Uniqueness for the two-dimensional Navier-Stokes equation with a measure as initial vorticity

Isabelle Gallagher
Centre de Mathématiques L. Schwartz
Ecole polytechnique
91128 Palaiseau, France
Isabelle.Gallagher@math.polytechnique.fr

Thierry Gallay
Institut Fourier
Université de Grenoble I
38402 Saint-Martin d'Hères, France
Thierry.Gallay@ujf-grenoble.fr

June 15, 2004

Abstract

We show that any solution of the two-dimensional Navier-Stokes equation whose vorticity distribution is uniformly bounded in $L^1(\mathbf{R}^2)$ for positive times is entirely determined by the trace of the vorticity at $t = 0$, which is a finite measure. When combined with previous existence results by Cottet, by Giga, Miyakawa, and Osada, and by Kato, this uniqueness property implies that the Cauchy problem for the vorticity equation in $\mathbf{R}^2$ is globally well-posed in the space of finite measures. In particular, this provides an example of a situation where the Navier-Stokes equation is well-posed for arbitrary data in a function space that is large enough to contain the initial data of some self-similar solutions.

1 Introduction

We consider the two-dimensional incompressible Navier-Stokes equation

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u \,=\, \Delta u - \nabla p \;, \quad \mathrm{div}\; u \,=\, 0 \;, \quad x \in \mathbf{R}^2 \;, \quad t > 0 \;, \tag{1.1}$$

where $u(x,t) \in \mathbf{R}^2$ denotes the velocity field of the fluid and $p(x,t) \in \mathbf{R}$ the pressure field. Since this system is very famous, we do not comment here on its derivation and rather refer to the monographs [7], [23], [28] for a general introduction. The first mathematical result on the Cauchy problem is due to Leray [24] who proved that, for any initial data $u_0 \in L^2(\mathbf{R}^2)$, system (1.1) has a unique global solution $u \in C^0([0,+\infty), L^2(\mathbf{R}^2))$ such that $u(\cdot,0) = u_0$ and $\nabla u \in L^2((0,+\infty), L^2(\mathbf{R}^2))$. The space $L^2(\mathbf{R}^2)$ is naturally associated with the Navier-Stokes equation for two different reasons. First it is the *energy space*, because the square of the L^2 norm of u is the total (kinetic) energy of the fluid, which is nonincreasing with time. Next, the space $L^2(\mathbf{R}^2)$ is *scale invariant*, in the sense that $\|\lambda u_0(\lambda\cdot)\|_{L^2(\mathbf{R}^2)} = \|u_0\|_{L^2(\mathbf{R}^2)}$ for any $u_0 \in L^2(\mathbf{R}^2)$ and any $\lambda > 0$. This is important because the transformation

$$u(x,t) \,\mapsto\, \lambda u(\lambda x, \lambda^2 t) \;, \quad p(x,t) \,\mapsto\, \lambda^2 p(\lambda x, \lambda^2 t) \;, \quad \lambda > 0 \;, \tag{1.2}$$

is a symmetry of (1.1). This invariance was used by Kato [20] to prove that the Navier-Stokes equation in the d-dimensional space $\mathbf{R}^d$ is locally well-posed for arbitrary data in $L^d(\mathbf{R}^d)$ and even globally well-posed for sufficiently small data in that space, see also [29], [14]. Kato's result was subsequently extended to larger scale invariant function spaces, such as the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbf{R}^d)$ with $s = -1 + \frac{d}{p}$ and $p < \infty$, see Cannone and Planchon [3], [4] and

Meyer [25]. A similar analysis was carried out for the vorticity equation in Morrey spaces by Giga and Miyakawa [17]. One interest of dealing with larger function spaces is that they may contain initial data which are homogeneous of degree -1 and therefore give rise to self-similar solutions of (1.1). This is the case of the Besov space above if $q = \infty$, or of the larger space $\mathrm{BMO}^{-1}(\mathbf{R}^d)$ introduced by Koch and Tataru [22]. In such spaces, however, it is not known if the Cauchy problem is well-posed for large data, even locally in time.

We now return to the two-dimensional case $d = 2$ which is simpler for several reasons. First, the a priori estimates allow in that case to prove that all solutions are global. For instance, in [9], F. Planchon and the first author proved that, for arbitrary data in $\dot{B}^s_{p,q}(\mathbf{R}^2)$ with $s = -1 + \frac{2}{p}$ and $p, q < \infty$, there exists a unique global solution to the Navier-Stokes equation (1.1). This result was recently extended by Germain [13] to the larger space $\mathrm{VMO}^{-1}(\mathbf{R}^2)$, which is the closure of $\mathcal{S}(\mathbf{R}^2)$ in $\mathrm{BMO}^{-1}(\mathbf{R}^2)$. To our knowledge, this is the largest space for the velocity field in which one can solve the Navier-Stokes equation for arbitrary data. Note however that $\mathrm{VMO}^{-1}(\mathbf{R}^2)$ does not contain any non-trivial homogeneous function of degree -1.

Another specificity of the two-dimensional case is that the vorticity $\omega \stackrel{\text{def}}{=} \partial_1 u_2 - \partial_2 u_1$ is a scalar quantity which satisfies a remarkably simple equation, namely

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega \;=\; \Delta \omega \;, \quad x \in \mathbf{R}^2 \;, \quad t > 0 \;. \tag{1.3}$$

The velocity field $u(x,t)$ can be reconstructed from the vorticity distribution $\omega(x,t)$ by the Biot-Savart law

$$u(x,t) \;=\; \frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\, \omega(y,t)\, \mathrm{d}y \;,$$

where $x^\perp = (x_1, x_2)^\perp \stackrel{\text{def}}{=} (-x_2, x_1)$. In terms of the vorticity, the invariance (1.2) reads

$$\omega(x,t) \;\mapsto\; \lambda^2 \omega(\lambda x, \lambda^2 t) \;. \tag{1.4}$$

A natural scale invariant space for the vorticity is thus $L^1(\mathbf{R}^2)$. The Cauchy problem for (1.3) in $L^1(\mathbf{R}^2)$ was studied for instance in [1], where results analogous to Leray's and Kato's theorems for the velocity field are obtained. However, it is important to realize that a vorticity in $L^1(\mathbf{R}^2)$ does not imply a velocity field in $L^2(\mathbf{R}^2)$. Indeed, if $u \in L^2(\mathbf{R}^2)$ and if $\omega = \partial_1 u_2 - \partial_2 u_1 \in L^1(\mathbf{R}^2)$, then it is easy to verify that necessarily $\int_{\mathbf{R}^2} \omega \,\mathrm{d}x = 0$. Since the integral of ω (which is the circulation of the velocity field at infinity) is conserved under the evolution of (1.3), it follows that if the initial vorticity has nonzero integral then the associated velocity field will never be of finite energy. This "discrepancy" between function spaces for the vorticity and the velocity is in fact specific to the two-dimensional case. Indeed, if for instance ω solves the vorticity equation in $L^{\frac{3}{2}}(\mathbf{R}^3)$, then the associated velocity field does solve the Navier-Stokes equation in $L^3(\mathbf{R}^3)$.

In this paper, we study the Cauchy problem for the vorticity equation (1.3) in $\mathcal{M}(\mathbf{R}^2)$, the space of all finite real measures on $\mathbf{R}^2$. If $\mu \in \mathcal{M}(\mathbf{R}^2)$, the total variation of μ is defined by

$$\|\mu\|_{\mathcal{M}} \;=\; \sup\left\{ \int_{\mathbf{R}^2} \varphi \,\mathrm{d}\mu \;\middle|\; \varphi \in C_0(\mathbf{R}^2)\,,\; \|\varphi\|_{L^\infty} \le 1 \right\} \;,$$

where $C_0(\mathbf{R}^2)$ is the set of all real-valued continuous functions on $\mathbf{R}^2$ vanishing at infinity. We recall that $\mathcal{M}(\mathbf{R}^2)$ equipped with the total variation norm is a Banach space, whose norm is invariant under the scaling transformation (1.4). Another useful topology on $\mathcal{M}(\mathbf{R}^2)$ is the weak$*$-topology which can be characterized as follows: a sequence $\{\mu_n\}$ in $\mathcal{M}(\mathbf{R}^2)$ converges weakly to μ if $\int_{\mathbf{R}^2} \varphi \,\mathrm{d}\mu_n \to \int_{\mathbf{R}^2} \varphi \,\mathrm{d}\mu$ as $n \to \infty$ for all $\varphi \in C_0(\mathbf{R}^2)$. In that case, we write $\mu_n \rightharpoonup \mu$.

Existence of solutions of (1.3) with initial data in $\mathcal{M}(\mathbf{R}^2)$ was first proved by Cottet [8], and independently by Giga, Miyakawa, and Osada [16]. Uniqueness is a more difficult problem.

Using a Gronwall-type argument, it is shown in [16] that uniqueness holds if the atomic part of the initial vorticity is sufficiently small, see also [21]. The fact that the size condition only involves the atomic part of the measure is a consequence of the key estimate (see [16])

$$\limsup_{t\to 0} t^{1-\frac{1}{q}} \|\mathrm{e}^{t\Delta}\mu\|_{L^q} \;\le\; C_q\|\mu\|_{\mathrm{pp}}\;, \quad 1 < q \le +\infty\;,$$

where $\|\mu\|_{\mathrm{pp}}$ denotes the total variation of the atomic part of $\mu \in \mathcal{M}(\mathbf{R}^2)$. On the other hand, the case of a large Dirac mass was solved recently by C.E. Wayne and the second author [12] using a completely different approach, which we now briefly describe. We first observe that, given any $\alpha \in \mathbf{R}$, equation (1.3) has an exact self-similar solution given by

$$\omega(x,t) \;=\; \frac{\alpha}{t}\, G\Big(\frac{x}{\sqrt{t}}\Big)\;, \quad u(x,t) \;=\; \frac{\alpha}{\sqrt{t}}\, v^G\Big(\frac{x}{\sqrt{t}}\Big)\;, \quad x \in \mathbf{R}^2\;, \quad t > 0\;, \tag{1.5}$$

where

$$G(\xi) \;=\; \frac{1}{4\pi}\,\mathrm{e}^{-|\xi|^2/4}\;, \quad v^G(\xi) \;=\; \frac{1}{2\pi}\,\frac{\xi^\perp}{|\xi|^2}\Big(1-\mathrm{e}^{-|\xi|^2/4}\Big)\;, \quad \xi \in \mathbf{R}^2\;. \tag{1.6}$$

This solution is often called the *Lamb-Oseen vortex* with total circulation α. In fact $\omega(x,t)$ is also a solution of the linear heat equation $\partial_t\omega = \Delta\omega$, because the nonlinearity in (1.3) vanishes identically due to radial symmetry (this is again specific to the two-dimensional case). The strategy of [12] consists in rewriting (1.3) into self-similar variables as in (2.9) below. Using a pair of Lyapunov functions, the authors show that the Oseen vortices αG ($\alpha \in \mathbf{R}$) are the only equilibria of the rescaled equation. By compactness arguments, they deduce that all solutions converge in $L^1(\mathbf{R}^2)$ to Oseen vortices as $t \to +\infty$, and as a byproduct that (1.5) is the unique solution of (1.3) such that $\|\omega(\cdot,t)\|_{L^1} \le K$ for all $t > 0$ and $\omega(\cdot,t) \rightharpoonup \alpha\delta_0$ as $t \to 0$, where δ_0 is the Dirac mass at the origin.

The goal of the present paper is to solve the uniqueness problem in the general case by combining the result of [16], which works when the initial measure has small atomic part, with the method of [12], which allows to handle large Dirac masses. Our main result is the following:

Theorem 1.1 *Let $\mu \in \mathcal{M}(\mathbf{R}^2)$, and fix $T > 0$, $K > 0$. Then the vorticity equation (1.3) has at most one solution*

$$\omega \in C^0((0,T), L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2))$$

such that $\|\omega(\cdot,t)\|_{L^1} \le K$ for all $t \in (0,T)$ and $\omega(\cdot,t) \rightharpoonup \mu$ as $t \to 0$.

Here and in the sequel, we say that $\omega(t) \equiv \omega(\cdot,t)$ is a (mild) solution of (1.3) on $(0,T)$ if the associated integral equation

$$\omega(t) \;=\; \mathrm{e}^{(t-t_0)\Delta}\omega(t_0) - \int_{t_0}^t \nabla\cdot\mathrm{e}^{(t-s)\Delta}\Big(u(s)\omega(s)\Big)\,\mathrm{d}s \tag{1.7}$$

is satisfied for all $0 < t_0 < t < T$.

If we combine Theorem 1.1 with the existence results in [8], [16], [21], we conclude that there is a unique global solution to (1.3) for any initial measure in $\mathcal{M}(\mathbf{R}^2)$. In fact the method we use to prove uniqueness also implies that this solution depends continuously on the data, so that the Cauchy problem for the vorticity equation (1.3) is globally well-posed in the space $\mathcal{M}(\mathbf{R}^2)$. If in addition we use the results in [12] on the long-time behavior of the solutions, we obtain the following final statement:

Theorem 1.2 *For any $\mu \in \mathcal{M}(\mathbf{R}^2)$, the vorticity equation (1.3) has a unique global solution*

$$\omega \in C^0((0,\infty), L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2))$$

such that $\|\omega(\cdot,t)\|_{L^1} \le \|\mu\|_{\mathcal{M}}$ for all $t > 0$ and $\omega(\cdot,t) \rightharpoonup \mu$ as $t \to 0$. This solution depends continuously on the initial measure μ in the norm topology of $\mathcal{M}(\mathbf{R}^2)$, uniformly in time on compact intervals. Moreover,

$$\int_{\mathbf{R}^2} \omega(x,t)\,\mathrm{d}x \;=\; \alpha \;\stackrel{def}{=}\; \mu(\mathbf{R}^2)\ , \quad \textit{for all } t > 0\ ,$$

and

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} \Big\| \omega(x,t) - \frac{\alpha}{t} G\Big(\frac{x}{\sqrt{t}}\Big)\Big\|_{L^p_x} \;=\; 0\ , \quad \textit{for all } p \in [1,\infty]\ . \tag{1.8}$$

Remark that the space $\mathcal{M}(\mathbf{R}^2)$ does contain nontrivial homogeneous distributions (the Dirac masses at the origin), hence Theorem 1.2 gives an example of a situation where the Navier-Stokes equation is well-posed for arbitrary data in a function space that is large enough to contain the initial data of some self-similar solutions (the Oseen vortices). Remark also that Oseen's vortex plays a double role in Theorem 1.2: it is the unique solution of (1.3) when the initial vorticity μ is a Dirac mass at the origin, and on the other hand it describes the long-time behavior of all solutions, see (1.8). In fact, it is possible to show that (1.8) is a consequence of the uniqueness of the solution when $\mu = \alpha\delta_0$, see [6] and [18].

The rest of this paper is devoted to the proof of Theorem 1.1 and of the continuity statement in Theorem 1.2. Before entering the details, let us give a short idea of the argument. Previous works on the subject assumed that the initial vorticity μ either has a small atomic part [16],[21], or consists of a single Dirac mass [12]. So it is natural to decompose μ into a finite sum of mutually singular Dirac masses, and a remainder whose atomic part is arbitrarily small (depending on the number of terms in the previous sum). The idea is then to use the methods of [12] to deal with the large Dirac masses, and the argument of [16],[21] to treat the remainder. The difficulty is of course that equation (1.3) is nonlinear so that the interactions between the various terms have to be controlled.

To implement these ideas, we start in Section 2.1 by recalling some general properties of convection-diffusion equations, of the heat semi-group in self-similar variables, and of the Biot-Savart law. The proof of Theorem 1.1 begins in Section 3, where we decompose the initial measure as explained above and show that the solution $\omega(x,t)$ also admits a natural decomposition into a sum of Oseen vortices and a remainder. In Section 4, we derive the integral equations satisfied by the remainder terms, and we state a few crucial estimates that will be proved in an appendix (Section 6). These results are used in Section 5, where Theorem 1.1 is proved by a Gronwall-type argument. The same techniques also establish the continuity claim in Theorem 1.2.

Notations. We denote by $K_0, K_1, \dots$ our main constants, the values of which are fixed throughout the paper. In contrast, we denote by $C_0, C_1, \dots$ local constants which can take different values in different paragraphs. Other positive constants (which are not used anywhere else in the text) will be generically denoted by C. As a general rule, we do not distinguish between scalars and vectors in function spaces: although $u(x,t)$ is a vector field, we write $u \in L^2(\mathbf{R}^2)$ and not $u \in L^2(\mathbf{R}^2)^2$. To simplify the notation, we denote the map $x \mapsto \omega(x,t)$ by $\omega(\cdot,t)$ or just by $\omega(t)$.

Acknowledgements. The authors thank D. Iftimie for fruitful discussions.

2 Preliminaries

This section is a collection of known results that will be used in the proof of Theorem 1.1.

2.1 Fundamental solution of a convection-diffusion equation

We consider the following linear convection-diffusion equation

$$\partial_t\omega(x,t) + U(x,t)\cdot\nabla\omega(x,t) \;=\; \Delta\omega(x,t)\ , \tag{2.1}$$

where $x \in \mathbf{R}^2$, $t \in (0,T)$, and $U : \mathbf{R}^2 \times (0,T) \to \mathbf{R}^2$ is a (given) time-dependent divergence-free vector field. The results collected here are due to Osada [26], and to Carlen and Loss [5].

Following [5], we suppose that $U \in C^0((0,T), L^\infty(\mathbf{R}^2))$ and that

$$\|U(\cdot,t)\|_{L^\infty(\mathbf{R}^2)} \;\le\; \frac{K_0}{\sqrt{t}}\ , \quad 0 < t < T\ , \tag{2.2}$$

for some $K_0 > 0$. According to [26], we also assume that $\Omega \stackrel{\rm def}{=} \partial_1 U_2 - \partial_2 U_1 \in C^0((0,T), L^1(\mathbf{R}^2))$ with

$$\|\Omega(\cdot,t)\|_{L^1(\mathbf{R}^2)} \;\le\; K_0\ , \quad 0 < t < T\ . \tag{2.3}$$

Then any solution $\omega(x,t)$ of (2.1) can be represented as

$$\omega(x,t) \;=\; \int_{\mathbf{R}^2} \Gamma_U(x,t;y,s)\omega(y,s)\,{\rm d}y\ , \quad x \in \mathbf{R}^2\ , \quad 0 < s < t < T\ , \tag{2.4}$$

where Γ_U is the fundamental solution of the convection-diffusion equation (2.1). The following properties of Γ_U will be useful:

• For any $\beta \in (0,1)$ there exists $K_1 > 0$ (depending only on K_0 and β) such that

$$0 \;<\; \Gamma_U(x,t;y,s) \;\le\; \frac{K_1}{t-s}\,\exp\Bigl(-\beta\frac{|x-y|^2}{4(t-s)}\Bigr)\ , \tag{2.5}$$

for $x, y \in \mathbf{R}^2$ and $0 < s < t < T$, see [5]. We also have a similar Gaussian lower bound, see [26].

• There exists $\gamma \in (0,1)$ (depending only on K_0) and, for any $\delta > 0$, there exists $K_2 > 0$ (depending only on K_0 and δ) such that

$$|\Gamma_U(x,t;y,s) - \Gamma_U(x',t';y',s')| \le K_2\Bigl(|x-x'|^\gamma + |t-t'|^{\gamma/2} + |y-y'|^\gamma + |s-s'|^{\gamma/2}\Bigr)\ , \tag{2.6}$$

whenever $t - s \ge \delta$ and $t' - s' \ge \delta$, see [26].

• For $0 < s < t < T$ and $x, y \in \mathbf{R}^2$,

$$\int_{\mathbf{R}^2} \Gamma_U(x,t;y,s)\,{\rm d}x \;=\; 1\ , \quad \int_{\mathbf{R}^2} \Gamma_U(x,t;y,s)\,{\rm d}y \;=\; 1\ . \tag{2.7}$$

For $0 < s < r < t < T$ and $x, y \in \mathbf{R}^2$,

$$\Gamma_U(x,t;y,s) \;=\; \int_{\mathbf{R}^2} \Gamma_U(x,t;z,r)\Gamma_U(z,r;y,s)\,{\rm d}z\ . \tag{2.8}$$

Remark 2.1 *If $x, y \in \mathbf{R}^2$ and $t > 0$, it follows from (2.6) that the function $s \mapsto \Gamma_U(x,t;y,s)$ can be continuously extended up to $s = 0$, and that this extension (still denoted by Γ_U) satisfies properties (2.5) to (2.8) with $s = 0$.*

2.2 The heat semiflow in self-similar variables

Let $\omega(x,t)$ be a solution of the linear heat equation $\partial_t\omega = \Delta\omega$ in $\mathbf{R}^2$. As is well-known, it is natural to rewrite this system in terms of the "self-similar variables" $\xi = \frac{x}{\sqrt{t}}$, $\tau = \log(t)$. If we set

$$\omega(x,t) \;=\; \frac{1}{t}\,w\Big(\frac{x}{\sqrt{t}}\,,\,\log(t)\Big)\;,\quad x\in\mathbf{R}^2\;,\quad t>0\;, \tag{2.9}$$

then the new function $w(\xi,\tau)$ is a solution of the rescaled equation $\partial_\tau w = \mathcal{L}w$, where $\mathcal{L}$ is the Fokker-Planck operator

$$\mathcal{L} \;=\; \Delta_\xi + \frac{1}{2}\xi\cdot\nabla_\xi + 1\;. \tag{2.10}$$

This operator is the generator of a C_0 semigroup $S(\tau) = \exp(\tau\mathcal{L})$ given by the explicit formula

$$\Big(S(\tau)f\Big)(\xi) \;=\; \frac{\mathrm{e}^\tau}{4\pi a(\tau)}\int_{\mathbf{R}^2}\mathrm{e}^{-\frac{|\xi-\xi'|^2}{4a(\tau)}}f(\xi'\mathrm{e}^{\frac{\tau}{2}})\,\mathrm{d}\xi'\;,\quad \xi\in\mathbf{R}^2\;,\quad \tau>0\;, \tag{2.11}$$

where $a(\tau) = 1-\mathrm{e}^{-\tau}$. The linear operators $\mathcal{L}$ and $S(\tau)$ are studied in detail in ([10], Appendix A). For the reader's convenience, we recall here the main properties that will be used in the proof of Theorem 1.1.

Following [10], we introduce for $q\ge 1$ and $m\ge 0$ the weighted Lebesgue space $L^q(m)$ defined by

$$L^q(m) \;=\; \Big\{w\in L^q(\mathbf{R}^2)\,\Big|\,\|w\|_{L^q(m)}<\infty\Big\}\;,\quad \text{where } \|w\|_{L^q(m)} \;=\; \|(1+|\xi|^2)^{\frac{m}{2}}w\|_{L^q}\;. \tag{2.12}$$

We shall mainly use the Hilbert space $L^2(m)$, which satisfies $L^2(m)\hookrightarrow L^1(\mathbf{R}^2)$ if $m>1$. In this case, we define the closed subspace

$$L^2_0(m) \;=\; \Big\{w\in L^2(m)\,\Big|\,\int_{\mathbf{R}^2}w(\xi)\,\mathrm{d}\xi = 0\Big\}\;,\quad m>1\;.$$

Proposition 2.2 *Fix $m>1$.*
i) There exists $K_3>0$ such that, for all $w\in L^2(m)$,

$$\|S(\tau)w\|_{L^2(m)} \;\le\; K_3\|w\|_{L^2(m)}\;,\quad \|\nabla S(\tau)w\|_{L^2(m)} \;\le\; \frac{K_3}{a(\tau)^{\frac{1}{2}}}\|w\|_{L^2(m)}\;, \tag{2.13}$$

for all $\tau>0$, where $a(\tau) = 1-\mathrm{e}^{-\tau}$.
ii) If moreover $m>2$ and $w\in L^2_0(m)$, then

$$\|S(\tau)w\|_{L^2(m)} \;\le\; K_3\mathrm{e}^{-\frac{\tau}{2}}\|w\|_{L^2(m)}\;,\quad \tau\ge 0\;. \tag{2.14}$$

iii) More generally, if $q\in[1,2]$ there exists $K_4>0$ such that, for all $w\in L^q(m)$,

$$\|S(\tau)w\|_{L^2(m)} \;\le\; \frac{K_4}{a(\tau)^{\frac{1}{q}-\frac{1}{2}}}\|w\|_{L^q(m)}\;,\quad \|\nabla S(\tau)w\|_{L^2(m)} \;\le\; \frac{K_4}{a(\tau)^{\frac{1}{q}}}\|w\|_{L^q(m)}\;, \tag{2.15}$$

for all $\tau>0$.

Proof: The bounds (2.13), (2.14) are proved in ([10], Proposition A.2). Estimate (2.15) follows from (2.13) if we use in addition ([10], Proposition A.5). □

Since the operator $\mathcal{L}$ has variable coefficients, it does not commute with spatial derivatives, nor does the associated semigroup $S(\tau)$. However, the following useful identity holds:

$$\nabla S(\tau) \;=\; \mathrm{e}^{\frac{\tau}{2}}S(\tau)\nabla\;,\quad \tau\ge 0\;. \tag{2.16}$$

2.3 The Biot-Savart law

Finally we list some basic properties of the Biot-Savart law

$$u(x) \;=\; \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\,\omega(y)\,\mathrm{d}y\;, \quad x \in \mathbf{R}^2\;. \tag{2.17}$$

We recall that $L^2(m)$ is the weighted Lebesgue space defined in (2.12).

Proposition 2.3 *Assume that $\omega \in L^p(\mathbf{R}^2)$ for some $p \in (1,2)$, and let u be the vector field defined by (2.17). Then*
i) $u \in L^q(\mathbf{R}^2)$ where $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$, and there exists $C > 0$ such that

$$\|u\|_{L^q} \;\le\; C\|\omega\|_{L^p}\;. \tag{2.18}$$

ii) $\nabla u \in L^p(\mathbf{R}^2)$ and there exists $C > 0$ such that

$$\|\nabla u\|_{L^p} \;\le\; C\|\omega\|_{L^p}\;. \tag{2.19}$$

In addition, $\mathrm{div}(u) = 0$ *and* $\partial_1 u_2 - \partial_2 u_1 = \omega$.
iii) Let $b(x) = (1+|x|^2)^{\frac{1}{2}}$. If $\omega \in L^2(m)$ for some $m \in (0,1)$, or $\omega \in L^2_0(m)$ for some $m \in (1,2)$, then $b^{m-\frac{2}{q}}u \in L^q(\mathbf{R}^2)$ for any $q \in (2,\infty)$ and there exists $C > 0$ such that

$$\|b^{m-\frac{2}{q}}u\|_{L^q} \;\le\; C\|\omega\|_{L^2(m)}\;. \tag{2.20}$$

Proof: The bound (2.18) is a direct consequence of the classical Hardy-Littlewood-Sobolev inequality, see for instance ([27], Chapter V, Theorem 1). Estimate (2.19) holds because ∇u is the convolution of ω with a singular integral kernel of Calderón-Zygmund type, see ([27], Chapter II, Theorem 3). Finally, the weighted inequality (2.20) is proved in ([10], Proposition B.1). □

3 Decomposition of the solution

After these preliminaries, we begin the proof of Theorem 1.1. We fix $\mu \in \mathcal{M}(\mathbf{R}^2)$, $T > 0$ and $K > 0$, and we assume that $\omega \in C^0((0,T), L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2))$ is a solution of the vorticity equation (1.3) satisfying $\|\omega(\cdot,t)\|_{L^1} \le K$ for all $t \in (0,T)$ and $\omega(\cdot,t) \rightharpoonup \mu$ as $t \to 0$. From [2] we know that $\omega(x,t)$ coincides for $t > 0$ with a classical solution of (1.3) in $\mathbf{R}^2$ as constructed for instance in [1]. In particular $\omega(x,t)$ is smooth for $t > 0$, and since the Cauchy problem for (1.3) is globally well-posed in $L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2)$, we could assume without loss of generality that $T = +\infty$. In the sequel, however, we keep $T > 0$ arbitrary.

Since $\mu \in \mathcal{M}(\mathbf{R}^2)$ is a finite measure, the set $E_{\mathrm{pp}} = \{x \in \mathbf{R}^2 \,|\, \mu(\{x\}) \neq 0\}$ of all atoms of μ is at most countable, and

$$\|\mu\|_{\mathrm{pp}} \;\stackrel{\mathrm{def}}{=}\; \sum_{x \in E_{\mathrm{pp}}} |\mu(\{x\})| \;\le\; \|\mu\|_{\mathcal{M}} \;<\; \infty\;.$$

Therefore, given any $\varepsilon > 0$, there exists $N \in \mathbf{N}$ and $z_1, \ldots, z_N \in E_{\mathrm{pp}}$ with $z_i \neq z_j$ for $i \neq j$ such that μ can be decomposed as

$$\mu \;=\; \sum_{i=1}^{N} \alpha_i \delta_{z_i} + \mu_0\;, \tag{3.1}$$

where $\alpha_i = \mu(\{z_i\}) \neq 0$ and $\|\mu_0\|_{\rm pp} \le \varepsilon$. Here δ_z denotes the Dirac mass located at $z \in \mathbf{R}^2$. Of course, it may happen that $N = 0$ so that $\mu = \mu_0$, but if the set $E_{\rm pp}$ is infinite we have to take N large if ε is small. From now on we fix $\varepsilon > 0$ and assume that (3.1) holds with $\|\mu_0\|_{\rm pp} \le \varepsilon$. We denote

$$M_{\rm pp} \,=\, \sum_{i=1}^N |\alpha_i| \,\le\, \|\mu\|_{\rm pp}\ , \quad \text{and} \quad d \,=\, \min\Big\{|z_i - z_j| \,\Big|\, i,j \in \{1,\dots,N\}\,,\ i \neq j\Big\}\ . \tag{3.2}$$

At the very end of the proof, in Section 5.2, we shall assume that ε is sufficiently small.

Let $u(x,t)$ be the velocity field obtained from $\omega(x,t)$ via the Biot-Savart law (2.17). Since for all $t \in (0,T)$ we have $\|\omega(\cdot,t)\|_{L^1} \le K$, it follows from ([5], Theorem 2) that $t^{\frac{1}{2}}\|u(\cdot,t)\|_{L^\infty} \le CK$ for all $t \in (0,T)$, where $C > 0$ is a universal constant. Thus $\omega(x,t)$ is a solution of the convection-diffusion equation (2.1) with $U(x,t) = u(x,t)$, and assumptions (2.2), (2.3) are satisfied. It follows that $\omega(x,t)$ can be represented as in (2.4), where the fundamental solution $\Gamma_u(x,t;y,s)$ satisfies (2.5) to (2.8). In particular, using Remark 2.1, we have for all $x \in \mathbf{R}^2$ and all $t \in (0,T)$,

$$\begin{aligned}\omega(x,t) \;&=\; \int_{\mathbf{R}^2} \Gamma_u(x,t;y,0)\omega(y,s)\,\mathrm{d}y \\ &+\; \int_{\mathbf{R}^2} (\Gamma_u(x,t;y,s) - \Gamma_u(x,t;y,0))\omega(y,s)\,\mathrm{d}y\ , \quad 0 < s < t\ .\end{aligned}$$

In view of (2.6), the second integral in the right-hand side converges to zero as s goes to zero. On the other hand, since $y \mapsto \Gamma_u(x,t;y,0)$ is continuous and vanishes at infinity, and since $\omega(\cdot,s) \rightharpoonup \mu$ as $s \to 0$, we can take the limit $s \to 0$ in the first integral and we obtain the following useful representation:

$$\omega(x,t) \,=\, \int_{\mathbf{R}^2} \Gamma_u(x,t;y,0)\,\mathrm{d}\mu(y)\ , \quad x \in \mathbf{R}^2\ , \quad 0 < t < T\ . \tag{3.3}$$

Since $\Gamma_u(x,t;y,0)$ is positive and satisfies (2.7), it follows that $\|\omega(\cdot,t)\|_{L^1} \le \|\mu\|_{\mathcal{M}}$ for all $t \in (0,T)$. Thus we can assume that $K = \|\mu\|_{\mathcal{M}}$ without loss of generality.

Inserting (3.1) into (3.3), we obtain the decomposition

$$\omega(x,t) \,=\, \sum_{i=1}^N \omega_i(x,t) + \tilde\omega_0(x,t)\ , \tag{3.4}$$

where

$$\omega_i(x,t) \,=\, \alpha_i \Gamma_u(x,t;z_i,0)\ , \quad x \in \mathbf{R}^2\,,\ t \in (0,T)\,, \tag{3.5}$$

and

$$\tilde\omega_0(x,t) \,=\, \int_{\mathbf{R}^2} \Gamma_u(x,t;y,0)\,\mathrm{d}\mu_0(y)\ , \quad x \in \mathbf{R}^2\,,\ t \in (0,T)\,. \tag{3.6}$$

Thus, although (1.3) is a nonlinear equation, we see that the decomposition (3.1) of the initial measure induces a natural decomposition of the solution $\omega(x,t)$. Using the properties of the fundamental solution Γ_u listed in Section 2.1, one easily obtains the following results:

• For all $i \in \{1,\dots,N\}$, $\omega_i \in C^0((0,T), L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2))$ is a solution of (2.1) with $U(x,t) = u(x,t)$, namely

$$\frac{\partial \omega_i}{\partial t} + u \cdot \nabla \omega_i \,=\, \Delta \omega_i\ , \quad t \in (0,T)\ . \tag{3.7}$$

For any $t \in (0,T)$, $\int_{\mathbf{R}^2} \omega_i(x,t)\,\mathrm{d}x = \alpha_i$, $\|\omega_i(\cdot,t)\|_{L^1} = |\alpha_i|$, and

$$|\omega_i(x,t)| \,\le\, \frac{K_1|\alpha_i|}{t}\,\mathrm{e}^{-\beta\frac{|x-z_i|^2}{4t}}\ , \quad x \in \mathbf{R}^2\ . \tag{3.8}$$

In particular, $\omega_i(\cdot,t) \rightharpoonup \alpha_i\delta_{z_i}$ as $t \to 0$.

• Similarly, $\tilde\omega_0 \in C^0((0,T), L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2))$ is a solution of

$$\frac{\partial\tilde\omega_0}{\partial t} + u\cdot\nabla\tilde\omega_0 \;=\; \Delta\tilde\omega_0\;, \quad t \in (0,T)\;. \tag{3.9}$$

Moreover, $\|\tilde\omega_0(\cdot,t)\|_{L^1} \le \|\mu_0\|_{\mathcal{M}} \le \|\mu\|_{\mathcal{M}}$ for all $t \in (0,T)$, and $\tilde\omega_0(\cdot,t) \rightharpoonup \mu_0$ as $t \to 0$.

Since $u(x,t)$ is smooth for $t > 0$, it is clear from (3.7), (3.9) that $\omega_i(x,t)$ and $\tilde\omega_0(x,t)$ are smooth functions of $x \in \mathbf{R}^2$ and $t \in (0,T)$.

For $i \in \{1,\dots,N\}$, we have seen that $\omega_i(x,t)$ is a solution of (3.7) with a Dirac mass $\alpha_i\delta_{z_i}$ as initial data. If we believe in uniqueness, we expect that $\omega_i(x,t)$ will be very close, for small times, to an Oseen vortex located at z_i with circulation α_i. Thus if we further decompose

$$\omega_i(x,t) \;=\; \frac{\alpha_i}{t}\,G\Big(\frac{x-z_i}{\sqrt{t}}\Big) + \alpha_i\tilde\omega_i(x,t)\;, \quad x \in \mathbf{R}^2\;, \quad t \in (0,T)\;, \tag{3.10}$$

where G is defined in (1.6), we expect that the remainder $\tilde\omega_i(x,t)$ will be small as $t \to 0$. Summarizing, we have

$$\omega(x,t) \;=\; \sum_{i=1}^N \frac{\alpha_i}{t}\,G\Big(\frac{x-z_i}{\sqrt{t}}\Big) + \tilde\omega(x,t)\;, \quad u(x,t) \;=\; \sum_{i=1}^N \frac{\alpha_i}{\sqrt{t}}\,v^G\Big(\frac{x-z_i}{\sqrt{t}}\Big) + \tilde u(x,t)\;, \tag{3.11}$$

where

$$\tilde\omega(x,t) \;=\; \tilde\omega_0(x,t) + \sum_{i=1}^N \alpha_i\tilde\omega_i(x,t)\;, \quad \tilde u(x,t) \;=\; \tilde u_0(x,t) + \sum_{i=1}^N \alpha_i\tilde u_i(x,t)\;,$$

and where (for $i \in \{0,\dots,N\}$) $\tilde u_i(x,t)$ denotes the velocity field associated to $\tilde\omega_i(x,t)$ via the Biot-Savart law. In (3.11), remark that the explicit terms in the sums depend only on the initial measure μ, not on the solution $\omega(x,t)$.

4 Integral equations and main estimates

In this section we derive integral equations for the remainder terms $\tilde\omega_i(x,t)$ defined in (3.6) and (3.10), and we also list a few important estimates which will be proved in Section 6. We start in Section 4.1 with $\tilde\omega_0(x,t)$, which we call the "diffuse part" because it is associated to the measure μ_0 which (by construction) has small or no atomic part. The remaining terms $\tilde\omega_i(x,t)$ ($i \in \{1,\dots,N\}$), which originate from the large atoms of the initial measure μ, will be dealt with in Section 4.2.

4.1 The diffuse part

Let $\tilde\omega_0(x,t)$ be defined by (3.6). Our first result shows that $\tilde\omega_0(x,t)$ is small in an appropriate sense as $t \to 0$, because the measure μ_0 has a small atomic part.

Lemma 4.1 *For any $p \in (1,\infty]$, there exists $K_5 > 0$ (depending only on p and K) such that*

$$\limsup_{t\to 0}\, t^{1-\frac{1}{p}}\|\tilde\omega_0(\cdot,t)\|_{L^p} \;\le\; K_5\|\mu_0\|_{\mathrm{pp}}\;. \tag{4.1}$$

Proof: This property is established in ([16], Lemma 4.4) in the particular case where $\tilde{\omega}_0(\cdot,t) = \mathrm{e}^{t\Delta}\mu_0$. By (2.5), the fundamental solution $\Gamma_u(x,t;y,s)$ satisfies a Gaussian upper bound which has the same form as the heat kernel $\mathrm{e}^{(t-s)\Delta}(x,y)$, so using the same arguments as in [16] we immediately obtain (4.1). □

Our next result reflects the fact that $\mu_0(\{z_i\}) = 0$ for $i \in \{1,\dots,N\}$.

Lemma 4.2 *Assume that $\chi : [0,+\infty) \to \mathbf{R}_+$ is continuous and nonincreasing, with $\chi(0) = 1$ and $\chi(r) \to 0$ as $r \to \infty$. Then for all $i \in \{1,\dots,N\}$, the following estimates hold:*

$$\lim_{t\to 0} t^{1-\frac{1}{p}} \|\tilde{\omega}_0(x,t)\chi(\tfrac{|x-z_i|^2}{t})\|_{L^p_x} \;=\; 0\;, \quad 1 \le p \le +\infty\;, \tag{4.2}$$

$$\lim_{t\to 0} t^{\frac{1}{2}-\frac{1}{q}} \|\tilde{u}_0(x,t)\chi(\tfrac{|x-z_i|^2}{t})\|_{L^q_x} \;=\; 0\;, \quad 2 < q \le +\infty\;. \tag{4.3}$$

Proof: See Section 6.1. □

We now derive an integral equation for $\tilde{\omega}_0(x,t)$. Replacing in (3.9) the velocity field $u(x,t)$ with its expression (3.11) and using Duhamel's formula, we obtain for $0 < s < t < T$ the integral representation

$$\tilde{\omega}_0(t) \;=\; S_N(t,s)\tilde{\omega}_0(s) - \int_s^t S_N(t,t')(\tilde{u}(t')\cdot\nabla\tilde{\omega}_0(t'))\,\mathrm{d}t'\;, \tag{4.4}$$

where $\tilde{\omega}_0(t) \equiv \tilde{\omega}_0(\cdot,t)$, $\tilde{u}(t) \equiv \tilde{u}(\cdot,t)$, and $S_N(t,s)$ is the evolution operator associated to the convection-diffusion equation (2.1) with $U(x,t) = \sum_{i=1}^N \frac{\alpha_i}{\sqrt{t}} v^G\Big(\frac{x-z_i}{\sqrt{t}}\Big)$. From Section 2.1 we know that

$$(S_N(t,s)f)(x) \;=\; \int_{\mathbf{R}^2} \Gamma_U(x,t;y,s)f(y)\,\mathrm{d}y\;, \quad x \in \mathbf{R}^2\;, \quad 0 < s < t\;,$$

where the fundamental solution Γ_U satisfies (2.5) to (2.8) for some constants K_1, K_2 depending on M_{pp} (but otherwise independent of N). By Remark 2.1, $\Gamma_U(x,t;y,s)$ can be continuously extended to $s = 0$, so that $S_N(t,s)$ is well-defined for $0 \le s < t$. The following properties of this operator will be useful:

Proposition 4.3 *Let $p \in [1,\infty]$.*
i) There exists $K_6 > 0$ (depending on M_{pp}) such that, for any measure $\nu \in \mathcal{M}(\mathbf{R}^2)$,

$$\|S_N(t,s)\nu\|_{L^p} \;\le\; \frac{K_6}{(t-s)^{1-\frac{1}{p}}}\,\|\nu\|_{\mathcal{M}}\;, \quad 0 \le s < t\;. \tag{4.5}$$

ii) For any $\gamma \in (0,\frac{1}{2})$, there exists $K_7 > 0$ (depending on M_{pp} and γ) and $t_0 > 0$ (depending also on d) such that, for any function $f \in L^1(\mathbf{R}^2)$,

$$\|S_N(t,s)\nabla f\|_{L^p} \;\le\; \frac{K_7}{(t-s)^{\frac{3}{2}-\frac{1}{p}}}\Big(\frac{t}{s}\Big)^{\gamma}\|f\|_{L^1}\;, \quad 0 < s < t < s + t_0\;. \tag{4.6}$$

Proof: See Section 6.2. □

Remark 4.4 *We believe that (4.6) holds for $\gamma = 0$ and $t_0 = +\infty$, but we were not able to prove that. In what follows, we assume without loss of generality that $t_0 \le T$.*

As a consequence, if we write $\tilde{u}(t')\cdot\nabla\tilde{\omega}_0(t') = \nabla\cdot(\tilde{u}(t')\tilde{\omega}_0(t'))$ in the right-hand side of (4.4) and if we use the bound (4.6), we see that the integral in (4.4) has a limit in $L^1(\mathbf{R}^2)$ as $s \to 0$. Moreover, proceeding as in the proof of (3.3), we obtain $S_N(t,s)\tilde{\omega}_0(s) \to S_N(t,0)\mu_0$ as $s \to 0$. Thus $\tilde{\omega}_0(t)$ satisfies the integral equation

$$\tilde{\omega}_0(t) \;=\; S_N(t,0)\mu_0 - \int_0^t S_N(t,s)\nabla\cdot(\tilde{u}(s)\tilde{\omega}_0(s))\,\mathrm{d}s\;, \quad 0 < t < T\;. \tag{4.7}$$

4.2 The atomic part

We now fix $i \in \{1, \ldots, N\}$ and consider the quantity $\tilde{\omega}_i(x,t)$ defined in (3.5), (3.10). Following [10], [12], we introduce the self-similar variables

$$\xi = \frac{x - z_i}{\sqrt{t}} , \quad \tau = \log(t) .$$

We define new functions $\tilde{w}_i(\xi,\tau)$, $\tilde{v}_i(\xi,\tau)$ by the relations

$$\tilde{\omega}_i(x,t) = \frac{1}{t}\,\tilde{w}_i\Big(\frac{x-z_i}{\sqrt{t}}, \log(t)\Big) , \quad \tilde{u}_i(x,t) = \frac{1}{\sqrt{t}}\,\tilde{v}_i\Big(\frac{x-z_i}{\sqrt{t}}, \log(t)\Big) , \tag{4.8}$$

where $x \in \mathbf{R}^2$, $t \in (0,T)$, hence $\xi \in \mathbf{R}^2$, $\tau \in (-\infty, \log(T))$. For notational convenience, we also define

$$w_i(\xi,\tau) = \alpha_i G(\xi) + \alpha_i \tilde{w}_i(\xi,\tau) , \quad v_i(\xi,\tau) = \alpha_i v^G(\xi) + \alpha_i \tilde{v}_i(\xi,\tau) , \tag{4.9}$$

where G and v^G are defined in (1.6). In view of (3.10), we thus have

$$\omega_i(x,t) = \frac{1}{t}\,w_i\Big(\frac{x-z_i}{\sqrt{t}}, \log(t)\Big) , \quad u_i(x,t) = \frac{1}{\sqrt{t}}\,v_i\Big(\frac{x-z_i}{\sqrt{t}}, \log(t)\Big) , \tag{4.10}$$

where u_i is the velocity field associated to ω_i via the Biot-Savart law.

Inserting these definitions into (3.7), we obtain the following evolution equation for w_i:

$$\frac{\partial w_i}{\partial \tau}(\xi,\tau) + v_i(\xi,\tau)\cdot\nabla w_i(\xi,\tau) + R_i(\xi,\tau)\cdot\nabla w_i(\xi,\tau) = (\mathcal{L}w_i)(\xi,\tau) , \tag{4.11}$$

where $\mathcal{L}$ is the Fokker-Planck operator (2.10) and

$$R_i(\xi,\tau) = \sum_{\substack{j=1\\ j\neq i}}^{N} v_j(\xi - (z_j - z_i)\mathrm{e}^{-\frac{\tau}{2}}, \tau) + \mathrm{e}^{\frac{\tau}{2}}\tilde{u}_0(\xi \mathrm{e}^{\frac{\tau}{2}} + z_i, \mathrm{e}^{\tau}) . \tag{4.12}$$

The corresponding integral equation reads:

$$w_i(\tau) = S(\tau-\tau_0)w_i(\tau_0) - \int_{\tau_0}^{\tau} S(\tau-\tau')\Big(v_i(\tau') + R_i(\tau')\Big)\cdot\nabla w_i(\tau')\,\mathrm{d}\tau' , \tag{4.13}$$

for $-\infty < \tau_0 < \tau < \log(T)$. Here $S(\tau) = \exp(\tau\mathcal{L})$ is the semigroup generated by $\mathcal{L}$, and $w_i(\tau) \equiv w_i(\cdot,\tau)$, $v_i(\tau) \equiv v_i(\cdot,\tau)$. Alternatively, using (2.16) and the fact that v_i, R_i are divergence-free vector fields, we have

$$w_i(\tau) = S(\tau-\tau_0)w_i(\tau_0) - \int_{\tau_0}^{\tau} \mathrm{e}^{-\frac{1}{2}(\tau-\tau')}\nabla\cdot S(\tau-\tau')\Big((v_i(\tau') + R_i(\tau'))w_i(\tau')\Big)\,\mathrm{d}\tau' . \tag{4.14}$$

It is clear from the definitions that $w_i \in C^0((-\infty,\log(T)), L^1(\mathbf{R}^2)\cap L^\infty(\mathbf{R}^2))$. Moreover, by (3.8), we have the pointwise bound

$$|w_i(\xi,\tau)| \le K_1|\alpha_i|\,\mathrm{e}^{-\beta|\xi|^2/4} , \quad \xi \in \mathbf{R}^2 , \quad -\infty < \tau < \log(T) . \tag{4.15}$$

In particular, for any $m > 1$, the trajectory $\{w_i(\tau)\}$ is bounded in the weighted space $L^2(m)$ defined in (2.12). Since $w_i(\xi,\tau)$ is smooth, it follows that $w_i \in C^0((-\infty,\log(T)), L^2(m))$ for any $m > 1$. Our next result shows that $w_i(\tau)$ actually converges to $\alpha_i G$ as $\tau \to -\infty$:

Proposition 4.5 *For any $i \in \{1, \ldots, N\}$ and any $m > 1$, $w_i(\tau) \to \alpha_i G$ in $L^2(m)$ as $\tau \to -\infty$.*

Proof: See Section 6.3. □

This result implies that $\tilde{w}_i(\tau)$ converges to zero in $L^2(m)$ for any $m > 1$. In particular, returning to the original variables, we obtain

$$\lim_{t \to 0} t^{1-\frac{1}{p}} \|\tilde{\omega}_i(\cdot, t)\|_{L^p} \;=\; 0 \;, \quad p \in [1, 2] \;. \tag{4.16}$$

We now derive an integral equation for the remainder $\tilde{w}_i(\xi, \tau)$. If we neglect for the moment the term $R_i \cdot \nabla w_i$ in (4.11), and if we replace in this equation the functions w_i, v_i by their expressions (4.9) and keep only the linear terms in $\tilde{w}_i, \tilde{v}_i$, we obtain the following equation:

$$\frac{\partial \tilde{w}_i}{\partial \tau} + \alpha_i (v^G \cdot \nabla \tilde{w}_i + \tilde{v}_i \cdot \nabla G) \;=\; \mathcal{L} w_i \;.$$

As is shown in [12], this system defines a C_0 semigroup in $L^2(m)$, which we denote by $T_{\alpha_i}(\tau)$. We have the following result, which generalizes Proposition 2.2:

Proposition 4.6 *Fix $\alpha \in \mathbf{R}$ and $m > 1$.*
i) There exists $K_8 > 0$ such that, for all $w \in L^2(m)$,

$$\|T_\alpha(\tau) w\|_{L^2(m)} \;\le\; K_8 \|w\|_{L^2(m)} \;, \quad \tau \ge 0 \;. \tag{4.17}$$

ii) If moreover $m > 2$ and $w \in L^2_0(m)$ then

$$\|T_\alpha(\tau) w\|_{L^2(m)} \;\le\; K_8 \, \mathrm{e}^{-\frac{\tau}{2}} \|w\|_{L^2(m)} \;, \quad \tau \ge 0 \;. \tag{4.18}$$

iii) Finally if $q \in [1, 2]$ and $m > 2$, then $T_\alpha(\tau)\nabla$ can be extended to a bounded operator from $L^q(m)$ to $L^2_0(m)$ and there exists $K_9 > 0$ such that

$$\|T_\alpha(\tau) \nabla w\|_{L^2(m)} \;\le\; K_9 \frac{\mathrm{e}^{-\frac{\tau}{2}}}{a(\tau)^{\frac{1}{q}}} \|w\|_{L^q(m)} \;, \quad \tau > 0 \;, \tag{4.19}$$

where $a(\tau) = 1 - \mathrm{e}^{-\tau}$.

Proof: See Section 6.4. □

Remark 4.7 *One can show that the constants K_8, K_9 in Proposition 4.6 are uniformly bounded for α in compact intervals.*

Now if we replace in (4.11) the functions w_i, v_i with their expressions (4.9) and if we use the above notation, we see that $\tilde{w}_i(\tau)$ is a solution of the integral equation

$$\tilde{w}_i(\tau) \;=\; T_{\alpha_i}(\tau - \tau_0) \tilde{w}_i(\tau_0) - \int_{\tau_0}^{\tau} T_{\alpha_i}(\tau - \tau') \Big(\alpha_i \tilde{v}_i \cdot \nabla \tilde{w}_i + R_i \cdot \nabla (G + \tilde{w}_i) \Big)(\tau') \, \mathrm{d}\tau' \;,$$

for $-\infty < \tau_0 < \tau < \log(T)$. By Proposition 4.5, $\tilde{w}_i(\tau) \to 0$ in $L^2(m)$ as $\tau \to -\infty$. Thus taking the limit $\tau_0 \to -\infty$ and using Proposition 4.6, we obtain the desired equation:

$$\tilde{w}_i(\tau) \;=\; -\int_{-\infty}^{\tau} T_{\alpha_i}(\tau - \tau') \nabla \cdot \Big(\alpha_i \tilde{v}_i(\tau') \tilde{w}_i(\tau') + R_i(\tau')(G + \tilde{w}_i(\tau')) \Big) \, \mathrm{d}\tau' \;. \tag{4.20}$$

5 The contraction argument

This section is devoted to the proof of Theorem 1.1 and of the continuity statement in Theorem 1.2. By (3.11), we know that our solution $\omega(x,t)$ can be decomposed into a finite sum of Oseen vortices and a remainder $\tilde\omega(x,t)$ which is small due to (4.1), (4.16). Thus a natural idea is to consider the equation satisfied by $\tilde\omega(x,t)$ and to apply a Gronwall argument as in [16]. However, this approach requires very precise estimates on the evolution operator associated to the linearized equation

$$\frac{\partial\tilde\omega}{\partial t} + \sum_{i=1}^N \Bigl(\frac{\alpha_i}{\sqrt{t}} v^G\Bigl(\frac{x-z_i}{\sqrt{t}}\Bigr)\cdot\nabla\tilde\omega + \tilde u\cdot\nabla\Bigl(\frac{\alpha_i}{t}G\Bigl(\frac{x-z_i}{\sqrt{t}}\Bigr)\Bigr)\Bigr) ~=~ \Delta\tilde\omega ~,$$

which are not easy to obtain. Instead we chose to apply a Gronwall argument directly to the set of equations (4.7) (4.20), because the evolution operators $S_N(t,s)$ and $T_{\alpha_i}(\tau)$ that appear in these equations are simpler to estimate and were already studied in [26], [12]. The price to pay with this approach is that (4.20) still contains some linear terms in the right-hand side, which will make the Gronwall argument more delicate.

To make the computations easier to follow, we first deal with a single solution in Section 5.1, and in Section 5.2 we deduce estimates on the difference of two solutions which will imply Theorem 1.1. In Section 5.3 this argument is adapted to prove the continuity statement in Theorem 1.2.

5.1 Estimates on a single solution

Let ω be a solution of (1.3) satisfying the assumptions of Theorem 1.1, and let u be the corresponding velocity field. We recall that the initial measure μ can be decomposed as in (3.1), with $\|\mu_0\|_{\mathrm{pp}} \le \varepsilon$ for some $\varepsilon > 0$ that will be fixed in Section 5.2. According to (3.11), ω and u can be decomposed as follows:

$$\omega(x,t) ~=~ \sum_{i=1}^N \frac{\alpha_i}{t} G\Bigl(\frac{x-z_i}{\sqrt{t}}\Bigr) + \tilde\omega(x,t) ~, \quad u(x,t) ~=~ \sum_{i=1}^N \frac{\alpha_i}{\sqrt{t}} v^G\Bigl(\frac{x-z_i}{\sqrt{t}}\Bigr) + \tilde u(x,t) ~,$$

where

$$\tilde\omega(x,t) ~=~ \tilde\omega_0(x,t) + \sum_{i=1}^N \alpha_i\tilde\omega_i(x,t) ~, \quad \tilde u(x,t) ~=~ \tilde u_0(x,t) + \sum_{i=1}^N \alpha_i\tilde u_i(x,t) ~.$$

Moreover, according to (4.7) and (4.20), the remainder terms $\tilde\omega_i$ satisfy the following integral equations:

- For $i = 0$ and $0 < t < T$,

$$\tilde\omega_0(t) ~=~ S_N(t,0)\mu_0 - \int_0^t S_N(t,s)\nabla\cdot(\tilde u(s)\tilde\omega_0(s))\,\mathrm{d}s ~. \tag{5.1}$$

- For $i \in \{1,\dots,N\}$ and $-\infty < \tau < \log(T)$,

$$\tilde w_i(\tau) ~=~ -\int_{-\infty}^\tau T_{\alpha_i}(\tau-\tau')\nabla\cdot\Bigl(\alpha_i\tilde v_i(\tau')\tilde w_i(\tau') + R_i(\tau')(G + \tilde w_i(\tau'))\Bigr)\,\mathrm{d}\tau' ~, \tag{5.2}$$

where according to (4.8)

$$\tilde w_i(\xi,\tau) ~=~ \mathrm{e}^\tau\tilde\omega_i(\xi\mathrm{e}^{\frac{\tau}{2}},\mathrm{e}^\tau) ~, \quad \tilde v_i(\xi,\tau) ~=~ \mathrm{e}^{\frac{\tau}{2}}\tilde u_i(\xi\mathrm{e}^{\frac{\tau}{2}},\mathrm{e}^\tau) ~.$$

Fix $m > 2$. For $t \in (0,T)$, we define $M(t) = \max\{M_0(t), M_1(t), \ldots, M_N(t)\}$, where

$$M_0(t) \;=\; \sup_{0<s\le t} s^{\frac14}\|\tilde\omega_0(s)\|_{L^{\frac43}}\;, \quad M_i(t) \;=\; \sup_{-\infty<\tau'\le\log(t)} \|\tilde w_i(\tau')\|_{L^2(m)}\;, \quad i\in\{1,\ldots,N\}\;.$$

We have the following results:

Proposition 5.1 *There exist positive constants K_{10}, K_{11} (depending only on $M_{\rm pp}$) such that*

$$M_0(t) \;\le\; \delta_1(t) + K_{11}M_0(t)M(t)\;, \quad 0<t<t_0\;,$$

where $t_0 > 0$ is as in Proposition 4.3 and $\delta_1(t) \le K_{10}\varepsilon$ for $t > 0$ small enough (depending on μ_0).

We recall that $M_{\rm pp}$ and d are the quantities defined in (3.2).

Proof: The first term in the right-hand side of (5.1) can be estimated as in Lemma 4.1, namely

$$\limsup_{t\to0} t^{\frac14}\|S_N(t,0)\mu_0\|_{L^{\frac43}} \;\le\; K_{10}\|\mu_0\|_{\rm pp} \;\le\; K_{10}\varepsilon\;, \tag{5.3}$$

where K_{10} depends only on $M_{\rm pp}$. To bound the integral in (5.1), we observe that $t^{\frac14}\|\tilde\omega_i(t)\|_{L^{\frac43}} \le CM_i(t)$ for $0<t<T$. This is obvious for $i=0$, whereas for $i\in\{1,\ldots,N\}$ we have

$$t^{\frac14}\|\tilde\omega_i(t)\|_{L^{\frac43}} \;=\; \|\tilde w_i(\log(t))\|_{L^{\frac43}} \;\le\; C\|\tilde w_i(\log(t))\|_{L^2(m)} \;\le\; CM_i(t)\;,$$

since $L^2(m) \hookrightarrow L^{\frac43}(\mathbf{R}^2)$. It follows that

$$t^{\frac14}\|\tilde\omega(t)\|_{L^{\frac43}} \;\le\; M_0(t) + C\sum_{i=1}^N|\alpha_i|M_i(t) \;\le\; C_1M(t)\;,$$

where $C_1 > 0$ depends only on $M_{\rm pp}$. As a consequence, using (2.18) and Hölder's inequality, we find

$$\|\tilde u(t)\tilde\omega_0(t)\|_{L^1} \;\le\; \|\tilde u(t)\|_{L^4}\|\tilde\omega_0(t)\|_{L^{\frac43}} \;\le\; C\|\tilde\omega(t)\|_{L^{\frac43}}\|\tilde\omega_0(t)\|_{L^{\frac43}} \;\le\; C\frac{M_0(t)M(t)}{t^{\frac12}}\;, \quad 0<t<T\;.$$

Now, using Proposition 4.3, we obtain for $t\in(0,t_0)$

$$\begin{aligned} t^{\frac14}\Big\|\int_0^t S_N(t,s)\nabla\cdot(\tilde u(s)\tilde\omega_0(s))\,{\rm d}s\Big\|_{L^{\frac43}} &\le t^{\frac14}\int_0^t \frac{K_7}{(t-s)^{\frac34}}\Big(\frac{t}{s}\Big)^\gamma\|\tilde u(s)\tilde\omega_0(s)\|_{L^1}\,{\rm d}s \\ &\le t^{\frac14}\int_0^t\frac{C}{(t-s)^{\frac34}}\Big(\frac{t}{s}\Big)^\gamma\frac{M_0(s)M(s)}{s^{\frac12}}\,{\rm d}s \;\le\; K_{11}M_0(t)M(t)\;. \end{aligned}$$

Combining this estimate with (5.3) we obtain the desired result. □

Proposition 5.2 *There exists a constant $K_{12} > 0$ depending only on $M_{\rm pp}$ such that, for all $i \in \{1,\ldots,N\}$ and all $t\in(0,T)$,*

$$M_i(t) \;\le\; \delta_2(t) + \eta(t)M(t) + K_{12}M_i(t)M(t)\;,$$

where $\eta(t)$ and $\delta_2(t)$ converge to zero as $t\to0$.

Proof: We fix $i \in \{1,\dots,N\}$ and estimate successively all terms in the right-hand side of (5.2). Using (4.12) and (4.9), we obtain

$$\int_{-\infty}^{\tau} T_{\alpha_i}(\tau-\tau')\nabla\cdot\Big(\alpha_i\tilde v_i(\tau')\tilde w_i(\tau') + R_i(\tau')(G+\tilde w_i(\tau'))\Big)\,\mathrm{d}\tau' \;=\; \sum_{k=1}^{6} F_{i,k}(\tau)\;, \qquad (5.4)$$

where

$$\begin{aligned}
F_{i,1}(\tau) &= \sum_{j\neq i}\int_{-\infty}^{\tau} T_{\alpha_i}(\tau-\tau')\nabla\cdot\Big(\alpha_j v^G(\xi-(z_j{-}z_i)\mathrm{e}^{-\frac{\tau'}{2}})G\Big)\,\mathrm{d}\tau'\;,\\
F_{i,2}(\tau) &= \sum_{j\neq i}\int_{-\infty}^{\tau} T_{\alpha_i}(\tau-\tau')\nabla\cdot\Big(\alpha_j \tilde v_j(\xi-(z_j{-}z_i)\mathrm{e}^{-\frac{\tau'}{2}},\tau')G\Big)\,\mathrm{d}\tau'\;,\\
F_{i,3}(\tau) &= \int_{-\infty}^{\tau} T_{\alpha_i}(\tau-\tau')\nabla\cdot\Big(\mathrm{e}^{\frac{\tau'}{2}}\tilde u_0(\xi\mathrm{e}^{\frac{\tau'}{2}}+z_i,\mathrm{e}^{\tau'})G\Big)\,\mathrm{d}\tau'\;,\\
F_{i,4}(\tau) &= \sum_{j\neq i}\int_{-\infty}^{\tau} T_{\alpha_i}(\tau-\tau')\nabla\cdot\Big(\alpha_j v^G(\xi-(z_j{-}z_i)\mathrm{e}^{-\frac{\tau'}{2}})\tilde w_i(\tau')\Big)\,\mathrm{d}\tau'\;,\\
F_{i,5}(\tau) &= \sum_{j=1}^{N}\int_{-\infty}^{\tau} T_{\alpha_i}(\tau-\tau')\nabla\cdot\Big(\alpha_j \tilde v_j(\xi-(z_j{-}z_i)\mathrm{e}^{-\frac{\tau'}{2}},\tau')\tilde w_i(\tau')\Big)\,\mathrm{d}\tau'\;,\\
F_{i,6}(\tau) &= \int_{-\infty}^{\tau} T_{\alpha_i}(\tau-\tau')\nabla\cdot\Big(\mathrm{e}^{\frac{\tau'}{2}}\tilde u_0(\xi\mathrm{e}^{\frac{\tau'}{2}}+z_i,\mathrm{e}^{\tau'})\tilde w_i(\tau')\Big)\,\mathrm{d}\tau'\;.
\end{aligned}$$

We start with $F_{i,1}$. Recalling that $\|w\|_{L^2(m)} = \|b^m w\|_{L^2}$ with $b(\xi) = (1+|\xi|^2)^{\frac{1}{2}}$, we find using Proposition 4.6

$$\begin{aligned}
\|F_{i,1}(\tau)\|_{L^2(m)} &\le K_9\sum_{j\neq i}|\alpha_j|\int_{-\infty}^{\tau}\frac{\mathrm{e}^{-\frac{\tau-\tau'}{2}}}{a(\tau-\tau')^{\frac{1}{2}}}\|v^G(\xi-(z_j{-}z_i)\mathrm{e}^{-\frac{\tau'}{2}})b^m G\|_{L^2}\,\mathrm{d}\tau'\\
&\le K_9\sum_{j\neq i}|\alpha_j|\int_{-\infty}^{\tau}\frac{\mathrm{e}^{-\frac{\tau-\tau'}{2}}}{a(\tau-\tau')^{\frac{1}{2}}}\,\theta_j(\tau')\|G\|_{L^2(m+1)}\,\mathrm{d}\tau'\;,
\end{aligned}$$

where

$$\theta_j(\tau) \stackrel{\mathrm{def}}{=} \sup_{\xi\in\mathbf{R}^2}|b(\xi)^{-1}v^G(\xi-(z_j{-}z_i)\mathrm{e}^{-\frac{\tau}{2}})|\;,\quad j\neq i\;.$$

Using the explicit expression (1.6), it is easy to verify that $\theta_j(\tau) \le C\mathrm{e}^{\frac{\tau}{2}}$ for some $C>0$ depending only on d. It follows that

$$\|F_{i,1}(\tau)\|_{L^2(m)} \le C_1\mathrm{e}^{\frac{\tau}{2}}\;,\quad -\infty<\tau<\log(T)\;, \qquad (5.5)$$

where $C_1>0$ depends on M_{pp} and d.

To estimate $F_{i,2}$, we write similarly

$$\|F_{i,2}(\tau)\|_{L^2(m)} \le K_9\sum_{j\neq i}|\alpha_j|\int_{-\infty}^{\tau}\frac{\mathrm{e}^{-\frac{\tau-\tau'}{2}}}{a(\tau-\tau')^{\frac{1}{2}}}\Big\|\tilde v_j(\xi-(z_j{-}z_i)\mathrm{e}^{-\frac{\tau'}{2}},\tau')b^m G\Big\|_{L^2}\,\mathrm{d}\tau'\;.$$

Next, we fix $q\in(2,\infty)$ and $\nu\in(0,1)$ such that $\nu>\frac{2}{q}$. Then, by Hölder's inequality,

$$\Big\|\tilde v_j(\xi-(z_j{-}z_i)\mathrm{e}^{-\frac{\tau'}{2}},\tau')b^m G\Big\|_{L^2} \le \Big\|b^{\nu-\frac{2}{q}}\tilde v_j(\tau')\Big\|_{L^q}\Big\|b(\xi-(z_j{-}z_i)\mathrm{e}^{-\frac{\tau'}{2}})^{\frac{2}{q}-\nu}b^m G\Big\|_{L^{\frac{2q}{q-2}}}\;.$$

In view of (2.20) the first factor in the right-hand side can be estimated by $C\|\tilde{w}_j(\tau')\|_{L^2(m)}$, and a direct calculation shows that the second one is bounded by $C\mathrm{e}^{\tau'(\frac{\nu}{2}-\frac{1}{q})}$, where $C>0$ depends on d. Thus

$$\begin{aligned}
\|F_{i,2}(\tau)\|_{L^2(m)} &\le C\sum_{j\neq i}|\alpha_j|\int_{-\infty}^{\tau}\frac{\mathrm{e}^{-\frac{\tau-\tau'}{2}}}{a(\tau-\tau')^{\frac{1}{2}}}\|\tilde{w}_j(\tau')\|_{L^2(m)}\mathrm{e}^{\tau'(\frac{\nu}{2}-\frac{1}{q})}\,\mathrm{d}\tau' \\
&\le C_2 M(\mathrm{e}^{\tau})\mathrm{e}^{\tau(\frac{\nu}{2}-\frac{1}{q})}\ ,
\end{aligned} \tag{5.6}$$

where $C_2>0$ depends on M_{pp} and d.

Now we consider $F_{i,3}$. Using Proposition 4.6 and Hölder's inequality, we obtain

$$\begin{aligned}
\|F_{i,3}(\tau)\|_{L^2(m)} &\le K_9\int_{-\infty}^{\tau}\frac{\mathrm{e}^{-\frac{\tau-\tau'}{2}}}{a(\tau-\tau')^{\frac{1}{2}}}\left\|\mathrm{e}^{\frac{\tau'}{2}}\tilde{u}_0(\xi\mathrm{e}^{\frac{\tau'}{2}}+z_i,\mathrm{e}^{\tau'})G^{\frac{1}{2}}\right\|_{L^4}\|b^mG^{\frac{1}{2}}\|_{L^4}\,\mathrm{d}\tau' \\
&\le C\int_{-\infty}^{\tau}\frac{\mathrm{e}^{-\frac{\tau-\tau'}{2}}}{a(\tau-\tau')^{\frac{1}{2}}}\lambda_i(\mathrm{e}^{\tau'})\,\mathrm{d}\tau' \le C_3\lambda_i(\mathrm{e}^{\tau})\ ,
\end{aligned} \tag{5.7}$$

where $C_3>0$ depends on M_{pp} and

$$\lambda_i(t) = \sup_{0<s\le t} s^{\frac{1}{4}}\|\tilde{u}_0(x,s)\mathrm{e}^{-\frac{|x-z_i|^2}{8s}}\|_{L^4}\ .$$

Applying Lemma 4.2 with $q=4$ and $\chi(r)=\exp(-r/8)$, we see that $\lambda_i(t)$ converges to zero as t goes to 0.

For the term $F_{i,4}$, we first remark that

$$\|F_{i,4}(\tau)\|_{L^2(m)} \le K_9\sum_{j\neq i}|\alpha_j|\int_{-\infty}^{\tau}\frac{\mathrm{e}^{-\frac{\tau-\tau'}{2}}}{a(\tau-\tau')^{\frac{1}{2}}}\|v^G(\xi-(z_j-z_i)\mathrm{e}^{-\frac{\tau'}{2}})\tilde{w}_i(\tau')\|_{L^2(m)}\,\mathrm{d}\tau'\ . \tag{5.8}$$

Since $v^G\in L^\infty(\mathbf{R}^2)$, it follows that

$$\|F_{i,4}(\tau)\|_{L^2(m)} \le C\sum_{j\neq i}|\alpha_j|\int_{-\infty}^{\tau}\frac{\mathrm{e}^{-\frac{\tau-\tau'}{2}}}{a(\tau-\tau')^{\frac{1}{2}}}\|\tilde{w}_i(\tau')\|_{L^2(m)}\,\mathrm{d}\tau' \le C_4' M_i(\mathrm{e}^{\tau})\ , \tag{5.9}$$

where $C_4'>0$ depends on M_{pp}. This bound will be used later on when estimating the difference of two solutions. It is not sufficient for our present purposes because, unlike in (5.6), the prefactor of $M_i(\mathrm{e}^{\tau})$ does not converge to zero as $\tau\to-\infty$. To estimate $F_{i,4}$ more precisely, we observe that, on the one hand,

$$|\xi|\le\frac{d}{2}\mathrm{e}^{-\frac{\tau'}{2}} \quad\Rightarrow\quad |v^G(\xi-(z_j-z_i)\mathrm{e}^{-\frac{\tau'}{2}})|\le C\mathrm{e}^{\frac{\tau'}{2}}\ ,$$

where $C>0$ depends on d. On the other hand the bound (4.15) on $w_i(\xi,\tau)$ implies that

$$|\xi|\ge\frac{d}{2}\mathrm{e}^{-\frac{\tau'}{2}} \quad\Rightarrow\quad |\tilde{w}_i(\xi,\tau')|\le C\mathrm{e}^{-\frac{\beta}{4}|\xi|^2} \le C\mathrm{e}^{-\frac{\beta}{8}|\xi|^2}\mathrm{e}^{-\frac{\beta d^2}{32}\mathrm{e}^{-\tau'}}\ .$$

It follows that

$$\|v^G(\xi-(z_j-z_i)\mathrm{e}^{-\frac{\tau'}{2}})\tilde{w}_i(\tau')\|_{L^2(m)} \le C\Big(\mathrm{e}^{\frac{\tau'}{2}}\|\tilde{w}_i(\tau')\|_{L^2(m)}+\zeta(\mathrm{e}^{\tau'})\Big)\ ,$$

where $C > 0$ depends on d and $\zeta(t) = \exp(-\rho/t)$ for some $\rho > 0$. Replacing into (5.8), we thus find

$$\|F_{i,4}(\tau)\|_{L^2(m)} \;\le\; C_4\Big(\mathrm{e}^{\frac{\tau}{2}} M_i(\mathrm{e}^\tau) + \zeta(\mathrm{e}^\tau)\Big)\;, \tag{5.10}$$

where $C_4 > 0$ depends on M_{pp} and d.

To estimate $F_{i,5}$ we have by Proposition 4.6, for $1 < p < 2$,

$$\|F_{i,5}(\tau)\|_{L^2(m)} \;\le\; K_9 \sum_{j=1}^N |\alpha_j| \int_{-\infty}^\tau \frac{\mathrm{e}^{-\frac{\tau-\tau'}{2}}}{a(\tau-\tau')^{\frac{1}{p}}} \|\tilde v_j(\xi - (z_j - z_i)\mathrm{e}^{-\frac{\tau'}{2}}, \tau')\tilde w_i(\tau')\|_{L^p(m)}\,\mathrm{d}\tau'\;.$$

If $b(\xi) = (1+|\xi|^2)^{\frac{1}{2}}$, we have using (2.18) and Hölder's inequality

$$\begin{aligned}
\|\tilde v_j \tilde w_i\|_{L^p(m)} \;=\; \|b^m \tilde v_j \tilde w_i\|_{L^p} &\;\le\; \|b^m \tilde w_i\|_{L^2} \|\tilde v_j\|_{L^{\frac{2p}{2-p}}} \\
&\;\le\; C\|\tilde w_i\|_{L^2(m)}\|\tilde w_j\|_{L^p} \;\le\; C\|\tilde w_i\|_{L^2(m)}\|\tilde w_j\|_{L^2(m)}\;.
\end{aligned}$$

It follows that

$$\|F_{i,5}(\tau)\|_{L^2(m)} \;\le\; C\sum_{j=1}^N |\alpha_j| M_j(\mathrm{e}^\tau) M_i(\mathrm{e}^\tau) \;\le\; C_5 M_i(\mathrm{e}^\tau) M(\mathrm{e}^\tau)\;, \tag{5.11}$$

where $C_5 > 0$ depends on M_{pp}.

Finally we consider the last term, $F_{i,6}$. Choosing $p = \frac{4}{3} \in (1,2)$, we obtain as above

$$\begin{aligned}
\|F_{i,6}(\tau)\|_{L^2(m)} &\;\le\; K_9 \int_{-\infty}^\tau \frac{\mathrm{e}^{-\frac{\tau-\tau'}{2}}}{a(\tau-\tau')^{\frac{1}{p}}} \Big\|\mathrm{e}^{\frac{\tau'}{2}} \tilde u_0(\xi \mathrm{e}^{\frac{\tau'}{2}} + z_i, \mathrm{e}^{\tau'})\tilde w_i(\tau')\Big\|_{L^p(m)}\,\mathrm{d}\tau' \\
&\;\le\; K_9 \int_{-\infty}^\tau \frac{\mathrm{e}^{-\frac{\tau-\tau'}{2}}}{a(\tau-\tau')^{\frac{1}{p}}} \Big\|\mathrm{e}^{\frac{\tau'}{2}} \tilde u_0(\xi \mathrm{e}^{\frac{\tau'}{2}} + z_i, \mathrm{e}^{\tau'})\Big\|_{L^{\frac{2p}{2-p}}} \|\tilde w_i(\tau')\|_{L^2(m)}\,\mathrm{d}\tau'\;.
\end{aligned}$$

Using (2.18), we find (with $t = \mathrm{e}^\tau$)

$$\|\mathrm{e}^{\frac{\tau}{2}} \tilde u_0(\xi \mathrm{e}^{\frac{\tau}{2}} + z_i, \mathrm{e}^\tau)\|_{L^{\frac{2p}{2-p}}} \;=\; t^{1-\frac{1}{p}} \|\tilde u_0(\cdot,t)\|_{L^{\frac{2p}{2-p}}} \;\le\; C t^{1-\frac{1}{p}} \|\tilde\omega_0(\cdot,t)\|_{L^p} \;\le\; C M_0(t)\;,$$

hence finally

$$\|F_{i,6}(\tau)\|_{L^2(m)} \;\le\; C_6 M_0(\mathrm{e}^\tau) M_i(\mathrm{e}^\tau)\;, \tag{5.12}$$

where $C_6 > 0$ depends on M_{pp}. Collecting estimates (5.5) to (5.12), we obtain the desired bound on $M_i(t)$. This concludes the proof of Proposition 5.2. □

Note that Propositions 5.1 and 5.2 together imply that

$$M(t) \;\le\; \delta(t) + \eta(t) M(t) + K_{13} M(t)^2\;, \quad 0 < t < t_0\;, \tag{5.13}$$

where $K_{13} > 0$ depends only on M_{pp}, $\eta(t)$ goes to zero as $t \to 0$, and $\delta(t) \le K_{10}\varepsilon$ if $t > 0$ is small enough. Both functions $\eta(t)$, $\delta(t)$ depend on the full initial measure μ, not only on M_{pp} and d.

5.2 The uniqueness proof

This section is devoted to the end of the proof of Theorem 1.1. Let $\omega^{(1)}$ and $\omega^{(2)}$ be two solutions of (1.3) satisfying the assumptions of Theorem 1.1 with the same initial measure μ. Each solution can be decomposed as in (3.11), namely

$$\omega^{(\ell)}(x,t) \;=\; \sum_{i=1}^{N} \frac{\alpha_i}{t}\, G\Big(\frac{x-z_i}{\sqrt{t}}\Big) + \tilde\omega^{(\ell)}(x,t)\;, \quad \tilde\omega^{(\ell)}(x,t) \;=\; \tilde\omega_0^{(\ell)}(x,t) + \sum_{i=1}^{N} \alpha_i \tilde\omega_i^{(\ell)}(x,t)\;,$$

for $\ell \in \{1,2\}$. Estimate (5.13) becomes, with obvious notations,

$$M^{(\ell)}(t) \;\le\; \delta(t) + \eta(t) M^{(\ell)}(t) + K_{13} M^{(\ell)}(t)^2\;, \quad \ell \in \{1,2\}\;, \quad 0 < t < t_0\;. \tag{5.14}$$

Now, we define $\Delta(t) = \max\{\Delta_0(t), \Delta_1(t), \dots, \Delta_N(t)\}$, where

$$\Delta_0(t) \;=\; \sup_{0<s\le t} s^{\frac14} \|\tilde\omega_0^{(1)}(s) - \tilde\omega_0^{(2)}(s)\|_{L^{\frac43}}\;,$$

and

$$\Delta_i(t) \;=\; \sup_{-\infty<\tau'\le \log(t)} \|\tilde w_i^{(1)}(\tau') - \tilde w_i^{(2)}(\tau')\|_{L^2(m)}\;, \quad i \in \{1,\dots,N\}\;. \tag{5.15}$$

Here and in the sequel, $\tilde w_i^{(\ell)}(\xi,\tau) \;=\; \mathrm{e}^{\tau} \tilde\omega_i^{(\ell)}(\xi \mathrm{e}^{\frac{\tau}{2}}, \mathrm{e}^{\tau})$ for $\ell \in \{1,2\}$. We have the following result:

Proposition 5.3 *There exists a constant $K_{14} > 0$ depending only on M_{pp} such that*

$$\Delta(t) \;\le\; \eta(t)\Delta(t) + K_{14}\Big(M^{(1)}(t) + M^{(2)}(t)\Big)\Delta(t) + \zeta(t)\;, \quad 0 < t < t_0\;,$$

where $\eta(t)$ goes to zero as $t \to 0$ and $\zeta(t) = C\mathrm{e}^{-\rho/t}$ for some $\rho > 0$. Moreover,

$$\Delta(t) \;\le\; \eta(t)\Delta(t) + K_{14}\Big(M^{(1)}(t) + M^{(2)}(t)\Big)\Delta(t) + K_{14}\int_0^t \frac{\Delta(s)}{(t-s)^{\frac12} s^{\frac12}}\,\mathrm{d}s\;.$$

Proof: The argument consists in mimicking the proofs of Propositions 5.1 and 5.2 above. We start by estimating $\Delta_0(t)$. We have of course

$$\tilde\omega_0^{(1)}(t) - \tilde\omega_0^{(2)}(t) \;=\; -\int_0^t S_N(t,s)\nabla\cdot\Big(\tilde u^{(1)}(s)\tilde\omega_0^{(1)}(s) - \tilde u^{(2)}(s)\tilde\omega_0^{(2)}(s)\Big)\,\mathrm{d}s\;.$$

If we write $\tilde u^{(1)}\tilde\omega_0^{(1)} - \tilde u^{(2)}\tilde\omega_0^{(2)} = (\tilde u^{(1)} - \tilde u^{(2)})\tilde\omega_0^{(1)} + \tilde u^{(2)}(\tilde\omega_0^{(1)} - \tilde\omega_0^{(2)})$ and if we proceed exactly as in the proof of Proposition 5.1, we obtain

$$\Delta_0(t) \;\le\; C_0 \Delta(t)\Big(M^{(1)}(t) + M^{(2)}(t)\Big)\;, \quad 0 < t < t_0\;,$$

where $C_0 > 0$ depends only on M_{pp}.

We now bound $\Delta_i(t)$ for $i \in \{1,\dots,N\}$. Let $G_{i,k}(\tau) = F_{i,k}^{(1)}(\tau) - F_{i,k}^{(2)}(\tau)$ for $k \in \{1,\dots,6\}$, where $F_{i,k}^{(1)}$ and $F_{i,k}^{(2)}$ are defined in analogy with (5.4). Then obviously $G_{i,1} = 0$. The quadratic terms $G_{i,k}$ for $k \in \{5,6\}$ can be estimated as in the case of Δ_0 above. In view of (5.11), (5.12), we thus find

$$\|G_{i,5}(\tau)\|_{L^2(m)} + \|G_{i,6}(\tau)\|_{L^2(m)} \;\le\; C_1 \Delta(\mathrm{e}^{\tau})\Big(M^{(1)}(\mathrm{e}^{\tau}) + M^{(2)}(\mathrm{e}^{\tau})\Big)\;,$$

for $\tau \in (-\infty, \log(T))$, where $C_1 > 0$ depends on $M_{\rm pp}$. It remains to bound the linear terms $G_{i,k}$ for $k \in \{2,3,4\}$. Proceeding as in the proofs of (5.6), (5.10), we obtain

$$\|G_{i,2}(\tau)\|_{L^2(m)} \le C_2\Delta(\mathrm{e}^\tau)\mathrm{e}^{\tau(\frac{\nu}{2}-\frac{1}{q})}\ , \quad \|G_{i,4}(\tau)\|_{L^2(m)} \le C_4\Big(\mathrm{e}^{\frac{\tau}{2}}\Delta_i(\mathrm{e}^\tau)+\zeta(\mathrm{e}^\tau)\Big)\ ,$$

where C_2, C_4 depend on $M_{\rm pp}$ and d, and $\zeta(t) = C\mathrm{e}^{-\rho/t}$ for some $\rho > 0$. Furthermore, using the analogue of (5.9), we have

$$\|G_{i,4}(\tau)\|_{L^2(m)} \le C_4' \int_{-\infty}^{\tau} \frac{\mathrm{e}^{-\frac{\tau-\tau'}{2}}}{a(\tau-\tau')^{\frac12}}\|\tilde{w}_i^{(1)}(\tau')-\tilde{w}_i^{(2)}(\tau')\|_{L^2(m)}\,\mathrm{d}\tau'\ ,$$

where C_4' depends on $M_{\rm pp}$. Returning to the original time variable $t = \mathrm{e}^\tau$, we thus find

$$\|G_{i,4}(\log(t))\|_{L^2(m)} \le C_4'\int_0^t \frac{\Delta_i(s)}{(t-s)^{\frac12}s^{\frac12}}\,\mathrm{d}s\ , \quad 0 < t < T\ .$$

Finally, according to (5.7), we have the bound

$$\begin{aligned}\|G_{i,3}(\tau)\|_{L^2(m)} &\le C\int_{-\infty}^{\tau}\frac{\mathrm{e}^{-\frac{\tau-\tau'}{2}}}{a(\tau-\tau')^{\frac12}}\Big\|\mathrm{e}^{\frac{\tau'}{2}}\tilde{u}_0^{(1)}(\xi\mathrm{e}^{\frac{\tau'}{2}}+z_i,\mathrm{e}^{\tau'})-\mathrm{e}^{\frac{\tau'}{2}}\tilde{u}_0^{(2)}(\xi\mathrm{e}^{\frac{\tau'}{2}}+z_i,\mathrm{e}^{\tau'})\Big\|_{L^4}\mathrm{d}\tau'\\ &\le C\int_{-\infty}^{\tau}\frac{\mathrm{e}^{-\frac{\tau-\tau'}{2}}}{a(\tau-\tau')^{\frac12}}\Delta_0(\mathrm{e}^{\tau'})\,\mathrm{d}\tau' \le C_3\Delta_0(\mathrm{e}^\tau)\ ,\end{aligned}$$

which is sufficient for our purposes since $\Delta_0(t) \le C_0\Delta(t)(M^{(1)}(t)+M^{(2)}(t))$. Collecting all these estimates, we obtain the desired bounds on $\Delta(t)$. This concludes the proof of Proposition 5.3. □

Proof of Theorem 1.1: Let $\tilde{K} = \max\{K_5, K_{10}\}$, where K_5 is as in Lemma 4.1 and K_{10} as in Proposition 5.1. Assume that $\varepsilon > 0$ is sufficiently small so that

$$16K_{13}\tilde{K}\varepsilon \le 1\ , \quad \text{and} \quad 16K_{14}\tilde{K}\varepsilon \le 1\ , \tag{5.16}$$

where K_{13} is as in (5.14) and K_{14} as in Proposition 5.3. Finally, choose $t_1 \in (0, t_0]$ sufficiently small so that

$$\eta(t) \le \frac14\ , \quad \text{and} \quad \delta(t) \le \tilde{K}\varepsilon\ , \quad \text{for } 0 < t \le t_1\ ,$$

where $\delta(t), \eta(t)$ are as in (5.14) and $\eta(t)$ appears in Proposition 5.3 as well. We shall prove that $\Delta(t) = 0$ for $\in (0, t_1]$, hence $\omega^{(1)}(t) = \omega^{(2)}(t)$ on this time interval. Since $\omega^{(\ell)}(t_1) \in L^1(\mathbf{R}^2)\cap L^\infty(\mathbf{R}^2)$ and since the Cauchy problem is well-posed in that space, it will follow that $\omega^{(1)}(t) = \omega^{(2)}(t)$ for all $t \in (0,T)$.

We claim that $M^{(\ell)}(t) \le 2\tilde{K}\varepsilon$ for $\ell \in \{1,2\}$ and $t \in (0, t_1]$. Indeed, by Lemma 4.1 and Proposition 4.5, this is true at least for $t > 0$ sufficiently small. On the other hand, it follows from (5.14) that

$$M^{(\ell)}(t) \le \tilde{K}\varepsilon + \frac14 M^{(\ell)}(t) + K_{13}M^{(\ell)}(t)^2\ , \quad 0 < t \le t_1\ ,$$

hence $M^{(\ell)}(t) < 2\tilde{K}\varepsilon$ as long as $K_{13}M^{(\ell)}(t) < \frac14$. Since $K_{13}(2\tilde{K}\varepsilon) \le \frac18$, this proves the claim.

Now, it follows from (5.16) and Proposition 5.3 that

$$\Delta(t) \le \frac12\Delta(t) + \zeta(t)\ , \quad \text{and} \quad \Delta(t) \le \frac12\Delta(t) + K_{14}\int_0^t \frac{\Delta(s)}{(t-s)^{\frac12}s^{\frac12}}\,\mathrm{d}s\ ,$$

for $t \in (0, t_1]$. The first inequality implies that $\Delta(t) = \mathcal{O}(t^\infty)$ as $t \to 0$. In view of Lemma 5.4 below, the second bound then implies that $\Delta(t) = 0$ for $t \in (0, t_1]$. This concludes the proof of Theorem 1.1. □

Lemma 5.4 *Let $f : [0,T] \to \mathbf{R}_+$ be a continuous function satisfying*

$$f(t) \;\le\; K \int_0^t \frac{f(s)}{(t-s)^{\frac{1}{2}} s^{\frac{1}{2}}}\,\mathrm{d}s\;, \quad 0 \le t \le T\;,$$

for some $K > 0$. If $f(t) = \mathcal{O}(t^\alpha)$ as $t \to 0$ for all $\alpha > 0$, then $f \equiv 0$.

Proof: Given $\alpha \ge 0$, we define

$$F_\alpha(t) \;=\; \sup_{0<s\le t} \frac{f(s)}{s^\alpha}\;, \quad 0 < t \le T\;.$$

If $F_\alpha(T) < \infty$, we have for $t \in (0,T]$:

$$\frac{f(t)}{t^\alpha} \;\le\; \frac{K}{t^\alpha} \int_0^t \frac{s^\alpha F_\alpha(t)}{(t-s)^{\frac{1}{2}} s^{\frac{1}{2}}}\,\mathrm{d}s \;=\; KB(\tfrac{1}{2}, \alpha{+}\tfrac{1}{2}) F_\alpha(t)\;,$$

where

$$B(p,q) \;=\; \int_0^1 (1{-}x)^{p-1} x^{q-1}\,\mathrm{d}x \;=\; \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\;, \quad p,q > 0\;.$$

It follows that $F_\alpha(T) \le KB(\frac{1}{2}, \alpha{+}\frac{1}{2}) F_\alpha(T)$. Now, if $f(t) = \mathcal{O}(t^\infty)$ we can take $\alpha > 0$ large enough so that $KB(\frac{1}{2}, \alpha{+}\frac{1}{2}) < 1$. Then $F_\alpha(T) = 0$, which implies $f \equiv 0$. □

5.3 The continuity proof

In this section we prove that the (unique) solution of (1.3) depends continuously on the initial data in the norm topology of $\mathcal{M}(\mathbf{R}^2)$, as stated in Theorem 1.2. The arguments are very similar to those leading to the uniqueness theorem, except for the fact that the initial measures associated to both solutions are now different. So we shall merely sketch the proof and emphasize where the arguments of the previous sections must be adapted to infer continuity.

Fix $\mu^{(1)} \in \mathcal{M}(\mathbf{R}^2)$, and assume that $\mu^{(2)}$ is another finite measure satisfying $\|\mu^{(1)} - \mu^{(2)}\|_{\mathcal{M}} \le \delta$ for some sufficiently small $\delta > 0$. This implies in particular that the large atoms of $\mu^{(1)}, \mu^{(2)}$ are located at the same points in $\mathbf{R}^2$. More precisely, we can assume that both measures are decomposed as in Section 3, namely

$$\mu^{(\ell)} \;=\; \sum_{i=1}^N \alpha_i^{(\ell)} \delta_{z_i} + \mu_0^{(\ell)}\;, \quad \ell \in \{1,2\}\;,$$

where $\alpha_i^{(\ell)} = \mu^{(\ell)}(\{z_i\}) \neq 0$ and $\|\mu_0^{(\ell)}\|_{\mathrm{pp}} \le \varepsilon$. The parameter $\varepsilon > 0$ is independent of δ and will be assumed to satisfy a smallness condition similar to (5.16). By construction, we have

$$\|\mu^{(1)} - \mu^{(2)}\|_{\mathcal{M}} \;=\; \sum_{i=1}^N |\alpha_i^{(1)} - \alpha_i^{(2)}| + \|\mu_0^{(1)} - \mu_0^{(2)}\|_{\mathcal{M}} \;\le\; \delta\;.$$

For $\ell \in \{1,2\}$, let $\omega^{(\ell)} \in C^0((0,+\infty), L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2))$ be the solution of (1.3) with initial data $\mu^{(\ell)}$. Each solution can be decomposed as in (3.11), namely

$$\omega^{(\ell)}(x,t) \;=\; \sum_{i=1}^N \frac{\alpha_i^{(\ell)}}{t} G\Bigl(\frac{x-z_i}{\sqrt{t}}\Bigr) + \tilde\omega^{(\ell)}(x,t)\;, \quad \tilde\omega^{(\ell)}(x,t) \;=\; \tilde\omega_0^{(\ell)}(x,t) + \sum_{i=1}^N \alpha_i^{(\ell)} \tilde\omega_i^{(\ell)}(x,t)\;.$$

Using the same notations as in the previous section, our goal is to control the quantity $\Delta(t) = \max\{\Delta_0(t), \Delta_1(t), \ldots, \Delta_N(t)\}$, where

$$\Delta_0(t) \;=\; \sup_{0<s\le t} \|\tilde{\omega}_0^{(1)}(s) - \tilde{\omega}_0^{(2)}(s)\|_{L^1} + \sup_{0<s\le t} s^{\frac{1}{4}} \|\tilde{\omega}_0^{(1)}(s) - \tilde{\omega}_0^{(2)}(s)\|_{L^{\frac{4}{3}}} \;,$$

and $\Delta_i(t)$ is defined by (5.15) for $i \in \{1, \ldots, N\}$. We shall prove that, for any $\nu \in (0,1)$, there exists $T > 0$ and $C > 0$ (both independent of δ) such that $\Delta(T) \le C\delta^\nu$. In particular, this implies

$$\sup_{0<t\le T} \|\omega^{(1)}(t) - \omega^{(2)}(t)\|_{L^1} \;\le\; C\delta^\nu \;. \tag{5.17}$$

Since the Cauchy problem for (1.3) in $L^1(\mathbf{R}^2)$ is globally well-posed and since the solution is a locally Lipschitz function of the initial data in that space, uniformly in time on compact intervals, it follows that (5.17) holds for any $T > 0$. This proves the continuity claim in Theorem 1.2.

To bound $\Delta_0(t)$ we write

$$\begin{aligned}
\tilde{\omega}_0^{(1)}(t) - \tilde{\omega}_0^{(2)}(t) \;&=\; \Big(S_N^{(1)}(t,0) - S_N^{(2)}(t,0)\Big)\mu_0^{(1)} + S_N^{(2)}(t,0)\Big(\mu_0^{(1)} - \mu_0^{(2)}\Big) \\
&-\; \int_0^t \Big(S_N^{(1)}(t,s) - S_N^{(2)}(t,s)\Big)\nabla\cdot(\tilde{u}^{(1)}(s)\tilde{\omega}_0^{(1)}(s))\,\mathrm{d}s \\
&-\; \int_0^t S_N^{(2)}(t,s)\nabla\cdot\Big(\tilde{u}^{(1)}(s)\tilde{\omega}_0^{(1)}(s) - \tilde{u}^{(2)}(s)\tilde{\omega}_0^{(2)}(s)\Big) \;,
\end{aligned}$$

where $S_N^{(\ell)}(t,s)$ denotes the evolution operator associated to the convection–diffusion equation (2.1) with $U(x,t) = \sum_{i=1}^N \frac{\alpha_i^{(\ell)}}{\sqrt{t}} v^G\Big(\frac{x-z_i}{\sqrt{t}}\Big)$. The difference $S_N^{(1)}(t,s) - S_N^{(2)}(t,s)$ is estimated using the following variant of Proposition 4.3, which can be proved by a standard perturbation argument (we omit the details).

Proposition 5.5 *Let $p \in [1,\infty]$ and let $S_N^{(1)}$ and $S_N^{(2)}$ be defined as above.*
i) There exists $K_{15} > 0$ independent of δ such that, for any measure $\nu \in \mathcal{M}(\mathbf{R}^2)$,

$$\Big\|\Big(S_N^{(1)}(t,s) - S_N^{(2)}(t,s)\Big)\nu\Big\|_{L^p} \;\le\; \frac{K_{15}\,\delta}{(t-s)^{1-\frac{1}{p}}}\,\|\nu\|_{\mathcal{M}} \;, \quad 0 \le s < t \;.$$

ii) For any $\gamma \in (0,\frac{1}{2})$, there exists $K_{16} > 0$ and $t_0 > 0$ (both independent of δ) such that, for any function $f \in L^1(\mathbf{R}^2)$,

$$\Big\|\Big(S_N^{(1)}(t,s) - S_N^{(2)}(t,s)\Big)\nabla f\Big\|_{L^p} \;\le\; \frac{K_{16}\,\delta}{(t-s)^{\frac{3}{2}-\frac{1}{p}}}\Big(\frac{t}{s}\Big)^\gamma \|f\|_{L^1} \;, \quad 0 < s < t < s + t_0 \;.$$

Using Propositions 4.3 and 5.5, and proceeding as in Section 5.2 above, it is not difficult to show that

$$\Delta_0(t) \;\le\; C_0\delta(1 + M^{(1)}(t)^2) + C_0\Big(M^{(1)}(t) + M^{(2)}(t)\Big)\Delta(t) \;, \quad 0 < t < t_0 \;,$$

where $C_0 > 0$ is independent of δ and $t_0 > 0$ is as in Proposition 5.5.

To bound $\Delta_i(t)$ for $i \in \{1, \ldots, N\}$, we consider the integral equations of the form (5.2) satisfied by the rescaled functions $\tilde{w}_i^{(1)}(\tau)$ and $\tilde{w}_i^{(2)}(\tau)$, and we estimate the difference of both expressions. To this end, we clearly need a bound on the linear operator $(T_\alpha(\tau) - T_{\alpha'}(\tau))\nabla$ with $\alpha \ne \alpha'$. This is the content of the following proposition, whose proof is again left to the reader.

Proposition 5.6 *Fix $\alpha \in \mathbf{R}$, $m > 2$, $\nu \in (0, \frac{1}{2})$ and $q \in [1,2]$. Then there exists $K_{17} > 0$ and $\beta_0 > 0$ such that if $|\beta| \leq \beta_0$ then, for all $f \in L^q(m)$,*

$$\left\| \Bigl(T_{\alpha+\beta}(\tau) - T_\alpha(\tau) \Bigr) \nabla f \right\|_{L^2(m)} \leq K_{17} |\beta| \frac{\mathrm{e}^{-\nu\tau}}{a(\tau)^{\frac{1}{q}}} \|f\|_{L^q(m)} , \quad \tau > 0 ,$$

where $a(\tau) = 1 - \mathrm{e}^{-\tau}$.

As in Section 5.2, we define $G_{i,k}(\tau) = F^{(1)}_{i,k}(\tau) - F^{(2)}_{i,k}(\tau)$ for $k \in \{1, \dots, 6\}$, where $F^{(1)}_{i,k}$ and $F^{(2)}_{i,k}$ are defined in analogy with (5.4). Arguing as in the previous sections, we find the following estimates:

$$\begin{aligned}
\|G_{i,1}(\tau)\|_{L^2(m)} &\leq C_1 \delta \, \mathrm{e}^{\nu\tau} , \\
\|G_{i,2}(\tau)\|_{L^2(m)} &\leq C_2 \delta \, \mathrm{e}^{\nu\tau} M^{(1)}(\mathrm{e}^\tau) + C_2 \mathrm{e}^{\nu\tau} \Delta(\mathrm{e}^\tau) , \\
\|G_{i,3}(\tau)\|_{L^2(m)} &\leq C_3 \delta M^{(1)}(\mathrm{e}^\tau) + C_3 \Delta_0(\mathrm{e}^\tau) , \\
\|G_{i,4}(\tau)\|_{L^2(m)} &\leq C_4 \delta M^{(1)}(\mathrm{e}^\tau) + C_4 \Bigl(\mathrm{e}^{\nu\tau} \Delta(\mathrm{e}^\tau) + \zeta(\mathrm{e}^\tau) \Bigr) ,
\end{aligned}$$

where $\nu \in (0, \frac{1}{2})$ and $\zeta(t) = C \mathrm{e}^{-\rho/t}$ for some $\rho > 0$. Here and in the sequel, all constants are independent of δ. As in Section 5.2, the term $G_{i,4}(\tau)$ can also be estimated as follows:

$$\|G_{i,4}(\log(t))\|_{L^2(m)} \leq C_4 \delta M^{(1)}(t) + C_4 \int_0^t \frac{\Delta_i(s)}{(t-s)^{\frac{1}{2}} s^{\frac{1}{2}}} \, \mathrm{d}s .$$

Finally

$$\sum_{k \in \{5,6\}} \|G_{i,k}(\tau)\|_{L^2(m)} \leq C_5 \delta M^{(1)}(\mathrm{e}^\tau)^2 + C_5 \Delta(\mathrm{e}^\tau) \Bigl(M^{(1)}(\mathrm{e}^\tau) + M^{(2)}(\mathrm{e}^\tau) \Bigr) .$$

Combining these estimates with the above bound on $\Delta_0(t)$, we finally obtain

$$\Delta(t) \leq K_{18} \delta (1 + M^{(1)}(t)^2) + \eta(t) \Delta(t) + K_{19} \Bigl(M^{(1)}(t) + M^{(2)}(t) \Bigr) \Delta(t) + \zeta(t) , \tag{5.18}$$

as well as

$$\begin{aligned}
\Delta(t) &\leq K_{18} \delta (1 + M^{(1)}(t)^2) + \eta(t) \Delta(t) + K_{19} \Bigl(M^{(1)}(t) + M^{(2)}(t) \Bigr) \Delta(t) \\
&+ K_{20} \int_0^t \frac{\Delta(s)}{(t-s)^{\frac{1}{2}} s^{\frac{1}{2}}} \, \mathrm{d}s .
\end{aligned} \tag{5.19}$$

Now, proceeding as in the proof of Theorem 1.1 in Section 5.2, we can choose $\varepsilon > 0$ sufficiently small and then $t_1 \in (0, t_0]$ sufficiently small (both ε and t_1 independent of δ) so that $\eta(t) \leq \frac{1}{4}$ and $K_{19}(M^{(1)}(t) + M^{(2)}(t)) \leq \frac{1}{4}$ for all $t \in (0, t_1]$. The bounds (5.18), (5.19) then imply that, for any $\nu \in (0,1)$, there exists $K_{21} > 0$ (independent of δ) such that $\Delta(t) \leq K_{21} \delta^\nu$ for all $t \in (0, t_1]$, which is the desired result. Indeed, we have the following lemma, which is a generalization of Lemma 5.4:

Lemma 5.7 *Let $f : [0,T] \to \mathbf{R}_+$ be a continuous function satisfying*

$$f(t) \leq C_1 \delta + C_2 \int_0^t \frac{f(s)}{(t-s)^{\frac{1}{2}} s^{\frac{1}{2}}} \, \mathrm{d}s , \quad 0 \leq t \leq T ,$$

for some $C_1, C_2 > 0$ and some $\delta \in [0,1]$. Suppose moreover that $f(t) \leq C_1 \delta + \zeta(t)$ for all $t \in [0,T]$, where $\zeta(t) = C_3 \exp(-\rho/t)$ for some $\rho > 0$. Then for any $\nu \in (0,1)$, there exists a constant $C_4 > 0$ (independent of δ) such that $f(t) \leq C_4 \delta^\nu$ for all $t \in [0,T]$.

Remark 5.8 *What the proof really shows is that $\Delta(t_1) \le C\delta\log(C/\delta)^\gamma$ for some large $\gamma > 0$. Thus our method fails to show that the solution of (1.3) is a locally Lipschitz function of the initial data. This is because we chose to apply the Gronwall argument directly to equation (4.20), see the discussion at the beginning of Section 5.*

6 Appendix

In this final section we prove the main estimates stated in Section 4.

6.1 Proof of Lemma 4.2

We argue as in the proof of ([16], Lemma 4.4). Fix $p \in [1,\infty]$, $q \in (2,\infty]$, and $i \in \{1,\dots,N\}$. Without loss of generality, we can assume that $z_i = 0$. Using the definition (3.6), the bound (2.5), and the properties of the Biot-Savart law, it is easy to show that there exists $C_1 > 0$ such that

$$\|\tilde\omega_0(\cdot,t)\|_{L^p} \le \frac{C_1}{t^{1-\frac{1}{p}}}\|\mu_0\| \;, \quad \|\tilde u_0(\cdot,t)\|_{L^q} \le \frac{C_1}{t^{\frac12-\frac1q}}\|\mu_0\| \;, \quad t > 0 \;. \tag{6.1}$$

Fix any $\delta > 0$. Since $\mu_0(\{0\}) = 0$ by assumption, there exists $r > 0$ such that $|\mu_0|(B_{4r}) \le \delta/C_1$, where $B_r = \{x \in \mathbf{R}^2 \,|\, |x| \le r\}$ and $|\mu_0|$ denotes the total variation measure associated with μ_0. We decompose

$$\tilde\omega_0(x,t) = \int_{B_{4r}} \Gamma_u(x,t;y,0)\,\mathrm{d}\mu_0(y) + \int_{\mathbf{R}^2\setminus B_{4r}} \Gamma_u(x,t;y,0)\,\mathrm{d}\mu_0(y) \stackrel{\text{def}}{=} \omega^{(1)}(x,t) + \omega^{(2)}(x,t) \;.$$

We also have $\tilde u_0(x,t) = u^{(1)}(x,t) + u^{(2)}(x,t)$, where $u^{(j)}$ is the velocity field obtained from $\omega^{(j)}$ via the Biot-Savart law (2.17). By construction,

$$\sup_{t>0} t^{1-\frac1p}\|\omega^{(1)}(\cdot,t)\|_{L^p} \le C_1|\mu_0|(B_{4r}) \le \delta \;, \quad \sup_{t>0} t^{\frac12-\frac1q}\|u^{(1)}(\cdot,t)\|_{L^q} \le C_1|\mu_0|(B_{4r}) \le \delta \;. \tag{6.2}$$

To bound $\omega^{(2)}(x,t)$, we further decompose

$$\omega^{(2)}(x,t) = \omega^{(2)}(x,t)\mathbf{1}_{B_{2r}}(x) + \omega^{(2)}(x,t)\mathbf{1}_{\mathbf{R}^2\setminus B_{2r}}(x) \stackrel{\text{def}}{=} \omega^{(3)}(x,t) + \omega^{(4)}(x,t) \;.$$

Accordingly we set $u^{(2)}(x,t) = u^{(3)}(x,t) + u^{(4)}(x,t)$. Using (2.5), we find

$$|\omega^{(3)}(x,t)| \le \mathbf{1}_{B_{2r}}(x) \int_{\mathbf{R}^2\setminus B_{4r}} \frac{K_1}{t}\,\mathrm{e}^{-\beta\frac{|x-y|^2}{4t}}\,\mathrm{d}|\mu_0|(y) \le \mathrm{e}^{-\beta\frac{r^2}{2t}} \int_{\mathbf{R}^2} \frac{K_1}{t}\,\mathrm{e}^{-\beta\frac{|x-y|^2}{8t}}\,\mathrm{d}|\mu_0|(y) \;,$$

since $|x-y| \ge 2r$ in the first integral. It follows that

$$t^{1-\frac1p}\|\omega^{(3)}(\cdot,t)\|_{L^p} + t^{\frac12-\frac1q}\|u^{(3)}(\cdot,t)\|_{L^q} \le C\,\mathrm{e}^{-\beta\frac{r^2}{2t}}\|\mu_0\|_{\mathcal{M}} \xrightarrow[t\to0]{} 0 \;. \tag{6.3}$$

Finally $\|\omega^{(4)}(x,t)\chi(|x|^2/t)\|_{L^p_x} \le \chi(4r^2/t)\|\omega^{(4)}(\cdot,t)\|_{L^p}$, hence

$$t^{1-\frac1p}\|\omega^{(4)}(x,t)\chi(|x|^2/t)\|_{L^p_x} \le C_1\chi(4r^2/t)\|\mu_0\|_{\mathcal{M}} \xrightarrow[t\to0]{} 0 \;. \tag{6.4}$$

Combining (6.2), (6.3), (6.4), we obtain $\limsup_{t\to0} t^{1-\frac1p}\|\tilde\omega_0(x,t)\chi(|x|^2/t)\|_{L^p_x} \le \delta$. Since $\delta > 0$ was arbitrary, this proves (4.2).

To bound $u^{(4)}(x,t)$, we use yet another decomposition:

$$u^{(4)}(x,t) \;=\; u^{(4)}(x,t)\mathbf{1}_{B_r}(x) + u^{(4)}(x,t)\mathbf{1}_{\mathbf{R}^2\setminus B_r}(x) \;\stackrel{\text{def}}{=}\; u^{(5)}(x,t) + u^{(6)}(x,t)\ .$$

Using the Biot-Savart law (2.17), we find

$$|u^{(5)}(x,t)| \;\le\; \mathbf{1}_{B_r}(x)\int_{\mathbf{R}^2\setminus B_{2r}} \frac{C}{|x-y|}|\omega^{(4)}(y,t)|\,\mathrm{d}y \;\le\; \frac{C}{r}\mathbf{1}_{B_r}(x)\|\omega^{(4)}(\cdot,t)\|_{L^1} \;\le\; \frac{C}{r}\mathbf{1}_{B_r}(x)\|\mu_0\|\ ,$$

hence $t^{\frac12-\frac1q}\|u^{(5)}(\cdot,t)\|_{L^q} \le C(t/r^2)^{\frac12-\frac1q}\|\mu_0\|_{\mathcal{M}} \to 0$ as $t\to 0$. Finally,

$$t^{\frac12-\frac1q}\|u^{(6)}(x,t)\chi(|x|^2/t)\|_{L^q_x} \;\le\; \chi(r^2/t)t^{\frac12-\frac1q}\|u^{(4)}(\cdot,t)\|_{L^q} \;\le\; C\chi(r^2/t)\|\mu_0\|_{\mathcal{M}} \xrightarrow[t\to 0]{} 0\ .$$

Summarizing, we have shown $\limsup_{t\to 0} t^{\frac12-\frac1q}\|\tilde u_0(x,t)\chi(|x|^2/t)\|_{L^q_x} \le \delta$, which implies (4.3). $\square$

6.2 Proof of Proposition 4.3

Estimate (4.5) follows immediately from the bound (2.5) on the integral kernel $\Gamma_U(x,t;y,s)$. To prove (4.6), we first remark that it is sufficient to establish this estimate for $p=1$. Indeed, once this is done, we obtain using (4.5):

$$\begin{aligned}\|S_N(t,s)\nabla f\|_{L^p} \;&=\; \Big\|S_N\Big(t,\frac{t+s}{2}\Big)S_N\Big(\frac{t+s}{2},s\Big)\nabla f\Big\|_{L^p} \;\le\; K_6\Big(\frac{2}{t-s}\Big)^{1-\frac1p}\Big\|S_N\Big(\frac{t+s}{2},s\Big)\nabla f\Big\|_{L^1}\\ &\le\; K_6\Big(\frac{2}{t-s}\Big)^{1-\frac1p}K_7\Big(\frac{2}{t-s}\Big)^{\frac12}\Big(\frac{1+t/s}{2}\Big)^{\gamma}\|f\|_{L^1} \;\le\; K_6K_7\Big(\frac{2}{t-s}\Big)^{\frac32-\frac1p}\Big(\frac{t}{s}\Big)^{\gamma}\|f\|_{L^1}\ .\end{aligned}$$

It remains to prove (4.6) for $p=1$. We proceed in two steps:

Step 1 : the case of 1 vortex

Fix $\alpha\in\mathbf{R}$, and let $S_1(t,s)$ be the evolution operator associated to the non-autonomous equation

$$\frac{\partial\omega}{\partial t}(x,t) + \frac{\alpha}{\sqrt{t}}v^G\Big(\frac{x}{\sqrt{t}}\Big)\cdot\nabla\omega(x,t) \;=\; \Delta\omega(x,t)\ ,\quad x\in\mathbf{R}^2\ ,\quad t>0\ .$$

Due to the particular form of the convection term, it is natural to rewrite this equation in the self-similar variables $\xi = x/\sqrt{t}$, $\tau=\log(t)$. Defining $w(\xi,\tau)$ as in (2.9), we obtain the equivalent equation

$$\frac{\partial w}{\partial\tau}(\xi,\tau) + \alpha v^G(\xi)\cdot\nabla w(\xi,\tau) \;=\; (\mathcal{L}w)(\xi,\tau)\ , \tag{6.5}$$

where $\mathcal{L}$ is given by (2.10). We shall show that the autonomous equation (6.5) defines a strongly continuous semigroup in $L^1(\mathbf{R}^2)$, which we denote by $\mathcal{S}_1(\tau)$. We claim that, for any $\tau>0$, the operator $\mathcal{S}_1(\tau)\nabla$ can be extended to a bounded operator on $L^1(\mathbf{R}^2)$. Moreover, for any $\gamma>0$, there exists $C_1>0$ (depending on γ and $|\alpha|$) such that, for any $w\in L^1(\mathbf{R}^2)$,

$$\|\mathcal{S}_1(\tau)\nabla w\|_{L^1} \;\le\; C_1\frac{\mathrm{e}^{-(\frac12-\gamma)\tau}}{a(\tau)^{\frac12}}\|w\|_{L^1}\ ,\quad \tau>0\ , \tag{6.6}$$

where $a(\tau) = 1-\mathrm{e}^{-\tau}$. If we return to the original variables, we see that (6.6) is equivalent to (4.6) with $N=1$, $p=1$, and $t_0=+\infty$.

To prove (6.6), we introduce the Banach space $X\hookrightarrow L^1(\mathbf{R}^2)$ defined by

$$X \;=\; \Big\{w\in L^1(\mathbf{R}^2)\;\Big|\; w=\partial_1 f_1+\partial_2 f_2 \text{ with } f_1,f_2\in L^1(\mathbf{R}^2)\Big\}\ ,$$

equipped with the norm

$$\|w\|_X \;=\; \|w\|_{L^1} + \inf\Big\{\|f_1\|_{L^1} + \|f_2\|_{L^1}\;\Big|\; w = \partial_1 f_1 + \partial_2 f_2\Big\}\;.$$

We also consider the auxiliary equation for the vector field $f = (f_1, f_2)$:

$$\frac{\partial f}{\partial \tau} + \alpha v^G \operatorname{div} f \;=\; \Big(\mathcal{L} - \frac{1}{2}\Big) f\;. \tag{6.7}$$

Using a fixed point argument as in the proof of Lemma 6.4 below, it is straightforward to show that (6.7) defines a strongly continuous semigroup in $L^1(\mathbf{R}^2)^2$, which we denote by $\mathcal{T}_1(\tau)$. Moreover, there exists $\tau_0 > 0$ and $C_2 > 0$ such that, for all $f \in L^1(\mathbf{R}^2)^2$,

$$\|\mathcal{T}_1(\tau) f\|_{L^1} \;\le\; C_2 \|f\|_{L^1}\;, \quad \|\nabla \mathcal{T}_1(\tau) f\|_{L^1} \;\le\; \frac{C_2}{a(\tau)^{\frac{1}{2}}} \|f\|_{L^1}\;, \quad 0 < \tau \le \tau_0\;. \tag{6.8}$$

The evolutions defined by (6.5) and (6.7) are related via

$$\operatorname{div}(\mathcal{T}_1(\tau) f) \;=\; \mathcal{S}_1(\tau) \operatorname{div} f\;, \quad f \in X\;, \quad \tau > 0\;.$$

This shows that, for $\tau \in (0, \tau_0]$, $\mathcal{S}_1(\tau)\nabla$ can be extended to a bounded operator from $L^1(\mathbf{R}^2)$ into X with bound $C_2 a(\tau)^{-\frac{1}{2}}$; in particular (6.6) holds for $\tau \in (0, \tau_0]$. Moreover $\mathcal{S}_1(\tau)$ is a strongly continuous semigroup in X. Thus, to prove (6.6) for all times, it remains to show that, for any $\gamma > 0$, there exists $C_3 > 0$ such that $\|\mathcal{S}_1(\tau)\|_{\mathcal{L}(X)} \le C_3\, \mathrm{e}^{-(\frac{1}{2}-\gamma)\tau}$. Equivalently, we shall show that the spectral radius of $\mathcal{S}_1(\tau)$ in X satisfies $\rho_{\rm sp}(\mathcal{S}_1(\tau)) \le \mathrm{e}^{-\frac{\tau}{2}}$ for all $\tau \ge 0$.

To prove this, we argue exactly as in ([12], Sections 4.1 and 4.2). We first observe that $\mathcal{S}_1(\tau)$ is a compact perturbation of $S(\tau) = \exp(\tau\mathcal{L})$, and it is easy to verify using (2.11) that the spectral radius of $S(\tau)$ in X satisfies $\rho_{\rm sp}(S(\tau)) \;=\; \mathrm{e}^{-\frac{\tau}{2}}$ for all $\tau \ge 0$. Thus, it remains to show that all eigenvalues of the generator $L = \mathcal{L} - \alpha v^G \cdot \nabla$ of $\mathcal{S}_1(\tau)$ are contained in the half-plane $\{\lambda \in \mathbf{C}\,|\,\mathrm{Re}\,(\lambda) \le -\frac{1}{2}\}$. Assume on the contrary that some $\lambda \in \mathbf{C}$ with $\mathrm{Re}\,(\lambda) > -\frac{1}{2}$ is an eigenvalue of L in X. Since L is rotation invariant, we can use polar coordinates in $\mathbf{R}^2$ and assume that the eigenfunction φ associated to λ has the form $\varphi(r\cos\theta, r\sin\theta) = \psi(r)\,\mathrm{e}^{in\theta}$ for some $n \in \mathbf{Z}$. If we study the differential equation satisfied by ψ, we find as in ([12], Lemma 4.5) that

$$\psi(r) \;\sim\; A r^{2\lambda-2} + B r^{-2\lambda}\,\mathrm{e}^{-r^2/4}\;, \quad r \to +\infty\;,$$

for some $A, B \in \mathbf{C}$. Now, since $\varphi \in X$ and $\mathrm{Re}\,(\lambda) > -\frac{1}{2}$, we must have $A = 0$, hence φ has Gaussian decay at infinity, and $\int_{\mathbf{R}^2} \varphi(\xi)\,\mathrm{d}\xi = 0$. In particular, φ lies in the Hilbert space

$$Y_0 \;=\; \Big\{w \in L^2(\mathbf{R}^2, \mathbf{C})\;\Big|\; \int_{\mathbf{R}^2} G^{-1} |w(\xi)|^2\,\mathrm{d}\xi < \infty\,,\; \int_{\mathbf{R}^2} w(\xi)\,\mathrm{d}\xi = 0\Big\}\;.$$

But it is proved in [12] that L is self-adjoint in Y_0 with spectrum $\{-\frac{n}{2}\,|\,n \in \mathbf{N}\,,\; n \ge 1\}$ and $v^G \cdot \nabla$ is skew-symmetric in the same space Y_0. Thus we necessarily have $\mathrm{Re}\,(\lambda) \le -\frac{1}{2}$, which is a contradiction.

Step 2 : the case of N vortices

We now assume that $N \ge 2$ and we study the evolution operator $S_N(t,s)$ associated to the equation

$$\frac{\partial \omega}{\partial t}(x,t) + \sum_{i=1}^{N} \frac{\alpha_i}{\sqrt{t}}\, v^G\Big(\frac{x - z_i}{\sqrt{t}}\Big) \cdot \nabla \omega(x,t) \;=\; \Delta \omega(x,t)\;. \tag{6.9}$$

As we shall see, if $t/d^2 \ll 1$ where $d = \min\{|z_i - z_j| \,|\, i \neq j\}$, the N convection terms in (6.9) are nearly decoupled, and we can bound $S_N(t,s)$ using the previous estimates on $S_1(t,s)$.

Let $\chi : \mathbf{R}^2 \to [0,1]$ be a smooth function equal to one for $|x| \le \frac{1}{4}$ and zero for $|x| \ge \frac{1}{3}$. For $i \in \{1,\dots,N\}$ we set $\chi_i(x) = \chi((x-z_i)/d)$ and we define χ_0 such that $\sum_{i=0}^N \chi_i(x) = 1$ for all $x \in \mathbf{R}^2$. Observe that $0 \le \chi_0 \le 1$ and that there exists $C_1 > 0$ (independent of d) such that $\| \sum_{i=0}^N |\nabla\chi_i| \,\|_{L^\infty} \le C_1 d^{-1}$ and $\| \sum_{i=0}^N |\Delta\chi_i| \,\|_{L^\infty} \le C_1 d^{-2}$ for all $i \in \{0,\dots,N\}$.

If $\omega(x,t)$ satisfies (6.9), then for all $i \in \{0,\dots,N\}$ the function $\omega_i(x,t) \stackrel{\mathrm{def}}{=} \chi_i(x)\omega(x,t)$ is a solution of

$$\frac{\partial\omega_i}{\partial t} + \frac{\alpha_i}{\sqrt{t}} v^G\Big(\frac{x-z_i}{\sqrt{t}}\Big)\cdot\nabla\omega_i \;=\; \Delta\omega_i - \mathrm{div}(R_i\omega) + Q_i\omega \;,$$

where $\alpha_0 = 0$ and

$$\begin{aligned} R_i(x,t) &= \sum_{j\neq i} \frac{\alpha_j}{\sqrt{t}} v^G\Big(\frac{x-z_j}{\sqrt{t}}\Big)\chi_i(x) + 2\nabla\chi_i(x) \;, \\ Q_i(x,t) &= \sum_{j=1}^N \frac{\alpha_j}{\sqrt{t}} v^G\Big(\frac{x-z_j}{\sqrt{t}}\Big)\cdot\nabla\chi_i(x) + \Delta\chi_i(x) \;. \end{aligned}$$

By construction, $R_i(x,t)$ and $Q_i(x,t)$ are smooth functions of $x \in \mathbf{R}^2$ and $t \ge 0$. Moreover, if $R(x,t) = \sum_{i=0}^N |R_i(x,t)|$ and $Q(x,t) = \sum_{i=0}^N |Q_i(x,t)|$, there exists $C_2 > 0$ (independent of d) such that

$$\|R(\cdot,t)\|_{L^\infty} \;\le\; \frac{C_2}{d} \;, \quad \|Q(\cdot,t)\|_{L^\infty} \;\le\; \frac{C_2}{d^2} \;, \quad t \ge 0 \;. \tag{6.10}$$

If we denote by $\tilde{S}_i(t,s)$ the evolution operator associated to the ith vortex, we find the following integral equation

$$\omega_i(t) \;=\; \tilde{S}_i(t,s)\omega_i(s) + \int_s^t \tilde{S}_i(t,t')\Big(-\mathrm{div}(R_i(t')\omega(t')) + Q_i(t')\omega(t')\Big)\,\mathrm{d}t' \;, \tag{6.11}$$

for $0 < s < t$.

Now, we fix $s > 0$, $T > 0$, and we assume that $\omega_i(s) = \chi_i\nabla f \equiv \nabla(\chi_i f) - (\nabla\chi_i)f$ for some $f \in L^1(\mathbf{R}^2)$. Using (6.11) together with the bounds (4.5), (4.6) (for one vortex), we obtain for $s < t < s+T$:

$$\begin{aligned} \|\omega_i(t)\|_{L^1} \;&\le\; \frac{K_7}{(t-s)^{\frac{1}{2}}}\Big(\frac{t}{s}\Big)^\gamma \|\chi_i f\|_{L^1} + K_6\|(\nabla\chi_i)f\|_{L^1} + \int_s^t K_6\|Q_i(t')\omega(t')\|_{L^1}\,\mathrm{d}t' \\ &+ \int_s^t \frac{K_7}{(t-t')^{\frac{1}{2}}}\Big(\frac{t}{t'}\Big)^\gamma \|R_i(t')\omega(t')\|_{L^1}\,\mathrm{d}t' \;. \end{aligned}$$

Summing over $i \in \{0,\dots,N\}$, we thus find

$$\begin{aligned} \|\omega(t)\|_{L^1} \;&\le\; \frac{K_7}{(t-s)^{\frac{1}{2}}}\Big(\frac{t}{s}\Big)^\gamma \|f\|_{L^1} + \frac{C_1K_6}{d}\|f\|_{L^1} + \int_s^t K_6\|Q(t')\|_{L^\infty}\|\omega(t')\|_{L^1}\,\mathrm{d}t' \\ &+ \int_s^t \frac{K_7}{(t-t')^{\frac{1}{2}}}\Big(\frac{t}{t'}\Big)^\gamma \|R(t')\|_{L^\infty}\|\omega(t')\|_{L^1}\,\mathrm{d}t' \;. \end{aligned}$$

Thus, if we define (for a fixed $s > 0$)

$$\|\omega\|_T \;=\; \sup_{s<t<s+T} (t-s)^{\frac{1}{2}}\Big(\frac{s}{t}\Big)^\gamma \|\omega(t)\|_{L^1} \;,$$

we obtain using (6.10)

$$\|\omega\|_T \le C_3\Big(1+\frac{T^{\frac{1}{2}}}{d}\Big)\|f\|_{L^1} + C_4\frac{T}{d^2}\|\omega\|_T + C_5\frac{T^{\frac{1}{2}}}{d}\|\omega\|_T ,$$

where $C_3, C_4, C_5 > 0$ are independent of d. If we now assume that T is sufficiently small so that

$$\frac{T}{d^2} \le \min\Big\{1\,,\, \frac{1}{4C_4}\,,\, \frac{1}{(4C_5)^2}\Big\} ,$$

then $\|\omega\|_T \le 4C_3\|f\|_{L^1}$, hence

$$\|S_N(t,s)\nabla f\|_{L^1} \le \frac{4C_3}{(t-s)^{\frac{1}{2}}}\Big(\frac{t}{s}\Big)^{\gamma}\|f\|_{L^1} , \quad s < t < s+T .$$

This concludes the proof of Proposition 4.3. □

6.3 Proof of Proposition 4.5

The proof follows the approach of [12]. Using parabolic regularization, we first show that the trajectory $\{w_i(\tau)\}$ is relatively compact in $L^2(m)$. We next prove that the α-limit set $\mathcal{A}_i$ of this trajectory is fully invariant under the evolution defined by the autonomous equation

$$\frac{\partial w_i}{\partial \tau}(\xi,\tau) + v_i(\xi,\tau)\cdot\nabla w_i(\xi,\tau) = (\mathcal{L}w_i)(\xi,\tau) , \tag{6.12}$$

which is obtained by setting $R_i = 0$ in (4.11). Using the main result of [12] we conclude that $\mathcal{A}_i = \{\alpha_i G\}$, which proves the claim. We start with the compactness result:

Lemma 6.1 *For any $i \in \{1,\dots,N\}$ and any $m > 1$, the trajectory $\{w_i(\tau)\}_{\tau<\log T}$ is relatively compact in $L^2(m)$.*

Proof: Fix $i \in \{1,\dots,N\}$. By (4.15), for any $m > 1$, there exists $C_m > 0$ such that $\|w_i(\tau)\|_{L^2(m)} \le C_m$ for all $\tau < \log(T)$. Let $H^1(m)$ be the weighted Sobolev space defined by

$$H^1(m) = \{w \in L^2(m)\,|\,\nabla w \in L^2(m)\} , \quad \|w\|^2_{H^1(m)} = \|w\|^2_{L^2(m)} + \|\nabla w\|^2_{L^2(m)} . \tag{6.13}$$

By Rellich's criterion, the inclusion $H^1(m{+}1) \hookrightarrow L^2(m)$ is compact. Therefore, to prove Lemma 6.1, it is sufficient to verify that $\nabla w_i(\tau)$ is bounded in $L^2(m)$ for all $m > 1$. To this end, we proceed as in ([12], Lemma 2.1). Fix $m > 1$, $\tau_0 < \log(T)$, and consider the integral equation

$$\nabla w_i(\tau) = \nabla S(\tau-\tau_0)w_i(\tau_0) - \int_{\tau_0}^{\tau} \nabla S(\tau-s)(v_i(s)\cdot\nabla w_i(s) + R_i(s)\cdot\nabla w_i(s))\,\mathrm{d}s ,$$

where R_i is given by (4.12). If $0 < \tau-\tau_0 \le 1$, we can bound, using Proposition 2.2,

$$\|\nabla w_i(\tau)\|_{L^2(m)} \le \frac{K_3}{a(\tau-\tau_0)^{\frac{1}{2}}}\|w_i(\tau_0)\|_{L^2(m)} + \int_{\tau_0}^{\tau}\frac{K_4}{a(\tau-s)^{\frac{1}{p}}}\|(v_i(s)+R_i(s))\cdot\nabla w_i(s)\|_{L^p(m)}\,\mathrm{d}s ,$$

where $1 < p < 2$. If $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$, we have according to (2.17)

$$\|v_j(s)\|_{L^q} \le C\|w_j(s)\|_{L^p} \le C\|w_j(s)\|_{L^2(m)} \le CC_m ,$$

and according to (6.1)

$$\|\mathrm{e}^{\frac{\tau}{2}}\tilde{u}_0(\xi\mathrm{e}^{\frac{\tau}{2}}+z_i,\mathrm{e}^\tau)\|_{L^q_\xi} \;=\; \mathrm{e}^{\tau(\frac12-\frac1q)}\|\tilde{u}_0(\cdot,\mathrm{e}^\tau)\|_{L^q} \;=\; t^{\frac12-\frac1q}\|\tilde{u}_0(\cdot,t)\|_{L^q} \;\le\; C_1\|\mu_0\|\ .$$

Therefore, we find

$$\|(v_i(s)+R_i(s))\cdot\nabla w_i(s)\|_{L^p(m)} \;\le\; \|v_i(s)+R_i(s)\|_{L^q}\|\nabla w_i(s)\|_{L^2(m)} \;\le\; C\|\nabla w_i(s)\|_{L^2(m)}\ ,$$

hence

$$\|\nabla w_i(\tau)\|_{L^2(m)} \;\le\; \frac{C_2}{a(\tau-\tau_0)^{\frac12}} + \int_{\tau_0}^{\tau}\frac{C_3}{a(\tau-s)^{\frac1p}}\|\nabla w_i(s)\|_{L^2(m)}\,\mathrm{d}s\ , \tag{6.14}$$

where $C_2, C_3>0$ are independent of τ_0. Now, choose $\tilde{T}>0$ small enough so that

$$\sup_{\tau_0<\tau\le\tau_0+\tilde{T}}\int_{\tau_0}^{\tau}\frac{C_3 a(\tau-\tau_0)^{\frac12}}{a(\tau-s)^{\frac1p}a(s-\tau_0)^{\frac12}}\,\mathrm{d}s \;\equiv\; \sup_{0<\tau\le\tilde{T}}\int_0^\tau\frac{C_3 a(\tau)^{\frac12}}{a(\tau-s)^{\frac1p}a(s)^{\frac12}}\,\mathrm{d}s \;\le\; \frac12\ .$$

Then (6.14) implies that $\|\nabla w_i(\tau)\|_{L^2(m)} \le 2C_2 a(\tau-\tau_0)^{-\frac12}$ for $\tau_0<\tau<\min(\tau_0+\tilde{T},\log(T))$. Since $\tau_0<\log(T)$ was arbitrary and since $\tilde{T}$ is independent of τ_0, there exists $C_4>0$ such that $\|\nabla w_i(\tau)\|_{L^2(m)}\le C_4$ for all $\tau<\log(T)$. □

We next show that the term $R_i(\xi,\tau)$ in (4.11) is negligible as $\tau\to-\infty$:

Lemma 6.2 *For any $i\in\{1,\dots,N\}$, any $m>1$, and any $p\in(1,2)$, the following holds:*

$$\lim_{\tau\to-\infty}\|R_i(\tau)w_i(\tau)\|_{L^p(m)} = 0\ .$$

Proof: Let $q\in(2,\infty)$ be such that $\frac1q=\frac1p-\frac12$. If $b(\xi)=(1+|\xi|^2)^{\frac12}$, we have for all $\tau<\log(T)$

$$\|R_i(\tau)w_i(\tau)\|_{L^p(m)} \;=\; \|b^m R_i(\tau)w_i(\tau)\|_{L^p} \;\le\; \|b^{-1}R_i(\tau)\|_{L^q}\|b^{m+1}w_i(\tau)\|_{L^2} \;\le\; C_1\|b^{-1}R_i(\tau)\|_{L^q}\ ,$$

where $C_1>0$ is independent of τ. We claim that the last term in the right-hand side converges to zero as $\tau\to-\infty$. Indeed, if $0<\nu<1-\frac2q$ and $j\neq i$, we have

$$\|b^{-1}v_j(\cdot-(z_j-z_i)\mathrm{e}^{-\frac{\tau}{2}},\tau)\|_{L^q} \;\le\; C\|b^{-1}b(\cdot-(z_j-z_i)\mathrm{e}^{-\frac{\tau}{2}})^{-\nu}\|_{L^\infty}\|b^\nu v_j(\tau)\|_{L^q}\ .$$

The first factor in the right-hand side is $\mathcal{O}(\mathrm{e}^{\nu\frac{\tau}{2}})$ as $\tau\to-\infty$, and using (2.20) the second one can be bounded by $C\|w_j(\tau)\|_{L^2(m)}\le C_2$, where $C_2>0$ is independent of τ. On the other hand, applying Lemma 4.2 with $\chi(r)=(1+r)^{-\frac12}$, we obtain (with $t=\mathrm{e}^\tau$)

$$\|b^{-1}\mathrm{e}^{\frac{\tau}{2}}\tilde{u}_0(\xi\mathrm{e}^{\frac{\tau}{2}}+z_i,\mathrm{e}^\tau)\|_{L^q_\xi} \;=\; t^{\frac12-\frac1q}\Big\|\tilde{u}_0(x,t)\Big(1+\frac{|x-z_i|^2}{t}\Big)^{-\frac12}\Big\|_{L^q_x} \xrightarrow[t\to0]{} 0\ ,$$

Thus $\|b^{-1}R_i(\tau)\|_{L^q}\to0$ as $\tau\to-\infty$. □

Now, fix $m>1$ and let $\mathcal{A}_i$ be the α-limit set in $L^2(m)$ of the trajectory $\{w_i(\tau)\}_{\tau<\log(T)}$. As is well known, $\mathcal{A}_i$ is nonempty, compact, and attracts $w_i(\tau)$ in the sense that $\mathrm{dist}_{L^2(m)}(w_i(\tau),\mathcal{A}_i)\to 0$ as $\tau\to-\infty$. Let also $\Phi(\tau)_{\tau\ge0}$ be the semiflow in $L^2(m)$ defined by the limiting equation (6.12), see ([10], Theorem 3.2). Our last result is:

Lemma 6.3 *For any $i\in\{1,\dots,N\}$ and any $\tau\ge0$, we have $\Phi(\tau)\mathcal{A}_i=\mathcal{A}_i$.*

Proof: It is clearly enough to prove the result for $0 \le \tau \le 1$. If $w_\infty \in \mathcal{A}_i$, there exists a sequence τ_n going to $-\infty$ such that $\|w_i(\tau_n) - w_\infty\|_{L^2(m)} \to 0$ as $n \to \infty$. From (4.14) we have

$$w_i(\tau_n + \tau) \;=\; S(\tau)w_i(\tau_n) - \int_0^\tau \mathrm{e}^{-\frac{1}{2}(\tau-\tau')}\nabla\cdot S(\tau-\tau')(v_i w_i + R_i w_i)(\tau_n+\tau')\,\mathrm{d}\tau' \;,$$

for $\tau \in [0,1]$. On the other hand, $W_i(\tau) \stackrel{\text{def}}{=} \Phi(\tau)w_\infty$ satisfies

$$W_i(\tau) \;=\; S(\tau)w_\infty - \int_0^\tau \mathrm{e}^{-\frac{1}{2}(\tau-\tau')}\nabla\cdot S(\tau-\tau')(V_i W_i)(\tau')\,\mathrm{d}\tau' \;, \quad 0 \le \tau \le 1 \;,$$

where $V_i(\tau)$ is the velocity field obtained from $W_i(\tau)$ via the Biot-Savart law. Now we compute the difference of both expressions. Using Proposition 2.2 and proceeding as in ([10], Lemma 3.1) we obtain, for any $p \in (1,2)$,

$$\begin{aligned}
\|w_i(\tau_n+\tau) - W_i(\tau)\|_{L^2(m)} \;\le\;& K_3\|w_i(\tau_n) - w_\infty\|_{L^2(m)} \\
&+K_4\int_0^\tau \mathrm{e}^{-\frac{1}{2}(\tau-\tau')}\frac{1}{a(\tau-\tau')^{\frac{1}{p}}}\|R_i(\tau_n+\tau')w_i(\tau_n+\tau')\|_{L^p(m)}\,\mathrm{d}\tau' \\
&+K_4\int_0^\tau \mathrm{e}^{-\frac{1}{2}(\tau-\tau')}\frac{C}{a(\tau-\tau')^{\frac{1}{p}}}\big(\|w_i(\tau_n+\tau')\|_{L^2(m)} + \|W_i(\tau')\|_{L^2(m)}\big) \\
&\qquad\qquad\times\|w_i(\tau_n+\tau') - W_i(\tau')\|_{L^2(m)}\,\mathrm{d}\tau' \;.
\end{aligned}$$

The first term in the right-hand side converges to zero as $n \to \infty$, and so does the second one (uniformly in $\tau \in [0,1]$) by Lemma 6.2. Thus using the uniform bound on $\|w_i(\tau)\|_{L^2(m)}$ we deduce that

$$\|w_i(\tau_n+\tau) - W_i(\tau)\|_{L^2(m)} \;\le\; \varepsilon(n) + C\int_0^\tau \frac{1}{a(\tau-\tau')^{\frac{1}{p}}}\|w_i(\tau_n+\tau') - W_i(\tau')\|_{L^2(m)}\,\mathrm{d}\tau' \;,$$

where $\varepsilon(n) \to 0$ as $n \to \infty$ uniformly in $\tau \in [0,1]$, and where $C > 0$ is independent of n and τ. Gronwall's lemma then implies

$$\lim_{n\to\infty}\sup_{0\le\tau\le1}\|w_i(\tau_n+\tau) - W_i(\tau)\|_{L^2(m)} \;=\; 0 \;.$$

In particular $W_i(\tau) \equiv \Phi(\tau)w_\infty \in \mathcal{A}_i$ for any $\tau \in [0,1]$, hence $\Phi(\tau)\mathcal{A}_i \subset \mathcal{A}_i$ for any $\tau \in [0,1]$.

Conversely let $w_\infty \in \mathcal{A}_i$ and $0 \le \tau \le 1$. There exists a sequence τ_n going to $-\infty$ such that $\|w_i(\tau_n) - w_\infty\|_{L^2(m)} \to 0$ as $n \to \infty$. Up to the extracting a subsequence, we can suppose that $w_i(\tau_n - \tau)$ converges in $L^2(m)$ towards some $W_\infty \in \mathcal{A}_i$ as n goes to infinity. By the argument above, we have $\Phi(\tau)W_\infty = w_\infty$, hence $\mathcal{A}_i \subset \Phi(\tau)\mathcal{A}_i$. □

In ([12], Lemma 3.3) it is shown that, if $\mathcal{A}$ is a bounded subset of $L^2(m)$ satisfying $\Phi(\tau)\mathcal{A} = \mathcal{A}$ for all $\tau \ge 0$, then necessarily $\mathcal{A} \subset \{\alpha G \,|\, \alpha \in \mathbf{R}\}$. Applying this result to the α-limit set of $\{w_i(\tau)\}$, we obtain $\mathcal{A}_i = \{\alpha_i G\}$, since any $w \in \mathcal{A}_i$ satisfies $\int_{\mathbf{R}^2} w(\xi)\,\mathrm{d}\xi = \alpha_i$. This concludes the proof of Proposition 4.5. □

6.4 Proof of Proposition 4.6

Estimates (4.17) and (4.18) are established in ([12], Section 4.2). To prove iii), we first observe that it is sufficient to establish (4.19) for $0 < \tau \le \tau_0$, where $\tau_0 > 0$ is arbitrary. Indeed, once this is done, we have for $\tau > \tau_0$:

$$\begin{aligned}
\|T_\alpha(\tau)\nabla w\|_{L^2(m)} \;&=\; \|T_\alpha(\tau-\tau_0)T_\alpha(\tau_0)\nabla w\|_{L^2(m)} \\
&\le\; K_8\,\mathrm{e}^{-\frac{(\tau-\tau_0)}{2}}\|T_\alpha(\tau_0)\nabla w\|_{L^2(m)} \;\le\; \frac{K_8K_9\,\mathrm{e}^{-\frac{\tau}{2}}}{a(\tau_0)^{\frac{1}{q}}}\|w\|_{L^q(m)} \;.
\end{aligned}$$

To prove (4.19) for small τ, we consider the auxiliary equation in $L^2(m)^2$:

$$\frac{\partial f}{\partial \tau} + \alpha\Big(v^G \operatorname{div} f + v^{\operatorname{div} f} G\Big) \;=\; \Big(\mathcal{L} - \frac{1}{2}\Big) f \;. \tag{6.15}$$

Here, $f(x,t) \in \mathbf{R}^2$ is a vector field, and $v^{\operatorname{div} f}$ denotes the velocity field obtained from the scalar $\operatorname{div} f$ via the Biot-Savart law (2.17). As we shall see, this equation defines a semigroup in $L^2(m)^2$, which we denote by $\hat{T}_\alpha(\tau)$. A straightforward calculation shows that the semigroups $T_\alpha(\tau)$ and $\hat{T}_\alpha(\tau)$ are related via

$$T_\alpha(\tau) \operatorname{div} f \;=\; \operatorname{div}(\hat{T}_\alpha(\tau) f) \;, \quad f \in H^1(m)^2 \;, \quad \tau \ge 0 \;,$$

where $H^1(m)$ is defined in (6.13). Assertion iii) in Proposition 4.6 is now a direct consequence of this identity and of the following result:

Lemma 6.4 *Equation (6.15) defines a strongly continuous semigroup $\hat{T}_\alpha(\tau)$ in $L^2(m)^2$ for any $m > 1$. If $q \in [1,2]$ and $\tau > 0$, $\hat{T}_\alpha(\tau)$ can be extended to a bounded operator from $L^q(m)^2$ to $H^1(m)^2$, and there exist $\tau_0 > 0$ and $C > 0$ such that*

$$\|\hat{T}_\alpha(\tau) f\|_{L^2(m)} \;\le\; \frac{C}{a(\tau)^{\frac{1}{q}-\frac{1}{2}}} \|f\|_{L^q(m)} \;, \quad \|\nabla \hat{T}_\alpha(\tau) f\|_{L^2(m)} \;\le\; \frac{C}{a(\tau)^{\frac{1}{q}}} \|f\|_{L^q(m)} \;, \tag{6.16}$$

for $\tau \in (0, \tau_0]$.

Proof. We consider the integral equation associated with (6.15), namely

$$f(\tau) \;=\; \mathrm{e}^{-\frac{1}{2}\tau} S(\tau) f_0 - \alpha \int_0^\tau \mathrm{e}^{-\frac{1}{2}(\tau-s)} S(\tau-s) \Big(v^G \operatorname{div} f(s) + v^{\operatorname{div} f(s)} G\Big) \,\mathrm{d}s \;. \tag{6.17}$$

We assume that $f_0 \in L^q(m)^2$ for some $q \in [1,2]$ and some $m > 1$. Given $\tau_0 > 0$, we shall solve (6.17) in the Banach space $X \stackrel{\text{def}}{=} \{f \in C^0((0,\tau_0], H^1(m)^2) \,|\, \|f\|_X < \infty\}$, where

$$\|f\|_X \;=\; \sup_{0<\tau\le\tau_0} a(\tau)^{\frac{1}{q}-\frac{1}{2}} \|f(\tau)\|_{L^2(m)} + \sup_{0<\tau\le\tau_0} a(\tau)^{\frac{1}{q}} \|\nabla f(\tau)\|_{L^2(m)} \;.$$

Let $f \in X$ and denote by $F(\tau)$ the expression in the right-hand side of (6.17). Using the estimates collected in Proposition 2.2, we obtain for $\tau \in (0, \tau_0]$:

$$\|F(\tau)\|_{L^2(m)} \;\le\; \frac{K_6\, \mathrm{e}^{-\frac{1}{2}\tau}}{a(\tau)^{\frac{1}{q}-\frac{1}{2}}} \|f_0\|_{L^q(m)} + |\alpha| \int_0^\tau \frac{K_6\, \mathrm{e}^{-\frac{1}{2}(\tau-s)}}{a(\tau-s)^{\frac{1}{p}-\frac{1}{2}}} \|v^G \operatorname{div} f(s) + v^{\operatorname{div} f(s)} G\|_{L^p(m)} \,\mathrm{d}s \;,$$

where $1 < p < 2$. If $\frac{1}{p'} = \frac{1}{p} - \frac{1}{2}$, we obtain using (2.18) and Hölder's inequality

$$\begin{aligned} \|v^G \operatorname{div} f(s)\|_{L^p(m)} \;&\le\; \|v^G\|_{L^{p'}} \|\operatorname{div} f(s)\|_{L^2(m)} \;\le\; C\|G\|_{L^p} \|\operatorname{div} f(s)\|_{L^2(m)} \;, \\ \|v^{\operatorname{div} f(s)} G\|_{L^p(m)} \;&\le\; C\|v^{\operatorname{div} f(s)}\|_{L^{p'}} \|G\|_{L^2(m)} \;\le\; \|\operatorname{div} f(s)\|_{L^p} \|G\|_{L^2(m)} \;, \end{aligned}$$

hence both terms can be bounded by $C\|G\|_{L^2(m)} \|\operatorname{div} f(s)\|_{L^2(m)}$. Therefore,

$$a(\tau)^{\frac{1}{q}-\frac{1}{2}} \|F(\tau)\|_{L^2(m)} \;\le\; C\mathrm{e}^{-\frac{1}{2}\tau} \|f_0\|_{L^q(m)} + C|\alpha| \Lambda_1(\tau) \|f\|_X \;,$$

where

$$\Lambda_1(\tau) \;=\; \int_0^\tau \mathrm{e}^{-\frac{1}{2}(\tau-s)} \frac{a(\tau)^{\frac{1}{q}-\frac{1}{2}}}{a(\tau-s)^{\frac{1}{p}-\frac{1}{2}} a(s)^{\frac{1}{q}}} \,\mathrm{d}s \;.$$

Using similar estimates, one finds

$$a(\tau)^{\frac{1}{q}} \|\nabla F(\tau)\|_{L^2(m)} \;\le\; C e^{-\frac{1}{2}\tau} \|f_0\|_{L^q(m)} + C|\alpha| \Lambda_2(\tau) \|f\|_X \;,$$

where

$$\Lambda_2(\tau) \;=\; \int_0^\tau e^{-\frac{1}{2}(\tau-s)} \frac{a(\tau)^{\frac{1}{q}}}{a(\tau-s)^{\frac{1}{p}} a(s)^{\frac{1}{q}}} \,\mathrm{d}s \;.$$

Thus, there exist positive constants C_1, C_2 such that, for all $f \in X$,

$$\|F\|_X \;\le\; C_1 \|f_0\|_{L^q(m)} + C_2 |\alpha| \Lambda(\tau_0) \|f\|_X \;,$$

where

$$\Lambda(\tau_0) \;=\; \sup_{0<\tau\le\tau_0} \Lambda_1(\tau) + \sup_{0<\tau\le\tau_0} \Lambda_2(\tau) \;.$$

If we now choose $\tau_0 > 0$ small enough so that $C_2|\alpha|\Lambda(\tau_0) \le \frac{1}{2}$, it follows from these estimates that (6.17) has a unique solution $f \in X$ such that $\|f\|_X \le 2C_1 \|f_0\|_{L^q(m)}$. This proves (6.16). $\square$

References

[1] M. Ben-Artzi. Global solutions of two-dimensional Navier-Stokes and Euler equations. *Arch. Rational Mech. Anal.* **128** (1994), 329–358.

[2] H. Brezis. Remarks on the preceding paper by M. Ben-Artzi: "Global solutions of two-dimensional Navier-Stokes and Euler equations". *Arch. Rational Mech. Anal.* **128** (1994), 359–360.

[3] M. Cannone. *Ondelettes, paraproduits et Navier-Stokes.* Diderot Editeur, Paris, 1995.

[4] M. Cannone and F. Planchon. Self-similar solutions for Navier-Stokes equations in $\mathbf{R}^3$. *Comm. Partial Differ. Equations* **21** (1996), 179–193.

[5] E. A. Carlen and M. Loss. Optimal smoothing and decay estimates for viscously damped conservation laws, with applications to the 2-D Navier-Stokes equation. *Duke Math. J.* **81** (1995), 135–157 (1996).

[6] A. Carpio. Asymptotic behavior for the vorticity equations in dimensions two and three. *Comm. Partial Differential Equations* **19** (1994), 827–872.

[7] A. Chorin and J. Marsden. *A mathematical introduction to fluid mechanics.* Second edition. Texts in Applied Mathematics, **4**. Springer-Verlag, New York, 1990.

[8] G.-H. Cottet. Equations de Navier-Stokes dans le plan avec tourbillon initial mesure. *C. R. Acad. Sci. Paris Sér. I Math.* **303** (1986), 105–108.

[9] I. Gallagher and F. Planchon. On global infinite energy solutions to the Navier–Stokes equations in two dimensions. *Arch. Rational Mech. Anal.* **161** (2002), 307–337.

[10] Th. Gallay and C. E. Wayne. Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbf{R}^2$. *Arch. Rational Mech. Anal.* **163** (2002), 209–258.

[11] Th. Gallay and C. E. Wayne. Long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbf{R}^3$. *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.* **360** (2002), 2155–2188.

[12] Th. Gallay and C. E. Wayne. Global stability of vortex solutions of the two-dimensional Navier-Stokes equation. *Comm. Math. Phys.*, to appear. Preprint version available at `http://www.arXiv.org/math.AP/0402449`.

[13] P. Germain. Existence globale de solutions de l'équation de Navier-Stokes 2D avec données initiales dans ∂BMO. In preparation.

[14] Y. Giga and T. Miyakawa. Solutions in L^r of the Navier-Stokes initial value problem. *Arch. Rational Mech. Anal.* **89** (1985), 267–281.

[15] Y. Giga and T. Kambe. Large time behavior of the vorticity of two-dimensional viscous flow and its application to vortex formation. *Comm. Math. Phys.* **117** (1988), 549–568.

[16] Y. Giga, T. Miyakawa, and H. Osada. Two-dimensional Navier-Stokes flow with measures as initial vorticity. *Arch. Rational Mech. Anal.* **104** (1988), 223–250.

[17] Y. Giga and T. Miyakawa. Navier-Stokes flow in $\mathbf{R}^3$ with measures as initial vorticity and Morrey spaces. *Comm. Partial Differential Equations* **14** (1989), 577–618.

[18] M.-H. Giga and Y. Giga. *Nonlinear partial differential equations – asymptotic behaviour of solutions and self-similar solutions.* Book in preparation.

[19] D. Henry. *Geometric theory of semilinear parabolic equations.* Springer-Verlag, Berlin, 1981.

[20] T. Kato. Strong L^p-solutions of the Navier-Stokes equation in $\mathbf{R}^m$, with applications to weak solutions. *Math. Z.* **187** (1984), 471–480.

[21] T. Kato. The Navier-Stokes equation for an incompressible fluid in $\mathbf{R}^2$ with a measure as the initial vorticity. *Differential Integral Equations* **7** (1994), 949–966.

[22] H. Koch and D. Tataru. Well-posedness for the Navier–Stokes equations. *Advances in Mathematics* **157** (2001), 22–35.

[23] O. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow.* Second English edition. Mathematics and its Applications, Vol. 2. Gordon and Breach, New York-London-Paris, 1969.

[24] J. Leray. Etude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique. *J. Math. Pures. Appl.* **12** (1933), 1–82.

[25] Y. Meyer. *Wavelets, Paraproducts and Navier–Stokes.* Current Developments in Mathematics, International Press, Cambridge, Massachussets, 1996.

[26] H. Osada. Diffusion processes with generators of generalized divergence form. *J. Math. Kyoto Univ.* **27** (1987), 597–619.

[27] E. Stein. *Harmonic Analysis.* Princeton University Press, 1993.

[28] R. Temam. *Navier-Stokes equations. Theory and numerical analysis.* Reprint of the 1984 edition. AMS Chelsea Publishing, Providence, RI, 2001.

[29] F. Weissler. The Navier-Stokes initial value problem in L^p. *Arch. Rational Mech. Anal.* **74** (1980), 219–230.