mathcal{B}_n}$ and is not an eigenvalue of $\mathcal{M}_{n-1}$ acting on $\mathcal{A}_{n-1,\mathcal{B}_{n-1}}$, then $Det^\flat(Id + \mathcal{D}'_{n-1,L}(z)) = 0$ and the order of the zero is at least the algebraic multiplicity of the eigenvalue.
(3) If $k \geq 2$ and $1/z$ (with $|z| < 1/R$) is an eigenvalue of $d\mathcal{S}'_{k-2}\mathcal{M}_{k-1}$ acting on $\mathcal{A}_{k-1,\mathcal{B}_{k-1}}$ then $Det^\flat(Id + \mathcal{D}'_{k-1,L}(z)) = 0$ and the order of the zero is at least the geometric multiplicity of the eigenvalue (which is thus finite and the eigenvalue isolated).

Proof of Lemma 11. By Lemma 10, the assumption $\mathrm{Det}^\flat(\mathrm{Id} + \mathcal{D}'_{k-1,L}(z)) = 0$ implies that $1/z \notin \mathrm{sp}(\mathcal{M}_{k-1})$ and there is a nonzero $\varphi \in \mathcal{A}_{k,\mathcal{B}_k}$ with $\mathcal{D}'_{k-1,L}(z)\varphi = -\varphi$. It follows that φ is *not* in the kernel of $\mathcal{S}'_{k-1}$. Writing $\varphi = d\mathcal{S}'_{k-1}\varphi + \mathcal{S}'_k d\varphi = \varphi_1 + \varphi_2$ (with $\mathcal{S}'_{k-1}\varphi_1 \neq 0$), our assumption implies

$$\begin{cases} d\mathcal{S}'_{k-1}\varphi_1 + d\mathcal{S}'_{k-1}\mathcal{N}_{k,L}(z)(\mathrm{Id} - \mathcal{M}_{k-1,L}(z))^{-1}\mathcal{S}'_{k-1}\varphi_1 = 0 \\ \varphi_2 = \mathcal{S}'_k d\varphi_2 = -\mathcal{S}'_k d\mathcal{N}_{k,L}(z)(\mathrm{Id} - \mathcal{M}_{k-1,L}(z))^{-1}\mathcal{S}'_{k-1}\varphi_1 \,. \end{cases} \tag{3.18}$$

The first equality in (3.18) is equivalent with

$$\begin{aligned} 0 &= \left(d + (d\mathcal{M}_{k-1,L}(z) - d\mathcal{S}'_{k-1}\mathcal{M}_{k,L}(z)d)(\mathrm{Id} - \mathcal{M}_{k-1,L}(z))^{-1}\right)\mathcal{S}'_{k-1}\varphi_1 \\ &= (\mathrm{Id} - d\mathcal{S}'_{k-1}\mathcal{M}_{k,L}(z))d(\mathrm{Id} - \mathcal{M}_{k-1,L}(z))^{-1}\mathcal{S}'_{k-1}\varphi_1 \,. \end{aligned}$$

Note that $d(\mathrm{Id} - \mathcal{M}_{k-1,L}(z))^{-1}\mathcal{S}'_{k-1}\varphi_1$ is in $\mathcal{A}_{k,\mathcal{B}_k}$ by the boundedness of the d operator in Axiom 1. Then it is not very difficult to check (see the proof of Lemma 5) that the bounds on the norm of $\mathcal{N}^{(j)}_{k-1}$ translate into exponentially decaying bounds for $j \mapsto \|\mathcal{N}_{k-1,L}(z)^{(j)}\|$.

Since $\chi_K\mathcal{S}'_{k-1}\varphi_1 \in \mathcal{A}_{k-1,\mathcal{B}_{k-1}}$ does not vanish, $(\mathrm{Id} - \mathcal{M}_{k-1,L}(z))^{-1}\mathcal{S}'_{k-1}\varphi_1 \neq 0$. If

$$\hat\phi = d(\mathrm{Id} - \mathcal{M}_{k-1,L}(z))^{-1}\mathcal{S}'_{k-1}\varphi_1 \neq 0 \,, \tag{3.19}$$

then we are done. Indeed, $\hat\phi \in \mathrm{Im}\, d \cap \mathcal{A}_{k,\mathcal{B}_k}$ would then be a nonzero fixed vector for $d\mathcal{S}'_{k-1}\mathcal{M}_{k,L}(z)$. Since the essential spectral radius of $d\mathcal{S}'_{k-1}\mathcal{M}_{k,L}(z)$ is smaller than one, is an easy algebraic exercise to see that the fixed vectors $d\mathcal{S}'_{k-1}\mathcal{M}_{k,L}(z)$ (if $|z| < 1/R$) are in bijection with the eigenvectors of $d\mathcal{S}'_{k-1}\mathcal{M}_k$ for the eigenvalue $1/z$ so that we are in the first case of the first claim. Note also at this point that if there exists $\hat\phi \in \mathcal{A}_{n,\mathcal{B}_n}$ with $d\mathcal{S}'_{n-1}\mathcal{M}_{n,L}(z)\hat\phi = \hat\phi$ then $\varphi_1 := d(\mathrm{Id} - \mathcal{M}_{n-1,L}(z))\mathcal{S}'_{n-1}\hat\phi \in \mathcal{A}_{n,\mathcal{B}_n}$ would satisfy the first identity in (3.18), while φ_2 may be defined by the second equality of (3.18). In this case $\mathrm{Det}^\flat(\mathrm{Id} + \mathcal{D}'_{n-1,L}(z)) = 0$ by Lemma 10.

Let us assume that (3.19) does not hold. Writing $\upsilon = (\mathrm{Id} - \mathcal{M}_{k-1,L}(z))^{-1}\mathcal{S}'_{k-1}\varphi_1$ we would then have $\upsilon \in \mathcal{A}_{k-1,\mathcal{B}_{k-1}}$ and $d\upsilon = 0$ and, applying $\mathcal{S}'_{k-1}d$ to both sides of $\mathcal{S}'_{k-1}\varphi_1 = d\mathcal{S}'_{k-2}\upsilon - \mathcal{M}_{k-1,L}(z)\upsilon$,

$$\mathcal{S}'_{k-1}\varphi_1 = -\mathcal{S}'_{k-1}d\mathcal{M}_{k-1,L}(z)\upsilon\,,$$

since $\mathcal{S}'_{k-1}\varphi_1 = (\mathrm{Id} - \mathcal{M}_{k-1,L}(z))\upsilon$, we find $(\mathrm{Id} - \mathcal{M}_{k-1,L}(z))\upsilon = -\mathcal{S}'_{k-1}d\mathcal{M}_{k-1,L}(z)\upsilon$, contradicting $1/z \notin \mathrm{sp}\,\mathcal{M}_{k-1}$ if $k = 1$, and if $k \ge 2$ implying

$$d\mathcal{S}'_{k-2}\upsilon = \upsilon = (\mathrm{Id}\,-\mathcal{S}'_{k-1}d)\mathcal{M}_{k-1,L}(z)\upsilon = d\mathcal{S}'_{k-2}\mathcal{M}_{k-1,L}(z)\upsilon\,.$$

For the converse (i.e., the last claim in Lemma 11), we see that if $\upsilon = d\mathcal{S}'_{k-2}\mathcal{M}_{k-1,L}(z)\upsilon$, we may take $\varphi_1 = -d\mathcal{M}_{k-1,L}\upsilon$ since then $(\mathrm{Id}\,-\mathcal{M}_{k-1,L})^{-1}\mathcal{S}'_{k-1}\varphi_1 = -\upsilon$ so that $d(\mathrm{Id}\,-\mathcal{M}_{k-1,L})^{-1}\mathcal{S}'_{k-1}\varphi_1 = 0$. $\square$

Lemma 12 (Modified homotopy operators $\mathcal{S}''$: perturbing nonintrinsic eigenvalues). *Assume that for some $2 \le k \le n-1$ there are $0 \ne \tilde\varphi_k \in \mathcal{A}_{k,\mathcal{B}_k}$ and $z \in \mathbb{C}$ with $|1/z| > R$ such that*

$$\tilde\varphi_k = \frac{1}{z}d\mathcal{S}'_{k-1}\mathcal{M}_k\tilde\varphi_k\,,$$

as an isolated eigenvalue of finite geometric multiplicity. Then, there are $z' \ne z$, arbitrarily close to z, and two rank-one operators of arbitrarily small norm

$$\mathcal{F}'_\ell : \mathcal{A}_{\ell+1,\mathcal{B}_{\ell+1}} \to \{\varphi \in \mathcal{A}_\ell \mid \chi_K\varphi \in \mathcal{A}_{\ell,\mathcal{B}_\ell}\}\,,\ \ell = k-1, k\,,$$

so that the perturbed operators $\mathcal{S}''_{k-1} = \mathcal{S}'_{k-1} + \mathcal{F}'_{k-1}$, $\mathcal{S}''_k = \mathcal{S}'_k + \mathcal{F}'_k$, $\mathcal{S}''_\ell = \mathcal{S}'_\ell$, $\ell \notin \{k-1, k\}$, still satisfy $\mathcal{S}''\mathcal{S}'' = 0$, $\mathcal{S}''d + d\mathcal{S}'' = 1$ and, additionally, $1/z'$ is an eigenvalue of $d\mathcal{S}''_{k-1}\mathcal{M}_k\varphi$ on $\mathcal{A}_{k,\mathcal{B}_k}$ while $d\mathcal{S}''_\ell = d\mathcal{S}'_\ell$ for all $\ell \ne k-1$.

Proof of Lemma 12. We take $\varphi'_{k-1} := \mathcal{S}'_{k-1}\mathcal{M}_k\tilde\varphi_k \in \mathcal{A}_{k-1,\mathcal{B}_{k-1}}$. By our assumptions, $d\varphi'_{k-1} \ne 0$. Let then ν'_k be a unit-norm continuous functional on $\mathcal{A}_{k,\mathcal{B}_k}$ which satisfies

$$\nu'_k \circ d = 0\,, \quad \alpha := \nu'_k(\mathcal{M}_k\tilde\varphi_k) \ne 0\,.$$

(Take $\nu'_k(\varphi) = \nu'_{k+1}(d\varphi)$ with ν'_{k+1} continuous on $\mathcal{A}_{k+1,\mathcal{B}_{k+1}}$ and $\nu'_{k+1}(d\mathcal{M}_k\tilde\varphi_k) \ne 0$.) We set, for small complex ϵ,

$$\mathcal{F}'_{k-1}(\varphi) = \epsilon\varphi'_{k-1}\cdot\nu'_k(\varphi)\,, \quad \mathcal{F}'_k(\varphi) = -\epsilon d\varphi'_k\cdot\nu'_k(\mathcal{S}_k\varphi)\,.$$

We have $d\mathcal{F}'_{k-1} = -\mathcal{F}'_k d$, $\mathcal{F}'_{k-1}d = 0$ and $d\mathcal{F}'_k = 0$ so that $d\mathcal{S}'' + \mathcal{S}''d = \mathrm{Id}$. Also, $\mathcal{F}'_k\mathcal{S}'_{k+1} = 0$, and $\mathcal{S}'_{k-1}\mathcal{F}'_k + \mathcal{F}'_{k-1}\mathcal{S}'_k + \mathcal{F}'_{k-1}\mathcal{F}'_k = 0$, guaranteeing $\mathcal{S}''\mathcal{S}'' = 0$. Clearly, $d\mathcal{F}'_k = 0$ so that $d\mathcal{S}''_k = d\mathcal{S}'_k$. Finally,

$$zd\mathcal{S}''_{k-1}\mathcal{M}_k\tilde\varphi_k = \tilde\varphi_k + \epsilon z\cdot(d\mathcal{S}'_{k-1}\mathcal{M}_k\tilde\varphi_k)\cdot\nu'_k(\mathcal{M}_k\tilde\varphi_k) = (1+\alpha\epsilon)\tilde\varphi_k\,. \quad \square$$

Main Result

Theorem 13. *Let ψ_ω, g_ω satisfy the assumptions of Section 2, let $\mathcal{B}_{k,t}$ and R satisfy Axioms 1 and 2–3. Then $Det^{\#}(Id - z\mathcal{M})$ is meromorphic in the disc $\{|z| < 1/R\}$. The order of z as a zero/pole of $Det^{\#}(Id - z\mathcal{M})$ coincides with the sum of the algebraic multiplicity of $1/z$ as an eigenvalue of the $\mathcal{M}_{2k}$ on $\mathcal{A}_{2k,\mathcal{B}_{2k}}$, for $0 \le 2k \le n$, minus the sum of the algebraic multiplicity of $1/z$ as an eigenvalue of the $\mathcal{M}_{2k+1}$ on $\mathcal{A}_{2k+1,\mathcal{B}_{2k+1}}$ for $1 \le 2k+1 \le n$.*

Proof of Theorem 13. Perturbing our family, we may assume that all eigenvalues of moduli $> R$ of the $\mathcal{M}_k$, $k = 0, \ldots, n-1$, are simple, and none of their eigenvectors belong to $\operatorname{Ker} d$ or $\operatorname{Ker} \mathcal{N}_k$. We may also assume that the eigenvalues of $\mathcal{M}_n$ are simple.

We apply Lemma 9 to construct adapted homotopy operators $\mathcal{S}'$ and let $\mathcal{D}'_{k,\ell}(z)$ be the kneading operators from (3.16), for large enough $\ell \ge L$.

We consider the modified equality (3.10) from Theorem 2 for the perturbed family and the adapted homotopy operators, using Lemma 10 to view it as an alternated product of meromorphic functions in the disc of radius $1/R$. (Later in the proof we shall use further finite rank perturbations of the $\mathcal{S}'$ given by Lemma 12 and thus satisfying the assumptions of Lemma 8.)

By Lemma 10, the determinant $\operatorname{Det}^\flat(\mathrm{Id} + \mathcal{D}'_{2k,\ell}(z))$ for $0 \le 2k \le n-1$ (which appears in the denominator) has poles in the disc of radius $1/R$ only at the inverse eigenvalues of $\mathcal{M}_{2k}$, with order exactly one. We have the same statement for the $\operatorname{Det}^\flat(\mathrm{Id} + \mathcal{D}'_{2k+1,\ell}(z))$ for $1 \le 2k+1 \le n-1$, which appear in the numerator.

Lemma 11 also says that if $\operatorname{Det}^\flat(\mathrm{Id} + \mathcal{D}'_{k-1,\ell}(z))$ for $1 \le k \le n$ (in the denominator for even $k-1$ and in the numerator for odd $k-1$) vanishes in this disc then $1/z$ is an eigenvalue of $d\mathcal{S}'_{k-1}\mathcal{M}_k$ or $k \ge 2$ and $1/z$ is an eigenvalue of $d\mathcal{S}'_{k-2}\mathcal{M}_{k-1}$. Also, whenever $1/z$ is an eigenvalue of $d\mathcal{S}'_{k-2}\mathcal{M}_{k-1}$ then $\operatorname{Det}^\flat(\mathrm{Id} + \mathcal{D}'_{k-1,\ell}(z)) = 0$, and if $1/z$ is an eigenvalue of $d\mathcal{S}\mathcal{M}_n = \mathcal{M}_n$ then $\operatorname{Det}^\flat(\mathrm{Id} + \mathcal{D}'_{n-1,\ell}(z)) = 0$.

To finish the proof, we will show that the zeroes of the flat determinants must cancel in the alternated product, except of course for the zeroes of $\operatorname{Det}^\flat(\mathrm{Id} + \mathcal{D}'_{n-1,\ell}(z))$ (in the denominator if $n-1$ is even, in the numerator if $n-1$ is odd) corresponding to $1/z$ being an eigenvalue of $\mathcal{M}_n$. Assume for a contradiction that $\operatorname{Det}^\#(1 - \hat{z}\mathcal{M}) = 0$ to order D strictly larger than the value claimed in Theorem 13, due (at least in part) to a factor $\operatorname{Det}^\flat(\mathrm{Id} + \mathcal{D}'_{\hat{k}-1,\ell}(\hat{z})) = 0$ vanishing to order $\hat{D} \ge 1$ for some odd $0 \le \hat{k} - 1 \le n-1$. (The case of poles and even $k-1$ is dealt similarly.) Lemma 11 says that $1/\hat{z}$ is then either an eigenvalue of $d\mathcal{S}'_{k-2}\mathcal{M}_{\hat{k}-1}$ (since $\hat{k} \ge 2$) or an eigenvalue of $d\mathcal{S}'_{\hat{k}-1}\mathcal{M}_{\hat{k}}$ (and $\hat{k} < n$). If $1/\hat{z}$ is an eigenvalue of $d\mathcal{S}'_{k-2}\mathcal{M}_{\hat{k}-1}$ (the other case is left to the reader), we may use Lemma 12 to perturb this eigenvalue to some $1/\hat{z}' \ne 1/\hat{z}$. Note that this does not modify the $\operatorname{Det}^\flat(\mathrm{Id} + \mathcal{D}'_{k-1,\ell}(z))$ except for $k = \hat{k}$ and $k = \hat{k}+1$. (For both of these determinants, only the set of zeroes may change, and only one of the determinants is in the numerator.) The perturbation $\mathcal{D}''_{\hat{k}-1,\ell}(z)$ of $\mathcal{D}'_{\hat{k}-1,\ell}(z)$ may be made as small as desired in (finite-rank) operator norm in a neighbourhood of $\hat{z}$ by taking small enough $\epsilon \neq 0$ in Lemma 12. Since $\hat{z} \neq \hat{z}'$, the continuous dependence of the regularised determinant on the operator together with Rouché's Theorem guarantee that the order of $\hat{z}$ as a zero of $\mathrm{Det}^\flat(\mathrm{Id} + \mathcal{D}''_{\hat{k}-1,\ell}(z))$ is strictly smaller than $\hat{D}$. Iterating this procedure (or the procedure associated to the other case) at most $\hat{D}$ times, we get that $\mathrm{Det}^\flat(\mathrm{Id} + \mathcal{D}'''_{\hat{k}-1,\ell}(\hat{z})) \neq 0$, while none of the zeroes of the other kneading determinants (in the numerator) or the poles (in the denominator or in fact also the numerator) have been altered. Then, either the order of $\hat{z}$ as a zero of $\mathrm{Det}^\#(1 - z\mathcal{M})$ is strictly smaller than D, a contradiction, or $\mathrm{Det}^\flat(\mathrm{Id} + \mathcal{D}'''_{k-1,\ell}(\hat{z})) = 0$ for some odd $k - 1 \neq \hat{k} - 1$. In the second case, we may proceed as above to ensure $\mathrm{Det}^\flat(\mathrm{Id} + \mathcal{D}''''_{k-1,\ell}(\hat{z})) \neq 0$ and eventually obtain the contradiction that the order of $\hat{z}$ as a zero of $\mathrm{Det}^\#(1 - z\mathcal{M})$ is strictly smaller than D. $\square$

A corollary of the proof of Theorem 13 is (we do not have to replace $\widetilde{R}$ by $\widetilde{R}^{1/n}$):

Theorem 14. *Assume Axioms 1 and 2. For each large enough L, the following alternated product of regularised determinants extends holomorphically to $\{|z| < 1/\widetilde{R} \mid 1/z \notin \cup_k \mathrm{sp}(\mathcal{M}_k)\}$:*

$$\prod_{k=0}^{n-1} Det^{reg}_{[n/2]+1}(Id + \mathcal{D}^{(r)}_{k,L}(z))^{(-1)^{k+1}} . \tag{3.20}$$

Although $\mathrm{Det}^{\mathrm{reg}}_{[n/2]+1}(\mathrm{Id} + \mathcal{D}^{(r)}_{k,L}(z)) = \exp - \sum_{\ell=[n/2]+1}^{\infty} \frac{z^\ell}{\ell} \mathrm{tr}^\flat(\mathcal{D}_{k,L}(z))^\ell$, it does not seem easy to relate (3.20) to a dynamical zeta function.

4. Application to expanding maps on compact manifolds

In this section, we apply Theorem 13 from Section 3 to C^r (locally) expanding endomorphisms on compact manifolds and C^r weights, giving a new (and completely different) proof of a result of Ruelle [Ru3], avoiding Markov structures. We do not recover the full strength of his statement except if the dynamics is C^∞ and if the weight is the inverse Jacobian (with respect to Lebesgue).

Let us start by stating precisely this result. Let M be a C^r ($r \geq 1$) compact manifold of dimension $n \geq 2$, and let $f : M \to M$ be C^r and (locally) uniformly expanding, that is, there is $\theta < 1$ such that $\|D_a f(\xi)\| \geq \theta^{-1}\|\xi\|$ for all ξ in the tangent space $T_a M$ at a, where $\|\cdot\|$ is the Euclidean norm on $T_a M$. Let $g : M \to \mathbb{C}$ be C^r. The transfer operator $\mathcal{L}_0 = \mathcal{L}_{f,g}$, acting on the Banach space of C^r functions $M \to \mathbb{C}$, is given by

$$\mathcal{L}_0\phi(a) = \sum_{b: f(b)=a} g(b)\phi(b) = \sum_j g_j(a)\phi(\psi_j(a)) , \tag{4.1}$$

where the ψ_j are the finitely many local inverse branches of f. Similarly, we can introduce operators acting on Banach spaces of k-forms on M with C^m coefficients, for $0 \leq m \leq r-1$, putting $\mathcal{L}_k\phi = \sum_j g_j \cdot (\psi_j^*)\phi$.

An expanding map is transversal, and the Lefschetz numbers of the inverse map at the periodic orbits are all positive, so that the Ruelle zeta function associated to f, g

$$\zeta_{f,g}(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{a \in \mathrm{Fix}\, f^m} \prod_{\ell=0}^{m-1} g(f^\ell(a))$$

can be viewed as a Lefschetz-Ruelle zeta function. Ruelle proved:

Theorem (Ruelle [Ru3]). *Let $P \in \mathbb{R}$ be the topological pressure of $\log |g|$ and f.*
(1) The spectral radius of $\mathcal{L}_0$ acting on C^m is at most e^P while its the essential spectral radius is at most $\theta^m e^P$ for $0 \le m \le r$. The spectral radius of $\mathcal{L}_k$ acting on forms with C^m coefficients is at most $\theta^k e^P$ while its the essential spectral radius is at most $\theta^{m+k} e^P$, for $1 \le k \le n$ and $0 \le m \le r-1$.
(2) The power series $\zeta_{f,g}(z)$ defines a meromorphic function in the disc of radius $\theta^{-r} e^{-P}$. In this disc, the order of z as a zero/pole of $\zeta_{f,g}(z)$ coincides with the sum of the algebraic multiplicity of $1/z$ as an eigenvalue of the $\mathcal{L}_{2k+1}$ acting on C^m for any $r-(2k+1) \le m \le r-1$ and $1 \le 2k+1 \le n$, minus the sum of the algebraic multiplicity of $1/z$ as an eigenvalue of the $\mathcal{L}_{2k}$ acting on C^m for $r-2k \le m \le r-1$ and $0 \le 2k \le n$ (for $k=0$ one can also let $\mathcal{L}_0$ act on C^r).

We will recover Ruelle's result if the system is in fact nr times differentiable, with the same estimates if $g(y) = 1/|\det Df(y)|$, (i.e., the weight giving rise to the absolutely continuous invariant measure), and only a weaker result, replacing e^P by $\theta^{-n+1} e^P$, in the case of an arbitrary smooth weight. There is certainly room for improvement here.

Once the first claim of the above theorem is established (see [GuLa] for better estimates), Ruelle [Ru3] associates to each $\mathcal{L}_k$ a Fredholm-like (flat) dynamical determinant

$$d_k(z) = \mathrm{Det}^\flat(\mathrm{Id} - z\mathcal{L}_k) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{a \in \mathrm{Fix}\, f^m} \prod_{\ell=0}^{m-1} \frac{g(f^\ell(a)) \cdot \mathrm{Tr}\, \Lambda^k(D_a f^{-m})}{\det(\mathrm{Id} - D_a f^{-m})}, \quad (4.2)$$

and proves that $d_k(z)$ is holomorphic in the disc of radius $\theta^{-k-r} e^{-P}$, where its zeroes correspond exactly the the inverse eigenvalues of $\mathcal{L}_k$ (outside of the disc of radius $\theta^{k+r} e^P$). Writing $\zeta_{f,g}(z)$ as an alternated product of the $d_k(z)$ gives the second claim.

The present approach does not allow us to analyze the independent factors $d_k(z)$. However, since the spectral radii of the operators $\mathcal{L}_k$ are strictly decreasing, the annulus $e^{-P} \le |z| < \theta^{-1} e^{-P}$, e.g., only contains inverse eigenvalues of $\mathcal{L}_0$ acting on $C^r(M)$.

From a manifold to $\mathbb{R}^n$ – Equivalent models

We wish to associate to f and g data $\{\psi_\omega, g_\omega\}_{\omega \in \Omega}$ in such a way that the operators $\mathcal{L}_k$ and $\mathcal{M}_k$ are conjugated. Their spectra on suitable spaces will thus coincide. For this, first choose a C^r atlas $\{V_j\}$ for M such that $f|_{V_j} \to f(V_j)$ is a diffeomorphism. Choose a C^r partition of the unity $\{\chi_j\}$, where each χ_j is supported in V_j. Denote by $\psi_j : f(V_j) \to V_j$ the inverse map of $f|_{V_j}$. Since $f(V_j)$ is not necessarily contained in some V_i, we refine the cover V_j by putting $V_{ji} = f(V_i) \cap V_j$ and set $\psi_{ji} = \psi_j|_{V_{ji}} : V_{ji} \to$

$\psi_j(V_{ji}) \subset V_i$. We choose for each j a C^r partition of the unity $\{\chi_{ji}\}$ on V_j such that each χ_{ji} is supported in V_{ji}. Finally, we set $g_{ji}(a) = \chi_i(\psi_j(a)) \cdot \tilde{\chi}_{ji}(a) \cdot g_j(a)$ (note that g_{ji} is compactly supported in V_{ji}), and we may rewrite our operator as

$$\mathcal{L}_0\phi(a) = \sum_{j,i} g_{ji}(a) \cdot \phi(\psi_{ji}(a)) \,. \tag{4.3}$$

The operators $\mathcal{L}_k$ acting on k-forms on M are similarly defined (replacing the composition with ψ_{ji} by the pullback). We next choose charts $\pi_j : U_j \to V_j$, where the U_j are bounded and two-by-two disjoint open subsets of $\mathbb{R}^n$. We denote by π the map $\pi|_{U_j} = \pi_j$ and set $U_{ji} = \pi_j^{-1}(V_{ji})$.

Let now φ be a form in $U = \cup_j U_j$. We set $(\tau\varphi)(a) = \sum_i \chi_i(a) \cdot (\pi_i^{-1})^*\varphi(a)$. Then, $\tau\pi^*$ is the identity on C^r forms in M (in particular π^* is injective). We now define $\mathcal{M}_k$ acting on k-forms in $U \subset \mathbb{R}^n$ as $\mathcal{M}_k = \pi^*\mathcal{L}_k\tau$, that is

$$\mathcal{M}_k\varphi(x) = \sum_{ji} g_{ji}(\pi(x))\chi_i(\psi_{ji} \circ \pi(x)) \cdot (\pi_i^{-1} \circ \psi_{ji} \circ \pi)^*\varphi(x) \,. \tag{4.4}$$

It is easy to choose Ω, ψ_ω and g_ω in order to view $\mathcal{M}_k$ as an operator of the form (2.3), with g_ω compactly supported in U_ω.

Sobolev spaces

We shall work with the Bessel potential $\mathcal{J}_1$, defined on $L^q(\mathbb{R}^n)$ by (see [St, Chapter V.3], $\Gamma(\cdot)$ denotes the Euler gamma function):

$$\mathcal{J}_1(\varphi)(x) = \frac{1}{\Gamma(1/2)\sqrt{4\pi}} \int_{\mathbb{R}^n} \int_0^\infty e^{-\pi|y|^2/t} e^{-t/4\pi} t^{(-n+1)/2} \frac{dt}{t} \, \varphi(x-y) \, dy \,.$$

Write $\mathcal{J}_\ell$ for the ℓth iterate of $\mathcal{J}_1$. It is a well-known (see [St, V.3.3-3.4]) and important result in the theory of Sobolev spaces that for all $1 < q < n$ and each $\ell \geq 1$, $\mathcal{J}_\ell$ is an isomorphism from the Sobolev space $L^q_m(\mathbb{R}^n) = W^{m,q}(\mathbb{R}^n)$ to the Sobolev space $L^q_{m+\ell}(\mathbb{R}^n)$ if $1 < q < \infty$, m is a nonnegative integer, and $\ell \geq 1$ [St, V.3.2-3.3]. In particular, $\mathcal{J}_1^{-1}\varphi \in L^q_{m-1}(\mathbb{R}^n)$ if $\varphi \in L^q_m$ and $m \geq 1$. We take $\widetilde{\mathcal{J}}_k = \mathcal{J}_{r-k}$.

We define the Sobolev spaces $W^{m,p}(M)$ [Ad] of the manifold M using our chart $\pi : U \to M$ (by compactness, other chart systems yield equivalent norms):

$$W^{m,p}(M) := \{\phi : M \to \mathbb{C} \mid \phi \circ \pi \in W^{m,p}(U)\}.$$

For $\phi \in \mathcal{A}_{k,W^{m,p}(M)}$, set $||\phi||_{\mathcal{A}_{k,W^{m,p}(M)}} := ||\pi^* u||_{\mathcal{A}_{k,W^{m,p}(U)}}$.

Clearly, $\mathcal{L}_k$ is bounded on $\mathcal{A}_{k,W^{m,p}(M)}$ for all $0 < p < 1$ and $m \leq r-1$ (for $k = 0$ we can take $r = m$). By definition, π^* is a Banach space isomorphism between $\mathcal{A}_{k,W^{m,p}(M)}$ and its image in $\mathcal{A}_{k,W^{m,p}(U)}$. The spectrum of $\mathcal{L}_k$ on $\mathcal{A}_{k,W^{m,p}(M)}$ coincides with its spectrum on the image $\mathcal{L}_k(\mathcal{A}_{k,W^{m,p}(M)})$, similarly for $\mathcal{M}_k$ and $\mathcal{A}_{k,W^{m,p}(U)}$. Also, π^* is an isomorphism between $\mathcal{L}_k(\mathcal{A}_{k,W^{m,p}(M)})$ and $\mathcal{M}_k(\mathcal{A}_{k,W^{m,p}(U)})$, with inverse τ, and which conjugates $\mathcal{M}_k$ and $\mathcal{L}_k$. By definition, $\mathrm{Det}^\flat(\mathrm{Id} - z\mathcal{L}_k) = \mathrm{Det}^\flat(\mathrm{Id} - z\mathcal{M}_k)$. We may henceforth concentrate on the $\mathcal{M}_k$ acting on $\mathcal{A}_{k,W^{m,p}(U)}$.

Checking Axioms 1 and 2–3

From now on, assume that the data ψ_ω, g_ω is $C^{\hat{r}}$ for some $\hat{r} \geq nr$ and satisfies the hypotheses of Section 2. Additionally, the ψ_ω are uniform θ-contractions with $\theta < 1$. We l use the notations of Sections 2 and 3, in particular the definitions of K , K' and χ_K, and we assume that $U = K'$. The Banach spaces $\mathcal{B}_{k,t} = W^{rn-k,t}(K')$ clearly satisfy Axiom 1. To establish the bounds in Axiom 2 the following estimate will be instrumental. We let $\mathcal{M}_k^+$ be the transfer operator with g replaced by $|g|$.

Lemma 15 (Lasota-Yorke inequality). *For $0 \leq k \leq n$, $0 \leq m \leq \hat{r} - 1$ ($0 \leq m \leq \hat{r}$ if $k = 0$), and $1 < t < \infty$ there are $C > 0$ and a sequence $\{\rho_\ell\}$ of positive real numbers so that for all ℓ and $\varphi \in W^{m,t}(K')$,*

$$||\mathcal{M}_k^\ell \varphi||_{m,t} \leq \rho_\ell ||\varphi||_{m-1,t} + C \cdot \theta^{\ell m} ||(\mathcal{M}_k^+)^\ell||_t \cdot ||\varphi||_{m,t} \,. \tag{4.5}$$

Proof of Lemma 15. The norm $\|\cdot\|_{m,t}$ on $W^{m,t}(K')$ is given by $\left(\sum_{|\alpha|\leq m} \|D^\alpha \varphi\|_t^t\right)^{1/t}$, where $\|\cdot\|_t$ is the $L^t(K')$ norm, $\alpha = (\alpha_1, \ldots, \alpha_p)$ is a multi-index of size $|\alpha| = p$ and $D^\alpha = \partial_{\alpha_1} \cdots \partial_{\alpha_p}$.

We begin with $\ell = 1$ and $k = 0$. Let α be a multi-index with $|\alpha| = m$. The Leibnitz rule gives $D^\alpha \mathcal{M}_0 \varphi = \sum_\omega g_\omega D^\alpha(\varphi \circ \psi_\omega) + H_\alpha \varphi$, where $H_\alpha \varphi$ is a sum of terms

$$D^\beta g_\omega \cdot D^{\beta'} (\varphi \circ \psi_\omega) \,, \tag{4.6}$$

with $|\beta| \leq |\alpha|$, $|\beta'| \leq |\alpha| - 1$. The data being at least C^m, all terms (4.6) are bounded by a constant multiple of $||\varphi||_{m-1,t}$. Now, $D^\alpha(\varphi \circ \psi_\omega)$ is a sum of, on the one hand, terms involving derivatives of φ of order $\leq |\alpha| - 1$ (which are also bounded by a constant multiple of $||\varphi||_{m-1,t}$), and, on the other hand, terms of the form ($(\psi_\omega)_j$ denotes the j-th coordinate of ψ_ω)

$$\sum_\omega g_\omega \cdot \Big(\prod_{i=1}^m \partial_{\alpha_i}((\psi_\omega)_{\gamma_i})\Big) \cdot (D^\alpha \varphi) \circ \psi_\omega \,. \tag{4.7}$$

The L^t norm to the power t of (4.7) is $\leq \theta^{mt} ||\mathcal{M}_0^+(D^\alpha \varphi)||_t^t$, by the contraction assumption. Since $\varphi \in W^{m,t}(K')$, $||D^\alpha \varphi||_t \leq ||\varphi||_{m,t}$. The number of terms (4.7) depends only on m and n. The result for $\ell = 1$ follows by summing over α with $|\alpha| \leq m$ (if $|\alpha| \leq m-1$, terms of the form (4.7) are also bounded by a constant multiple of $||\varphi||_{m-1,t}$).

For $\ell > 1$, just use that $\|(\mathcal{M}_0^\ell)^+\|_t \leq \|(\mathcal{M}_0^+)^\ell\|_t$.

The claims for $k \geq 1$ are obvious by the contraction property of the pullback. $\square$

Lemma 16 (Axiom 2). *Let $1 < t < \infty$.*
(1) $\rho_{ess}(\mathcal{M}_0|_{W^{m,t}(K')}) \le \theta^m \rho(\mathcal{M}_0^+|_{L^t(K')})$ for all $0 \le m \le \hat r$.
(2) For all $0 \le m \le \hat r - 1$ we have $\rho(\mathcal{M}_k|_{\mathcal{A}_{k,L^t(K')}}) \le \theta^k \rho(\mathcal{M}_0^+|_{L^t(K')})$ and

$$\rho(\mathcal{M}_k|_{\mathcal{A}_{k,W^{m,t}(K')}}) \le \theta^k \rho(\mathcal{M}_0^+|_{L^t(K')})\,,\ \rho_{ess}(\mathcal{M}_k|_{\mathcal{A}_{k,W^{m,t}(K')}}) \le \theta^{m+k} \rho(\mathcal{M}_0^+|_{L^t(K')})\,.$$

(3) $\rho(\mathcal{M}_0^+|_{L^t(K')}) \le 1 = e^P$ if $g_\omega = |\det D\psi_\omega|$. Otherwise $\rho(\mathcal{M}_0^+|_{L^t(K')}) \le \theta^{-n/t} e^P$.

The spaces $\mathcal{B}_{k,t} = W^{\hat r - k,t}(K')$ for $k = 0, \dots, n$, together with the operators $\mathcal{M}_k$, satisfy Axiom 2 for $\widetilde R = \theta^{\hat r}$ if $g_\omega = |\det D\psi_\omega|$, and (after renormalising so that $e^{P(\log|g|} = 1$) and $\widetilde R > \theta^{\hat r - (n-1)}$ otherwise (if t is not much smaller than $n/(n-1)$).

Proof of Lemma 16. For the first claim, since $W^{m-1,t}(K')$ is compactly embedded in $W^{m,t}(K')$ [Ad], we may combine Lemma 15 and the Hennion formula ([He]) to deduce $\rho_{ess}(\mathcal{M}_0|_{\mathcal{A}_{0,W^{m,t}(K')}}) \le \theta^m \rho(\mathcal{M}_0^+|_{\mathcal{A}_{0,L^t(K')}})$.

The inequalities in the second claim are then obvious, by the action of the pullback.

To prove the third claim, it is more convenient to estimate the spectral radius of $\mathcal{L}_0^+$ acting on $L^t(M)$ (see [MS] for an analogous bound in a more specific situation). The Hölder inequality for finite sums gives, for $1 < t' < t$ so that $t^{-1} + (t')^{-1} = 1$, and writing $\mathcal{L}_{0,t'}$ for the operator associated to $|\det(D\psi_\omega)| \cdot (|g_\omega|/|\det D\psi_\omega|)^{t'}$

$$\begin{aligned}
\|(\mathcal{L}_0^+)^\ell \varphi\|^t_{L^t(M)} &= \int_M \big(\sum_{\vec\omega \in \Omega^\ell} \prod_{j=1}^{\ell} |g_{\omega_j} \circ \psi^{j-1}_{\vec\omega}| \cdot |\varphi \circ \psi^\ell_{\vec\omega}|\big)^t \, dx \\
&\le \int_M \Big(\sum_{\vec\omega \in \Omega^\ell} |\varphi \circ \psi^\ell_{\vec\omega}|^t \cdot |\det D\psi^\ell_{\vec\omega}|\Big) \Big(\sum_{\vec\omega \in \Omega^\ell} \big(\prod_{j=1}^{\ell} |g_{\omega_j} \circ \psi^{j-1}_{\vec\omega}| \cdot |\det D\psi^\ell_{\vec\omega}|^{-1/t}\big)^{t'}\Big)^{t/t'} dx \\
&\le \big(\int_M |\varphi|^t \, dx\big) \cdot \sup_M \Big(\sum_{\vec\omega \in \Omega^\ell} |\det D\psi^\ell_{\vec\omega}| \cdot \big(\frac{\prod_{j=1}^{\ell} |g_{\omega_j} \circ \psi^{j-1}_{\vec\omega}|}{|\det D\psi^\ell_{\vec\omega}|}\big)^{t'}\Big)^{t/t'} dx \\
&\le \|\varphi\|^t_{L^t} \cdot \big(\sup_M ((\mathcal{L}_{0,t'})^\ell(1))\big)^{t/t'}\,, \qquad \forall \ell\,, \forall \varphi \in L^t(M)\,.
\end{aligned} \tag{4.8}$$

Taking the t-th root of the above inequality implies

$$\rho(\mathcal{M}_0|_{L^t(K')}) = \rho(\mathcal{L}_0|_{L^t(M)}) \le 1 = e^{P(\log g)} \text{ if } g_\omega = |\det D\psi_\omega|\,.$$

If $g_\omega \ne |\det D\psi_\omega|$, we get

$$\rho(\mathcal{L}_0^+|_{L^t(M)}) \le \lim_{\ell \to \infty} \big(\sup_M \big(\sum_{\vec\omega \in \Omega^\ell} |\det D\psi^\ell_{\vec\omega}|^{-1+1/t'} \cdot \prod_{j=1}^{\ell} |g_{\omega_j} \circ \psi^{j-1}_{\vec\omega}|\big)\big)^{1/\ell} \le \theta^{-n/t} \cdot e^{P(\log|g|)}\,.$$

This immediately implies Axiom 2 for the $\mathcal{M}_k$ and the claimed value of $\widetilde R$. $\square$

The properties in Axiom 3 hold by uniform contraction and bounded distortion.

Completing the new proof of Ruelle's theorem

The proofs of Lemmas 15–16 adapt to the C^m setting, giving for all f, g the well-known bound for all $0 \le m \le \hat r - 1$ ($0 \le m \le \hat r$ if $k = 0$)

$$\rho_{ess}(\mathcal{L}_k|_{\mathcal{A}_{k,C^m(M)}}) = \rho_{ess}(\mathcal{M}_k|_{\mathcal{A}_{k,C^m(\overline{K'})}}) \le \theta^{m+k} e^P .$$

Indeed, when proving the Lasota-Yorke inequality, we may work with $\mathcal{M}_0$ acting on C^0 instead of L^t and the extraneous $\theta^{-n/t}$ factor does not appear. (This also proves the estimates as $t \to \infty$ in Axiom 2(2).)

Clearly, a generalised eigenfunction in $\mathcal{A}_{k,C^{\hat r-k}(\overline{K'})}$ is in $\mathcal{A}_{k,W^{\hat r-k,t}(K')}$. The converse is true, working with eigenfunctions of the dual. Hence, if $\hat r \ge rn$ and f and g are $C^{\hat r}$, the result of Theorem 13 on the Sobolev spaces in fact implies the statement given above of Ruelle's theorem if $g(y) = 1/|\det Df(y)|$ and a weaker statement, replacing $\theta^{-r}e^{-P}$ by $\theta^{-(r-1+1/n)}e^{-P}$, for general $C^{\hat r}$ weights.

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