TWO ANALYTIC CONTINUATIONS OF THE LIPPMANN-SCHWINGER EIGENFUNCTIONS

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Abstract. We first present two possible analytic continuations of the Lippmann-Schwinger eigenfunctions to the second sheet of the Riemann surface, and then we compare the different Gamow vectors that are obtained through each analytic continuation.

1. Introduction

Gamow vectors should be related with the analytic continuation of the Lippmann-Schwinger eigenfunctions to the resonance energy. However, the analytic continuation of the Lippmann-Schwinger eigenfunctions to the second sheet is not unique. In this paper, we address this non-uniqueness by constructing and comparing two possible analytic continuations.

2. Preliminaries

Before proceeding with their analytic continuations, we recall some of the basic properties of the Lippmann-Schwinger eigenfunctions. Let us take a simple example, such as the spherical shell potential for zero angular momentum:

$$V(r)=\left\{ \begin{array}{ll} 0 & 0<r<a \\ V_0 & a<r<b \\ 0 & b<r<\infty\,. \end{array} \right. \tag{1}$$

For this potential, the Lippmann-Schwinger equation

$$|E^{\pm}\rangle = |E\rangle + \frac{1}{E - H_0 \pm i\epsilon} V |E^{\pm}\rangle \tag{2}$$

has the following solutions in the radial position representation for zero angular momentum [1, 2]:

$$\langle r|E^{\pm}\rangle \equiv \chi^{\pm}(r;E) = \kappa(E)\, \frac{\chi(r;E)}{\mathcal{J}_{\pm}(E)} \,, \tag{3}$$

where $\kappa(E)$ is a normalization factor

$$\kappa(E) = \sqrt{\frac{1}{\pi} \frac{2m/\hbar^2}{\sqrt{2m/\hbar^2\, E}}} \ , \tag{4}$$

$\chi(r;E)$ is the regular solution,

$$\chi(r;E) = \begin{cases} \sin(\sqrt{\frac{2m}{\hbar^2}E}\; r) & 0 < r < a \\ \mathcal{J}_1(E) e^{i\sqrt{\frac{2m}{\hbar^2}(E-V_0)}\; r} + \mathcal{J}_2(E) e^{-i\sqrt{\frac{2m}{\hbar^2}(E-V_0)}\; r} & a < r < b \\ \mathcal{J}_3(E) e^{i\sqrt{\frac{2m}{\hbar^2}E}\; r} + \mathcal{J}_4(E) e^{-i\sqrt{\frac{2m}{\hbar^2}E}\; r} & b < r < \infty \,, \end{cases} \tag{5}$$

and $\mathcal{J}_{\pm}(E)$ are the Jost functions,

$$\mathcal{J}_{+}(E) = -2i\mathcal{J}_4(E) \,, \tag{6}$$

$$\mathcal{J}_{-}(E) = 2i\mathcal{J}_3(E) \,. \tag{7}$$

In terms of the Jost functions, the S matrix is given by

$$S(E) = \frac{\mathcal{J}_{-}(E)}{\mathcal{J}_{+}(E)} \,. \tag{8}$$

The $\pm i\epsilon$ impose certain boundary conditions on the solutions of Eq. (2). In the position representation, these boundary conditions determine the asymptotic behavior of the Lippmann-Schwinger eigenfunctions:

$$\langle r|E^{+}\rangle \sim e^{-ikr} - S(E) e^{ikr} \,, \quad \text{as } r \to \infty \,, \tag{9}$$

$$\langle r|E^{-}\rangle \sim e^{ikr} - S^{*}(E) e^{-ikr} \,, \quad \text{as } r \to \infty \,, \tag{10}$$

where

$$k = \sqrt{\frac{2m}{\hbar^2} E} > 0 \tag{11}$$

is the wave number.

In terms of E, the boundary conditions imposed by the $\pm i\epsilon$ are tantamount to obtaining boundary values from above ($+i\epsilon$) and from below ($-i\epsilon$) the spectrum of H. Thus, the Lippmann-Schwinger eigenfunctions are to be seen as boundary values from above and from below the cut. This is analogous to the calculation of the advanced and retarded Green functions $G^{\pm}(r,s;E)$. These functions are boundary values on the upper and lower rims of the cut of the Green function $G(r,s;z)$:

$$G^{\pm}(r,s;E) = \lim_{\mathrm{Im}(z)\to 0} G(r,s;z)\,, \qquad z = E \pm i\mathrm{Im}(z),\ \mathrm{Im}(z) > 0\,. \tag{12}$$

As is well known, for the potential (1) all these functions (the Green function, the S matrix, and the Lippmann-Schwinger eigenfunctions) depend explicitly not on E but on k. This is why it is convenient to work with k rather than with E. In terms of k, the Lippmann-Schwinger eigenfunctions (3) read as

$$\chi^{\pm}(r;k) = \sqrt{\frac{1}{\pi}\,\frac{2m/\hbar^2}{k}}\,\frac{\chi(r;k)}{\mathcal{J}_{\pm}(k)}\,. \tag{13}$$

These eigenfunctions are normalized to $\delta(E-E')$. In order to normalize them to $\delta(k-k')$, we define

$$\phi^{\pm}(r;k) \equiv \sqrt{\frac{\hbar^2}{2m}2k}\,\chi^{\pm}(r;k) = \sqrt{\frac{2}{\pi}}\,\frac{\chi(r;k)}{\mathcal{J}_{\pm}(k)}\,. \tag{14}$$

It is important to notice that in this equation, as well as in Eqs. (9) and (10), k is *positive.*

We also can write Eq. (12) in terms of k. If we denote the complex wave number by $q \equiv k + i\mathrm{Im}(q)$, we have that

$$G^{+}(r,s;k) = \lim_{\mathrm{Im}(q)\to 0} G(r,s;q)\,, \qquad q = k + i\mathrm{Im}(q),\ \mathrm{Im}(q), k > 0\,. \tag{15}$$

Thus, G^{+} is the boundary value of G on the positive real line. In the same vein, G^{-} is the boundary value of G on the negative real line:

$$G^{-}(r,s;-k) = \lim_{\mathrm{Im}(q)\to 0} G(r,s;q)\,, \qquad q = -k + i\mathrm{Im}(q),\ \mathrm{Im}(q), k > 0\,. \tag{16}$$

To finish this section, we recall that it is possible to obtain one of the Jost functions from the other one by way of the following relation:

$$\mathcal{J}_{+}(k) = \mathcal{J}_{-}(-k)\,, \quad k > 0\,. \tag{17}$$

It also holds that

$$-\chi(r;k) = \chi(r;-k)\,, \quad k>0\,. \tag{18}$$

Equations (14), (17) and (18) yield

$$\phi^+(r;k) = -\phi^-(r;-k)\,, \quad k>0\,; \tag{19}$$

that is, we can obtain one of the Lippmann-Schwinger eigenfunctions from the other one.

3. Two possible analytic continuations

After recalling some basic results in the previous section, we are in a position to analytically continue the Lippmann-Schwinger eigenfunctions into the second sheet of the Riemann surface. We shall examine two possible analytic continuations, which will be referred to as Continuation 1 and Continuation 2.

3.1. CONTINUATION 1

By means of Eq. (19), we can obtain one of the Lippmann-Schwinger eigenfunctions in terms of the other one. What is more, in analogy with Eqs. (15) and (16), the Lippmann-Schwinger eigenfunctions should be seen as limiting values on the positive and on the negative real k-axis of one and the same function, which we shall denote by ϕ_1. In order to construct ϕ_1, let us define the Jost function $\mathcal{J}_1(q)$ in such a way that $\mathcal{J}_1(q)$ coincides with $\mathcal{J}_+(k)$ on the positive k-axis and with $\mathcal{J}_-(k)$ on the negative k-axis.[1] The eigenfunction ϕ_1 is defined as follows:[2]

$$\phi_1(r;q) := \sqrt{\frac{2}{\pi}}\,\frac{\chi(r;q)}{\mathcal{J}_1(q)}\,. \tag{20}$$

Then,

$$\phi_1^+(r;k) = \lim_{\mathrm{Im}(q)\to 0} \phi_1(r;q)\,, \quad q=k+i\mathrm{Im}(q)\,,\ k>0\,, \tag{21}$$

and

$$\phi_1^-(r;-k) = \lim_{\mathrm{Im}(q)\to 0} -\phi_1(r;q)\,, \quad q=-k+i\mathrm{Im}(q)\,,\ k>0\,. \tag{22}$$

That is, we have obtained the Lippmann-Schwinger eigenfunctions as boundary values of a unique eigenfunction $\phi_1(r;q)$ that is defined for all complex q.

[1] Obviously, $\mathcal{J}_1(q)$ is just $\mathcal{J}_+(q)$, although it is better to use the subindexes $+/-$ only to refer to boundary values.

[2] The subindex 1 refers to Continuation 1.

The function $\phi_1^+(r;k)$ is the boundary value on the positive k-axis, whereas $\phi_1^-(r;-k)$ is the boundary value on the negative k-axis.

Now, to reach the resonance poles, we continue analytically $\phi_1^+(r;k)$ into the fourth quadrant and $\phi_1^-(r;k)$ into the third quadrant. Clearly, this looks like the natural analytic continuation. However, resonances seem to need a different analytic continuation, as explained in the next subsection.

3.2. CONTINUATION 2

The second analytic continuation consists of continuing $\chi^+(r;E)$ not to the lower half-plane of the second sheet as in Continuation 1, but to the upper half-plane of the second sheet. In terms of k, this means that instead of continuing $\phi^+(r;k)$ into the fourth quadrant as in Continuation 1, we analytically continue $\phi^+(r;k)$ into the third quadrant. For the out states, Continuation 2 means that $\chi^-(r;E)$ is not continued into the upper but into the lower half-plane of the second sheet. In terms of k, $\phi^-(r;k)$ is not continued into the third but into the fourth quadrant.

If we denote the analytic continuation of $\mathcal{J}_-(k)$ by $\mathcal{J}_2(q)$,[3] we can define the eigenfunction[4]

$$\phi_2(r;q) := \sqrt{\frac{2}{\pi}}\,\frac{\chi(r;q)}{\mathcal{J}_2(q)} \tag{23}$$

so that

$$\phi_2^+(r;-k) = \lim_{\mathrm{Im}(q)\to 0} -\phi_2(r;q)\,, \quad q=-k-i\mathrm{Im}(q)\,,\ k>0\,, \tag{24}$$

and

$$\phi_2^-(r;k) = \lim_{\mathrm{Im}(q)\to 0} \phi_2(r;q)\,, \quad q=k-i\mathrm{Im}(q)\,,\ k>0\,. \tag{25}$$

Notice that

$$\phi_2^+(r;-k) = \phi_1^-(r;k)\,, \tag{26}$$

$$\phi_2^-(r;k) = \phi_1^-(r;-k)\,; \tag{27}$$

that is, when the energy is real or lies on the first sheet, Continuations 1 and 2 yield the same Lippmann-Schwinger eigenfunctions. However, when the energy lies on the second sheet, Continuations 1 and 2 yield different Lippmann-Schwinger eigenfunctions.

4. Differences between Continuations 1 and 2

Although Continuations 1 and 2 yield the same results for real energies (thus, the standard results of scattering theory are unchanged), they do

[3]Obviously, $\mathcal{J}_2(q)$ is just $\mathcal{J}_-(q)$.
[4]The subindex 2 refers to Continuation 2.

yield different Gamow vectors and different complex basis expansions, as we are going to outline in this section.

4.1. FIRST DIFFERENCE: GAMOW VECTORS

The analytic continuation of $\phi_1^+(r;k)$ to the fourth quadrant has poles. These poles are the same as those of the S matrix. The Gamow vector corresponding to the resonance pole k_n is given by

$$\langle r|k_n^+\rangle_1 = A_{n,1}\,\mathrm{res}\left[\phi_1^+(r;q)\right]_{q=k_n} , \tag{28}$$

where k_n is a zero of $\mathcal{J}_+(q) \equiv \mathcal{J}_1(q)$ in the fourth quadrant, and $A_{n,1}$ is a normalization factor. The Gamow vector corresponding to a zero $-k_n^*$ of $\mathcal{J}_+(q)$ in the third quadrant is given by

$$\langle r|-k_n^{*\,-}\rangle = B_{n,1}\,\mathrm{res}\left[\phi_1^-(r;q)\right]_{q=-k_n^*} , \tag{29}$$

where $B_{n,1}$ is a normalization factor.

By contrast, the continuation of $\phi_2^-(r;k)$ is analytic on the whole fourth quadrant. According to Continuation 2, the Gamow vector corresponding to the wave number k_n is given by

$$\langle r|k_n^-\rangle_2 = A_{n,2}\,\phi_2^-(r;q=k_n)\,. \tag{30}$$

The Gamow vector corresponding to $-k_n^*$ is, according to Continuation 2, given by

$$\langle r|-k_n^{*\,+}\rangle_2 = B_{n,2}\,\phi_2^+(r;q=-k_n^*)\,. \tag{31}$$

$A_{n,2}$ and $B_{n,2}$ are normalization factors.

For the potential we are considering, $\langle r|k_n^+\rangle_1$ and $\langle r|k_n^-\rangle_2$ are proportional to each other. However, their wave-number representations are quite different. In fact,

$$\langle^+k|k_n^+\rangle_1 \equiv \text{a somewhat awkward distribution,} \tag{32}$$

whereas [3]

$$\langle^-k|k_n^-\rangle_2 \equiv c_n\delta(k-k_n) \equiv d_n\frac{1}{k^2-k_n^2}\,; \tag{33}$$

that is, Continuation 2 yields Gamow vectors whose wave-number representations are given by the complex delta function and by the Breit-Wigner amplitude (up to normalization constants c_n and d_n, and up to some technical details).

4.2. SECOND DIFFERENCE: COMPLEX BASIS EXPANSIONS

Continuation 2 enables the expansion of the wave functions in terms of Gamow vectors and a background. This expansion is possible because certain integral at infinity vanishes. However, Continuation 1 does not enable this expansion in a clear way.

5. Conclusion

The solutions of the Lippmann-Schwinger equation are to be interpreted as boundary values on the upper and lower rims of the cut. Since the upper (lower) rim of the cut corresponds to the positive (negative) k-axis, these boundary values have an easier interpretation in the k-plane.

The Lippmann-Schwinger eigenfunctions can be analytically continued into the second sheet in two different ways:

–According to Continuation 1, the eigenfunction $\langle r|k^+\rangle_1$ (respectively $\langle r|-k^-\rangle_1$) is analytically continued into the fourth (respectively third) quadrant of the complex k-plane. Continuation 1, although very natural, yields Gamow vectors whose wave-number representation is given by an awkward distribution.

–According to Continuation 2, the eigenfunction $\langle r|k^-\rangle_2$ (respectively $\langle r|-k^{*+}\rangle_2$) is analytically continued into the fourth (respectively third) quadrant of the complex k-plane. Continuation 2 yields Gamow vectors whose wave-number representation is given by the complex delta function and by the Breit-Wigner amplitude!

Either continuation is mathematically meaningful and has both advantages and drawbacks. Continuation 1 has the advantage of providing a very natural analytical continuation. Continuation 2 implies an awkward analytic continuation, but it provides a link between the Gamow vectors and the Breit-Wigner amplitude.

As of November 21, 2003, it is not known which of these solutions is the right one, and why the other should not be used. Work on this direction is under way.

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