Topological method for symmetric periodic orbits for maps with a reversing symmetry

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Abstract

We present a topological method of obtaining the existence of infinite number of symmetric periodic orbits for systems with reversing symmetry. The method is based on covering relations. We apply the method to a four-dimensional reversible map.

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1 Introduction

The goal of this paper is to present a topological method, which allows to establish, in finite computation, the existence of an infinite number of symmetric periodic orbits for dynamical systems with a reversing symmetry.

The role and impact of time reversing symmetries in dynamical systems has been extensively covered in the literature, see [La, La1, D1, D2] and references given there. In this paper we will restrict ourselves to the discrete time case. Before we outline our results we need first to introduce a few definitions.

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Definition 1 *An invertible transformation $M : \Omega \longrightarrow \Omega$ is called a* symmetry *of the discrete dynamical system induced by $f : \Omega \longrightarrow \Omega$ if*

$$M \circ f = f \circ M.$$

It is easy to see that a symmetry of a discrete dynamical system transforms trajectories into trajectories.

Alternatively, a dynamical system may admit transformations $S : \Omega \to \Omega$ that map trajectories into other trajectories reversing the time direction.

Definition 2 *[La1] An invertible transformation $S : \Omega \longrightarrow \Omega$ is called a reversing symmetry for the discrete dynamical system induced by $f : \Omega \longrightarrow \Omega$ if*

$$S \circ f = f^{-1} \circ S,$$

or equivalently $S^{-1} \circ f \circ S \circ f = \mathrm{Id}|_\Omega$.

Definition 3 *Let $\varphi : \mathbb{T} \times \Omega \to \Omega$ (where $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$) be a dynamical system. For each point $x_0 \in \Omega$, we define its orbit $\Gamma(x_0)$ by*

$$\Gamma(x_0) = \{x \mid x = \varphi(t, x_0) \textit{ for some } t \in \mathbb{T}\}$$

To describe the symmetry properties of orbits, following [GSS, La1] we introduce the notion of isotropy subgroup.

Definition 4 *Consider a dynamical system on a phase space Ω with a symmetry (reversing symmetry) S.*

If $S(\Gamma(x_0)) = \Gamma(x_0)$ for some $x_0 \in \Omega$, then we say that the orbit $\Gamma(x_0)$ is S-symmetric. If the symmetric orbit $\Gamma(x_0)$ is periodic, then we will say that x_0 is a symmetric periodic point.

If S is clear from the context, then we will often drop it and speak of a symmetric orbit.

Let $P : X \to X$ be a map with a reversing symmetry S. Let $\mathrm{Fix}\,(S) = \{x \mid S(x) = x\}$ be the fixed point set for S.

A standard method for studying symmetric periodic orbits for systems with a reversing symmetry is the Fixed Set Iteration (FSI) method [LQ, La1] (also known as DeVogelaere method [DV]). This method is based on the intersections of the iterates of $\mathrm{Fix}\,(S)$ and $\mathrm{Fix}\,(P \circ S)$ (see [La1, Prop. 1.2.2] for more details). Below we give its simplified version.

Theorem 1 *If $y_1 \in \mathrm{Fix}\,(S)$ and $P^k(y_1) \in \mathrm{Fix}\,(S)$ for some $k \in \mathbb{Z}_+$, then y_1 is a symmetric periodic point for P and its principal period divides $2k$.*

Theorem 1 shows that to obtain a symmetric periodic orbit of period k, we have to examine the $k/2$-iterate of P, which is an apparent obstacle for obtaining infinite number of S-symmetric periodic orbits with unbounded periods in a finite computation.

One way to overcome this problem is the method proposed by Devaney in [D2]. To describe the Devaney's method, assume that $P : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a C^1-map with an reversing symmetry S and $\mathrm{Fix}\,(S)$ is an n-dimensional manifold. Now, if we have $p \in \mathrm{Fix}\,(S)$, which is a fixed (periodic) point for P and the unstable manifold of p, $W^u(p)$, intersects transversally $\mathrm{Fix}\,(S)$, then by symmetry $W^s(p) = S(W^u(p))$, hence $W^s(p)$ intersects $\mathrm{Fix}\,(S)$ transversally at q. Now we apply P^k to the neighborhood of q and it is easy to see, that for k large enough there exists a disk D_k in $\mathrm{Fix}\,(S)$, such that $P^k(D_k)$ intersects $\mathrm{Fix}\,(S)$ transversally in the vicinity of p. Now by Theorem 1 one obtains periodic points of arbitrary high periods.

The method proposed in this paper bears some resemblance with the Devaney's method, as it also is based on some kind of transversality, which is preserved under the iteration of the map. Our method is purely topological and is based on the notion of covering relation (see [GiZ]) as a tool for the propagation of the topological transversality. Here we will informally outline our approach. We say that a cube N P-covers a cube M, denoted by $N \overset{P}{\Longrightarrow} M$, if the image of the cube N under the map P is stretched across the cube M in a topologically nontrivial manner (see Definition 7). These cubes together with the corresponding choices of coordinate systems will be referred to as h-sets (the letter h suggesting the hyperbolic-like directions). Now in each h-set N we can define horizontal and vertical disks (see Definitions 10 and 11). In Section 3 we prove (see Theorem 3) that if we have the chain of covering relations

$$N = N_0 \overset{P}{\Longrightarrow} N_1 \overset{P}{\Longrightarrow} \dots \overset{P}{\Longrightarrow} N_k = M, \tag{1}$$

then for any horizontal disk H in N_0 and any vertical disk V in N_k, there exists $x \in H$, such that $P^i(x) \in N_i$ for $i = 1, \dots, k$ and $P^k(x) \in V$. Now if $\mathrm{Fix}\,(S)$ forms a horizontal disk in N and $\mathrm{Fix}\,(S)$ forms a vertical disk in M, then from Theorem 1 we obtain symmetric periodic point. Observe that if we can build a chain of covering relations (1) linking N and M of arbitrary length, then we will have symmetric periodic points of arbitrary high period. For example it was shown that this happens for suitable 2-dimensional Poincaré maps for the Michelson system arising from the Kuramoto-Sivashinsky PDE [W], the planar restricted three body problem modelling the motion of the Oterma comet in the Sun-Jupiter system [WZ2] or the Henon-Heiles Hamiltonian [AZ]. In the above mentioned applications we had one unstable and one stable directions. In this paper we apply our method to a four-dimensional reversible map to prove the existence of symbolic dynamics and an infinite number of symmetric periodic orbits. The main feature, which makes this example interesting, is the fact that both stable and unstable directions are two-dimensional. The proof is computer assisted, i.e., rigorous numerics is was used to verify assumptions of abstract theorems.

The proposed method was introduced in [W] in the planar case and direct coverings. In this case the proof of the transversality theorem (corresponding to Theorem 3) is very simple and is based on the connectivity argument. Unfortunately this proof cannot be generalized to higher dimension or to include coverings induced by inverse mappings.

The content of this paper can be described as follows. In Section 2 we define topological notions: h-sets, covering relations and backcoverings relations. In Section 3 we prove the main transversality theorem (Theorem 3). In Section 4 describe how it can be applied to reversible dynamical systems in general. In Section 5 we consider an application of our method to a four-dimensional map with an reversing symmetry. In Section 6 we describe how to verify the existence of covering relations by computer.

2 Topological tools: h-sets and covering relations

In this section we present main topological tools used in this paper. The crucial notion is that of *covering relation* [GiZ].

2.1 h-sets

Notation: For a given norm in $\mathbb{R}^n$ by $B_n(c,r)$ we will denote an open ball of radius r centered at $c \in \mathbb{R}^n$. When the dimension n is obvious from the context we will drop the subscript n. Let $S^n(c,r) = \partial B_{n+1}(c,r)$, by the symbol S^n we will denote $S^n(0,1)$. We set $\mathbb{R}^0 = \{0\}$, $B_0(0,r) = \{0\}$, $\partial B_0(0,r) = \emptyset$.

For a given set Z, by $\operatorname{int} Z$, $\overline{Z}$, ∂Z we denote the interior, the closure and the boundary of Z, respectively. For the map $h : [0,1] \times Z \to \mathbb{R}^n$ we set $h_t = h(t,\cdot)$. By Id we denote the identity map. For a map f, by $\operatorname{dom}(f)$ we will denote the domain of f. Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map, then we will say that $X \subset \operatorname{dom}(f^{-1})$ if the map $f^{-1} : X \to \mathbb{R}^n$ is well defined and continuous. For $N \subset \Omega$, N-open and $c \in \mathbb{R}^n$ by $\deg(f,N,c)$ we denote the local Brouwer degree. For the properties of this notion we refer the reader to [L] (see also Appendix in [GiZ]).

Definition 5 *[GiZ, Definition 1] A h-set, N, is the object consisting of the following data*

- $|N|$ *- a compact subset of* $\mathbb{R}^n$
- $u(N), s(N) \in \{0,1,2,\dots\}$*, such that* $u(N) + s(N) = n$
- *a homeomorphism* $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$*, such that*
$$c_N(|N|) = \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1).$$

We set

$$\begin{aligned}
N_c &= \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1),\\
N_c^- &= \partial\overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1)\\
N_c^+ &= \overline{B_{u(N)}}(0,1) \times \partial\overline{B_{s(N)}}(0,1)\\
N^- &= c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+)
\end{aligned}$$

Hence a h-set, N, is a product of two closed balls in some coordinate system. The numbers, $u(N)$ and $s(N)$, stand for the dimensions of nominally unstable and stable directions, respectively. The subscript c refers to the new coordinates given by homeomorphism c_N. Observe that if $u(N) = 0$, then $N^- = \emptyset$ and if $s(N) = 0$, then $N^+ = \emptyset$. In the sequel to make notation less cumbersome we will drop the bars in the symbol $|N|$ and we will use N to denote both the h-sets and its support.

Definition 6 *[GiZ, Definition 3] Let N be a h-set. We define a h-set N^T as follows*

- $|N^T| = |N|$
- $u(N^T) = s(N)$, $s(N^T) = u(N)$
- *We define a homeomorphism $c_{N^T} : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N^T)} \times \mathbb{R}^{s(N^T)}$, by*

  $$c_{N^T}(x) = j(c_N(x)),$$

  where $j : \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)} \to \mathbb{R}^{s(N)} \times \mathbb{R}^{u(N)}$ is given by $j(p, q) = (q, p)$.

∎

Observe that $N^{T,+} = N^-$ and $N^{T,-} = N^+$. This operation is useful in the context of inverse maps.

2.2 Covering relations

Definition 7 *[GiZ, Definition 6] Assume that N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : N \to \mathbb{R}^n$ be a continuous map. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$. Let w be a nonzero integer. We say that*

$$N \stackrel{f,w}{\Longrightarrow} M$$

(N f-covers M with degree w) iff the following conditions are satisfied

1. *there exists a continuous homotopy $h : [0, 1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, such that the following conditions hold true*

$$h_0 = f_c, \tag{2}$$
$$h([0, 1], N_c^-) \cap M_c = \emptyset, \tag{3}$$
$$h([0, 1], N_c) \cap M_c^+ = \emptyset. \tag{4}$$

2. *There exists a map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that*

$$h_1(p, q) = (A(p), 0), \text{ for } p \in \overline{B_u}(0, 1) \text{ and } q \in \overline{B_s}(0, 1), \tag{5}$$
$$A(\partial B_u(0, 1)) \subset \mathbb{R}^u \setminus \overline{B_u}(0, 1). \tag{6}$$

Moreover, we require that

$$\deg(A, \overline{B_u}(0, 1), 0) = w,$$

Note that in the case $u = 0$, if $N \overset{f,w}{\Longrightarrow} M$, then $f(|N|) \subset \mathrm{int}\,|M|$ and $w = 1$.

Intuitively, $N \overset{f}{\Longrightarrow} M$ if f stretches N in the 'nominally unstable' direction, so that its projection onto 'unstable' direction in M covers in topologically nontrivial manner projection of M. In the 'nominally stable' direction N is contracted by f. As a result N is mapped across M in the unstable direction, without touching M^+. It is also very helpful to note that the degree w in the covering relation depends only on $A_{|\partial B_u(0,1)}$.

Definition 8 *[GiZ, Definition 7] Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. Assume that $g^{-1} : |M| \to \mathbb{R}^n$ is well defined and continuous. We say that $N \overset{g,w}{\Longleftarrow} M$ (N g-backcovers M with degree w) iff $M^T \overset{g^{-1},w}{\Longrightarrow} N^T$.*

The following theorem was proved in [GiZ].

Theorem 2 *[GiZ, Theorem 9] Assume N_i, $i = 0, \ldots, k$, $N_k = N_0$ are h-sets and for each $i = 1, \ldots, k$ we have either*

$$N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i \tag{7}$$

or $|N_i| \subset \mathrm{dom}\,(f_i^{-1})$ and

$$N_{i-1} \overset{f_i,w_i}{\Longleftarrow} N_i. \tag{8}$$

Then there exists a point $x \in \mathrm{int}\,|N_0|$, such that

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \in \mathrm{int}\,|N_i|, \quad i = 1, \ldots, k \tag{9}$$

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x \tag{10}$$

Obviously we cannot make any claim about the uniqueness of x in Theorem 2.

Theorem 2 shows that both the direct and the inverse covering relation can be treated on the same footing. This justifies the following definition.

Definition 9 *Assume N, M are h-sets and P is a continuous map. We say that*

$$N \overset{P,w}{\Longleftrightarrow} M$$

If one of the two following conditions is satisfied

$$N \subset \mathrm{dom}\,(P) \quad \textit{and} \quad N \overset{P,w}{\Longrightarrow} M$$
$$|M| \subset \mathrm{dom}\,(P^{-1}) \quad \textit{and} \quad N \overset{P,w}{\Longleftarrow} M.$$

We would like to stress, that the relation $N \overset{P,w}{\Longleftrightarrow} M$ is not symmetric.

3 The topological transversality theorem

The goal of this section is to state and prove the main topological transversality theorem for chain of covering relations. For this end we need first to define the notions of vertical and horizontal disks in an h-set.

Definition 10 *Let N be an h-set. Let $b : \overline{B_{u(N)}}(0,1) \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is* a horizontal disk in N *if there exists a continuous homotopy $h : [0,1] \times \overline{B_{u(N)}}(0,1) \to N_c$, such that*

$$\begin{aligned} h_0 &= b_c && (11)\\ h_1(x) &= (x,0), \qquad \text{for all } x \in \overline{B_{u(N)}}(0,1) && (12)\\ h(t,x) &\in N_c^-, \qquad \text{for all } t \in [0,1] \text{ and } x \in \partial\overline{B_{u(N)}}(0,1) && (13) \end{aligned}$$

Definition 11 *Let N be an h-set. Let $b : \overline{B_{s(N)}}(0,1) \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is* a vertical disk in N *if there exists a continuous homotopy $h : [0,1] \times \overline{B_{s(N)}}(0,1) \to N_c$, such that*

$$\begin{aligned} h_0 &= b_c \\ h_1(x) &= (0,x), \qquad \text{for all } x \in \overline{B_{s(N)}}(0,1) \\ h(t,x) &\in N_c^+, \qquad \text{for all } t \in [0,1] \text{ and } x \in \partial\overline{B_{s(N)}}(0,1). \end{aligned}$$

It is easy to see that b is the horizontal disk in N iff b is the vertical disk in N^T.

We would like to remark here that the horizontal disk in N can be at the same time also vertical in N. An example of such disk is shown on Fig. 1. In case homotopies used in the definitions of horizontal and vertical disks are different. The existence of such disks, which are both vertical and horizonal will play very important role in our method for detection of an infinite number symmetric periodic orbits for maps with reversal symmetry.

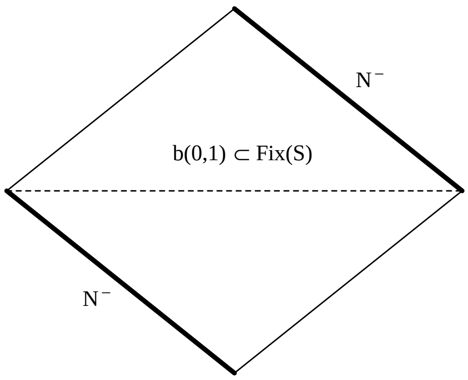

Figure 1: The curve b is both horizontal and vertical disk in N. In this example $u(N) = s(N) = 1$.

Now we are ready to state and prove the main topological transversality theorem. A simplified version of this theorem was given in [W] for the case of one unstable direction and covering relations chain without backcoverings. The argument in [W], which was quite simple and was based on the connectivity only, cannot be carried over to larger number of unstable directions or to the situation when both covering and backcovering relations are present.

Theorem 3 *Let $k \geq 1$. Assume N_i, $i = 0, \dots, k$, are h-sets and for each $i = 1, \dots, k$ we have either*

$$N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i \tag{14}$$

or $|N_i| \subset \operatorname{dom}(f_i^{-1})$ and

$$N_{i-1} \overset{f_j,w_i}{\Longleftarrow} N_i. \tag{15}$$

Assume that b_0 is a horizontal disk in N_0 and b_e is a vertical disk in N_k.

Then there exists a point $x \in \operatorname{int}|N_0|$, such that

$$\begin{aligned} x &= b_0(t), \quad \textit{for some } t \in B_{u(N_0)}(0,1) && (16)\\ f_i \circ f_{i-1} \circ \cdots \circ f_1(x) &\in \operatorname{int}|N_i|, \quad i = 1, \dots, k && (17)\\ f_k \circ f_{k-1} \circ \cdots \circ f_1(x) &= b_e(z), \quad \textit{for some } z \in B_{s(N_k)}(0,1) && (18) \end{aligned}$$

Proof: Without any loss generality we can assume that

$$c_{N_i} = \mathrm{Id}, \qquad \text{for } i = 0, \dots, k.$$

Then

$$f_i = f_{i,c}, \qquad \text{for } i = 1, \dots, k,$$
$$N_i = N_{c,i}, \quad N_i^{\pm} = N_{i,c}^{\pm} \quad \text{for } i = 0, \dots, k$$

We define $g_i = f_i^{-1}$, for those i for which we have the back-covering relation $N_{i-1} \overset{f_j,w_i}{\Longleftarrow} N_i$.

Notice that from the definition of covering relation, it follows immediately that there are $u \geq 0$, $s \geq 0$, such that $u(N_i) = u$ and $s(N_i) = s$, for all $i = 0, \dots, k$.

The idea of the proof is to rewrite our problem as a zero finding problem for a suitable map, then to compute its local Brouwer degree to infer the existence of a solution.

As a tool for keeping track of the occurrences of coverings and backcoverings, we define the map $\operatorname{dir} : \{1, \dots, k\} \to \{0, 1\}$ by $\operatorname{dir}(i) = 1$ if $N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i$ and $\operatorname{dir}(i) = 0$ if $N_{i-1} \overset{f_j,w_i}{\Longleftarrow} N_i$. For $i = 1, \dots, k$ let h_i be a homotopy map from the definition of covering relation for $N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i$ or $N_{i-1} \overset{f_j,w_i}{\Longleftarrow} N_i$. In the case of a direct covering (i.e. $\operatorname{dir}(i) = 1$), the homotopy h_i satisfies

$$\begin{aligned} h_i(0,x) &= f_i(x), \quad \text{where } x \in \mathbb{R}^{u+s}, && (19)\\ h_i(1,(p,q)) &= (A_i(p), 0), \quad \text{where } p \in \mathbb{R}^u \text{ and } q \in \mathbb{R}^s, && (20)\\ h_i([0,1], N_{i-1}^-) \cap N_i &= \emptyset, && (21)\\ h_i([0,1], N_{i-1}) \cap N_i^+ &= \emptyset. && (22) \end{aligned}$$

In the case of a backcovering (i.e. $\operatorname{dir}(i)=0$), the homotopy h_i satisfies

$$\begin{aligned} h_i(0,x) &= g_i(x), \quad \text{where } x\in\mathbb{R}^{u+s}, & (23)\\ h_i(1,(p,q)) &= (0,A_i(q)), \quad \text{where } p\in\mathbb{R}^u \text{ and } q\in\mathbb{R}^s, & (24)\\ h_i([0,1],N_i^+)\cap N_{i-1} &= \emptyset, & (25)\\ h_i([0,1],N_i)\cap N_{i-1}^- &= \emptyset. & (26) \end{aligned}$$

Let h_t and h_z be the homotopies appearing in the definition of the horizontal and vertical disk for b_0 and b_e, respectively.

It is enough to prove that there exists $t\in B_u(0,1)$, $z\in B_s(0,1)$ and $x_i\in \operatorname{int} N_i$ for $i=1,\dots,k$ such that

$$\begin{aligned} b_0(t) &= x_0,\\ f_i(x_{i-1}) &= x_i, \quad \text{if } \operatorname{dir}(i)=1,\\ g_i(x_i) &= x_{i-1}, \quad \text{if } \operatorname{dir}(i)=0,\\ b_e(z) &= x_k. \end{aligned} \tag{27}$$

We will treat (27) as a multidimensional system of equations to be solved. To this end, let us define

$$\Pi = \overline{B_u}(0,1)\times N_1\times\cdots\times N_{k-1}\times\overline{B_s}(0,1)$$

A point $x\in\Pi$ will be represented by $x=(t,x_1,\dots,x_{k-1},z)$.

We define a map $F=(F_1,\dots,F_k):\Pi\to\mathbb{R}^{(u+s)k}$ as follows: for $i=2,\dots,k-1$ we set

$$F_i(t,x_1,\dots,x_{k-1},z)=\begin{cases} x_i-f_i(x_{i-1}) & \text{if } \operatorname{dir}(i)=1,\\ x_{i-1}-g_i(x_i) & \text{if } \operatorname{dir}(i)=0.\end{cases}$$

For $i=1$ we set

$$F_1(t,x_1,\dots,x_{k-1},z)=\begin{cases} x_1-f_1(b_0(t)) & \text{if } \operatorname{dir}(1)=1,\\ b_0(t)-g_1(x_1) & \text{if } \operatorname{dir}(1)=0.\end{cases}$$

For $i=k$ we define

$$F_k(t,x_1,\dots,x_{k-1},z)=\begin{cases} b_e(z)-f_k(x_{k-1}) & \text{if } \operatorname{dir}(k)=1,\\ x_{k-1}-g_k(b_e(z)) & \text{if } \operatorname{dir}(k)=0.\end{cases}$$

With this notation, solving the system (27) is equivalent to solving the equation $F(x)=0$ in $\operatorname{int}\Pi$.

We define a homotopy $H=(H_1,\dots,H_k):[0,1]\times\Pi\to\mathbb{R}^{(u+s)k}$ as follows. For $i=2,\dots,k-1$ we set

$$H_i(\lambda,t,x_1,\dots,x_{k-1},z)=\begin{cases} x_i-h_i(\lambda,x_{i-1}) & \text{if } \operatorname{dir}(i)=1,\\ x_{i-1}-h_i(\lambda,x_i) & \text{if } \operatorname{dir}(i)=0.\end{cases}$$

For $i = 1$ we set

$$H_1(\lambda, t, x_1, \ldots, x_{k-1}, z) = \begin{cases} x_1 - h_1(\lambda, h_t(\lambda, t)) & \text{if dir}\,(1) = 1, \\ h_t(\lambda, t) - h_1(\lambda, x_1) & \text{if dir}\,(1) = 0. \end{cases}$$

For $i = k$ we define

$$H_k(\lambda, t, x_1, \ldots, x_{k-1}, z) = \begin{cases} h_z(\lambda, z) - h_k(\lambda, x_{k-1}) & \text{if dir}\,(k) = 1, \\ x_{k-1} - h_k(\lambda, h_z(\lambda, z)) & \text{if dir}\,(k) = 0. \end{cases}$$

Notice that $H(0, x) = F(x)$. The assertion of the theorem is a consequence of the following two lemmas, which will be proved after we complete the current proof.

Lemma 4 *For all $\lambda \in [0, 1]$ the local Brouwer degree* $\deg(H(\lambda, \cdot), \operatorname{int} \Pi, 0)$ *is well defined and does not depend on λ. Namely, for all $\lambda \in [0, 1]$ we have*

$$\deg(H(\lambda, \cdot), \operatorname{int} \Pi, 0) = \deg(H(1, \cdot), \operatorname{int} \Pi, 0).$$

Lemma 5

$$|\deg(H(1, \cdot), \operatorname{int} \Pi, 0)| = |w_1 \cdot w_2 \cdot \cdots \cdot w_k|$$

We continue the proof of Theorem 3. Since $F = H(0, \cdot)$, from the above lemmas it follows immediately that

$$\deg(F, \operatorname{int} \Pi, 0) = \deg(H(0, \cdot), \operatorname{int} \Pi, 0) = \deg(H(1, \cdot), \operatorname{int} \Pi, 0) \neq 0.$$

Hence there exists $x \in \Pi$ such that $F(x) = 0$. ∎

Proof of Lemma 4: From the homotopy property of the local Brouwer degree (see Appendix in [GiZ]) it is enough to prove that

$$H(\lambda, x) \neq 0, \quad \text{for all } x \in \partial(\Pi) \text{ and } \lambda \in [0, 1]. \tag{28}$$

In order to prove (28), let us fix $x = (t, x_1, \ldots, x_{k-1}, k) \in \partial\Pi$. It is easy to see that one of the following conditions must be satisfied

$$\begin{aligned} t &\in \partial B_u(0, 1), && (29) \\ z &\in \partial B_s(0, 1), && (30) \\ x_i &\in N_i^+, \quad \text{for some } i = 1, \ldots, k-1, && (31) \\ x_i &\in N_i^- \quad \text{for some } i = 1, \ldots, k-1. && (32) \end{aligned}$$

We will deal with all above cases separately.

Consider first (29). Let us fix $\lambda \in [0, 1]$. Let $x_0 = h_t(\lambda, t)$. From Def. 10 it follows that $x_0 \in N_0^-$. There are now two possibilities: either dir $(1) = 1$ (direct covering) or dir $(1) = 0$ (backcovering). Assume that dir $(1) = 1$. From condition (21) it follows that $h_1(\lambda, x_0) \notin N_1$, hence $H_1(\lambda, x) \neq 0$. Assume now that dir $(1) = 0$. We have $h_t(\lambda, t) \in N_0^-$ and since by (26) $h_1(\lambda, N_1) \cap N_0^- = \emptyset$, hence $H_1(\lambda, x) \neq 0$.

Consider now (30). Let us fix $\lambda \in [0,1]$ and let $x_k = h_z(\lambda, z)$. From Def. 11 it follows that $x_k \in N_k^+$. Now if dir $(k) = 1$, then from condition (22) $h_k(\lambda, N_{k-1}) \cap N_k^+ = \emptyset$, hence $H_k(\lambda, x) \neq 0$. If dir $(k) = 0$, then from (25) if follows, that $h_k(\lambda, x_k) \notin N_{k-1}$, hence $H_k(\lambda, x) \neq 0$.

For each of cases (31) and (32) we have to consider the following four possibilities

$$N_{i-1} \overset{f_i}{\Longrightarrow} N_i \overset{f_{i+1}}{\Longrightarrow} N_{i+1}, \tag{33}$$

$$N_{i-1} \overset{f_i}{\Longrightarrow} N_i \overset{f_{i+1}}{\Longleftarrow} N_{i+1}, \tag{34}$$

$$N_{i-1} \overset{f_i}{\Longleftarrow} N_i \overset{f_{i+1}}{\Longrightarrow} N_{i+1}, \tag{35}$$

$$N_{i-1} \overset{f_i}{\Longleftarrow} N_i \overset{f_{i+1}}{\Longleftarrow} N_{i+1}. \tag{36}$$

Assume first that $x_i \in N_i^+$. If (33) or (34) holds true, then from (22) we obtain

$$h_i(\lambda, x_{i-1}) \neq x_i,$$

for every $\lambda \in [0,1]$ and every $x_{i-1} \in N_{i-1}$. If (35) or (36) is satisfied, then from (25) it results that

$$h_i(\lambda, x_i) \neq x_{i-1},$$

for every $\lambda \in [0,1]$ and every $x_{i-1} \in |N_{i-1}|$. This proves if $x_i \in N_i^+$, then $H(\lambda, x) \neq 0$ for any $\lambda \in [0,1]$.

Assume now that $x_i \in N_i^-$. If (33) or (35) holds true, then from (21) it follows that for every $\lambda \in [0,1]$ and every $x_{i+1} \in N_{i+1}$ we have

$$h_{i+1}(\lambda, x_i) \neq x_{i+1}.$$

If (34) or (36) is satisfied, then from (26) we obtain

$$h_{i+1}(\lambda, x_{i+1}) \neq x_i,$$

for every $\lambda \in [0,1]$ and every $x_{i+1} \in N_{i+1}$. This proves that if $x_i \in N_i^-$, then $H(\lambda, x) \neq 0$ for any $\lambda \in [0,1]$. ∎

Proof of Lemma 5: Let us represent x_i for $i = 1, \ldots, k-1$ as a pair $x_i = (p_i, q_i)$, where $p_i \in \mathbb{R}^u$ and $q_i \in \mathbb{R}^s$. In this representation the map $H(1, t, p_1, q_1, \ldots, p_{k-1}, q_{k-1}, z) = (\tilde{p}_1, \tilde{q}_1, \ldots, \tilde{p}_k, \tilde{q}_k)$ has the following form (for $\alpha = 0$)

- if $i = 2, \ldots, k-1$ then

$$\text{if dir}(i) = 1, \text{ then} \qquad \tilde{p}_i = (1-\alpha)p_i - A_i(p_{i-1}), \qquad \tilde{q}_i = q_i, \tag{37}$$

$$\text{if dir}(i) = 0, \text{ then} \qquad \tilde{p}_i = p_{i-1}, \qquad \tilde{q}_i = (1-\alpha)q_{i-1} - A_i(q_i) \tag{38}$$

- if $i = 1$, then

$$\text{if dir}\,(1) = 1, \text{ then} \qquad \tilde{p}_1 = (1-\alpha)p_1 - A_1(t), \qquad \tilde{q}_1 = q_1 \tag{39}$$

$$\text{if dir}\,(1) = 0, \text{ then} \qquad \tilde{p}_1 = t, \qquad \tilde{q}_1 = -A_1(q_1) \tag{40}$$

- if $i = k$, then

$$\text{if dir}\,(k) = 1, \text{ then} \qquad \tilde{p}_k = -A_k(p_{k-1}), \qquad \tilde{q}_k = z \tag{41}$$

$$\text{if dir}\,(k) = 0, \text{ then} \qquad \tilde{p}_k = p_{k-1}, \qquad \tilde{q}_k = (1-\alpha)q_{k-1} - A_k(z) \tag{42}$$

The above equations define a homotopy $C : [0,1] \times \Pi \to \mathbb{R}^{(u+s)k}$. We will show that $\deg(C(\alpha,\cdot), \operatorname{int}\Pi, 0)$ is independent of α and then we compute the degree of $C(1,\cdot)$.

Lemma 6 *For any* $\alpha \in [0,1]$

$$\deg(C(\alpha,\cdot), \operatorname{int}\Pi, 0) = \deg(C(1,\cdot), \operatorname{int}\Pi, 0).$$

Proof: From the homotopy property of the local degree (see Appendix in [GiZ]), it follows that it is enough to prove that

$$C(\alpha, x) \neq 0, \quad \text{for all } x \in \partial\Pi \text{ and } \alpha \in [0,1]. \tag{43}$$

Let us take $x = (t, p_1, q_1, \ldots, p_{k-1}, q_{k-1}, z) \in \partial\Pi$. One of the following conditions holds true

$$\begin{aligned} t &\in S^u, \\ z &\in S^s, \\ p_i &\in S^u, \quad \text{for some } i = 2, \ldots, k-1 \\ q_i &\in S^s, \quad \text{for some } i = 2, \ldots, k-1. \end{aligned}$$

Assume that $t \in S^u$. If dir $(1) = 1$, then $\|A_1(t)\| > 1$, hence $\|\tilde{p}_1\| \geq \|A_1(t)\| - \|p_1\| > 0$. If dir $(1) = 0$, then $\tilde{p}_1 = t \neq 0$.

Assume that $z \in S^s$. If dir $(k) = 1$, then $\tilde{q}_k = z \neq 0$. If dir $(k) = 0$, then $\|A_k(z)\| > 1$ and we obtain $\|\tilde{q}_k\| \geq \|A_k(z)\| - \|q_{k-1}\| > 0$.

Assume that $p_i \in S^u$. If dir $(i+1) = 1$, then $\tilde{p}_{i+1} \neq 0$, because from condition (6) it follows that

$$\|A_{i+1}(p_i)\| > 1 \geq \|(1-\alpha)p_{i+1}\|, \tag{44}$$

for any $p_{i+1} \in \overline{B_u}(0,1)$.

If dir $(i+1) = 0$, then obviously $\tilde{p}_{i+1} = p_i \neq 0$.

The argument for the case $q_i \in S^s$ is similar. ∎

Now we turn to the computation of the degree of $C(1,\cdot)$. Observe that $C(1,\cdot)$ has the following form: for $i = 2, \ldots, k-1$

$$\tilde{p}_i = -A_i(p_{i-1}),\ \tilde{q}_i = q_i \quad \text{if dir}\,(i) = 1 \tag{45}$$

$$\tilde{p}_i = p_{i-1},\ \tilde{q}_i = -A_i(q_i) \quad \text{if dir}\,(i) = 0, \tag{46}$$

for $i = 1$

$$\tilde{p}_1 = -A_1(t),\ \tilde{q}_1 = q_1 \quad \text{if dir}\,(1) = 1 \tag{47}$$

$$\tilde{p}_1 = t,\ \tilde{q}_1 = -A_1(q_1) \quad \text{if dir}\,(1) = 0, \tag{48}$$

and for $i = k$

$$\tilde{p}_k = -A_k(p_{k-1}),\ \tilde{q}_k = z \quad \text{if dir}\,(1) = 1 \tag{49}$$

$$\tilde{p}_k = t,\ \tilde{q}_k = -A_k(z) \quad \text{if dir}\,(1) = 0. \tag{50}$$

From the product property of the degree (see Appendix in [GiZ]) it follows that

$$|\deg(C(1,\cdot),\Pi,0)| = \\ \left|\Pi_{i\in \text{dir}^{-1}(1)} \deg(-A_i, \overline{B_u}(0,1),0) \cdot \Pi_{i\in\text{dir}^{-1}(0)} \deg(-A_i, \overline{B_s}(0,1),0)\right|.$$

In the formula above if $\text{dir}^{-1}(j) = \emptyset$ (for $j = 0,1$), then the corresponding product is set to be equal to 1. Similarly if $u = 0$ or $s = 0$, then the corresponding product is also set equal to 1.

From Collorary 18 in [GiZ] it follows that

$$\deg(-A, U, 0) = (-1)^u \deg(A, U, 0). \tag{51}$$

This finishes the proof. ∎

4 Reversing symmetry and covering relations.

In this section we apply the tools developed in previous sections to the study of symmetric periodic orbits for reversible maps.

Theorem 7 *Let S be a reversing symmetry for the local dynamical system induced by the map P. Assume that*

$$M_0 \overset{P}{\Longleftrightarrow} M_1 \overset{P}{\Longleftrightarrow} M_2 \cdots \overset{P}{\Longleftrightarrow} M_{n-1} \overset{P}{\Longleftrightarrow} M_n,$$

and

1. *there exists a horizontal disk in M_0 contained in* $\text{Fix}\,(S)$
2. *there exists a vertical disk in M_n contained in* $\text{Fix}\,(S)$

Then there exists $x \in M_0$, such that

$$S(x) = x,\ P^{2n}(x) = x$$
$$\textit{the orbit of } x \textit{ is } S\textit{-symmetric}$$
$$P^i(x) \in M_i \textit{ for } i = 1,\ldots,n,$$
$$P^{n+i}(x) \in S(M_{n-i}) \textit{ for } i = 1,\ldots,n.$$

Proof: From Theorem 3 it follows that there exists $x \in \mathrm{Fix}\,(S)$, such that

$$P^i(x) \in M_i \text{ for } i = 1, \ldots, n,$$
$$P^n(x) \in \mathrm{Fix}\,(S).$$

The assertion now follows from the reversing symmetry of P. ∎

Now we turn our attention to the action of symmetry on h-sets and covering relations.

Definition 12 *Let N be an h-set in $\mathbb{R}^n$. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism. We define an h-set $L * N$ as follows*

- $|L * N| = L(|N|)$,
- $u(L * N) = u(N)$ *and* $s(L * N) = s(N)$,
- $c_{L*N} = c_N \circ L^{-1}$.

We define an h-set $L^T * N$ by

$$L^T * N = (L * N)^T$$

Informally speaking, $L * N$ is just a natural symmetric image of N and $L^T * N$ is the symmetric image of N, but we additionally switch the 'expanding' and 'contracting' directions.

We have the following

Lemma 8 *Let $S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a reversing symmetry for a map P and $N \overset{P}{\Longleftrightarrow} M$, then $S^T * M \overset{P}{\Longleftrightarrow} S^T * N$.*

Proof: From the definition of covering relations and reversing symmetry it follows immediately, that

$$\text{if} \quad N \overset{P}{\Longrightarrow} M, \quad \text{then} \quad S^T * M \overset{P}{\Longleftarrow} S^T * N$$
$$\text{if} \quad N \overset{P}{\Longleftarrow} M, \quad \text{then} \quad S^T * M \overset{P}{\Longrightarrow} S^T * N$$

∎

The reversing symmetry maps, in a natural way, horizontal disks to vertical disks and vice versa. Namely, we have the following obvious lemma.

Lemma 9 *Let $S : \mathbb{R}^n \to \mathbb{R}^n$ a homeomorphism, N be an h-set and γ be a horizontal (vertical) disk in N.*

Then $S(\gamma)$ is a vertical (horizontal) disk in $S^T * N$.

In the context of proving the existence of an infinite number of symmetric periodic orbits symmetric h-sets are of special importance.

Definition 13 *Let S be a reversing symmetry. We say that an h-set N is* S^T-symmetric *if $S^T * N = N$.*

It is easy to see, that if N is S^T-symmetric h-set, then $u(N) = s(N)$ and the dimension of the phase space $n = u(N) + s(N)$ must be even.

The following theorem, which is an easy consequence of Theorem 7, illustrates our method of proving of the existence of an infinite number of symmetric periodic orbits.

Theorem 10 *Let S be a reversing symmetry for the local dynamical system induced by the map P. Let M_i for $i = 0, 1, 2, 3$ be h-sets and $M_i \cap M_j = \emptyset$ for $i \neq j$. Assume that*

$$M_0 \overset{P}{\Longleftrightarrow} M_1 \overset{P}{\Longleftrightarrow} M_2,$$
$$M_0 \overset{P}{\Longleftrightarrow} M_0$$
$$M_0 \textit{ and } M_2 \textit{ are } S^T\textit{-symmetric}$$
$$M_3 = S^T * M_1$$
$$\textit{there exists a horizontal disk in } M_0 \textit{ contained in } \mathrm{Fix}\,(S)$$
$$\textit{there exists a horizontal disk in } M_2 \textit{ contained in } \mathrm{Fix}\,(S)$$

Then for any sequence $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \{0, 1, 2, 3\}^{n+1}$, satisfying the following conditions

$$\alpha_0 \in \{0, 2\}, \quad \alpha_n \in \{0, 2\}$$
$$\textit{if } \alpha_i = 0, \textit{ then } \quad \alpha_{i+1} \in \{0, 1\}$$
$$\textit{if } \alpha_i = 1, \textit{ then } \quad \alpha_{i+1} = 2$$
$$\textit{if } \alpha_i = 2, \textit{ then } \quad \alpha_{i+1} = 3$$
$$\textit{if } \alpha_i = 3, \textit{ then } \quad \alpha_{i+1} = 1,$$

there exists $x \in |M_{\alpha_0}|$, such that

$$S(x) = x, \; P^{2n}(x) = x$$
$$\textit{the orbit of } x \textit{ is } S\textit{-symmetric}$$
$$P^i(x) \in |M_i| \textit{ for } i = 1, \ldots, n,$$
$$P^{n+i}(x) \in S(|M_{n-i}|) \textit{ for } i = 1, \ldots, n.$$

Proof: From Lemma 9 it follows that in M_0 and M_2 there exist vertical disks contained in $\mathrm{Fix}\,(S)$. We can now apply Theorem 7 to the chain of covering relations

$$M_{\alpha_0} \overset{P}{\Longleftrightarrow} M_{\alpha_1} \overset{P}{\Longleftrightarrow} \cdots \overset{P}{\Longleftrightarrow} M_{\alpha_n}$$

∎

5 One four-dimensional reversible example.

In this section we present the application of the method introduced throughout the paper to a four-dimensional reversible map. As a consequence we obtain the existence of chaotic dynamics and the existence of an infinite number of symmetric periodic orbits for a certain iteration of such map. The proof is computer assisted, i.e., rigorous numerics is used to verify assumptions of abstract theorems. The main feature, which makes this example interesting is the fact that both stable and unstable directions are two-dimensional and the map itself is not close to a product of two two-dimensional maps, with unstable dimension each.

5.1 An example of four-dimensional reversible map.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $n > 0$ be a continuous map, $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a map defined by

$$F \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y + f(x) \\ x \end{bmatrix}$$

and let $S(x, y) = (y, x)$. It is straightforward to show that $S \circ F \circ S \circ F = \mathrm{Id}$. Therefore F is a reversible homeomorphism of $\mathbb{R}^{2n}$. In suitable coordinates we may rewrite F as follows

$$F \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y + \frac{1}{2}f(x+y) \\ x + \frac{1}{2}f(x+y) \end{bmatrix} \tag{52}$$

and the reversing symmetry S will be given by $S(x, y) = (-x, y)$.

Let us fix $n = 2$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1(1-x_1) + 4 - x_2 \\ x_2(1-x_2) + 4 + x_1 \end{bmatrix}. \tag{53}$$

In the remainder of this section we will investigate the map F given by (52), where f is as above. The map $S(x_1, x_2, y_1, y_2) = (-x_1, -x_2, y_1, y_2)$ is a reversing symmetry of F.

It is easy to verify that each solution of

$$F(x_1, x_2, y_1, y_2) = (x_1, x_2, y_1, y_2) \tag{54}$$

satisfies $x_1 = x_2 = 0$, hence all fixed points of F are symmetric. Solving of Eq. (54) leads to the following system of equations

$$\begin{cases} y_1^2 + (y_2+1)^2 = 9 \\ (y_1+1)^2 - y_2^2 = 1 \end{cases}$$

describing an intersection of a circle with a hyperbola in four points as was shown in Fig. 2. Fixed points P_1, P_2 are hyperbolic with real eigenvalues, the fixed point P_3 is hyperbolic with four complex eigenvalues, the point P_4 possesses two complex eigenvalues on the unit circle and two real eigenvalues, i.e., it is of elliptic-hyperbolic type.

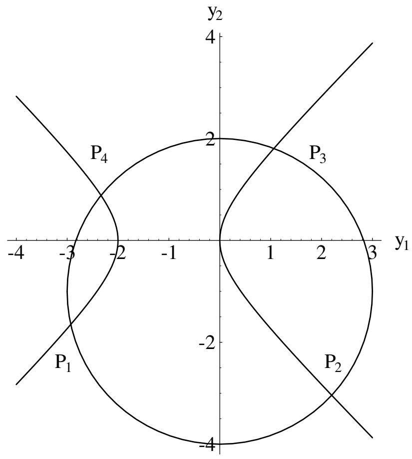

Figure 2: The location of the fixed points of F.

The fixed points may be exactly computed (for example using Mathematica). However, for our further consideration it is sufficient to use two approximate fixed points, which we will still denote by P_i, given by

$$
\begin{aligned}
P_1 &= (0, 0, -2.9288690017630725, -1.649404627725545), & (55)\\
P_2 &= (0, 0, 2.199939462565084, -3.0396731015162355). & (56)
\end{aligned}
$$

We will show that in the vicinity of P_1 and P_2 the map F^7 has symbolic dynamics on two symbols and there exist an infinite number of symmetric periodic points of an arbitrary large period.

Before we proceed with the statement of main results for F we need to discuss how we represent h-sets.

5.2 Representation of h-sets in $\mathbb{R}^n$.

To define a h-set N we need to specify a homeomorphism c_N of $\mathbb{R}^n$ and two numbers $u(N)$ and $s(N)$ (see Def. 5). Since we will use the computer in order to verify covering relations, the homeomorphism must be representable by the machine. The simplest case is to take an affine map. We will use the maximum norm on $\mathbb{R}^n$, i.e. $\|x\| = \max_i |x_i|$ and we treat vectors as columns.

Let $x, u_1, \ldots, u_k, s_1, \ldots, s_{n-k} \in \mathbb{R}^n$, $0 \leq k \leq n$ be such that the vectors $u_1, \ldots, u_k, s_1, \ldots, s_{n-k}$ are linearly independent. We define a matrix $M \in \mathbb{R}^{n \times n}$ by

$$M = [u_1, \ldots, u_k, s_1, \ldots, s_{n-k}].$$

We define a h-set

$$N = \mathfrak{h}(x, u_1, \ldots, u_k, s_1, \ldots, s_{n-k})$$

as follows

$$
\begin{aligned}
u(N) &= k,\\
s(N) &= n-k,\\
|N| &= M(\overline{B_n}(0,1))+x = M(\overline{B_{u(N)}}(0,1)\times\overline{B_{s(N)}}(0,1))+x,\\
c_N(v) &= M^{-1}(v-x), \quad \text{for } v\in\mathbb{R}^n
\end{aligned}
$$

Hence, the h-set N defined above is a parallelepiped centered at x.

In the sequel we will work in $\mathbb{R}^{2k}$ with $s(N)=u(N)=k$. In this case we will use also the notation $N=\mathfrak{h}(x,M)$, where M is a linear isomorphism of $\mathbb{R}^{2k}$. The first k columns of M correspond to unstable directions and the last k columns of M correspond to stable directions.

5.3 Important h-sets and covering relations between them

As was mentioned before P_1 and P_2 are good numerical approximations to two hyperbolic fixed points with two-dimensional stable and unstable manifolds. We choose $u^i_j, s^i_j\in\mathbb{R}^4$, $i,j=1,2$ to be a good numerical approximations of unstable and stable eigenvectors of $DF(P_i)$. Put

$$
\begin{aligned}
u^1_1 &= \begin{bmatrix}0.527847408170044\\0.254065286036574\\0.730261232439584\\0.351491787265563\end{bmatrix}, \quad u^2_1 = \begin{bmatrix}-0.05726452423754\\0.594572575636284\\-0.0768865282444865\\0.7983061369797889\end{bmatrix},\\
u^1_2 &= \begin{bmatrix}0.233876807615845\\0.485903716548415\\0.365235930520818\\0.758816138574061\end{bmatrix}, \quad u^2_2 = \begin{bmatrix}0.8918103319483236\\-0.0858921121857865\\0.4421352370808943\\-0.0425829663821858)\end{bmatrix}
\end{aligned}
\tag{57}
$$

and put $s^j_i = S(u^j_i)$, $i,j=1,2$. We define matrices M_i for $i=1,2$ by

$$
M_1 = 0.012[u^1_1,u^1_2,s^1_1,s^1_2], \quad M_2 = 0.31[u^2_1,u^2_2,s^2_1,s^2_2]. \tag{58}
$$

We define two S^T symmetric h-sets centered at P_1 and P_2 by

$$
N_1=\mathfrak{h}(P_1,M_1), \quad N_2=\mathfrak{h}(P_2,M_2). \tag{59}
$$

The important remark is that the sets $|N_1|$ and $|N_2|$ are disjoint. The projection of $|N_1|$ and $|N_2|$ onto (y_1,y_2) coordinates is presented in Fig. 3. The following lemma was proved with a computer assistance.

Lemma 11 *The following covering relations hold:*

$$
N_1 \overset{F,1}{\Longrightarrow} N_1, \quad N_2 \overset{F,-1}{\Longrightarrow} N_2.
$$

The details of the proof will be presented in Section 6.

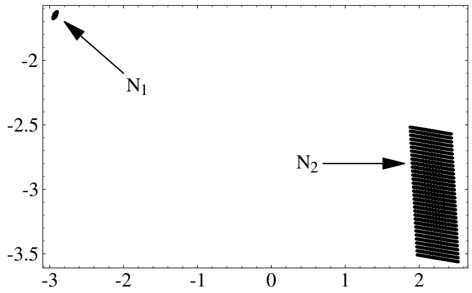

Figure 3: The projection of the sets $|N_1|$ and $|N_2|$ onto (y_1, y_2) coordinates.

Remark 12 *From above lemma and Theorem 2 it follows immediately, that there exists fixed points in N_1 and N_2. However, we cannot claim that they are unique in N_i nor S-symmetric.*

Now we will construct a dynamical link between sets N_1 and N_2. More precisely, we construct a chain of covering relations connecting N_1 and N_2. For this purpose we look for a point Q_1 in the neighborhood of P_1 on the unstable manifold of P_1 and such that $F^k(Q_1)$ (here $k = 10$) is close to P_2. Then we define additional points Q_i by taking some forward iterates of Q_1.

To be specific we set

$$\begin{aligned} Q_1 &= P_1 + 0.0330092432u_1^1 - 0.048949u_1^2 + 0.0004931s_1^1, \\ Q_2 &= F^4(Q_1), \\ Q_3 &= F(Q_2). \end{aligned}$$

We have

$$\begin{gathered} F^{-1}(Q_1) \in |N_1|, \\ \|F^{-1}(Q_1) - P_1\| < 0.006, \\ \|F^{10}(Q_1) - P_2\| < 0.001. \end{gathered}$$

The points Q_1, Q_2 and Q_3 will be used as the centers of new h-sets. We define

$$\begin{aligned} H_1 &= \mathfrak{h}(Q_1, .001u_1^1, .00175u_2^1, .005s_1^1, 0.005s_2^1), \\ H_2 &= \mathfrak{h}(Q_2, .28u_1^2, .28u_2^2, .2s_1^2, .38s_2^2), \\ H_3 &= \mathfrak{h}(Q_3, .15u_1^2, .15u_2^2, .12s_1^2, .42s_2^2). \end{aligned}$$

Lemma 13 *The following covering relations hold*

$$N_1 \overset{F,1}{\Longrightarrow} H_1 \overset{F^4,-1}{\Longrightarrow} H_2 \overset{F,-1}{\Longrightarrow} H_3 \overset{F,-1}{\Longrightarrow} N_2.$$

The details of the proof will be presented in Section 6.

Let us comment briefly about the spatial relations between N_i and H_j. The set H_1 is close to N_1, this is the reason for using the same stable and unstable directions in the definitions of these sets. Sets H_2 and H_3 are close to N_2 and as in the previous case we used the same stable and unstable directions for them.

The numerical evidence of the existence of covering relation $H_1 \overset{F^4,-1}{\Longrightarrow} H_2$ is presented in Fig. 4.

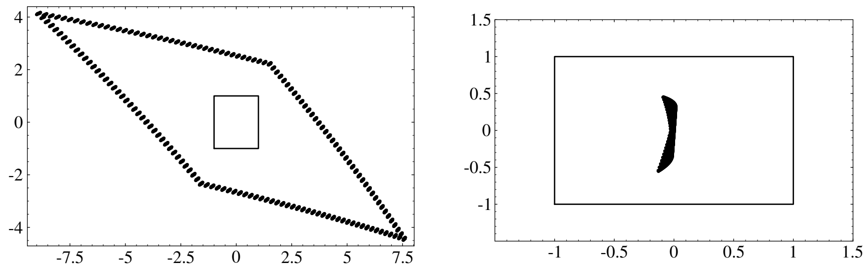

Figure 4: The numerical evidence of the existence of covering relation $H_1 \overset{F^4,-1}{\Longrightarrow} H_2$, (left) the set $F_c^4((H_1)_c^-)$ projected onto unstable directions of $(H_2)_c$ is outside the unit ball, (right) the set $F_c^4((H_1)_c)$ projected onto stable directions of $(H_2)_c$ is inside the unit ball.

5.4 Chaotic dynamics of F.

Theorem 14 *The discrete dynamical system induced by the map F^7 is semiconjugated with the full shift on two symbols, i.e. for an arbitrary $(i_j)_{j\in\mathbb{Z}} \in \{1,2\}^{\mathbb{Z}}$ there exists a point $x_0 \in |N_{i_0}|$ such that*

$$F^{7j}(x_0) \in |N_{i_j}|, \quad j \in \mathbb{Z}. \tag{60}$$

Moreover, if the sequence $(i_j)_{j\in\mathbb{Z}}$ is periodic, then the point may be chosen as a periodic point for F^7 with the same principal period.

Proof: From Lemma 13 we obtain that

$$N_1 \overset{F,1}{\Longrightarrow} H_1 \overset{F^4,-1}{\Longrightarrow} H_2 \overset{F,-1}{\Longrightarrow} H_3 \overset{F,-1}{\Longrightarrow} N_2. \tag{61}$$

Since the h-sets N_1 and N_2 are symmetric (by their definition), the reversing symmetry property of F implies

$$\begin{aligned} N_2 = S^T \star N_2 \overset{F,-1}{\Longleftarrow} S^T \star H_3 \overset{F,-1}{\Longleftarrow} S^T \star H_2 \\ S^T \star H_2 \overset{F^4,-1}{\Longleftarrow} S^T \star H_1 \overset{F,1}{\Longleftarrow} S^T \star N_1 = N_1. \end{aligned} \tag{62}$$

From Lemma 11 we get

$$N_1 \overbrace{\overset{F,1}{\Longrightarrow} N_1 \overset{F,1}{\Longrightarrow} \cdots \overset{F,1}{\Longrightarrow}}^{7 \text{ times}} N_1, \tag{63}$$

$$N_2 \overbrace{\overset{F,-1}{\Longrightarrow} N_2 \overset{F,-1}{\Longrightarrow} \cdots \overset{F,-1}{\Longrightarrow}}^{7 \text{ times}} N_2\,. \tag{64}$$

Let $(i_j)_{j\in\mathbb{Z}} \in \{1,2\}^{\mathbb{Z}}$ be a periodic sequence od symbols, i.e. $i_{j+k} = i_j$ for $j \in \mathbb{Z}$ and a certain $k > 0$. Let

$$N_{i_0} \Longleftrightarrow N_{i_1} \cdots \Longleftrightarrow N_{i_k} = N_{i_0} \tag{65}$$

be a periodic sequence of covering relations, where by $N_{i_j} \Longleftrightarrow N_{i_{j+1}}$ we mean a corresponding sequence (61-64). Now, Theorem 2 implies that there exists a k-periodic point $x_0 \in |N_{i_0}|$ for F^7, such that assertion (60) is satisfied.

Let $(i_j)_{j\in\mathbb{Z}} \in \{1,2\}^{\mathbb{Z}}$ be a nonperiodic sequence of symbols. We define the periodic sequences

$$(i^k_j)_{j\in\mathbb{Z}} = \{\ldots,\{i_{-k},\ldots,i_0,\ldots,i_k\},\{i_{-k},\ldots,i_0,\ldots,i_k\},\ldots\} \in \{1,2\}^{\mathbb{Z}}$$

where $i^k_0 = i_0$, $k > 0$. Now, for $k > 0$ we can find $(2k+1)-$periodic point x_k for F^7 such that $F^{7j}(x_k) \in |N_{i^k_j}|$, $k > 0$, $j \in \mathbb{Z}$. Since N_{i_0} is a compact set, we can find a condensation point $x_0 \in |N_{i_0}|$ of $\{x_k\}_{k>0}$. Obviously, x_0 satisfies assertion (60). ∎

5.5 Symmetric periodic points for F.

Theorem 15 *There exists an infinite number of symmetric periodic points for F with an arbitrary large principal periods.*

The proof of Theorem 15 is a direct consequence of the following lemma.

Lemma 16 *Let*

$$V_0 \overset{g_0}{\Longleftrightarrow} V_1 \overset{g_1}{\Longleftrightarrow} \cdots \overset{g_{k-1}}{\Longleftrightarrow} V_k$$

be a sequence of covering relations, where

$$\begin{aligned} V_0, V_k &\in \{N_1, N_2\}, \\ V_i \in \{N_1, N_2, H_1, H_2, H_3\}, &\quad \text{for } j = 1,\ldots,k-1, \\ g_i = F \text{ or } g_i = F^4, &\quad \text{for } i = 0,\ldots,k-1. \end{aligned}$$

Then there exists a symmetric periodic point x_0 of F, such that

$$\begin{aligned} x_0 &\in |V_0| \cap \mathrm{Fix}\,(S), \\ (g_i \circ \cdots \circ g_1 \circ g_0)(x_0) \in |V_{i+1}|, &\quad i = 0,\ldots,k-2, \\ (g_{k-1} \circ \cdots \circ g_1 \circ g_0)(x_0) &\in |V_k| \cap \mathrm{Fix}\,(S) \end{aligned}$$

Proof: Recall, that the sets N_1, N_2 are symmetric and defined by vectors u^j_i, s^j_i, $i,j = 1,2$ – see(57) and (59). For $i = 1,2$ we define a map $b_i : \overline{B_2}(0,1) \longrightarrow |N_i|$ by

$$b_i(p,q) \quad = \quad M_i[p,q,p,q]^T + P_i, \quad i = 1,2.$$

We will show, that b_i is a horizontal disk in N_i.

First observe that

$$b_{i,c} = c_{N_i} \circ b_i(p,q) = (p,q,p,q). \tag{66}$$

We define the homotopy h_i by

$$h_i(t,p,q) = (p,q,(1-t)p,(1-t)q).$$

It is easy to see, that conditions (11-13) from Definition 10 are satisfied. Namely, we have

$$\begin{aligned}
h_i(0,p,q) &= (p,q,p,q) = (b_i)_c(p,q), \quad \text{for } (p,q) \in \overline{B_2}(0,1), \\
h_i(1,p,q) &= (p,q,0,0), \quad \text{for } (p,q) \in \overline{B_2}(0,1), \\
h_i(t,p,q) &\in (N_i)_c^- = \partial\overline{B_2}(0,1) \times \overline{B_2}(0,1), \quad \text{for } (p,q) \in \partial\overline{B_2}(0,1), t \in [0,1].
\end{aligned}$$

This proves that b_i is a horizontal disk in N_i.

Let us remind the reader that $S(P_i) = P_i$ and $S(u^i_j) = s^i_j$. Hence we obtain

$$\begin{aligned}
S(b_i(p,q)) = S(k_i(pu^i_1 + qu^i_2 + ps^i_1 + qs^i_2) + P_i) = \\
= k_i\left(pS(u^i_1) + qS(u^i_2) + pS(s^i_1) + qS(s^i_2)\right) + S(P_i) = b_i(p,q).
\end{aligned}$$

where $k_1 = 0.012$, $k_2 = 0.31$ are the coefficients used in (58) to define N_i. This proves that b_i are contained in $\mathrm{Fix}\,(S)$.

We will show that b_i is also vertical disk in N_i. Namely, if follows from Lemma 9 that $S(b_i) = b_i$ is a vertical disk in $S^T * N_i = N_i$.

Since $V_0, V_k \in \{N_1, N_2\}$ we conclude that there exists a horizontal disk contained in $\mathrm{Fix}\,(S) \cap |V_0|$ and there exists vertical disk contained in $\mathrm{Fix}\,(S) \cap |V_k|$. Now, the assertion is a direct consequence of Theorem 3. ∎

Proof of Theorem 15: The assertion follows from the fact that the sets $|N_1|$ and $|N_2|$ are disjoint and we can construct an arbitrary number of sequences satisfying assumptions of Lemma 16, for example

$$\overbrace{N_1 \overset{F,1}{\Longrightarrow} N_1 \overset{F,1}{\Longrightarrow} \ldots \overset{F,1}{\Longrightarrow} N_1}^{k \text{ times}} \overset{F,1}{\Longrightarrow} H_1 \overset{F^4,-1}{\Longrightarrow} H_2 \overset{F,-1}{\Longrightarrow} H_3 \overset{F,-1}{\Longrightarrow} H_4 \overset{F,-1}{\Longrightarrow} N_2,$$

for $k > 0$. ∎

6 How to verify covering relations with a computer assistance.

In this section we discuss some numerical aspects of the verification of covering relations.

Let N, M be a h-sets in $\mathbb{R}^n$ such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$ and let $f : |N| \to \mathbb{R}^n$ be a continuous. In order to prove that the covering relation $N \overset{f}{\Longrightarrow} M$ holds, it is necessary to find the homotopy $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$ and a map $A : \mathbb{R}^u \to \mathbb{R}^u$ satisfying conditions (2-5).

Since in our example $f_c = c_M \circ f \circ c_N^{-1}$ is a diffeomorphism we can try to find a homotopy between f_c and its derivative computed in the center of set and projected onto unstable directions, i.e., we define

$$A : \mathbb{R}^u \ni p \to \pi_u(Df_c(0)(p,0)) \in \mathbb{R}^u,$$

where $\pi_u : \mathbb{R}^n \to \mathbb{R}^u$ is a projection onto first u variables. We require A to be an isomorphism, then we have

$$\deg(A, \overline{B_u}(0,1), 0) = \operatorname{sgn}(\det A) = \pm 1.$$

Now we define the homotopy between f_c and $(p,q) \to (A(p), 0)$ by

$$h(t,p,q) = (1-t)f_c + t(A(p),0), \quad \text{for } (p,q) \in \overline{B_u}(0,1) \times \overline{B_s}(0,1). \tag{67}$$

Obviously the homotopy (67) satisfies conditions (2) and (5). We need to check if the homotopy (67) satisfies conditions (3-4).

Below we describe precise algorithms.

Definition 14 *Let $U \subset \mathbb{R}^n$ be a bounded set. We say that $\mathcal{G} \subset 2^{\mathbb{R}^n}$ is a grid of U if*

1. *$\mathcal{G}$ is a finite set*
2. *$U \subset \bigcup_{G \in \mathcal{G}} G$*
3. *each $G \in \mathcal{G}$ can be represent in a computer*

Definition 15 *Let $U \subset \mathbb{R}$ be a bounded set. By $(U)_I$ we denote the interval enclosure of the set, i.e., the set $(U)_I$ is the smallest representable interval containing U or $[-\infty, \infty]$ if there is not a representable interval containing U.*

Let $U \subset \mathbb{R}^n$ be a bounded set. By $(U)_I$ we denote $(\pi_1(U))_I \times \cdots \times (\pi_n(U))_I$ where π_i is a projection onto i-th variable.

In the algorithms presented bellow all computations are performed in interval arithmetic [Mo].

First we discuss how we check condition (3).

Algorithm 1

function ComputeUnstableWall($\mathcal{G}_1$: grid, $\mathcal{G}_2$: grid) : **bool**
var
    X, LX, Z : representable sets;
begin
    foreach $G_1 \in \mathcal{G}_1$
    begin
        $X := G_1 \times 0$; // $0 \in \mathbb{R}^s$

$LX := (Df_c(0)(X))_I$;
foreach $G_2 \in \mathcal{G}_2$
begin
$Z := (f_c(G_1 \times G_2) \cup LX)_I$;
if not $\pi_u(Z) \subset \mathbb{R}^u \setminus \overline{B_u}(0,1)$ **return False**;
end;
end;
return True;
end.

Lemma 17 *Assume N, M be a h-sets and $f : |N| \to \mathbb{R}^n$ be such that f_c is smooth. Let $\mathcal{G}_1$ be a grid of $\partial B_u(0,1)$ and let $\mathcal{G}_2$ be a grid of $\overline{B_s}(0,1)$. If Algorithm 1 is called with arguments $(\mathcal{G}_1, \mathcal{G}_2)$ and returns* **True** *then the homotopy defined in (67) satisfies condition (3).*

Proof: Let $(p,q) \in N_c^-$. Since $\mathcal{G}_1$ is a grid of $\partial B_u(0,1)$ and $\mathcal{G}_2$ is a grid of $\overline{B_s}(0,1)$ then

$$\mathcal{G}_1 \times \mathcal{G}_2 := \{G_1 \times G_2 \mid G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2\}$$

is a grid of N_c^-. Therefore $(p,q) \in G_1 \times G_2$ for some $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$. Since the Algorithm stops and returns **True** the condition

$$\pi_u((f_c(G_1 \times G_2) \cup LX)_I) \subset \mathbb{R}^u \setminus \overline{B_u}(0,1)$$

is satisfied, which implies that for $t \in [0,1]$, $h(t,p,q) \notin M_c$. ∎

Now we discuss how we verify condition (4). The main point of our approach is that it is enough to compute $f_c(\partial N_c)$.

Algorithm 2

function ComputeBoundary($\mathcal{G}$: grid) : **bool**
var
X : representable set;
begin
foreach $G \in \mathcal{G}$
begin
$X := (f_c(G))_I$;
if not $\pi_s(X) \subset \operatorname{int} B_s(0,1)$ **return False**;
end;
return True;
end.

Lemma 18 *Assume N, M be a h-sets, $f : \mathbb{R}^n \to \mathbb{R}^n$ be such that f_c is a diffeomorphism. Let $\mathcal{G}$ be a grid of ∂N_c. If Algorithm 2 is called with argument $\mathcal{G}$ and returns* **True** *then the homotopy defined by (67) satisfies condition (4).*

Proof: Let $x \in \partial N_c$. Since Algorithm stops and returns **True** we obtain that $\pi_s(f_c(x)) \in B_s(0,1)$. From Eq. (67) it follows that for $x \in \partial N_c$ and $t \in [0,1]$ $\pi_s(h(t,x)) \in B_s(0,1)$. Therefore for $x \in \partial N_c$ and $t \in [0,1]$

$$h(t,x) \notin \overline{B_u}(0,1) \times \partial B_s(0,1) = M_c^+. \tag{68}$$

There remains to prove that Eq. (68) is satisfied for $x \in N_c$ and $t \in [0,1]$. Since f_c is a diffeomorphism, the Brouwer–Jordan Theorem implies that $\pi_s(f_c(N_c)) \subset B_s(0,1)$. Hence, for $t \in [0,1]$

$$\pi_s(h(t, N_c)) \subset B_s(0,1)$$

which implies that $h(t, N_c) \cap M_c^+ = \emptyset$. ∎

6.1 Technical data.

The grids used in the numerical proof of Lemma 11 and Lemma 13 always consist of "boxes", i.e., products of representable intervals. The total number of boxes used in the proof is approximately $2.2 \cdot 10^8$. The numerical proof of Lemma 11 and Lemma 13 took approximately 36 minutes on 2.4GHz processor under PLD Linux Distribution.

The C++ sources with a short description how to run the program are available at [W1].

References

[AZ] G. Arioli and P. Zgliczyński, *Periodic, homoclinic and heteroclinic orbits for Hénon Heiles Hamiltonian near the critical energy level*, Nonlinearity, vol. 16, No. 5 (2003), 1833–1852

[D1] R. L. Devaney, *Reversible diffeomorphisms and flows.*, Trans. Amer. Math. Soc. 218 (1976), 89–113.

[D2] R.L. Devaney, *Homoclinic bifurcation and the area conserving Hénon mapping*, J. Diff. Eq 51, 2, 254-266, 1984

[DV] R. DeVogelaere, *On the structure of symmetric periodic solutions of conservative systems*, in: Contribution to the theory of nonlinear oscillations, vol. 4, Princeton, Princeton University Press, 1958

[GiZ] M. Gidea and P. Zgliczyński, *Covering relations for multidimensional dynamical systems*, *http://www.im.uj.edu.pl/˜zgliczyn*, Journal of Differential Equations,

[GSS] M. Golubitsky, I. Stewart, D.G. Schaeffer, *Singularietes and Groups in Bifurcation Theory vol.2*, Applied Mathematical Sciences vol. 42, Springer New York ,1987

[La] J.S.W. Lamb, *Reversing symmetries in dynamical systems*, J. Phys. A:Math. Gen. 25, 925–937 , 1992

[La1] J.S.W. Lamb, *Reversing symmetries in dynamical systems*, PhD Thesis, Amsterdam University, 1994

[LQ] J.S.W. Lamb and G.R. Quispel, *Reversing k-symmetries in dynamical systems*, Physica D 73, 277–304, 1994

[L] N. G. Lloyd, *Degree theory*, Cambridge Tracts in Math., No. 73, Cambridge Univ. Press, London, 1978

[Mo] R.E. Moore, *Interval Analysis.* Prentice Hall, Englewood Cliffs, N.J., 1966

[W] D. Wilczak, *Chaos in the Kuramoto-Sivashinsky equations - a computer assisted proof,* J. Diff. Eq., 194(2003), 433–459

[W1] D. Wilczak, `http://www.wsb-nlu.edu.pl/~dwilczak.`

[WZ] D. Wilczak and P. Zgliczyński, *Heteroclinic Connections between Periodic Orbits in Planar Restricted Circular Three Body Problem - A Computer Assisted Proof*, Comm. Math. Phys., 234 (2003) 1, 37-75

[WZ2] D. Wilczak and P. Zgliczyński, *Heteroclinic Connections between Periodic Orbits in Planar Restricted Circular Three Body Problem - Part II*, `http://www.wsb-nlu.edu.pl/~dwilczak,` submitted