A simple Hamiltonian model of runaway particle with singular interaction

P. Buttà*, F. Manzo*, C. Marchioro*

May 27, 2004

Abstract

We consider a Hamiltonian system given by a charged particle under the action of a constant electric field and interacting with a medium, which is described as a Vlasov fluid. We assume that the action of the charged particle on the fluid is negligible and that the latter has one dimensional symmetry. We prove that if the singularity of the particle/medium interaction is integrable and the electric field intensity is large enough then the particle escapes to infinity with a quasi-uniformly accelerated motion.

Keywords: Runaway particle; Vlasov equation; singular interaction.
AMS Subject Classification: 82D10, 82C21, 76X05.

1 Introduction

The paper is written in the framework of fully Hamiltonian models of non-equilibrium statistical mechanics. We consider a charged particle, with a large initial velocity, moving in a constant electric field and interacting with an infinitely extended medium. We study the dynamics at very long time and prove that, if the interaction particle/medium is not too strong, the particle escapes to infinity with a quasi-uniformly accelerated motion.

This effect has been widely studied in kinetic theory to explain the so-called "runaway electrons" observed in plasma physics [8]. The conditions under which this effect takes place are related to a fast decrease in the scattering cross section of the particle/medium interaction.

A rigorous derivation of this behavior in the framework of fully Hamiltonian models, where the medium consists of an infinite number of interacting particles, is difficult. Recently, in the case of a one-dimensional system of classical

*Dipartimento di Matematica Università di Roma 'La Sapienza', Piazzale Aldo Moro 2, 00185 Rome, Italy. E-mail addresses: butta@mat.uniroma1.it, manzof@mat.uniroma1.it, marchior@mat.uniroma1.it.
Work partially supported by the GNFM-INDAM and the Italian Ministry of the University

particles interacting via a finite range, non-negative potential, it was proved that a sufficient condition for the runaway effect is that the particle/medium interaction is bounded [4]. The investigation has been carried out also in the case of non strictly one-dimensional systems (particles confined in an infinite tube), where a similar result was obtained for large electric field [3].

Actually, heuristic arguments suggest that the result can be extended to singular interactions provided the singularity is integrable (the most important example being Coulomb interaction), see the discussion in [3, Sect. 5].

At the present state of knowledge, a rigorous proof of this conjecture is too difficult. Indeed, in the case of singular interactions, the charged particle can be drastically slowed down by a single *central* collision. Clearly, such a collision is an exceptional event, but taking into account this fact requires a too deep control of the dynamics.

To overcome the above difficulties, we consider a simplified model where the medium is described by the solution to the Vlasov equation, corresponding to a "mean field approximation" of a particle system [1, 2, 7, 9, 12]. Even in this approximation, the existence of time evolution and, above all, the long time behavior of the system [particle] + [Vlasov fluid with infinite mass] are open problems. Here, we consider a simpler model, to be seen as a perturbative approximation, where we neglect the effect of the charged particle on the fluid. As further technical assumptions, we consider a Vlasov fluid with bounded, non-negative, finite range interaction, and we restrict ourselves to initial data with planar symmetry, which makes the problem one-dimensional. We study this model for large electric fields and show the *runaway effect* for any singular but integrable particle/fluid interaction.

It would be reasonable to study more sophisticated models enriched in three directions: i) taking into account the full particle/fluid interaction; ii) considering more general initial data, possibly introducing an external confining potential; iii) considering a singular interaction among the fluid particles (for instance Coulomb-like at short distance). To give an idea of the difficulties, we remark that the existence of the Vlasov fluid with infinite mass and Coulomb-like interaction is a well-known and unsolved problem (for bounded mass see the celebrated papers [10, 11, 13]). A result for a particular state with unbounded mass has been recently obtained [6]. The existence of the full dynamics [charged particle] + [Vlasov fluid] is another problem that needs attention and a careful statement, since, a priori, blow-up of the solution may occur.

The plan of the paper is the following. In Section 2 the model is defined and the main results are stated. The proofs are given in the remaining sections.

2 Definitions and results

We consider a *Vlasov fluid* in three dimensions, namely, the pair of functions:

$$(\boldsymbol{x}, \boldsymbol{v}) \mapsto (\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{v}; t), \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{v}; t)), \quad \hat{f}_0(\boldsymbol{x}, \boldsymbol{v}) \mapsto \hat{f}(\boldsymbol{x}, \boldsymbol{v}; t),$$

where $(\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$, $t \in \mathbb{R}$, solution to:

$$\begin{cases} \dot{\boldsymbol{X}}(\boldsymbol{x}, \boldsymbol{v}; t) = \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{v}; t), \\ \dot{\boldsymbol{V}}(\boldsymbol{x}, \boldsymbol{v}; t) = (\boldsymbol{F} * \hat{f})\ (\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{v}; t); t), \\ \big(\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{v}; 0), \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{v}; 0)\big) = (\boldsymbol{x}, \boldsymbol{v}), \\ \hat{f}(\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{v}; t), \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{v}; t); t) = \hat{f}_0(\boldsymbol{x}, \boldsymbol{v}), \end{cases} \tag{2.1}$$

where $(\boldsymbol{F} * \hat{f})(\boldsymbol{y}; t) \doteq \int d\boldsymbol{v}' \int d\boldsymbol{x}'\ \boldsymbol{F}(\boldsymbol{y} - \boldsymbol{x}')\hat{f}(\boldsymbol{x}', \boldsymbol{v}'; t)$ and the force field $\boldsymbol{F}(\boldsymbol{x}) = -\nabla\Phi(|\boldsymbol{x}|)$ with the potential $\Phi \in C^2(\mathbb{R})$ a non-negative, even function such that:

$$\Phi(0) > 0 \text{ and } \Phi(r) = 0 \text{ if } r > 1.$$

In the case of smooth initial data $\hat{f}(\boldsymbol{X}(\boldsymbol{x}, \boldsymbol{v}; t), \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{v}; t); t)$ satisfies the differential equation:

$$\Big(\partial_t + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} + (\boldsymbol{F} * \hat{f})\ (\boldsymbol{x}; t) \cdot \nabla_{\boldsymbol{v}}\Big)\ \hat{f}(\boldsymbol{x}, \boldsymbol{v}; t) = 0. \tag{2.2}$$

We recall that the time evolution defined by Eq. (2.1) preserves the measure $d\boldsymbol{x}\, d\boldsymbol{v}$, i.e. the Jacobian $J(\boldsymbol{X}, \boldsymbol{V}|\boldsymbol{x}, \boldsymbol{v}) = 1$ (Liouville's theorem).

A charged particle with unitary mass and charge is subjected to an electric field $\boldsymbol{E}$ and interacts with the Vlasov fluid via a potential Ψ. Calling $\boldsymbol{\xi}$ the coordinates of the charged particle, the equations of motion are:

$$\begin{aligned} &\ddot{\boldsymbol{\xi}}(t) = \boldsymbol{E} - \int d\boldsymbol{x}\, d\boldsymbol{v}\ \nabla\Psi(|\boldsymbol{\xi}(t) - \boldsymbol{x}|)\ \hat{f}(\boldsymbol{x}, \boldsymbol{v}; t), \\ &\boldsymbol{\xi}(0) = 0, \qquad \dot{\boldsymbol{\xi}}(0) = \boldsymbol{\nu}. \end{aligned} \tag{2.3}$$

We assume that the action of the charged particle on the fluid is negligible, i.e. that $\hat{f}(\boldsymbol{x}, \boldsymbol{v}; t)$ satisfies Eq. (2.2) also in the presence of the charged particle. The electric field is directed along the first coordinate axis, i.e. $\boldsymbol{E} = (E, 0, 0)$, and the potential $\Psi \in C^2(\mathbb{R} \setminus \{0\})$ is an even function such that:

$$\Psi(r) = 0 \text{ if } r > 1.$$

The initial condition is smooth and satisfies, for suitable constants $\hat{K}_0, \lambda > 0$,

$$0 \leq \hat{f}_0(\boldsymbol{x}, \boldsymbol{v}) \leq \hat{K}_0\, e^{-\lambda \boldsymbol{v}^2}. \tag{2.4}$$

Existence and uniqueness of the solution to Eqs. (2.1) were proved in [5] for a wide class of initial conditions. Under the above hypotheses we can state the following

Conjecture. *Assume that the force $-\nabla\Psi(|x|)$ is integrable and let $\boldsymbol{\nu} = (\nu, 0, 0)$ be the initial velocity. Then there exist constants $K, K' \geq 0$ such that, for ν and E large enough,*

$$|\dot{\boldsymbol{\xi}}(t) - \boldsymbol{E}t - \boldsymbol{\nu}| \leq Kt + K'.$$

We are not able to prove this conjecture in its generality, mainly because we are not able to control the time evolution of spatial regions with very hight density of the Vlasov fluid. We can prove the conjecture only for the particular class of initial data with *planar symmetry*, i.e. when:

$$\hat{f}_0(\boldsymbol{x}, \boldsymbol{v}) = \hat{f}_0((x_1, 0, 0), \boldsymbol{v}) \qquad \forall\, (\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{R}^3 \times \mathbb{R}^3, \tag{2.5}$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$.

The planar symmetry is readily seen to be preserved by Vlasov equation: $\hat{f}_0(\boldsymbol{x}, \boldsymbol{v}; t) = \hat{f}((x_1, 0, 0), \boldsymbol{v}; t)$. In particular, the function

$$f(x, v; t) \doteq \int dv_2\, dv_3\, \hat{f}((x_1, 0, 0), (v, v_2, v_3); t), \qquad (x, v) \in \mathbb{R} \times \mathbb{R},$$

is the unique solution to the one-dimensional Vlasov equation:

$$\begin{cases} \dot{X}(x, v; t) = V(x, v; t), \\ \dot{V}(x, v; t) = (F * f)\;(X(x, v; t); t), \\ \big(X(x, v; 0), V(x, v; 0)\big) = (x, v), \\ f(X(x, v; t), V(x, v; t); t) = f_0(x, v), \end{cases} \tag{2.6}$$

where $(F * f)(y; t) \doteq \int dv' \int dx'\, F(y - x') f(x', v'; t)$ and $F(x) = -\varphi'(x)$ with $\varphi(x) \doteq 2\pi \int_0^1 dy\, y\, \Phi(\sqrt{x^2 + y^2})$. Moreover, by (2.4), for some $K_0 > 0$,

$$0 \le f_0(x, v) \le K_0\, e^{-\lambda v^2}. \tag{2.7}$$

Under the planar symmetry the charged particle moves along the first coordinate axis (recall $\boldsymbol{\nu} = (\nu, 0, 0)$ and $\boldsymbol{E} = (E, 0, 0)$), $\boldsymbol{\xi}(t) = (\xi(t), 0, 0)$, via the equation:

$$\begin{aligned} &\ddot{\xi}(t) = E - \int dx\, dv\, \psi'(\xi(t) - x)\, f(x, v; t), \\ &\xi(0) = 0, \qquad \dot{\xi}(0) = \nu. \end{aligned} \tag{2.8}$$

where $\psi(\xi) \doteq 2\pi \int_0^1 d\eta\, \eta\, \Psi(\sqrt{\xi^2 + \eta^2})$.

We can now state our main results:

Proposition 2.1 *For $f_0(x, v)$ satisfying* (2.7) *there exist $a, K_1 > 0$ such that:*

$$|V(x, v; t)| \le K_1 + 2|v| + at \qquad \forall\, t \ge 0. \tag{2.9}$$

Theorem 2.2 *If ψ' is integrable, there exist $K_2, K_3 > 0$ and $\nu_0, E_0 > 0$ such that, for any $\nu \ge \nu_0$ and $E \ge E_0$,*

$$|\dot{\xi}(t) - Et - \nu| \le K_2 t + K_3. \tag{2.10}$$

Remark 2.3 It is natural to ask what happens if ν and/or E are small. It is trivial to consider small initial velocities and correspondingly high electric fields. Indeed, the boundedness of the force due to the fluid implies that, for large enough electric fields, the particle speed becomes large within a finite time; as a result, Eq. (2.10) holds with a suitable constant K_3 depending on ν. Other cases are harder.

3 Proof of Proposition 2.1

Proposition 2.1 is obtained proving first an analogous result in the case of a cut-off version of the Vlasov equation and then removing the cut-off.

For $N, M > 0$ let $f_0^{N,M}(x,v) \doteq f_0(x,v)\chi(|x| \le M)\chi(|v| \le N)$ where $\chi(A)$ denotes the characteristic function of the set A. We introduce the sequence of cut-off problems:

$$\begin{cases} \dot{X}^{N,M}(x,v;t) = V^{N,M}(x,v;t), \\ \dot{V}^{N,M}(x,v;t) = (F * f^{N,M})\,(X^{N,M}(x,v;t);t), \\ \big(X^{N,M}(x,v;0), V^{N,M}(x,v;0)\big) = (x,v), \quad |x| \le M,\ |v| \le N, \\ f^{N,M}(X^{N,M}(x,v;t), V^{N,M}(x,v;t);t) = f_0^{N,M}(x,v). \end{cases} \tag{3.1}$$

We next set:

$$\mathcal{V}^{N,M}(t) \doteq \sup_{s\in[0,t]} \sup_{|x|\le M,\ |v|\le N} \left|V^{N,M}(x,v;s)\right| \tag{3.2}$$

and

$$R^{N,M}(t) \doteq 1 + \int_0^t ds\, \mathcal{V}^{N,M}(s). \tag{3.3}$$

Given a density $f(x,v)$, $\mu \in \mathbb{R}$ and $R > 0$, we call *smoothed energy* of the ball with center μ and radius R the quantity:

$$\begin{aligned} \mathcal{W}(f;\mu,R) \doteq \frac{1}{2}\int dx\, dv\, g^{\mu,R}(x)\, f(x,v)\, v^2 \\ + \frac{1}{2}\int dx\, dv\, g^{\mu,R}(x)\, f(x,v) \int dy\, dw\, f(y,w)\, \varphi(x-y), \end{aligned} \tag{3.4}$$

where $g^{\mu,R}(x) \doteq g(|x-\mu|/R)$ with $g \in C^\infty(\mathbb{R}_+)$ such that $g(x) = 1$ for $x \in [0,1]$, $g(x) = 0$ for $x \in [2,\infty)$, and $g'(x) \in [-2,0]$. Note that by (2.7) and since the potential φ is bounded, there exists $Q_0 > 0$ such that:

$$\sup_\mu \mathcal{W}(f_0;\mu,R) \le Q_0\, R \qquad \forall\, R > 0. \tag{3.5}$$

Lemma 3.1 below, the one-dimensional version of [5, Prop. 4.2], shows that the energy at time t of a region of size $R^{M,N}(t)$ is bounded in terms of the size of the region. In Lemma 3.2 we then use this result to control the maximum speed of the Vlasov fluid, showing it cannot grow more than linearly in time.

Lemma 3.1 *For $f_0(x,v)$ satisfying (2.7) there exists $K_4 > 0$ such that, for any $N, M > 0$,*

$$\sup_\mu \mathcal{W}(f^{N,M}(t),\mu,R^{N,M}(t)) \le K_4\, R^{N,M}(t).$$

The proof of Lemma 3.1 is given in the Appendix.

Lemma 3.2 *For $f_0(x,v)$ satisfying (2.7) there exists $K_5 > 0$ such that, for any $N \geq K_5$, $M > 0$, and $t \geq 0$,*

$$\mathcal{V}^{N,M}(t) < 2(N + K_5^2 t). \tag{3.6}$$

Proof. We prove the lemma by contradiction. We assume that the velocity of a fluid particle becomes larger than $2(N + K_5^2 t)$ at a given time t. We evaluate the force acting on this particle (during its backward time-evolution) due to "fast" and "slow" particles, respectively. By Lemma 3.1 we can show that the mass of the fast particles at each time is finite so that they produce a bounded force. On the other hand, slow particles interact with the tagged one only for a short time, so that we can bound their average (in time) force by a constant. Therefore, by choosing K_5^2 large enough we get the contradiction.

Define $\mathcal{U}(t) \doteq 2(N + K_5^2 t)$ and suppose, ab absurdo, that for every $K_5 > 0$,

$$\tau \doteq \inf\{t > 0 : \mathcal{V}^{N,M}(t) \geq \mathcal{U}(t)\} < \infty, \tag{3.7}$$

for some M and $N \geq K_5^2$. By continuity, we can pick (x,v), $|x| \leq M$, $|v| \leq N$, such that $|V^{N,M}(x,v;\tau)| = \mathcal{U}(\tau)$. Moreover, we assume, without loss of generality, that $V^{N,M}(x,v;\tau) = \mathcal{U}(\tau)$. Let

$$\tau^* \doteq \sup\Big\{t > 0 : V^{N,M}(x,v;t) < \frac{3}{4}\mathcal{U}(\tau)\Big\}. \tag{3.8}$$

We simplify notation by calling:

$$\begin{aligned} X_s &\doteq X^{N,M}(x,v;s), \qquad & V_s &\doteq V^{N,M}(x,v;s), \\ Y_s &\doteq X^{N,M}(y,w;s), \qquad & W_s &\doteq V^{N,M}(y,w;s). \end{aligned}$$

By our hypotheses on the initial condition, $\tau > 0$ and, by continuity, τ^* is well defined and strictly smaller than τ. Thus, under the hypothesis that τ is finite, we have:

$$\begin{aligned} \frac{N + K_5^2\tau}{2} &= V_\tau - V_{\tau^*} = \left| \int_{\tau^*}^{\tau} ds \int dy\, dw\, f_0^{N,M}(y,w)\, \varphi'(X_s - Y_s)| \right| \\ &\leq ||\varphi'||_\infty \int dy\, dw\, f_0^{N,M}(y,w) \int_{\tau^*}^{\tau} ds\, \chi(|X_s - Y_s| \leq 1). \end{aligned} \tag{3.9}$$

We introduce the sets:

$$\Delta_{y,w}^+ \doteq \Big\{s \in [\tau^*, \tau] : |W_s| > N + K_5^2\tau,\ |X_s - Y_s| \leq 1\Big\}, \tag{3.10}$$

$$\Delta_{y,w}^- \doteq \Big\{s \in [\tau^*, \tau] : |W_s| \leq N + K_5^2\tau,\ |X_s - Y_s| \leq 1\Big\}. \tag{3.11}$$

Clearly:

$$V_s - W_s \geq \frac{N + K_5^2\tau}{2} \qquad \text{if } s \in \Delta_{y,w}^-, \tag{3.12}$$

$$|V_s - W_s| \leq 4(N + K_5^2\tau) \qquad \text{if } s \in \Delta_{y,w}^+. \tag{3.13}$$

Let $\delta(t) \doteq X_t - Y_t$, $t_1 \doteq \inf\{t \in [\tau^*, \tau] : |\delta(t)| \le 1\}$, $t_2 \doteq \sup\{t \in [\tau^*, \tau] : |\delta(t)| \le 1\}$. By (3.12) we have $\dot\delta(t) > 0$ inside $\Delta^-_{y,w}$, so that:

$$
\begin{aligned}
2 \quad &\ge \quad \left| \int_{t_1}^{t_2} ds\, \dot\delta(s) \right| = \left| \int_{\Delta^-_{y,w}} ds\, \dot\delta(s) + \int_{\Delta^+_{y,w}} ds\, \dot\delta(s) \right| \\
&\ge \quad \int_{\Delta^-_{y,w}} ds\, \dot\delta(s) - \int_{\Delta^+_{y,w}} ds\, |\dot\delta(s)|,
\end{aligned}
$$

whence, from (3.12) and (3.13) we get:

$$
2 \ge \frac{N + K_5^2 \tau}{2} \left(|\Delta^-_{y,w}| - 8 |\Delta^+_{y,w}| \right),
$$

namely:

$$
|\Delta^-_{y,w}| \le \frac{4}{N + K_5^2 \tau} + 8 |\Delta^+_{y,w}|.
$$

We can then decompose $\Delta^-_{y,w} = \Delta^\dagger_{y,w} \cup \Delta^*_{y,w}$, with $\Delta^\dagger_{y,w} \cap \Delta^*_{y,w} = \emptyset$, $|\Delta^\dagger_{y,w}| \le 8|\Delta^+_{y,w}|$, and $|\Delta^*_{y,w}| \le 4/(N + K_5^2\tau)$.

We now notice that, by (3.3), $\Delta^+_{y,w} \cup \Delta^-_{y,w} \ne \emptyset$ only if $y \in [x - 2R^{N,M}(\tau), x + 2R^{N,M}(\tau)]$ and that, by definition (3.7), $R^{N,M}(\tau) \le 1 + 2(N + K_5^2 \tau)\tau$. Then, using $N \ge K_5$ and setting $\bar K_0 \doteq K_0 \sqrt{\pi/\lambda}$, we have:

$$
\begin{aligned}
\int dy\, dw\, f_0^{N,M}(y,w)\, |\Delta^*_{y,w}| \quad &\le \quad \frac{16 \bar K_0 R^{N,M}(\tau)}{N + K_5^2 \tau} \le 32 \bar K_0 \frac{1 + (N + K_5^2\tau)\tau}{N + K_5^2 \tau} \\
&\le \quad 32 \bar K_0 \left(\frac{1}{K_5} + \tau \right). \qquad (3.14)
\end{aligned}
$$

On the other hand, letting $N^>(s) \doteq \int dy\, dw\, f_0^{N,M}(y,w)\, \chi(\Delta^+_{y,w})$, by Liouville's theorem, Lemma 3.1, and definition (3.10),

$$
\begin{aligned}
\frac{(N + K_5^2 s)^2}{2} N^>(s) \quad &\le \quad \int dY\, dW\, f^{N,M}(Y,W;s)\, \frac{W^2}{2}\, \chi(|X_s - Y| \le 1) \\
&\le \quad \mathcal{W}(f^{N,M}(s); X_s, R^{N,M}(s)) \;\le\; K_4 R^{N,M}(s),
\end{aligned}
$$

and hence (using again $N \ge K_5$):

$$
\begin{aligned}
&\int dy\, dw\, f_0^{N,M}(y,w)\, |\Delta^+_{y,w}| = \int_0^\tau ds\, N^>(s) \\
&\qquad \le \frac{2K_4 \tau R^{N,M}(\tau)}{(N + K_5^2 \tau)^2} \le 4 K_4 \tau \frac{1 + (N + K_5^2\tau)\tau}{(N + K_5^2\tau)^2} \le \frac{8K_4}{K_5^2}\tau. \qquad (3.15)
\end{aligned}
$$

By using (3.14) and (3.15) in (3.9), we get:

$$
\begin{aligned}
\frac{N + K_5^2\tau}{2} \quad &\le \quad ||\varphi'||_\infty \int dy\, dw f_0^{N,M}(y,w)\, \left(|\Delta^*_{y,w}| + 9 |\Delta^+_{y,w}| \right) \\
&\le \quad 8\, ||\varphi'||_\infty \left[\frac{4\bar K_0}{K_5} + \left(4\bar K_0 + \frac{9K_4}{K_5^2} \right) \tau \right],
\end{aligned}
$$

whence a contradiction for K_5 sufficiently large. $\square$

Proof of Proposition 2.1. Following [5] the solution to the Vlasov equation (2.6) can be obtained from the solutions to the cut-off problem (3.1) as $(N,M)\to\infty$ in a suitable way. Given $(x_0,v_0)\in\mathbb{R}^2$ let $k_0=\lfloor |x_0|\rfloor+1$. We choose $M=k_0+N^5$ and use the shorthand notation, with e.g. $X^N(x,v;t)$, $V^N(x,v;t)$ in place of $X^{N,k_0+N^5}(x,v;t)$, $V^{N,k_0+N^5}(x,v;t)$. We define (below $k\in[k_0,k_0+N^5]$):

$$\delta^N(x,v;t)\doteq|V^{N+1}(x,v;t)-V^N(x,v;t)|,\quad u_k^N(t)\doteq\sup_{|x|\le k,\,|v|\le N}\delta^N(x,v;t). \tag{3.16}$$

We have:

$$\begin{aligned}\delta^N(x,v;t) \quad\le\quad & \int_0^t ds\,\Big|\int dy\,dw\,f_0^{N+1}(y,w)\,\varphi'(X^{N+1}(x,v;s)-X^{N+1}(y,w;s))\\ &-\int dy\,dw\,f_0^N(y,w)\,\varphi'(X^N(x,v;s)-X^N(y,w;s))\Big|.\end{aligned} \tag{3.17}$$

By Lemma 3.2 the dy-integrals are restricted to $|y-x|\le p_s^N$ where $p_s^N\doteq 1+2(N+1+K_5^2s)s$. The contribution at time s in (3.17) can thus be bounded by:

$$\begin{aligned}&\int dy\,dw\,f_0^N(y,w)\,\big|\varphi'(X^{N+1}(x,v;s)-X^{N+1}(y,w;s))\\ &\qquad\qquad-\varphi'(X^N(x,v;s)-X^N(y,w;s))\big|\\ &+\int dy\,dw\,\big|f_0^{N+1}(y,w)-f_0^N(y,w)\big|\,\big|\varphi'(X^{N+1}(x,v;s)-X^{N+1}(y,w;s))\big|\\ &\le 2\bar K_0\,\|\varphi''\|_\infty\,p_s^N\sup_{\substack{|y-x|\le p_s^N\\|w|\le N}}\int_0^s d\tau\,\delta^N(y,w,\tau)\\ &+\,K_0\|\varphi'\|_\infty p_s^N\int_N^\infty dw\,e^{-\lambda w^2},\end{aligned} \tag{3.18}$$

where in the last inequality we used (2.7) and $\bar K_0=K_0\sqrt{\pi/\lambda}$. Let us define:

$$N_\gamma(t)\doteq|v_0|+\gamma(1+t), \tag{3.19}$$

where the parameter $\gamma\ge1$ will be chosen later on. We consider $N\ge N_\gamma(t)$ so that $t\le N/\gamma$ and $p_t^N\le C_1N^2/\gamma$ for a suitable positive constant C_1. By (3.17) and (3.18) we get, for some $C_2,C_3>0$,

$$u_{k_0}^N(t)\le\frac{C_2N^3}{\gamma^2}e^{-\lambda N^2}+\frac{C_3N^2}{\gamma}\int_0^t dt_1\int_0^{t_1}dt_2\,u^N_{k_0+C_1N^2/\gamma}(t_2). \tag{3.20}$$

We iterate (3.20) ℓ times with:

$$\ell\doteq\left\lfloor\frac{\gamma(M-k_0)}{C_1N^2}\right\rfloor=\left\lfloor\frac{\gamma N^3}{C_1}\right\rfloor, \tag{3.21}$$

getting:

$$u_{k_0}^N(t) \le \frac{C_2 N^3}{\gamma^2} e^{-\lambda N^2} \sum_{i=0}^{\ell-1} \frac{(C_3 N^2 t^2)^i}{\gamma^i (2i)!} + 4(N+1+K_5^2 t) \frac{(C_3 N^2)^\ell t^{2\ell+1}}{\gamma^\ell (2\ell)!}, \quad (3.22)$$

where we used that $u_{k'}^N(s) \le 4(N+1+K_5^2 t)$ for any $k' \le M$ and $s \in [0,t]$, see Lemma 3.2. Recalling $t \le N/\gamma$, the first term in (3.22) is bounded by:

$$\frac{C_2 N^3}{\gamma^2} \exp\left[-\lambda N^2 + \sqrt{\frac{C_3}{\gamma^3}} N^2\right] \le \exp\left[-\frac{\lambda}{2} N^2\right],$$

for sufficiently large γ. By our choice (3.21) of ℓ and using Stirling's formula, the last term in (3.22) is easily seen to be bounded by $\exp[-C_4 N^3]$ for γ large enough and a positive constant $C_4 > 0$. Hence, $u_{k_0}^N(t) \to 0$ as $N \to \infty$ in a summable way. Therefore, $V^N(x_0, v_0, t)$ converges as $N \to \infty$ and, for any $N \ge N_\gamma(t)$,

$$|V^N(x_0, v_0, t) - V^{N_\gamma(t)}(x_0, v_0, t)| \le \sum_{n \ge N_\gamma(t)} \left(e^{-\lambda n^2/2} + e^{-C_4 n^3}\right) \le C_5,$$

for a suitable $C_5 > 0$. By Lemma 3.2 and (3.19), we finally get:

$$|V^N(x_0, v_0, t)| \le C_5 + 2\gamma + 2|v_0| + 2(\gamma + K_5^2)t.$$

Taking the limit $N \to \infty$ the estimate (2.9) follows. □

4 Proof of Theorem 2.2

By (2.8) and Liouville's theorem, for any $t \ge 0$, we have:

$$|\dot{\xi}(t) - Et - \nu| \;\le\; \mathcal{H}_1(t) + \mathcal{H}_2(t), \quad (4.1)$$

with:

$$\begin{aligned}
\mathcal{H}_1(t) &= \int_0^t ds \int_{|v|>as} dv \int dx\, f_0(x,v)\, |\psi'(\xi(s) - X_s)|, \\
\mathcal{H}_2(t) &= \int_0^t ds \int_{|v|\le as} dv \int dx\, f_0(x,v)\, |\psi'(\xi(s) - X_s)|,
\end{aligned}$$

where $X_s \doteq X(x, v; s)$. Recalling (2.7), (2.9), and setting $V_s \doteq V(x, v; s)$, we have, for $|v| > as$,

$$f(X_s, V_s; s) = f_0(x,v) \le G(V_s) \doteq K_0 \min\left\{e^{-\frac{\lambda}{4}\left(|V_s| - K_1 - as\right)^2}; e^{-\lambda a^2 s^2}\right\}.$$

Then, by Liouville's theorem, for a suitable constant $C_6 > 0$ and any $t \geq 0$,

$$
\begin{aligned}
\mathcal{H}_1(t) &\leq \int_0^t ds \int dX\, dV\, G(V)\, |\psi'(\xi(s) - X)| \;\leq\; K_0\, \|\psi'\|_1 \int_0^t ds \\
&\quad \times \left[4(K_1 + as) e^{-\lambda a^2 s^2} + \int_{|V| > 2(K_1 + as)} dV\, e^{-\frac{\lambda}{4}\left(|V| - K_1 - as\right)^2} \right] \\
&\leq C_6. \qquad (4.2)
\end{aligned}
$$

We now investigate $\mathcal{H}_2(t)$, i.e. the contribution of the slow particles (w.r.t. the charged one). The main point is that the time average of the force can be controlled by the L_1-norm of the force itself divided by the velocity gap. More precisely, assume:

$$\nu \geq 8K_1, \qquad E \geq 8a, \tag{4.3}$$

and define:

$$T \doteq \inf\left\{ t > 0 : |\dot{\xi}(t) - Et - \nu| > \frac{Et + \nu}{8} \right\}. \tag{4.4}$$

By continuity $T > 0$. We look for an upper bound to $\mathcal{H}_2(t)$ when $t \in [0, T)$. We write:

$$\mathcal{H}_2(t) = \int_{|v| \leq at} dv \int dx\, f_0(x, v) \int_{|v|/a}^t ds\, |\psi'(\xi(s) - X_s)|. \tag{4.5}$$

By (2.9) we have, for any $s \in [0, T)$,

$$\frac{7(Es + \nu)}{8} - K_1 - as - 2|v| \leq \dot{\xi}(s) - V_s \leq \frac{9(Es + \nu)}{8} + K_1 + as + 2|v|,$$

and therefore, by (4.3),

$$\frac{Es + \nu}{2} \leq \dot{\xi}(s) - V_s \leq \frac{3(Es + \nu)}{2} \qquad \forall\, s \in [|v|/a, T). \tag{4.6}$$

Let $r(s) \doteq \xi(s) - X_s$ and $\tau_{x,v} \doteq \inf\{t \in [0, T) : |\xi(t) - X_t| \leq 1\}$ (setting $\tau_{x,v} \doteq \infty$ if the above set is empty). Since $\dot{r}(s) = \dot{\xi}(s) - V_s \geq (E\tau_{x,v} + \nu)/2$ for $s \in [|v|/a, T)$, we have:

$$\int_{|v|/a}^t ds\, |\psi'(\xi(s) - X_s)| \leq \int_{-1}^1 dr\, \frac{2|\psi'(r)|}{E\tau_{x,v} + \nu} = \frac{2\|\psi'\|_1}{E\tau_{x,v} + \nu}.$$

Then, by (2.7) and (4.5),

$$\mathcal{H}_2(t) \leq 2K_0 \|\psi'\|_1 \int_{|v| \leq at} dv\, e^{-\lambda v^2} \int dx\, \frac{\chi(\tau_{x,v} \leq t)}{E\tau_{x,v} + \nu}.$$

By (4.6) we have $\tau_{x,v} < T$ for $|v| \leq at$ only if:

$$|x| \leq \frac{3E\tau_{x,v}^2}{4} + \frac{3\nu\tau_{x,v}}{2} + 1 = \frac{3(E\tau_{x,v} + \nu)^2}{4E} - \frac{3\nu^2}{4E} + 1.$$

In particular, if $|x| > 4$ then $E\tau_{x,v} + \nu \geq \sqrt{E|x| + \nu^2}$, so that:

$$\begin{aligned} \int dx\, \frac{\chi(\tau_{x,v} \leq t)}{E\tau_{x,v} + \nu} &\leq \int \frac{dx}{\nu}\, \chi(|x| \leq 4) \\ &\quad + \int \frac{dx}{\sqrt{E|x| + \nu^2}}\, \chi\Big(E|x| \leq (Et + \nu)^2 - \nu^2\Big) \\ &\leq 4t + \frac{8}{\nu}. \end{aligned}$$

Therefore, by (4.3), for suitable constants $C_7, C_8 > 0$ and any $t \in [0, T)$,

$$\mathcal{H}_2(t) \leq C_7\, t + C_8. \tag{4.7}$$

By (4.1), (4.2), and (4.7), we finally obtain that, if ν and E satisfy (4.3) then:

$$|\dot{\xi}(t) - Et - \nu| \;\leq\; C_7\, t + C_6 + C_8 \qquad \forall\, t \in [0, T). \tag{4.8}$$

Now fix $\nu_0 > 8 \max\{K_1; C_6 + C_8\}$ and $E_0 > 8 \max\{a; C_7\}$. By continuity and definition (4.4), if $\nu \geq \nu_0$ and $E \geq E_0$ then $T = \infty$. Bound (4.8) is therefore valid in this case for any positive time and (2.10) follows with $K_2 = C_7$ and $K_3 = C_6 + C_8$. □

Appendix

Proof of Lemma 3.1. The main tool here is energy conservation. Let us define:

$$R(t,s) \doteq R^{N,M}(t) + \int_s^t d\tau\, \mathcal{V}^{N,M}(\tau), \tag{A.1}$$

so that:

$$R(t,t) = R^{N,M}(t), \quad R(t,0) \leq 2R^{N,M}(t), \quad \partial_s R(t,s) = -\mathcal{V}^{N,M}(s). \tag{A.2}$$

In order to simplify notation, in the sequel we drop the cut-off indexes and set:

$$\begin{aligned} X_t &\doteq X^{N,M}(x,v;t), \qquad & V_t &\doteq V^{N,M}(x,v;t), \\ Y_t &\doteq X^{N,M}(y,w;t), \qquad & W_t &\doteq V^{N,M}(y,w;t). \end{aligned}$$

By definition (3.4) of W and Liouville's theorem,

$$\begin{aligned} \mathcal{W}(f(s); \mu, R(t,s)) &= \frac{1}{2} \int dx\, dv\, g^{\mu, R(t,s)}(X_s) f_0(x,v) V_s^2 \\ &\quad + \frac{1}{2} \int dx\, dv\, dy\, dw\, f_0(x,v)\, f_0(y,w)\, g^{\mu,R(t,s)}(X_s)\, \varphi(X_s - Y_s). \end{aligned}$$

For $s < t$, we have:

$$\begin{aligned}
\partial_s \mathcal{W}(f(s);\mu,R(t,s)) &= \int dx\, dv\, f_0(x,v)\, g^{\mu,R(t,s)}(X_s) \\
&\times \left\{ V_s \dot{V}_s + \frac{1}{2}\int dy\, dw\, f_0(y,w)\ (V_s - W_s)\ \varphi'(X_s - Y_s)\right\} \\
&+ \frac{1}{2}\int dx\, dv\, f_0(x,v)\, \partial_s g^{\mu,R(t,s)}(X_s) \\
&\times \left\{ V_s^2 + \int dy\, dw\, f_0(y,w)\, \varphi(X_s - Y_s)\right\}. \qquad \text{(A.3)}
\end{aligned}$$

It is easy to see that $\partial_s g^{\mu,R(t,s)}(X_s) \le 0$, whence the last term on the r.h.s. of (A.3) is non positive and can be neglected. Thus, by (3.1) and (A.3) we get:

$$\begin{aligned}
\partial_s \mathcal{W}(f(s);\mu,R(t,s)) \;\; &\le \;\; -\frac{1}{2}\int dx\, dv\, dy\, dw\, f_0(x,v)\, f_0(y,w)\, g^{\mu,R(t,s)}(X_s) \\
&\quad \times (V_s + W_s)\, \varphi'(X_s - Y_s) \\
&= \;\; \frac{1}{2}\int dx\, dv\, dy\, dw\, f_0(x,v)\, f_0(y,w)\, \varphi'(X_s - Y_s) V_s \\
&\quad \times \left(g^{\mu,R(t,s)}(Y_s) - g^{\mu,R(t,s)}(X_s)\right). \qquad \text{(A.4)}
\end{aligned}$$

Now, since $|g(x) - g(y)| \le 2|x-y|$, by using $|V_s| \le \mathcal{V}^{N,M} = -\partial_s R(t,s)$ and (2.6), we have:

$$\mathcal{W}(f(s);\mu,R(t,s)) \le -\frac{\partial_s R(t,s)}{R(t,s)} \|\varphi'\|_\infty \int dx\, dv\, dy\, dw\, f_0(x,v)\, f_0(y,w)\, \chi_s,$$

where $\chi_s \doteq \chi(|X_s - Y_s| \le 1)\, [\chi(|X_s - \mu| \le 2R(t,s)) + \chi(|Y_s - \mu| \le 2R(t,s))]$. By symmetry and Liouville's theorem, we get:

$$\begin{aligned}
\mathcal{W}(f(s);\mu,R(t,s)) \;\le\; &-2\,\frac{\partial_s R(t,s)}{R(t,s)} \|\varphi'\|_\infty \int dx\, dv\, dy\, dw\, f(x,y;s) \\
&\times f(y,w;s)\, \chi\big(|x-y| \le 1\big)\, \chi\big(|x-\mu| \le 2R(t,s)\big). \qquad \text{(A.5)}
\end{aligned}$$

Consider now the partition of $\mathbb{R}$ into intervals $\Delta_i = [i\delta, (i+1)\delta)$, $i \in \mathbb{Z}$, with $\delta \doteq \min\{x > 0 : \varphi(x) \le \varphi(0)/2\}$ and define:

$$\rho_s(x) \doteq \int dv\, f(x,y;s), \qquad \rho_s^R(x) \doteq \rho_s(x)\, \chi(|x-\mu| \le 2R(t,s)+1).$$

The integral on the r.h.s. of (A.5) can be bounded in the following way:

$$\begin{aligned}
&\int dx\, dv\, dy\, dw\, f(x,y;s)\, f(y,w;s)\, \chi\big(|x-y|\le 1\big)\, \chi\big(|x-\mu|\le 2R(t,s)\big) \\
&= \sum_{i,j} \int_{\Delta_i} dx \int_{\Delta_j} dy\, \rho_s^R(x)\, \rho_s^R(y)\, \chi\big(|x-y|\le 1\big) \\
&\le \frac{1}{2} \sum_{i,j} \left[\left(\int_{\Delta_i} dx\, \rho_s^R(x) \right)^2 + \left(\int_{\Delta_j} dy\, \rho_s^R(y) \right)^2 \right] \chi\big(|i-j|\le 1+\delta^{-1}\big) \\
&= \left(3+\frac{2}{\delta}\right) \sum_i \left(\int_{\Delta_i} dx\, \rho_s^R(x) \right)^2 \\
&\le \frac{2}{\varphi(0)} \left(3+\frac{2}{\delta}\right) \sum_i \int_{\Delta_i} dx \int_{\Delta_i} dy\, \rho_s^R(x)\, \rho_s^R(y)\, \varphi(x-y) \\
&\le \frac{2}{\varphi(0)} \left(3+\frac{2}{\delta}\right) \int dx\, dy\, \rho_s(x)\, \rho_s(y)\, \varphi(x-y)\, \chi\big(|x-\mu|\le 2R(t,s)+1\big) \\
&\le \frac{2}{\varphi(0)} \left(3+\frac{2}{\delta}\right) \mathcal{W}(f(s);\mu,3R(t,s)).
\end{aligned}$$

On the other hand, by the positivity of the potential,

$$\mathcal{W}(f;\mu,3R) \le \sum_{k=-2}^{2} \mathcal{W}(f;\mu+kR,R) \le 5 \sup_\eta \mathcal{W}(f,\eta,R),$$

and hence, by (A.5) and the previous estimates, for some $C_9>0$,

$$\partial_s \mathcal{W}(f(s);\mu,R(t,s)) \;\le\; -C_9\, \frac{\partial_s R(t,s)}{R(t,s)}\, \mathcal{W}(f(s);R(t,s)),$$

where $\mathcal{W}(f;R) \doteq \sup_\eta \mathcal{W}(f;\eta,R)$. By integrating the differential inequality and taking the supremum over μ we get:

$$\mathcal{W}(f(s);R(t,s)) \le \mathcal{W}(f(0);R(t,0)) - C_9 \int_0^s d\tau\, \mathcal{W}(f(\tau);R(t,\tau))\, \partial_\tau \log R(t,\tau),$$

from which, recalling (3.5) and (A.2),

$$\begin{aligned}
\mathcal{W}(f(t);R(t)) \;&=\; \mathcal{W}(f(t);R(t,t)) \;\le\; \mathcal{W}(f(0);R(t,0))\, e^{C_9 \log \frac{R(t,0)}{R(t,t)}} \\
&\le\; 2^{C_9}\, \mathcal{W}(f_0;2R(t)) \;\le\; 2^{C_9+1}\, Q_0\, R(t).
\end{aligned}$$

Lemma 3.1 is thus proved. □

References

[1] W. Braun and K. Hepp, The Vlasov dynamics and its fluctuations in the $1/N$ limit of interacting classical particles, *Comm. Math. Phys.* **56** (1977) 101–113.

[2] P. Buttà, E. Caglioti and C. Marchioro, On the long time behavior of infinitely extended systems of particles interacting via Kac potentials, *J. Stat. Phys.* **108** (2002) 317–339.

[3] P. Buttà, E. Caglioti and C. Marchioro, On the motion of a charged particle interacting with an infinitely extended system, *Comm. Math. Phys.* **233** (2003) 545–569.

[4] P. Buttà, E. Caglioti and C. Marchioro, On the violation of Ohm's law for bounded interactions: a one dimensional system, *Comm. Math. Phys.* (2004), to appear.

[5] E. Caglioti, S. Caprino, C. Marchioro and M. Pulvirenti, The Vlasov equation with infinite mass, *Arch. Rational Mech. Anal.* **159** (2001) 85–108.

[6] S. Caprino, A Vlasov-Poisson plasma model with non-L_1 data, *Math. Methods Appl. Sci.* (2004), to appear.

[7] R.L. Dobrushin, Vlasov equations, *Soviet Journ. Functional Anal.* **13** (1979) 115–123.

[8] L.D. Landau and E.M. Lifshitz, *Physical Kinetics*, Course of Theoretical Physics, Vol. 10 (Pergamon Press, 1981).

[9] H. Neunzert, An introduction to the nonlinear Boltzmann-Vlasov equation, Kinetic theories and the Boltzmann equation (Montecatini, 1981), 60–110. *Lecture Notes in Math.* **1048**, Springer, Berlin, 1984.

[10] K. Pfaffelmoser, Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data, *Journ. Diff. Eq.* **95** (1992) 281–303.

[11] J. Shaeffer, Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions, *Comm. Partial Differential Equations* **16** (1991) 1313–1335.

[12] H. Spohn, On the Vlasov hierarchy, *Math. Methods Appl. Sci.* **3** (1981) 445–455.

[13] S. Wollman, Global-in-time solutions to the three-dimensional Vlasov-Poisson system, *Journ. Math. Anal. Appl.* **176** (1993) 76–91.