A Time–Independent Born–Oppenheimer Approximation with Exponentially Accurate Error Estimates

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Abstract

We consider a simple molecular–type quantum system in which the nuclei have one degree of freedom and the electrons have two levels. The Hamiltonian has the form

$$H(\epsilon) \;=\; -\,\frac{\epsilon^4}{2}\,\frac{\partial^2}{\partial y^2} \;+\; h(y),$$

where $h(y)$ is a 2×2 real symmetric matrix. Near a local minimum of an electron level $\mathcal{E}(y)$ that is not at a level crossing, we construct quasimodes that are exponentially accurate in the square of the Born–Oppenheimer parameter ϵ by optimal truncation of the Rayleigh–Schrödinger series. That is, we construct E_ϵ and Ψ_ϵ, such that $\|\Psi_\epsilon\| = O(1)$ and

$$\|\,(H(\epsilon)\,-\,E_\epsilon))\,\Psi_\epsilon\,\| \;<\; \Lambda\,\exp\left(\,-\,\Gamma/\epsilon^2\,\right), \qquad \text{where} \quad \Gamma > 0.$$

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1 Introduction

The goal of this paper is to construct exponentially accurate quasimodes for the time–independent Schrödinger equation for a simple molecular system. The small parameter that governs the approximation is the usual Born–Oppenheimer parameter ϵ, where ϵ^4 is the electron mass divided by the mean nuclear mass. Under appropriate circumstances, the quasimodes we produce correspond exactly to the low–lying energy levels of the system. In that case, the exact eigenvalues and our quasimode energies differ by at most $\Lambda\,\exp\,(\,-\,\Gamma/\epsilon^2\,)$. A bound of the same form holds for the norm of the difference between the quasimodes and the exact eigenvectors.

Hamiltonians for molecular systems can generally be put in the form

$$H(\epsilon)\ =\ -\frac{\epsilon^4}{2}\,\Delta_y\ +\ h(y),$$

where the variable y describes the nuclear configuration vector and the operator $h(y)$ is the electron Hamiltonian. In this paper we look at the special case of operators of this type, where y is a single real variable, and $h(y)$ is a 2×2 real symmetric matrix. Although this is a very simple model, to the best of our knowledge, there are no prior rigorous exponentially accurate results for the time–independent Born–Oppenheimer approximation. There are such results for the time–dependent Born–Oppenheimer approximation [7, 13, 15] and for lifetimes of resonances [12, 14].

To make a precise statement of our results, we need some notation and hypotheses. For small ϵ, we study the eigenvalue problem

$$\left[\,-\,\frac{\epsilon^4}{2}\,\frac{d^2}{dy^2}\,+\,h(y)\right]\,\Psi(\epsilon,\,y)\ =\ E(\epsilon)\ \Psi(\epsilon,\,y),\tag{1}$$

where $\Psi\in L^2(\mathbb{R},\mathbb{C}^2)$.

We rewrite the operator $h(y)$ as

$$h(y)\ =\ R(y)\,\begin{pmatrix}\mathcal{E}_1(y) & 0\\ 0 & \mathcal{E}_2(y)\end{pmatrix}\,R^T(y),\qquad \text{where}\quad R(y)\ =\ \begin{pmatrix}\cos\Theta(y) & \sin\Theta(y)\\ -\sin\Theta(y) & \cos\Theta(y)\end{pmatrix}.$$

Here we assume $\mathcal{E}_1(y)$, $\mathcal{E}_2(y)$, and $\Omega(y)=\Theta'(y)$ satisfy the following requirements:

1. $\mathcal{E}_1$, $\mathcal{E}_2$, and Ω are C^∞ real–valued functions on $\mathbb{R}$.

2. $\mathcal{E}_1$ has a non-degenerate minimun. Without loss of generality, we assume it is at $y=0$ with $\mathcal{E}_1(0)=0$, $\mathcal{E}_1'(0)=0$, and $\mathcal{E}_1''(0)=1$.

3. This minimum does not occur at an electron energy level crossing, *i.e.*, $\mathcal{E}_2(0)\neq 0$.

4. $\mathcal{E}_1$, $\mathcal{E}_2$, and Ω have analytic extensions to a neighborhood of the region $S_{\delta'}\ =\ \{\,z\in\mathbb{C}:|\mathrm{Im}\,z|<\delta'+\nu\,\}$ for some $\delta'>0$ and any arbitrarily small $\nu>0$. Without loss we may assume that $\delta'\leq 1$.

5. We assume $\mathcal{E}_1$ and $\mathcal{E}_2$ are bounded below, and that $|\mathcal{E}_1(z)|$, $|\mathcal{E}_2(z)|$, $|\Omega(z)|$, $|\Omega'(z)|$, and $|\Omega(z)|^2$ are all bounded by $M\,\exp\,(\tau|z|^2)$, for $z\in S_\delta$, for some positive $\delta\le\delta'$, M, and τ.

Under these hypotheses, we have the following result:

Theorem 1 *Assume the above hypotheses, and let α be a fixed non-negative integer. Then, using optimal truncation of a perturbation expansion for small ϵ, we can construct E_ϵ and Ψ_ϵ, such that $\|\Psi_\epsilon\| = O(1)$, $E_\epsilon = (\alpha+1/2)\,\epsilon^2 + O(\epsilon^4)$, and*

$$\|\,(H(\epsilon)-E_\epsilon)\,\Psi_\epsilon\,\| \;<\; \Lambda\,\exp\,\big(-\Gamma/\epsilon^2\,\big)\,.$$

This quasimode is associated with the α^{th} vibrational energy level in the local well of $\mathcal{E}_1$ near $y=0$.

Remarks. 1. Our hypotheses allow level crossings of $\mathcal{E}_1$ and $\mathcal{E}_2$, as long as they do not occur at the bottom of the well of $\mathcal{E}_1$ where Ψ_ϵ is concentrated.
2. Our quasienergy E_ϵ may lie within the essential spectrum. In that case, one would expect a resonance near our quasienergy associated with the system being temporarily trapped in the well.
3. The bottom of the essential spectrum is greater than or equal to the minimum of the lim inf of $\mathcal{E}_1(y)$ and $\mathcal{E}_2(y)$ as y tends to plus or minus infinity. For a proof, see [16].
4. Suppose the bottom of the essential spectrum is strictly positive. If $\mathcal{E}_2$ is bounded below by some positive number and $\mathcal{E}_1$ has a unique global minimum at zero, then for small ϵ, the quasimode of the theorem is an exponentially accurate approximation to the $\alpha^{\rm th}$ eigenvalue of $H(\epsilon)$. This is proved by combining our results with those of [2].
5. The assumption that $\mathcal{E}_1$ and $\mathcal{E}_2$ are bounded below is just used to guarantee self–adjointness of $H(\epsilon)$. It never enters directly into our calculations, and it could be weakened.

The time–independent Born–Oppenheimer approximation has a long history that dates back to [1]. The first mathematically rigorous result about its validity for low–lying states is [2], in which the expansion for the energy is proved to be accurate through fourth order in ϵ. Rigorous expansions to all orders are developed for systems with smooth potentials [3], diatomic Coulomb systems [4], and general Coulomb systems [10].

Among these papers, our techniques are closest to those of [3]. However, we tried to derive exponentially accurate results by using several variations of the perturbation expansions of [3] without success. The expansion we use to prove Theorem 1 is different, and it does not rely on the multiple scales technique that is central to [3].

Various other authors have studied time–independent Born–Oppenheimer limits for other problems. Sordoni [17] has extended the results mentioned above to include high angular momentum states of diatomic molecules. Herrin and Howland [8] have studied a situation in which the bundle of eigenvectors for the electron Hamiltonian has a non–trivial Berry phase. Rousse

[16] has constructed quasimodes at fixed energies above the bottoms of wells when the nuclei have one degree of freedom. His results lead to Bohr–Sommerfeld rules, and he can handle level crossings and avoided crossings whose gaps have certain ϵ dependence. Klein and Martinez and Messirdi [9, 11, 12, 14] have studied resonances whose lifetimes are finite because of non–adiabatic transitions of the electrons.

The paper is organized as follows: In Section 2, we derive the formal perturbation expansion that we use to prove Theorem 1. Section 3 contains some technical lemmas that we use throughout the rest of the paper. In Section 4, we study the growth of quantities that occur in the perturbation expansion. We then use that information in Section 5 to prove the error bounds that imply Theorem 1. A description of the explicit calculation of the error term is presented in the Appendix.

2 The Perturbation Expansion

It is convenient to restate equation (1) in the rescaled variable $x = y/\epsilon$. The resulting equation is

$$\left[-\,\frac{\epsilon^2}{2}\,\frac{d^2}{dx^2} \;+\; h(\epsilon x) \right] \,\Psi(\epsilon,\, x) \;=\; E(\epsilon)\;\Psi(\epsilon,\, x), \tag{2}$$

We write $\Psi(\epsilon,\, x)$ as a linear combination of the eigenvectors $\Phi_1(\epsilon x)$ and $\Phi_2(\epsilon x)$ of $h(\epsilon x)$. That is,

$$\Psi(\epsilon, x) \;=\; w(\epsilon, x)\,\Phi_1(\epsilon x) \,+\, v(\epsilon, x)\,\Phi_2(\epsilon x).$$

We substitute this expression into equation (2) and project the resulting equation into the $\Phi_1(\epsilon x)$ and $\Phi_2(\epsilon x)$ directions. This transforms equation (2) into the pair of scalar equations,

$$-\,\frac{\epsilon^2}{2}\,\partial_x^2\,w \,-\, \epsilon^3\,\Omega\,\partial_x\,v \,+\, \frac{\epsilon^4}{2}\,\Omega^2\,w \,-\, \frac{\epsilon^4}{2}\,\Omega'\,v \,+\, \mathcal{E}_1\,w \;=\; E\,w, \tag{3}$$

and

$$-\,\frac{\epsilon^2}{2}\,\partial_x^2\,v \,+\, \epsilon^3\,\Omega\,\partial_x\,w \,+\, \frac{\epsilon^4}{2}\,\Omega^2\,v \,+\, \frac{\epsilon^4}{2}\,\Omega'\,w \,+\, \mathcal{E}_2\,v \;=\; E\,v, \tag{4}$$

where we have used

$$\frac{d}{dx}\,\Phi_1(\epsilon x) \;=\; -\,\epsilon\,\Omega\,\Phi_2(\epsilon x), \quad \text{and} \quad \frac{d}{dx}\,\Phi_2(\epsilon x) \;=\; \epsilon\,\Omega\,\Phi_1(\epsilon x).$$

We now replace $E(\epsilon)$, $w(\epsilon, x)$, and $v(\epsilon, x)$ by their formal Rayleigh–Schrödinger series in powers of ϵ,

$$E(\epsilon) \;=\; \sum_{n=0}^{\infty} E_n\,\epsilon^n,$$

$$
\begin{aligned}
w(\epsilon,\, x) &= \sum_{n=0}^{\infty} w_n(x)\, \epsilon^n, \qquad \text{and} \\
v(\epsilon,\, x) &= \sum_{n=0}^{\infty} v_n(x)\, \epsilon^n,
\end{aligned}
$$

We also replace the functions Ω, Ω', $\mathcal{E}_1$, and $\mathcal{E}_2$ by their convergent Taylor series in $\epsilon\, x$, where we employ the following notation:

$$
\begin{aligned}
\Omega(\epsilon x) &= \sum_{n=0}^{\infty} a_n\, \epsilon^n\, x^n, \\
\Omega^2(\epsilon x) &= \sum_{n=0}^{\infty} b_n\, \epsilon^n\, x^n, \\
\Omega'(\epsilon x) &= \sum_{n=0}^{\infty} c_n\, \epsilon^n\, x^n, \\
\mathcal{E}_1(\epsilon x) &= \frac{1}{2}\, \epsilon^2 x^2 + \sum_{n=3}^{\infty} d_n\, \epsilon^n\, x^n, \qquad \text{and} \\
\mathcal{E}_2(\epsilon x) &= \sum_{n=0}^{\infty} r_n\, \epsilon^n\, x^n.
\end{aligned}
$$

We insert all these series into the equations (3) and (4) and equate terms of the same orders in ϵ on the two sides of the equations. This yields the following:

Order 0 :

$$
\begin{aligned}
E_0\, w_0 &= 0, \qquad \text{and} \\
r_0\, v_0 &= E_0\, v_0.
\end{aligned}
$$

We want $w_0 \neq 0$, so the first equation requires $E_0 = 0$. Then, since $r_0 \neq 0$, the second equation requires $v_0 = 0$.

Order 1 :

$$
\begin{aligned}
E_0\, w_1 + E_1\, w_0 &= 0, \quad \text{and} \\
r_0\, v_1 + r_1\, y\, v_0 &= E_0\, v_1 + E_1\, v_0.
\end{aligned}
$$

From our earlier conclusions, the first equation yields $E_1 = 0$. The second equation then forces $v_1 = 0$.

Order 2 :

Using $E_0 = E_1 = 0$, $v_0 = v_1 = 0$ and $d_2 = 1/2$ we have

$$\begin{aligned} -\frac{1}{2}\,\partial_x^2\,w_0 \,+\, \frac{1}{2}\,x^2\,w_0 &= E_2\,w_0, \quad \text{and} \\ r_0\,v_2 &= 0. \end{aligned}$$

From the first equation here, E_2 and w_0 must be solutions to the eigenvalue problem for the harmonic oscillator Hamiltonian

$$H_0 \;=\; -\,\frac{1}{2}\,\frac{d^2}{dx^2} \,+\, \frac{1}{2}\,x^2,$$

whose eigenvalues and normalized eigenfunctions are $(\alpha+1/2)$ and $\phi_\alpha(x)$, where $\alpha = 0, 1, 2, 3, \ldots$ We fix a choice of α and then have $E_2 = (\alpha + 1/2)$, and $w_0 = \phi_\alpha$. The second equation clearly requires $v_2 = 0$.

Order 3 :

$$\begin{aligned} (H_0 - E_2)\,w_1 \,-\, a_0\,\partial_x\,v_0 \,+\, d_3\,x^3\,w_0 &= E_3\,w_0, \qquad \text{and} \\ r_0\,v_3 &= -\,a_0\,\partial_x\,w_0. \end{aligned}$$

Equating components of the first equation in the direction of ϕ_α, we see that $E_3 = 0$. The components orthogonal to ϕ_α then require

$$w_1 \;=\; -\,d_3\,(H_0 - E_2)_\perp^{-1}\,x^3\,w_0,$$

where $(H_0 - E_2)_\perp^{-1}$ denotes the inverse of the restriction of $(H_0 - E_2)$ to the subspace orthogonal to ϕ_α. Note that we henceforth arbitrarily assume that $w_n \perp \phi_\alpha$ for all $n > 0$. We do not worry about normalization during these calculations.

The second equation at Order 3 simply yields

$$v_3 \;=\; -\,\frac{a_0}{r_0}\,\partial_x\,w_0.$$

Remark In the Order 3 calculations, E_3 vanishes because $\langle\,\phi_\alpha,\, x^3\,\phi_\alpha\,\rangle \;=\; 0$. The symmetry that forces this inner product to vanish also causes $E_n = 0$ for all odd values of n in the higher order calculations. We do not make use of this fact in our estimates.

Order n $\geq$ 4 :

$$\begin{aligned} &(H_0 - E_2)\,w_{n-2} \,-\, \sum_{m=0}^{n-3} a_m\,x^m\,\partial_x\,v_{n-m-3} \,+\, \frac{1}{2}\sum_{m=0}^{n-4} b_m\,x^m\,w_{n-m-4} \,-\, \frac{1}{2}\sum_{m=0}^{n-4} c_m\,x^m\,v_{n-m-4} \\ &+\, \sum_{m=0}^{n} d_m\,x^m\,w_{n-m} \;=\; \sum_{m=0}^{n} E_m\,w_{n-m}, \end{aligned}$$

and

$$-\frac{1}{2}\,\partial_x^2\,v_{n-2}\;+\;\sum_{m=0}^{n-3}a_m\,x^m\,\partial_x\,w_{n-m-3}\;+\;\frac{1}{2}\sum_{m=0}^{n-4}b_m\,x^m\,v_{n-m-4}\;+\;\frac{1}{2}\sum_{m=0}^{n-4}c_m\,x^m\,w_{n-m-4}$$

$$+\;\sum_{m=0}^{n}r_m\,x^m\,v_{n-m}\;=\;\sum_{m=0}^{n}E_m\,v_{n-m}.$$

We project the two sides of the first equation into the ϕ_α direction to obtain a formula for E_n. Then projecting onto the subspace orthogonal to ϕ_α, we obtain an equation for $(H_0-E_2)\,w_{n-2}$. We determine w_{n-2} by applying the operator $(H_0-E_2)_\perp^{-1}$. We then solve the second equation for v_n by dividing by r_0.

Thus, for $n\geq 4$, we have

$$\begin{aligned}
E_n\;&=\;-\sum_{m=0}^{n-3}a_m\,\langle\,\phi_\alpha,\,x^m\,\partial_x\,v_{n-m-3}\,\rangle\;+\;\frac{1}{2}\sum_{m=0}^{n-4}b_m\,\langle\,\phi_\alpha,\,x^m\,w_{n-m-4}\,\rangle\\
&\quad-\;\frac{1}{2}\sum_{m=0}^{n-4}c_m\,\langle\,\phi_\alpha,\,x^m\,v_{n-m-4}\,\rangle\;+\;\sum_{m=0}^{n}d_m\,\langle\,\phi_\alpha,\,x^m\,w_{n-m}\rangle,
\end{aligned}$$

$$\begin{aligned}
w_{n-2}\;&=\;(H_0-E_2)_\perp^{-1}\;\Bigg[\;\sum_{m=0}^{n-3}a_m\,x^m\,\partial_x\,v_{n-m-3}\;-\;\frac{1}{2}\sum_{m=0}^{n-4}b_m\,x^m\,w_{n-m-4}\\
&\qquad+\;\frac{1}{2}\sum_{m=0}^{n-4}c_m\,x^m\,v_{n-m-4}\;-\;\sum_{m=0}^{n}d_m\,x^m\,w_{n-m}\;+\;\sum_{m=0}^{n}E_m\,w_{n-m}\;\Bigg],
\end{aligned}$$

and

$$\begin{aligned}
v_n\;&=\;\frac{1}{r_0}\;\Bigg[\;\frac{1}{2}\,\partial_x^2\,v_{n-2}\;-\;\sum_{m=0}^{n-3}a_m\,x^m\,\partial_x\,w_{n-m-3}\;-\;\frac{1}{2}\sum_{m=0}^{n-4}b_m\,x^m\,v_{n-m-4}\\
&\qquad-\;\frac{1}{2}\sum_{m=0}^{n-4}c_m\,x^m\,w_{n-m-4}\;-\;\sum_{m=1}^{n}r_m\,x^m\,v_{n-m}\;+\;\sum_{m=0}^{n}E_m\,v_{n-m}\;\Bigg].
\end{aligned}$$

To facilitate certain estimations, we introduce a bounded operator A that was originally introduced in [18, 19]. This operator depends on the choice of α and is diagonal in the basis of Harmonic Oscillator eigenfunctions:

$$A\,\phi_\beta\;=\;\begin{cases}\phi_\beta & \text{if}\quad\beta=\alpha,\\ |\beta-\alpha|^{-1/2}\,\phi_\beta & \text{if}\quad\beta\neq\alpha.\end{cases}$$

This operator is useful for the following reasons: The estimate $\|(H_0 - E_2)_\perp^{-1}\,\psi\| \;\leq\; \|\psi\|$ is true for all ψ, but it is a very poor estimate if $\psi \;=\; \sum_{\beta\geq B} k_\beta\,\phi_\beta$, where B is large. However, the analogous estimate for the inverse of the restriction of $A\,(H_0 - E_2)\,A$ to the orthogonal complement of its kernel is optimal because $\|\,A\,(H_0 - E_2)\,A\,\psi\,\| \;=\; \|\,\psi\,\|$ for any $\psi\;\perp\;\phi_\alpha$.

Rather than estimating norms of w_n and v_n directly, we study $\hat{w}_n$ and $\hat{v}_n$, where $w_n \;=\; A\,\hat{w}_n$ and $v_n \;=\; A\,\hat{v}_n$. In addition, we multiply equation (3) by A. Note that A^{-1} is an unbounded, densely defined linear operator, but that any finite linear combination of ϕ_β's is in its domain.

After a little bit of rearranging, we obtain the following:

$$\begin{aligned}
E_0 &= E_1 \;=\; E_3 \;=\; 0,\\
E_2 &= \alpha + 1/2,\\
\hat{w}_0 &= \phi_\alpha,\\
\hat{w}_1 &= -\,d_3\,[A(H_0 - E_2)A]_\perp^{-1}\,A\,x^3\,A\,\hat{w}_0,\\
\hat{v}_0 &= \hat{v}_1 \;=\; \hat{v}_2 \;=0,\\
\hat{v}_3 &= -\,\frac{a_0}{r_0}\;A^{-1}\,\partial_x\,A\,\hat{w}_0,
\end{aligned}$$

and for $n \geq 4$,

$$\begin{aligned}
E_n \;=\; & -\sum_{m=0}^{n-3} a_m\,\langle\,A\,\partial_x\,x^m\,A\,\phi_\alpha,\;\hat{v}_{n-m-3}\,\rangle \;+\; \frac{1}{2}\sum_{m=0}^{n-4} b_m\,\langle\,A\,x^m\,A\,\phi_\alpha,\;\hat{w}_{n-m-4}\,\rangle\\
& -\frac{1}{2}\sum_{m=0}^{n-4} c_m\,\langle\,A\,x^m\,A\,\phi_\alpha,\;\hat{v}_{n-m-4}\,\rangle \;+\; \sum_{m=0}^{n} d_m\,\langle\,A\,x^m\,A\,\phi_\alpha,\;\hat{w}_{n-m}\rangle,
\end{aligned}$$

$$\begin{aligned}
\hat{w}_{n-2} \;=\; [A(H_0 - E_2)A]_\perp^{-1}\;\Bigg[\; & \sum_{m=0}^{n-3} a_m\,A\,x^m\,\partial_x\,A\,\hat{v}_{n-m-3} \;-\; \frac{1}{2}\sum_{m=0}^{n-4} b_m\,A\,x^m\,A\,\hat{w}_{n-m-4}\\
& +\;\frac{1}{2}\sum_{m=0}^{n-4} c_m\,A\,x^m\,A\,\hat{v}_{n-m-4} \;-\; \sum_{m=0}^{n} d_m\,A\,x^m\,A\,\hat{w}_{n-m}\\
& +\;\sum_{m=0}^{n} E_m\,A^2\,\hat{w}_{n-m}\;\Bigg],
\end{aligned}$$

and

$$\hat{v}_n \;=\; \frac{1}{r_0}\;\Bigg[\;\frac{1}{2}\,A^{-1}\,\partial_x^2\,A\hat{v}_{n-2} \;-\; \sum_{m=0}^{n-3} a_m\,A^{-1}\,x^m\,\partial_x\,A\hat{w}_{n-m-3}$$

$$- \frac{1}{2} \sum_{m=0}^{n-4} b_m \, A^{-1} \, x^m \, A \, \hat{v}_{n-m-4} \; - \; \frac{1}{2} \sum_{m=0}^{n-4} c_m \, A^{-1} \, x^m \, A \, \hat{w}_{n-m-4}$$

$$- \sum_{m=1}^{n} r_m \, A^{-1} \, x^m \, A \, \hat{v}_{n-m} + \sum_{m=0}^{n} E_m \, \hat{v}_{n-m} \Bigg] .$$

We conclude this section with a crucial observation: If we let $P_{i \le n}$ denote the projection onto the span of ϕ_i with $i \le n$, then for $n \ge 0$ we have $\hat{w}_n \in \mathrm{Ran}(P_{i \le 3n+\alpha})$, and for $n \ge 3$ we have $\hat{v}_n \in \mathrm{Ran}(P_{i \le 3n+\alpha-8})$. We prove this by first using our explicit formulas above to see that $\hat{w}_0 \in \mathrm{Ran}(P_{i \le \alpha})$, $\hat{w}_1 \in \mathrm{Ran}(P_{i \le 3+\alpha})$, and $\hat{v}_3 \in \mathrm{Ran}(P_{i \le 1+\alpha})$. We then use an easy induction to prove the general results by using our recursive formulas for $\hat{w}_{n-2}$ and $\hat{v}_n$ for $n \ge 4$.

3 Preliminary Estimates

In this section we state and prove a number of technical results that we need later.

Lemma 1 *Let $\#_x$ denote either x or ∂_x, and let $(\#_x)^l$ denote any product of l factors, each of which is either x or ∂_x. Let A be the operator defined in the previous section. For $l \ge 1$ and $n \ge 0$, the following inequalities hold.*

(i) $$\| A (\#_x)^l A \, P_{i \le n} \| \; \le \; \begin{cases} [\, 2 \, (2+\alpha) \,]^{1/2}, & \textit{when} \quad l = 1, \\ (2+\alpha) \, 2^{l/2} \left[\dfrac{(n+l-1)!}{(n+1)!} \right]^{1/2}, & \textit{when} \quad l > 1. \end{cases}$$

(ii) $$\| A^{-1} (\#_x)^l A \, P_{i \le n} \| \; \le \; [\, \alpha \, (2+\alpha) \,]^{1/2} \, 2^{l/2} \left[\frac{(n+l+1)!}{(n+1)!} \right]^{1/2} .$$

Proof: Statement (i) has been proved in Lemma 1 of [18]. To prove (ii), we note that

$$\| A^{-1} (\#_x)^l A \, P_{i \le n} \| \; \le \; \| A^{-1} P_{i \le n+l} \| \, \| (\#_x)^{l-1} P_{i \le n+1} \| \, \| \#_x P_{i \le n} \| \tag{5}$$

From the definition of the operator A, it is clear that

$$\| A^{-1} P_{i \le n+l} \| \; = \; |n + l - \alpha|^{1/2} .$$

By Lemma 5.1 of [6], the second factor in (5) satisfies the bound

$$\| (\#_x)^{l-1} P_{i \le n+1} \| \; \le \; 2^{(l-1)/2} \left[\frac{(n+l)!}{(n+1)!} \right]^{1/2} .$$

Finally, the proof of Lemma 1 of [18] shows that

$$\|\#_x P_{i\leq n}\| \ \leq\ [\,2\,(2+\alpha)\,]^{1/2}.$$

Therefore,

$$\|A^{-1}(\#_x)^l A P_{i\leq n}\| \ \leq\ (2+\alpha)^{1/2}\,2^{l/2}\ \left[\frac{(n+l+1)!}{(n+1)!}\right]^{1/2}\ \max_{n,l\geq 0}\ \left[\frac{|n+l-\alpha|}{n+l+1}\right]^{1/2}.$$

This last expression is clearly bounded by the right hand side of (ii). $\square$

Lemma 2 *Consider any real number $\kappa\geq 2$.*

(i) For each $\alpha\geq 0$ there exists a constant $C_1=C_1(\alpha)$ such that, for all $m\geq 0$,

$$\sum_{l=0}^{m}\left[\frac{(m-l+\alpha+1)!\,(l+\alpha+1)!}{(m+\alpha+1)!}\right]^{1/2}\ \leq\ C_1.$$

(ii) For all $\alpha\geq -1$ there exists a constant C_2 such that, for all $m\geq 0$,

$$\sum_{l=0}^{m}\kappa^{-5l/2}\left[\frac{(3m-2l+\alpha+1)!\,(m-l+\alpha+1)!}{(3m-3l+\alpha+1)!\,(m+\alpha+1)!}\right]^{1/2}\ \leq\ C_2.$$

(iii) For each $\alpha\geq 0$ there exists a constant $C_3=C_3(\alpha)$ such that, for all $m\geq 0$,

$$\sum_{l=1}^{m}\kappa^{-5l/2}\left[\frac{(m-l+\alpha+1)!\,(l+\alpha+1)!}{(\alpha+1)!\,(m+\alpha)!}\right]^{1/2}\ \leq\ C_3.$$

Proof: See Lemma 4 of [19].

Lemma 3 *Let f be a C^∞ real-valued function that has an analytic extension to the region*

$$S_\delta\ =\ \{z\in\mathbb{C}: |Im\, z|<\delta+\nu\}$$

for some $\delta>0$, and any arbitrarily small $\nu>0$. Furthermore, assume that f is exponentially bounded on S_δ, in the sense that there exist certain constants $M>0$ and $\tau>0$ for which

$$|f(z)|\ \leq\ M\exp\left(\tau|z|^2\right)$$

for all $z\in S_\delta$. Then,

(i) For each $\epsilon\in(0,\,(4\tau)^{-1/2})$,

$$\|\,f(\epsilon x)\,P_{i\leq n}\,\|\ <\ M\left[e^{1/4}\ +\ \left(\frac{32}{9}\right)^{1/4}\right]2^n.$$

(ii) For each $\epsilon \in (0, (8\tau)^{-1/2})$*,* $\eta \in (-\epsilon|x|, \epsilon|x|)$*, and any pair of integers* j *and* l *that satisfy* $j \geq 1$*,* $j + l \geq 0$*,*

$$\left\| \frac{f^{(j)}(\eta)}{j!}\, x^{j+l}\, P_{i\leq n} \right\| \;<\; \delta^{-j}\, M\, e^{2\tau\delta^2} \left[e^{1/4} + \left(\frac{32}{9}\right)^{1/4} \right] 2^{\frac{2n+j+l}{2}} \left[\frac{(n+j+l)!}{n!} \right]^{1/2}.$$

Proof: Set $R = \sqrt{2n+1}$. Let χ_R be the characteristic function of the interval $[-R, R]$. We have

$$\| f(\epsilon x)\, P_{i\leq n} \| \;\leq\; \| f(\epsilon x)\, [1 - \chi_R(x)]\, P_{i\leq n} \| \;+\; \| f(\epsilon x)\, \chi_R(x)\, P_{i\leq n} \|. \tag{6}$$

We begin by estimating the first term. Consider any harmonic oscillator eigenfunction ϕ_i with $0 \leq i \leq n$. We have

$$\| f(\epsilon x)\, [1 - \chi_R(x)]\, \phi_i(x) \|^2 \;=\; \int_R^\infty |f(\epsilon x)|^2\, |\phi_i(x)|^2\, dx \;+\; \int_{-\infty}^{-R} |f(\epsilon x)|^2\, |\phi_i(x)|^2\, dx.$$

By Lemma 3.1 of [5], the Hermite polynomials of order $i \leq n$ satisfy $|H_i(x)| \leq 2^i |x|^i$ whenever $|x| \geq R$. Therefore,

$$\begin{aligned}
\| f(\epsilon x)\, [1 - \chi_R(x)]\, \phi_i(x) \|^2 \;&\leq\; M^2\, \frac{2^{i+1}}{\pi^{1/2}\, i!} \int_R^\infty e^{-(1-2\tau\epsilon^2)x^2}\, x^{2i}\, dx \\
&<\; M^2\, \frac{2^{i+1}}{\pi^{1/2}\, i!} \int_0^\infty e^{-\frac{1}{2}x^2}\, x^{2i}\, dx \qquad (7) \\
&=\; \sqrt{\frac{2}{\pi}}\, M^2\, \frac{4^i}{i!}\, \Gamma(i + 1/2).
\end{aligned}$$

In (7), we use that $\epsilon^2 \leq (4\tau)^{-1}$ implies $1 - 2\tau\epsilon^2 \geq 1/2$. The Schwarz inequality implies

$$\| f(\epsilon x)\, [1 - \chi_R(x)]\, P_{i\leq n} \|^2 \;\leq\; \sqrt{\frac{2}{\pi}}\, M^2 \sum_{i=0}^{n} \frac{4^i}{i!}\, \Gamma(i + 1/2).$$

We now use the following crude approximation to estimate the sum in this expression:

$$\begin{aligned}
\sum_{i=0}^{n} \frac{4^i}{i!}\, \Gamma(i+1/2) \;&=\; \Gamma(1/2) \;+\; \sum_{i=1}^{n} \frac{4^i}{i!}\, \Gamma(i+1/2) \\
&<\; \sqrt{\pi} \;+\; \sum_{i=1}^{n} \frac{4^i}{i!}\, \Gamma(i+1) \\
&<\; \sqrt{\pi} \sum_{i=0}^{n} 4^i \\
&<\; \sqrt{\pi}\, \frac{4^{n+1}}{3}.
\end{aligned} \tag{8}$$

Therefore,

$$\| f(\epsilon x)\,[1-\chi_R(x)]\,P_{i\le n}\,\| \;<\; M\left(\frac{32}{9}\right)^{1/4} 2^n. \tag{9}$$

To estimate the second term in (6), we notice that the growth condition imposed to the function f implies that

$$|\, f(\epsilon x)\,\chi_R(x)\,| \;\le\; M\,e^{\tau\epsilon^2(2n+1)}$$

Hence,

$$\| f(\epsilon x)\,\chi_R(x)\,P_{i\le n}\,\| \;\le\; M\,e^{\tau\epsilon^2(2n+1)}\,\|P_{i\le n}\| \;\le\; M\,e^{(2n+1)/4}. \tag{10}$$

The inequality (i) then follows from (9) and (10).

To prove (ii), we proceed as in the proof of (i), with minor changes. As before, we have

$$\left\|\frac{f^{(j)}(\eta)}{j!}x^{j+l}P_{i\le n}\right\| \;\le\; \left\|\frac{f^{(j)}(\eta)}{j!}x^{j+l}[1-\chi_R(x)]P_{i\le n}\right\| \;+\; \left\|\frac{f^{(j)}(\eta)}{j!}x^{j+l}\chi_R(x)P_{i\le n}\right\|. \tag{11}$$

We apply the Cauchy integral formula to f and use the growth condition imposed on f to see that

$$\left|\frac{f^{(j)}(\eta)}{j!}\right| \;\le\; \delta^{-j}\,M\,e^{\tau(\epsilon|x|+\delta)^2} \;\le\; \delta^{-j}\,M\,e^{2\tau\delta^2}\,e^{2\tau\epsilon^2x^2}, \tag{12}$$

for $\eta\in(-\epsilon|x|,\,\epsilon|x|)$. We now use this inequality to estimate the first term in (11). We have

$$\begin{aligned}
\left\|\frac{f^{(j)}(\eta)}{j!}\,x^{j+l}\,[1-\chi_R(x)]\,\phi_i(x)\right\|^2 \;&\le\; \delta^{-2j}\,M^2\,e^{4\tau\delta^2}\,\frac{2^{i+1}}{\pi^{1/2}\,i!}\int_R^\infty e^{-(1-4\tau\epsilon^2)x^2}\,x^{2(i+j+l)}\,dx\\
&\le\; \delta^{-2j}\,M^2\,e^{4\tau\delta^2}\,\frac{2^{i+1}}{\pi^{1/2}\,i!}\int_0^\infty e^{-\frac{1}{2}x^2}\,x^{2(i+j+l)}\,dx\\
&=\; \sqrt{\frac{2}{\pi}}\,\delta^{-2j}\,M^2\,e^{4\tau\delta^2}\,\frac{2^{2i+j+l}}{i!}\,\Gamma(i+j+l+1/2).
\end{aligned}$$

Consequently,

$$\left\|\frac{f^{(j)}(\eta)}{j!}x^{j+l}[1-\chi_R(x)]P_{i\le n}\right\| \;\le\; \delta^{-j}Me^{2\tau\delta^2}\left(\frac{2}{\pi}\right)^{1/4} 2^{(j+l)/2}\left[\sum_{i=0}^n 4^i\,\frac{\Gamma(i+j+l+1/2)}{i!}\right]^{1/2}.$$

We now consider two cases to handle the sum in this expression. If $j+l=0$, we use the inequality (8). If $j+l\ge 1$, we have

$$\begin{aligned}
\sum_{i=0}^n 4^i\,\frac{\Gamma(i+j+l+1/2)}{i!} \;&<\; \sum_{i=0}^n 4^i\,\frac{(i+j+l)!}{i!}\\
&<\; \frac{(n+j+l)!}{n!}\sum_{i=0}^n 4^i\\
&<\; \frac{4}{3}\,4^n\,\frac{(n+j+l)!}{n!}.
\end{aligned}$$

When $j+l=0$, this inequality differs from (8) by a factor $\sqrt{\pi}$. Therefore, in general we have

$$\left\| \frac{f^{(j)}(\eta)}{j!}\, x^{j+l}\, [1-\chi_R(x)]\, P_{i\leq n} \right\| \;\leq\; \delta^{-j} M e^{2\tau\delta^2} \left(\frac{32}{9}\right)^{1/4} 2^{(2n+j+l)/2} \left[\frac{(n+j+l)!}{n!}\right]^{1/2}.$$

For the second term in (11) we have

$$\begin{aligned} \left\| \frac{f^{(j)}(\eta)}{j!}\, x^{j+l}\, \chi_R(x)\, P_{i\leq n} \right\| \;&\leq\; \delta^{-j}\, M\, e^{2\tau\delta^2}\, e^{2\tau\epsilon^2 R^2}\, \left\| x^{j+l} P_{i\leq n} \right\| && (13) \\ &\leq\; \delta^{-j}\, M\, e^{2\tau\delta^2}\, e^{(2n+1)/4}\, 2^{(j+l)/2} \left[\frac{(n+j+l)!}{n!}\right]^{1/2} && (14) \\ &<\; \delta^{-j}\, M\, e^{2\tau\delta^2}\, e^{1/4}\, 2^{(2n+j+l)/2} \left[\frac{(n+j+l)!}{n!}\right]^{1/2}, && (15) \end{aligned}$$

where we have used (12) to get (13), Lemma 5.1 of [6] to get (14), and the fact that $e^{1/2} < 2$ to finally obtain (15). □

4 Growth of the Perturbation Coefficients

In this section we prove that the coefficients of the formal Rayleigh–Schrödinger series for the energy $E(\epsilon)$ and the scalar functions $w(\epsilon x)$ and $v(\epsilon x)$ obey the following upper bounds.

Theorem 2 *Define $\kappa := 2/\delta^2$. There exist constants b_1, b_2, and b_3, such that for $b \geq \{b_1, b_2, b_3, 1\}$ and $n \geq 3$ we have*

$$\begin{aligned} |E_n| \;&<\; b_1\, \kappa^{3(n-2)}\, b^{2(n-3)}\, [(n+\alpha-2)!]^{1/2}, \\ \|\hat{w}_{n-2}\| \;&<\; b_2\, \kappa^{3(n-2)}\, b^{2(n-2)}\, [(n+\alpha-1)!]^{1/2}, \qquad \textit{and} \\ \|\hat{v}_n\| \;&<\; b_3\, \kappa^{3n}\, b^{2(n-1)}\, [(n+\alpha-1)!]^{1/2}. \end{aligned}$$

It follows that

$$\begin{aligned} |E_n| \;&<\; \kappa^{3(n-2)}\, b^{2n-5}\, [(n+\alpha-2)!]^{1/2}, \\ \|\hat{w}_{n-2}\| \;&<\; \kappa^{3(n-2)}\, b^{2n-3}\, [(n+\alpha-1)!]^{1/2}, \qquad \textit{and} \\ \|\hat{v}_n\| \;&<\; \kappa^{3n}\, b^{2n-1}\, [(n+\alpha-1)!]^{1/2}. \end{aligned}$$

We prove these estimates by mimicking the techniques used in [19] to study the semiclassical limit for the time-independent Schrödinger equation. We begin by using the recursive formulas of Section 2 to obtain the following set of inequalities:

$$\begin{aligned}
|E_n| \;\; &\le \;\; \sum_{m=0}^{n-3} |a_m| \, \|A \partial_x x^m A \phi_\alpha\| \, \|\hat{v}_{n-m-3}\| \; + \; \frac{1}{2} \sum_{m=0}^{n-4} |b_m| \, \|A x^m A \phi_\alpha\| \, \|\hat{w}_{n-m-4}\| \\
&\quad + \frac{1}{2} \sum_{m=0}^{n-4} |c_m| \, \|A x^m A \phi_\alpha\| \, \|\hat{v}_{n-m-4}\| \; + \; \sum_{m=3}^{n} |d_m| \, \|A x^m A \phi_\alpha\| \, \|\hat{w}_{n-m}\|,
\end{aligned}$$

$$\begin{aligned}
\|\hat{w}_{n-2}\| \;\; &\le \;\; \sum_{m=0}^{n-3} |a_m| \, \|A x^m \partial_x A \hat{v}_{n-m-3}\| \; + \; \frac{1}{2} \sum_{m=0}^{n-4} |b_m| \, \|A x^m A \hat{w}_{n-m-4}\| \\
&\quad + \frac{1}{2} \sum_{m=0}^{n-4} |c_m| \, \|A x^m A \hat{v}_{n-m-4}\| \; + \; \sum_{m=3}^{n} |d_m| \, \|A x^m A \hat{w}_{n-m}\| \; + \; \sum_{m=3}^{n} |E_m| \, \|\hat{w}_{n-m}\|,
\end{aligned}$$

$$\begin{aligned}
\|\hat{v}_n\| \;\; &\le \;\; |r_0|^{-1} \, \Bigg[\, \frac{1}{2} \, \|A^{-1} \partial_x^2 A \hat{v}_{n-2}\| \; + \; \sum_{m=0}^{n-3} |a_m| \, \|A^{-1} x^m \partial_x A \hat{w}_{n-m-3}\| \\
&\quad + \frac{1}{2} \sum_{m=0}^{n-4} |b_m| \, \|A^{-1} x^m A \hat{v}_{n-m-4}\| \; + \; \frac{1}{2} \sum_{m=0}^{n-4} |c_m| \, \|A^{-1} x^m A \hat{w}_{n-m-4}\| \\
&\quad + \sum_{m=1}^{n-3} |r_m| \, \|A^{-1} x^m A \hat{v}_{n-m}\| \; + \; \sum_{m=1}^{n-3} |E_m| \, \|\hat{v}_{n-m}\| \, \Bigg].
\end{aligned}$$

Because of hypothesis (4), we can use the Cauchy integral formula to show that there exists a constant $D > 0$ such that

$$\begin{aligned}
|a_n| \;\; &\le \;\; D \, \delta^{-n}, \\
|b_n| \;\; &\le \;\; (n+1) \, D \, \delta^{-n}, \\
|c_n| \;\; &\le \;\; (n+1) \, D \, \delta^{-n-1}, \\
|d_n| \;\; &\le \;\; D \, \delta^{-n}, \qquad \text{and} \\
|r_n| \;\; &\le \;\; D \, \delta^{-n},
\end{aligned}$$

for all $n \ge 0$. From the end of Section 2, we know that $\hat{w}_n \in \mathrm{Ran}(P_{i \le 3n+\alpha})$ and $\hat{v}_n \in \mathrm{Ran}(P_{i \le 3n+\alpha-8})$, the former for $n \ge 0$ and the latter for $n \ge 3$. Therefore, we have

$$\begin{aligned}
|E_n| \;\; &\le \;\; D \sum_{m=0}^{n-6} \delta^{-m} \, \|A \partial_x x^m A P_{i \le \alpha}\| \, \|\hat{v}_{n-m-3}\| \\
&\quad + \frac{D}{2} \sum_{m=0}^{n-4} (m+1) \, \delta^{-m} \, \|A x^m A P_{i \le \alpha}\| \, \|\hat{w}_{n-m-4}\| \\
&\quad + \frac{D}{2} \, \delta^{-1} \sum_{m=0}^{n-7} (m+1) \, \delta^{-m} \, \|A x^m A P_{i \le \alpha}\| \, \|\hat{v}_{n-m-4}\|
\end{aligned}$$

$$
\begin{aligned}
&\quad + D \sum_{m=3}^{n} \delta^{-m} \, \|Ax^m AP_{i\le\alpha}\| \, \|\hat{w}_{n-m}\|, \\
\|\hat{w}_{n-2}\| \;&\le\; D \sum_{m=0}^{n-6} \delta^{-m} \, \|Ax^m \partial_x AP_{i\le 3(n-m-3)+\alpha-8}\| \, \|\hat{v}_{n-m-3}\| \\
&\quad + \frac{D}{2} \sum_{m=0}^{n-4} (m+1) \, \delta^{-m} \, \|Ax^m AP_{i\le 3(n-m-4)+\alpha}\| \, \|\hat{w}_{n-m-4}\| \\
&\quad + \frac{D}{2} \, \delta^{-1} \sum_{m=0}^{n-7} (m+1) \, \delta^{-m} \, \|Ax^m AP_{i\le 3(n-m-4)+\alpha-8}\| \, \|\hat{v}_{n-m-4}\| \\
&\quad + D \sum_{m=3}^{n} \delta^{-m} \, \|Ax^m AP_{i\le 3(n-m)+\alpha}\| \, \|\hat{w}_{n-m}\| \;+\; \sum_{m=3}^{n} |E_m| \, \|\hat{w}_{n-m}\|, \\
\|\hat{v}_n\| \;&\le\; \frac{1}{2} \, |r_0|^{-1} \, \|A^{-1}\partial_x^2 AP_{i\le 3(n-2)+\alpha-8}\| \, \|\hat{v}_{n-2}\| \\
&\quad + D \, |r_0|^{-1} \sum_{m=0}^{n-3} \delta^{-m} \, \|A^{-1}x^m \partial_x AP_{i\le 3(n-m-3)+\alpha}\| \, \|\hat{w}_{n-m-3}\| \\
&\quad + \frac{D}{2} \, |r_0|^{-1} \sum_{m=0}^{n-7} (m+1) \, \delta^{-m} \, \|A^{-1}x^m AP_{i\le 3(n-m-4)+\alpha-8}\| \, \|\hat{v}_{n-m-4}\| \qquad (16) \\
&\quad + \frac{D}{2} \, |r_0|^{-1} \, \delta^{-1} \sum_{m=0}^{n-4} (m+1) \, \delta^{-m} \, \|A^{-1}x^m AP_{i\le 3(n-m-4)+\alpha}\| \, \|\hat{w}_{n-m-4}\| \\
&\quad + D \, |r_0|^{-1} \sum_{m=1}^{n-3} \delta^{-m} \, \|A^{-1}x^m AP_{i\le 3(n-m)+\alpha-8}\| \, \|\hat{v}_{n-m}\| \;+\; |r_0|^{-1} \sum_{m=1}^{n-3} |E_m| \, \|\hat{v}_{n-m}\|.
\end{aligned}
$$

To obtain these inequalities we have used $E_0 = E_1 = 0$ and $\hat{v}_0 = \hat{v}_1 = \hat{v}_2 = 0$. Also, empty sums whose upper limits are less than their lower limits are regarded as zero. We also regard the first term on the right hand side of (16) to be equal to zero for $n \le 4$.

Using Lemma 1 and recalling that $\kappa = 2/\delta^2$, we obtain

$$
\begin{aligned}
|E_n| \;&\le\; (2+\alpha) \, 2^{1/2} \, D \sum_{m=1}^{n-6} \kappa^{m/2} \left[\frac{(m+\alpha)!}{(\alpha+1)!} \right]^{1/2} \|\hat{v}_{n-m-3}\| \\
&\quad + (4+2\alpha)^{1/2} \, D \, \|\hat{v}_{n-3}\| \\
&\quad + (2+\alpha) \, 2^{-1} \, D \sum_{m=2}^{n-4} \kappa^{m/2} \, (m+1) \left[\frac{(m+\alpha-1)!}{(\alpha+1)!} \right]^{1/2} \|\hat{w}_{n-m-4}\| \\
&\quad + (2+\alpha)^{1/2} \, 2^{-1/2} \, D \, \delta^{-1} \sum_{m=0}^{1} (m+1) \, \|\hat{w}_{n-m-4}\|
\end{aligned}
$$

$$
\begin{aligned}
&+\,(2+\alpha)\,2^{-1}\,D\,\delta^{-1}\sum_{m=2}^{n-7}\kappa^{m/2}\,(m+1)\,\left[\frac{(m+\alpha-1)!}{(\alpha+1)!}\right]^{1/2}\|\hat{v}_{n-m-4}\|\\
&+\,(2+\alpha)^{1/2}\,2^{-1/2}\,D\,\delta^{-2}\sum_{m=0}^{1}(m+1)\,\|\hat{v}_{n-m-4}\|\\
&+\,(2+\alpha)\,D\sum_{m=3}^{n}\kappa^{m/2}\left[\frac{(m+\alpha-1)!}{(\alpha+1)!}\right]^{1/2}\|\hat{w}_{n-m}\|, \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
\|\hat{w}_{n-2}\| \;\leq\;& (2+\alpha)\,2^{1/2}\,D\sum_{m=1}^{n-6}\kappa^{m/2}\left[\frac{(3(n-3)-2m+\alpha-8)!}{(3(n-3)-3m+\alpha-7)!}\right]^{1/2}\|\hat{v}_{n-m-3}\|\\
&+\,(4+2\alpha)^{1/2}\,D\,\|\hat{v}_{n-3}\|\\
&+\,(2+\alpha)\,2^{-1}\,D\sum_{m=2}^{n-4}\kappa^{m/2}\,(m+1)\,\left[\frac{(3(n-4)-2m+\alpha-1)!}{(3(n-4)-3m+\alpha+1)!}\right]^{1/2}\|\hat{w}_{n-m-4}\|\\
&+\,(2+\alpha)^{1/2}\,2^{-1/2}\,D\,\delta^{-1}\sum_{m=0}^{1}(m+1)\,\|\hat{w}_{n-m-4}\|\\
&+\,(2+\alpha)\,2^{-1}\,D\,\delta^{-1}\sum_{m=2}^{n-7}\kappa^{m/2}\,(m+1)\left[\frac{(3(n-4)-2m+\alpha-9)!}{(3(n-4)-3m+\alpha-7)!}\right]^{1/2}\|\hat{v}_{n-m-4}\|\\
&+\,(2+\alpha)^{1/2}\,2^{-1/2}\,D\,\delta^{-2}\sum_{m=0}^{1}(m+1)\,\|\hat{v}_{n-m-4}\|\\
&+\,(2+\alpha)\,D\sum_{m=3}^{n}\kappa^{m/2}\left[\frac{(3n-2m+\alpha-1)!}{(3n-3m+\alpha+1)!}\right]^{1/2}\|\hat{w}_{n-m}\|\\
&+\,\sum_{m=3}^{n}|E_m|\,\|\hat{w}_{n-m}\|, \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
\|\hat{v}_{n}\| \;\leq\;& [\alpha(2+\alpha)]^{1/2}\,\frac{1}{|r_0|}\left[\frac{(3n+\alpha-11)!}{(3n+\alpha-13)!}\right]^{1/2}\|\hat{v}_{n-2}\|\\
&+\,[\alpha\,(4+2\alpha)]^{1/2}\,\frac{D}{|r_0|}\sum_{m=0}^{n-3}\kappa^{m/2}\left[\frac{(3(n-3)-2m+\alpha+2)!}{(3(n-3)-3m+\alpha+1)!}\right]^{1/2}\|\hat{w}_{n-m-3}\|\\
&+\,[\alpha\,(2+\alpha)]^{1/2}\,\frac{D}{2|r_0|}\sum_{m=0}^{n-7}\kappa^{m/2}\,(m+1)\left[\frac{(3(n-4)-2m+\alpha-7)!}{(3(n-4)-3m+\alpha-7)!}\right]^{1/2}\|\hat{v}_{n-m-4}\|\\
&+\,[\alpha\,(2+\alpha)]^{1/2}\,\frac{D}{2\delta|r_0|}\sum_{m=0}^{n-4}\kappa^{m/2}\,(m+1)\left[\frac{(3(n-4)-2m+\alpha+1)!}{(3(n-4)-3m+\alpha+1)!}\right]^{1/2}\|\hat{w}_{n-m-4}\|
\end{aligned}
$$

$$
\begin{aligned}
&+ \left[\alpha\,(2+\alpha)\right]^{1/2} \frac{D}{|r_0|} \sum_{m=1}^{n-3} \kappa^{m/2} \left[\frac{(3n-2m+\alpha-7)!}{(3n-3m+\alpha-7)!}\right]^{1/2} \|\hat{v}_{n-m}\| \\
&+ \frac{1}{|r_0|} \sum_{m=2}^{n-3} |E_m|\,\|\hat{v}_{n-m}\|. \qquad (19)
\end{aligned}
$$

Proof of Theorem 2: Since α is fixed, and we can increase b_1, b_2, and b_3 if necessary, we can easily prove Theorem 2 for $n = 3$. To prove the general result, we use induction and inequalities (17), (18), and (19).

Suppose the estimates are true for $m = 3, \cdots, n-1$, and consider first the bound (17) for $|E_n|$. Temporarily ignoring trivial factors, we have the following estimate for the first term in (17):

$$
\begin{aligned}
&\sum_{m=1}^{n-6} \kappa^{m/2} \left[\frac{(m+\alpha)!}{(\alpha+1)!}\right]^{1/2} \|\hat{v}_{n-m-3}\| \\
&\quad < \; b_3\,\kappa^{3(n-3)}\,b^{2(n-5)} \sum_{m=1}^{n-6} \kappa^{-5m/2} \left[\frac{((n-4)-m+\alpha)!\,(m+\alpha)!}{(\alpha+1)!}\right]^{1/2} \\
&\quad < \; b_3\,\kappa^{3(n-3)}\,b^{2(n-5)} \left[(n+\alpha-4)!\right]^{1/2} \sum_{m=1}^{n-4} \kappa^{-5m/2} \left[\frac{((n-4)-m+\alpha+1)!\,(m+\alpha+1)!}{(\alpha+1)!(n+\alpha-4)!}\right]^{1/2} \\
&\quad \le \; C_3\,\kappa^{3(n-3)}\,b^{2n-9} \left[(n+\alpha-4)!\right]^{1/2},
\end{aligned}
$$

where the last line follows from part (iii) of Lemma 2 and the inequality $b_3 \le b$.

We treat the last term in (17) in a similar fashion:

$$
\begin{aligned}
&\sum_{m=3}^{n} \kappa^{m/2} \left[\frac{(m+\alpha-1)!}{(\alpha+1)!}\right]^{1/2} \|\hat{w}_{n-m}\| \\
&\quad < \; b_2\,\kappa^{3n}\,b^{2(n-3)} \sum_{m=3}^{n} \kappa^{-5m/2} \left[\frac{((n-m+\alpha+1)!\,(m+\alpha-1)!}{(\alpha+1)!}\right]^{1/2} \\
&\quad \le \; b_2\,\kappa^{3n-5}\,b^{2(n-3)} \left[(n+\alpha-2)!\right]^{1/2} \sum_{l=1}^{n-2} \kappa^{-5l/2} \left[\frac{((n-2)-l+\alpha+1)!\,(l+\alpha+1)!}{(\alpha+1)!(n+\alpha-2)!}\right]^{1/2} \\
&\quad \le \; b_2 C_3\,\kappa^{3n-5}\,b^{2(n-3)} \left[(n+\alpha-2)!\right]^{1/2},
\end{aligned}
$$

where we have performed the change of index $l = m - 2$ in the third line above.

We now skip to the third term.

$$
\begin{aligned}
&\sum_{m=2}^{n-4} \kappa^{m/2}\,(m+1) \left[\frac{(m+\alpha-1)!}{(\alpha+1)!}\right]^{1/2} \|\hat{w}_{n-m-4}\| \\
&\quad < \; b_2\,\kappa^{3(n-4)}\,b^{2(n-6)}\,(n-3) \sum_{m=2}^{n-4} \kappa^{-5m/2} \left[\frac{((n-m+\alpha-3)!\,(m+\alpha-1)!}{(\alpha+1)!}\right]^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \; b_2\,\kappa^{3n-17}\,b^{2(n-6)}\,(n-3)\,\left[\frac{(n+\alpha-5)!}{(\alpha+1)!}\right]^{1/2} \\
&\quad \times \sum_{l=0}^{n-6} \kappa^{-5l/2}\,\left[\frac{((n-6)-l+\alpha+1)!\,(l+\alpha+1)!}{((n-6)+\alpha+1)!}\right]^{1/2} \\
&< \; C_1\,\kappa^{3n-17}\,b^{2n-11}\,\left[\frac{(n+\alpha-2)!}{(\alpha+1)!}\right]^{1/2}.
\end{aligned}
$$

The last inequality follows from part (i) of Lemma 2, along with the inequality $(n-3) < [(n+\alpha-3)(n+\alpha-2)]^{1/2}$.

Mimicking this calculation with minor changes, we obtain an estimate for the fifth term. The result is

$$
\sum_{m=2}^{n-7} \kappa^{m/2}\,(m+1)\,\left[\frac{(m+\alpha-1)!}{(\alpha+1)!}\right]^{1/2} \|\hat{v}_{n-m-4}\| \;<\; C_1\,\kappa^{3n-17}\,b^{2n-13}\,\left[\frac{(n+\alpha-2)!}{(\alpha+1)!}\right]^{1/2}.
$$

The second term satisfies

$$
\|\hat{v}_{n-3}\| \;<\; b_3\,\kappa^{3(n-3)}\,b^{2(n-4)}\,[(n+\alpha-4)!]^{1/2} \;<\; \kappa^{3(n-3)}\,b^{2n-7}\,[(n+\alpha-4)!]^{1/2}.
$$

Finally, because the estimates on $\|\hat{w}_n\|$ and $\|\hat{v}_n\|$ are increasing in n, the fourth and sixth terms satisfy

$$
\sum_{m=0}^{1} (m+1)\,\|\hat{w}_{n-m-4}\| \;<\; \|\hat{w}_{n-4}\| \,+\, 2\,\|\hat{w}_{n-5}\| \;<\; 3\,\kappa^{3(n-4)}\,b^{2n-7}\,[(n+\alpha-3)!]^{1/2},
$$

and

$$
\sum_{m=0}^{1} (m+1)\,\|\hat{v}_{n-m-4}\| \;<\; \|\hat{v}_{n-4}\| \,+\, 2\,\|\hat{v}_{n-5}\| \;<\; 3\,\kappa^{3(n-4)}\,b^{2n-9}\,[(n+\alpha-5)!]^{1/2}.
$$

Putting all these pieces together and including the trivial factors we ignored in each term, we get the bound for $|E_n|$ in the theorem if we define b_1 to be the maximum of its chosen value for $n=3$ and

$$
\begin{aligned}
&(2^{1/2}+b_2\kappa^4)\,(2+\alpha)\,\kappa^{-3}\,C_3\,D \,+\, (4+2\alpha)^{1/2}\,\kappa^{-3}\,D] \\
&\quad +\,(1+\delta^{-1})\,(2+\alpha)\,2^{-1}\,\kappa^{-11}\,[(\alpha+1)]^{-1/2}\,C_1\,D \,+\, 3\,(\delta^{-1}+\delta^{-2})\,(2+\alpha)^{1/2}\,2^{-1/2}\,\kappa^{-6}\,D.
\end{aligned}
$$

Note that our definition of b_1 depends on b_2. As we shall see below, b_2 does not depend on b_1 or b_3, and thus, b_1 is well defined.

We now turn to the bound on $\|\hat{w}_{n-2}\|$. For the sum in the first term in (18), we have

$$\begin{aligned}
&\sum_{m=1}^{n-6} \kappa^{m/2} \left[\frac{(3(n-3)-2m+\alpha-8)!}{(3(n-3)-3m+\alpha-7)!}\right]^{1/2} \|\hat{v}_{n-m-3}\| \\
&< \; b_3\, \kappa^{3(n-3)}\, b^{2(n-5)} \sum_{m=1}^{n-6} \kappa^{-5m/2} \left[\frac{(3(n-3)-2m+\alpha-8)!\,(n-m+\alpha-4)!}{(3(n-3)-3m+\alpha-7)!}\right]^{1/2} \\
&= \; b_3\, \kappa^{3(n-3)-5/2}\, b^{2(n-5)} \sum_{l=0}^{n-7} \kappa^{-5l/2} \left[\frac{(3(n-7)-2l+\alpha+2)!\,((n-7)-l+\alpha+2)!}{(3(n-7)-3l+\alpha+2)!}\right]^{1/2} \\
&\leq \; C_2\, \kappa^{3(n-3)-5/2}\, b^{2n-9}\, [(n+\alpha-5)!]^{1/2}.
\end{aligned}$$

We have made a change of index $l = m-1$ and have used part (ii) of Lemma 2 to obtain the last inequality.

We handle the seventh term in (18) in the same way. The result is

$$\sum_{m=3}^{n} \kappa^{m/2} \left[\frac{(3n-2m+\alpha-1)!}{(3n-3m+\alpha+1)!}\right]^{1/2} \|\hat{w}_{n-m}\| \; < \; C_2\, \kappa^{3n-5}\, b^{2n-5}\, [(n+\alpha-1)!]^{1/2}.$$

The sum in the third term is bounded as follows:

$$\begin{aligned}
&\sum_{m=2}^{n-4} \kappa^{m/2}\, (m+1) \left[\frac{(3(n-4)-2m+\alpha-1)!}{(3(n-4)-3m+\alpha+1)!}\right]^{1/2} \|\hat{w}_{n-m-4}\| \\
&< \; b_2\, \kappa^{3(n-4)}\, b^{2(n-6)}\, (n-3) \sum_{m=2}^{n-4} \kappa^{-5m/2} \left[\frac{(3(n-4)-2m+\alpha+1)!((n-4)-m+\alpha+1)!}{(3(n-4)-3m+\alpha+1)!}\right]^{1/2} \\
&\leq \; C_2\, b_2\, \kappa^{3(n-4)}\, b^{2(n-6)}\, (n-3)\, [(n+\alpha-3)!]^{1/2} \\
&< \; C_2\, \kappa^{3(n-4)}\, b^{2n-11}\, [(n+\alpha-1)!]^{1/2}.
\end{aligned}$$

A similar computation allows us to bound the sum in the fifth term. The result is

$$\sum_{m=2}^{n-7} \kappa^{m/2}\, (m+1) \left[\frac{(3(n-4)-2m+\alpha-9)!}{(3(n-4)-3m+\alpha-7)!}\right]^{1/2} \|\hat{v}_{n-m-4}\| \; < \; C_2\, \kappa^{3(n-4)}\, b^{2n-13}\, [(n+\alpha-3)!]^{1/2}.$$

The last term in (18) is bounded as follows:

$$\begin{aligned}
\sum_{m=3}^{n} |E_m|\, \|\hat{w}_{n-m}\| \; &< \; b_1\, b_2\, \kappa^{3(n-2)}\, b^{2(n-3)} \sum_{m=3}^{n} [(m+\alpha-2)!\,(n-m+\alpha+1)!]^{1/2} \\
&< \; b_1\, b_2\, \kappa^{3(n-2)}\, b^{2(n-3)} \sum_{l=1}^{n-2} [(l+\alpha+1)!\,((n-2)-l+\alpha+1)!]^{1/2} \\
&\leq \; C_1\, \kappa^{3(n-2)}\, b^{2(n-2)}\, [(n-\alpha-1)!]^{1/2}.
\end{aligned}$$

The last inequality follows after applying part (i) of Lemma 2.

The second, fourth, and sixth terms in (18) are identical to those in (17). Therefore, the bound for $\|\hat{w}_{n-2}\|$ in the theorem is proved if we define b_2 to be the maximum of its chosen value for $n = 3$ and

$$\begin{aligned} &C_1 \,+\, (4+2\alpha)^{1/2}\,\kappa^{-6}\,D \,+\, (2+\alpha)\,(1+2^{1/2}\kappa^{-13/2})\,\kappa\,C_2\,D \\ &\quad +\, 3\,(\delta^{-1}+\delta^{-2})\,(2+\alpha)^{1/2}\,2^{-1/2}\,\kappa^{-6}\,D \,+\, 2^{-1}\,(2+\alpha)\,(1+\delta^{-1})\,\kappa^{-6}\,C_2\,D. \end{aligned}$$

Finally, we verify the inequality for $\|\hat{v}_n\|$ in the theorem using inequality (19). We start with the first term. We have

$$\begin{aligned} &\left[\frac{(3n+\alpha-11)!}{(3n+\alpha-13)!}\right]^{1/2} \|\hat{v}_{n-2}\| \\ &\quad <\; b_3\,\kappa^{3(n-2)}\,b^{2(n-3)}\,[(n+\alpha-1)!]^{1/2} \sup_{n\ge 5}\left[\frac{(3n+\alpha-11)\,(3n+\alpha-12)}{(n+\alpha-1)\,(n+\alpha-2)}\right]^{1/2} \\ &\quad \le\; g_1\,\kappa^{3(n-2)}\,b^{2n-5}\,[(n+\alpha-1)!]^{1/2}, \end{aligned} \tag{20}$$

where g_1 stands for sup-value in (20).

For the sum in the second term, we have

$$\begin{aligned} &\sum_{m=0}^{n-3} \kappa^{m/2}\left[\frac{(3(n-3)-2m+\alpha+2)!}{(3(n-3)-3m+\alpha+1)!}\right]^{1/2} \|\hat{w}_{n-m-3}\| \\ &\quad <\; b_2\,\kappa^{3(n-3)}\,b^{2(n-3)} \sum_{m=0}^{n-3} \kappa^{-5m/2}\left[\frac{(3(n-3)-2m+\alpha+2)!\,(n-m+\alpha-2)!}{(3(n-3)-3m+\alpha+1)!}\right]^{1/2} \\ &\quad =\; b_2\,\kappa^{3(n-3)}\,b^{2(n-3)}\,(3(n-3)+\alpha+2)^{1/2} \\ &\quad\quad \times \sum_{m=0}^{n-3} \kappa^{-5m/2}\left[\frac{(3(n-3)-2m+\alpha+1)!\,((n-3)-m+\alpha+1)!}{(3(n-3)-3m+\alpha+1)!}\right]^{1/2} \\ &\quad \le\; C_2\,\kappa^{3(n-3)}\,b^{2n-5}\,[(n+\alpha-1)!]^{1/2} \sup_{n\ge 3}\left[\frac{(3(n-3)+\alpha+2)}{((n-3)+\alpha+2)}\right]^{1/2} \\ &\quad \le\; g_2\,C_2\,\kappa^{3(n-3)}\,b^{2n-5}\,[(n+\alpha-1)!]^{1/2}, \end{aligned} \tag{21}$$

where g_2 is defined by the sup-value in (21).

The fifth term can be treated in the same way. The bound for the sum in it is

$$\sum_{m=1}^{n-3} \kappa^{m/2}\left[\frac{(3n-2m+\alpha-7)!}{(3n-3m+\alpha-7)!}\right]^{1/2} \|\hat{v}_{n-m}\| \;<\; C_2\,\kappa^{3n}\,b^{2n-3}\,[(n+\alpha-1)!]^{1/2}.$$

The sums in third and fourth terms of inequality (19) are almost identical to those in the fifth and third terms of inequality (18). For that reason we just write down the corresponding bounds:

$$\sum_{m=0}^{n-7} \kappa^{m/2}\,(m+1)\,\left[\frac{(3(n-4)-2m+\alpha-7)!}{(3(n-4)-3m+\alpha-7)!}\right]^{1/2}\,\|\hat{v}_{n-m-4}\| \;<\; C_2\,\kappa^{3(n-4)}\,b^{2n-9}\,[(n+\alpha-2)!]^{1/2},$$

and

$$\sum_{m=0}^{n-4} \kappa^{m/2}\,(m+1)\,\left[\frac{(3(n-4)-2m+\alpha+1)!}{(3(n-4)-3m+\alpha+1)!}\right]^{1/2}\,\|\hat{w}_{n-m-4}\| \;<\; C_2\,\kappa^{3(n-4)}\,b^{2n-7}\,[(n+\alpha-1)!]^{1/2}.$$

To deal with the last term in (19), we have,

$$\begin{aligned}
\sum_{m=2}^{n-3} |E_m|\,\|\hat{v}_{n-m}\| \;&=\; |E_2|\,\|\hat{v}_{n-2}\| \;+\; \sum_{m=3}^{n-3} |E_m|\,\|\hat{v}_{n-m}\| \\
&<\; (\alpha+1/2)\,b_3\,\kappa^{3(n-2)}\,b^{2(n-3)}\,[(n+\alpha-3)!]^{1/2} \\
&\quad +\; b_1\,b_3\,\kappa^{3(n-2)}\,b^{2(n-4)} \sum_{m=3}^{n-3} [(m+\alpha-2)!\,(n-m+\alpha-1)!]^{1/2} \\
&\leq\; (\alpha+1/2)\,\kappa^{3(n-2)}\,b^{2n-5}\,[(n+\alpha-3)!]^{1/2} \\
&\quad +\; \kappa^{3(n-2)}\,b^{2(n-2)} \sum_{l=0}^{n-6} [(l+\alpha+1)!\,((n-5)-l+\alpha+1)!]^{1/2} \\
&<\; (\alpha+1/2+C_1)\,\kappa^{3(n-2)}\,b^{2(n-2)}\,[(n-\alpha-3)!]^{1/2}.
\end{aligned}$$

The estimate for $\|\hat{v}_n\|$ is then satisfied if we define b_3 to be the maximum of its chosen value for $n=3$ and

$$\begin{aligned}
&[\alpha(2+\alpha)]^{1/2}\,|r_0|^{-1}\,\kappa^{-6}\,g_1 \;+\; [\alpha(4+2\alpha)]^{1/2}\,|r_0|^{-1}\,\kappa^{-9}\,g_2\,C_2\,D \\
&\quad +\; [\alpha(2+\alpha)]^{1/2}\,|r_0|^{-1}\,C_2\,D \;+\; 2^{-1}\,(1+\delta^{-1})\,[\alpha(2+\alpha)]^{1/2}\,|r_0|^{-1}\,\kappa^{-12}\,C_2\,D \\
&\quad +\; (\alpha+1/2+C_1)\,|r_0|^{-1}\,\kappa^{-6}.
\end{aligned}$$

The theorem's first three inequalities (that we have now proved) immediately imply the second three inequalities in the theorem. □

5 Estimates for the Norm of the Error

The upper bounds we have proved for $|E_n|$, $\|\hat{w}_n\|$, and $\|\hat{v}_n\|$ in the previous section allow us to estimate the error made when truncated series are inserted into equation (2). To do this, we mimic the arguments in [18, 19], which are based on a technique developed in [6]. Concretely, for $N \geq 1$ define

$$
\begin{aligned}
E^N &:= \sum_{n=0}^{N+2} \epsilon^n E_n, \\
w^N &:= \sum_{n=0}^{N} \epsilon^n w_n, \\
v^N &:= \sum_{n=0}^{N+2} \epsilon^n v_n, \qquad \text{and} \\
\Psi^N &:= w^N \Phi_1 + v^N \Phi_2.
\end{aligned}
$$

Let $H(\epsilon)$ be the self–adjoint operator in equation (2). The error commited when truncated Rayleigh–Schrödinger series replace the actual solution to equation (2) is given by

$$
\left(H(\epsilon) - E^N\right) \Psi^N = S_N \Phi_1 + T_N \Phi_2, \tag{22}
$$

where the residual error terms S_N and T_N are given explicitly by

$$
\begin{aligned}
S_N &:= -\frac{\epsilon^2}{2} \partial_x^2 w^N - \epsilon^3 \Omega \partial_x v^N + \frac{\epsilon^4}{2} \Omega^2 w^N - \frac{\epsilon^4}{2} \Omega' v^N + \mathcal{E}_1 w^N - E^N w^N, \qquad \text{and} \\
T_N &:= -\frac{\epsilon^2}{2} \partial_x^2 v^N + \epsilon^3 \Omega \partial_x w^N + \frac{\epsilon^4}{2} \Omega^2 v^N + \frac{\epsilon^4}{2} \Omega' w^N + \mathcal{E}_2 v^N - E^N v^N.
\end{aligned}
$$

Since the perturbation coefficients are the solutions to the recursive equations computed in Section 2, all terms of order $n \leq N+2$ cancel in both S_N and T_N. By a computation that is described in the Appendix, the residual terms can be written as

$$
\begin{aligned}
S_N = & - \sum_{m=0}^{N} \epsilon^{m+3} \mathcal{A}^{[N-m-1]} \partial_x v_m - \epsilon^{N+4} \mathcal{A}^{[-1]} \partial_x v_{N+1} - \epsilon^{N+5} \mathcal{A}^{[-1]} \partial_x v_{N+2} \\
& + \frac{1}{2} \sum_{m=0}^{N-1} \epsilon^{m+4} \mathcal{B}^{[N-m-2]} w_m + \frac{\epsilon^{N+4}}{2} \mathcal{B}^{[-1]} w_N \\
& - \frac{1}{2} \sum_{m=0}^{N-1} \epsilon^{m+4} \mathcal{C}^{[N-m-2]} v_m - \frac{\epsilon^{N+4}}{2} \mathcal{C}^{[-1]} v_N - \frac{\epsilon^{N+5}}{2} \mathcal{C}^{[-1]} v_{N+1} - \frac{\epsilon^{N+6}}{2} \mathcal{C}^{[-1]} v_{N+2} \\
& + \sum_{m=0}^{N} \epsilon^m \mathcal{D}^{[N-m+2]} w_m \\
& - \sum_{l=N+3}^{2N+2} \epsilon^l \sum_{m=l-N-2}^{N} E_{l-m} w_m,
\end{aligned} \tag{23}
$$

and

$$
\begin{aligned}
T_N \;=\;& -\frac{1}{2}\,\epsilon^{N+3}\,\partial_x^2\,v_{N+1} \;-\; \frac{1}{2}\,\epsilon^{N+4}\,\partial_x^2\,v_{N+2} \\
& + \sum_{m=0}^{N} \epsilon^{m+3}\,\mathcal{A}^{[N-m-1]}\,\partial_x\,w_m \\
& + \frac{1}{2}\sum_{m=0}^{N-1} \epsilon^{m+4}\,\mathcal{B}^{[N-m-2]}\,v_m \;+\; \frac{\epsilon^{N+4}}{2}\,\mathcal{B}^{[-1]}\,v_N \;+\; \frac{\epsilon^{N+5}}{2}\,\mathcal{B}^{[-1]}\,v_{N+1} \;+\; \frac{\epsilon^{N+6}}{2}\,\mathcal{B}^{[-1]}\,v_{N+2} \\
& + \frac{1}{2}\sum_{m=0}^{N-1} \epsilon^{m+4}\,\mathcal{C}^{[N-m-2]}\,w_m \;+\; \frac{\epsilon^{N+4}}{2}\,\mathcal{C}^{[-1]}\,w_N \\
& + \sum_{m=0}^{N+2} \epsilon^{m}\,\mathcal{R}^{[N-m+2]}\,v_m \\
& - \sum_{l=N+3}^{2N+4} \epsilon^{l} \sum_{m=l-N-2}^{N+2} E_{l-m}\,v_m. \qquad (24)
\end{aligned}
$$

In these expressions we have used the following conventions.

$$
\begin{aligned}
\mathcal{A}(z) &= \Omega(z), && (25)\\
\mathcal{B}(z) &= \Omega(z)^2, && (26)\\
\mathcal{C}(z) &= \Omega'(z), && (27)\\
\mathcal{D}(z) &= \mathcal{E}_1(z), \qquad \text{and} && (28)\\
\mathcal{R}(z) &= \mathcal{E}_2(z). && (29)
\end{aligned}
$$

Also, we have used the following notation for exact Taylor series errors for any function $f(z)$ expanded around $z=0$:

$$
f^{[n]}(z) \;=\; \begin{cases} f(z) \;-\; \displaystyle\sum_{m=0}^{n} \frac{f^{(m)}}{m!}\,z^m & \text{if} \quad n \geq 0, \\[2ex] \qquad f(z) & \text{if} \quad n = -1. \end{cases}
$$

Note that in this notation, the integer n may never be less than -1, and that $f^{[n]}$ is $O(\epsilon^{n+1})$ since $z = \epsilon x$.

Our main result in this work relies on an upper bound for the L^2-norm of (22). Note that, since $\Phi_1(x)$ and $\Phi_2(x)$ are orthogonal for each $x \in \mathbb{R}$, we have

$$
\left\|\, \left(H(\epsilon) - E^N\right)\, \Psi^N \,\right\|^2 \;=\; \|S_N\|^2 \;+\; \|T_N\|^2\,.
$$

We recall that hypothesis (5) ensures that the absolute values of the functions (25)–(29) are bounded above by $M\exp(\tau|z|^2)$.

Theorem 3 *There exist constants G_1, G_2, F_1, and F_2 such that, for every $N \geq 3$ and $\epsilon \in (0,\,(8\tau)^{-1/2})$,*

$$\|S_N\| \;<\; \sum_{m=N+3}^{2N+2} G_1\, F_1^m\, \epsilon^m\, [(m+\alpha+1)]^{1/2}, \qquad \textit{and}$$

$$\|T_N\| \;<\; \sum_{m=N+3}^{2N+4} G_2\, F_2^m\, \epsilon^m\, [(m+\alpha+1)]^{1/2}.$$

As a consequence,

$$\left\| \left(H(\epsilon) - E^N\right)\, \Psi^N \right\| \;<\; \sum_{m=N+3}^{2N+4} G\, F^m\, \epsilon^m\, [(m+\alpha+1)]^{1/2},$$

for some constants G and F.

Proof: Recall that $w_m = A\hat{w}_m$, $v_m = A\hat{v}_m$, and that these perturbation coefficients belong to $\mathrm{Ran}(P_{i\leq 3m+\alpha})$ and $\mathrm{Ran}(P_{i\leq 3m+\alpha-8})$, respectively. Also, $E_0 = E_1 = 0$ and $\hat{v}_0 = \hat{v}_1 = \hat{v}_2 = 0$. In what follows, $\tilde{g}_i$, $\tilde{f}_i$, g_i, f_i, $\tilde{a}_i$, $\tilde{b}_i$, a_i, and b_i, with $i = 1, 2, \cdots$, denote certain constants that will appear along the way. We also regard all the empty sums as zero.

Taking norms in the expression for S_N, we get

$$\begin{aligned}
\|S_N\| \;\leq\; & \sum_{m=3}^{N-1} \epsilon^{m+3}\, \|\mathcal{A}^{[N-m-1]} P_{i\leq 3m+\alpha-7}\|\, \|\partial_x A\|\, \|\hat{v}_m\| \;+\; \epsilon^{N+3}\, \|\mathcal{A} P_{i\leq 3N+\alpha-7}\|\, \|\partial_x A\|\, \|\hat{v}_N\| \\
& +\; \epsilon^{N+4}\, \|\mathcal{A} P_{i\leq 3(N+1)+\alpha-7}\|\, \|\partial_x A\|\, \|\hat{v}_{N+1}\| \;+\; \epsilon^{N+5}\, \|\mathcal{A} P_{i\leq 3(N+2)+\alpha-7}\|\, \|\partial_x A\|\, \|\hat{v}_{N+2}\| \\
& +\; \frac{1}{2} \sum_{m=0}^{N-2} \epsilon^{m+4}\, \|\mathcal{B}^{[N-m-2]} A P_{i\leq 3m+\alpha}\|\, \|\hat{w}_m\| \\
& +\; \frac{\epsilon^{N+3}}{2}\, \|\mathcal{B} P_{i\leq 3(N-1)+\alpha}\|\, \|\hat{w}_{N-1}\| \;+\; \frac{\epsilon^{N+4}}{2}\, \|\mathcal{B} P_{i\leq 3N+\alpha}\|\, \|\hat{w}_N\| \\
& +\; \frac{1}{2} \sum_{m=3}^{N-2} \epsilon^{m+4}\, \|\mathcal{C}^{[N-m-2]} A P_{i\leq 3m+\alpha-8}\|\, \|\hat{v}_m\| \\
& +\; \frac{\epsilon^{N+3}}{2}\, \|\mathcal{C} P_{i\leq 3(N-1)+\alpha-8}\|\, \|\hat{v}_{N-1}\| \;+\; \frac{\epsilon^{N+4}}{2}\, \|\mathcal{C} P_{i\leq 3N+\alpha-8}\|\, \|\hat{v}_N\| \\
& +\; \frac{\epsilon^{N+5}}{2}\, \|\mathcal{C} P_{i\leq 3(N+1)+\alpha-8}\|\, \|\hat{v}_{N+1}\| \;+\; \frac{\epsilon^{N+6}}{2}\, \|\mathcal{C} P_{i\leq 3(N+2)+\alpha-8}\|\, \|\hat{v}_{N+2}\| \\
& +\; \sum_{m=0}^{N} \epsilon^m\, \|\mathcal{D}^{[N-m+2]} A P_{i\leq 3m+\alpha}\|\, \|\hat{w}_m\| \\
& +\; \sum_{m=N+3}^{2N+2} \epsilon^m \sum_{l=m-N}^{N-2} |E_l|\, \|\hat{w}_{m-l}\|.
\end{aligned} \tag{30}$$

Note that the operator A has been dropped in several terms.

We apply Taylor's theorem to the function $\mathcal{A}$, to see that

$$\mathcal{A}^{[N-m-1]}(\epsilon x) \;=\; \frac{\mathcal{A}^{(N-m)}(\eta)}{(N-n)!}\,\epsilon^{N-n}\,x^{N-n},$$

for some $\eta \in (-\epsilon|x|,\,\epsilon|x|)$. Also, from the proof of the Lemma 1, we know that

$$\|\partial_x A\| \;\leq\; (4+2\alpha)^{1/2},$$

and that the same bound holds for $\|xA\|$. Thus, the first term in (30) satisfies

$$\begin{aligned}&\sum_{m=3}^{N-1} \epsilon^{m+3}\,\|\mathcal{A}^{[N-m-1]}P_{i\leq 3m+\alpha-7}\|\,\|\partial_x A\|\,\|\hat{v}_m\| \\ &\quad\leq\; (4+2\alpha)^{1/2}\,\epsilon^{N+3}\sum_{m=3}^{N-1}\left\|\frac{\mathcal{A}^{(N-m)}(\eta)}{(N-m)!}\,x^{N-m}\,P_{i\leq 3m+\alpha-7}\right\|\,\|\hat{v}_m\|.\end{aligned}$$

We now apply part (ii) of Lemma 3 and the estimate for $\|\hat{v}_m\|$. This yields

$$\begin{aligned}&\sum_{m=3}^{N-1} \epsilon^{m+3}\,\|\mathcal{A}^{[N-m-1]}P_{i\leq 3m+\alpha-7}\|\,\|\partial_x A\|\,\|\hat{v}_m\| \\ &\quad<\; \epsilon^{N+3}\,\tilde{g}_1\,\tilde{f}_1^N\sum_{m=3}^{N-1}\kappa^{3m}\left[\frac{(N+2m+\alpha-7)!\,(m+\alpha-1)!}{(3m+\alpha-7)!}\right]^{1/2} \\ &\quad=\; \epsilon^{N+3}\,\tilde{g}_1\,\tilde{f}_1^N\,\kappa^{3N}\sum_{l=1}^{N-3}\kappa^{-3l}\left[\frac{(3(N-3)-2l+\alpha+2)!\,((N-3)-l+\alpha+2)!}{(3(N-3)-3l+\alpha+2)!}\right]^{1/2}.\end{aligned} \tag{31}$$

Now applying Lemma 2, we obtain

$$\sum_{m=3}^{N-1} \epsilon^{m+3}\,\|\mathcal{A}^{[N-m-1]}P_{i\leq 3m+\alpha-7}\|\,\|\partial_x A\|\,\|\hat{v}_m\| \;<\; g_1\,f_1^{N+3}\,\epsilon^{N+3}\,[(N+\alpha-1)!]^{1/2}.$$

The second term in inequality (30) can be estimated using part (i) of Lemma 3. This yields

$$\begin{aligned}&\epsilon^{N+3}\,\|\mathcal{A}P_{i\leq 3N+\alpha-7}\|\,\|\partial_x A\|\,\|\hat{v}_N\| \\ &\quad<\; \epsilon^{N+3}\,(4+2\alpha)^{1/2}\,M\left[e^{1/4}+\left(\frac{32}{9}\right)^{1/4}\right]2^{3N+\alpha-7}\,\kappa^{3N}\,b^{2N-1}\,[(N+\alpha-1)!]^{1/2} \\ &\quad=\; g_2\,f_2^{N+3}\,\epsilon^{N+3}\,[(N+\alpha-1)!]^{1/2},\end{aligned}$$

The third and fourth terms yield similar bounds. We obtain,

$$\epsilon^{N+4}\,\|\mathcal{A}P_{i\leq 3(N+1)+\alpha-7}\|\,\|\partial_x A\|\,\|\hat{v}_{N+1}\| \;<\; g_3\,f_3^{N+4}\,\epsilon^{N+4}\,[(N+\alpha)!]^{1/2},$$

and

$$\epsilon^{N+5}\,\|\mathcal{A}P_{i\leq 3(N+2)+\alpha-7}\|\,\|\partial_x A\|\,\|\hat{v}_{N+2}\| \;<\; g_4\,f_4^{N+5}\,\epsilon^{N+5}\,[(N+\alpha+1)!]^{1/2}.$$

We use similar arguments to treat the terms that involve the function $\mathcal{B}$. Thus, for the fifth term in (30), we have

$$\begin{aligned}
&\frac{1}{2}\sum_{m=0}^{N-2}\epsilon^{m+4}\,\|\mathcal{B}^{[N-m-2]}AP_{i\leq 3m+\alpha}\|\,\|\hat{w}_m\| \\
&\leq\; \frac{\epsilon^{N+3}}{2}\sum_{m=0}^{N-2}\left\|\frac{\mathcal{B}^{(N-m-1)}(\eta)}{(N-m-1)!}\,x^{N-m-2}\,P_{i\leq 3m+\alpha+1}\right\|\,\|xA\|\,\|\hat{w}_m\| \\
&<\; \epsilon^{N+3}\,\tilde{g}_2\,\tilde{f}_2^N\,\kappa^{3N}\sum_{l=2}^{N}\kappa^{-3l}\left[\frac{(3N-2l+\alpha+1)!\,(N-l+\alpha+1)!}{(3N-3l+\alpha+1)!}\right]^{1/2} \\
&<\; g_5\,f_5^{N+3}\,\epsilon^{N+3}\,[(N+\alpha+1)!]^{1/2}.
\end{aligned} \tag{32}$$

Also,

$$\frac{\epsilon^{N+3}}{2}\,\|\mathcal{B}P_{i\leq 3(N-1)+\alpha}\|\,\|\hat{w}_{N-1}\| \;<\; g_6\,f_6^{N+3}\,\epsilon^{N+3}\,[(N+\alpha)!]^{1/2},$$

and

$$\frac{\epsilon^{N+4}}{2}\,\|\mathcal{B}P_{i\leq 3N+\alpha}\|\,\|\hat{w}_N\| \;<\; g_7\,f_7^{N+4}\,\epsilon^{N+4}\,[(N+\alpha+1)!]^{1/2}.$$

The terms involving the function $\mathcal{C}$ in (30) can be handled in much the same way as we did with the previous ones. Thus,

$$\frac{1}{2}\sum_{m=3}^{N-2}\epsilon^{m+4}\,\|\mathcal{C}^{[N-m-2]}AP_{i\leq 3m+\alpha-8}\|\,\|\hat{v}_m\| \;<\; g_8\,f_8^{N+3}\,\epsilon^{N+3}\,[(N+\alpha-1)!]^{1/2},$$

$$\frac{\epsilon^{N+3}}{2}\,\|\mathcal{C}P_{i\leq 3(N-1)+\alpha-8}\|\,\|\hat{v}_{N-1}\| \;<\; g_9\,f_9^{N+3}\,\epsilon^{N+3}\,[(N+\alpha-2)!]^{1/2},$$

$$\frac{\epsilon^{N+4}}{2}\,\|\mathcal{C}P_{i\leq 3N+\alpha-8}\|\,\|\hat{v}_N\| \;<\; g_{10}\,f_{10}^{N+4}\,\epsilon^{N+4}\,[(N+\alpha-1)!]^{1/2},$$

$$\frac{\epsilon^{N+5}}{2}\,\|\mathcal{C}P_{i\leq 3(N+1)+\alpha-8}\|\,\|\hat{v}_{N+1}\| \;<\; g_{11}\,f_{11}^{N+5}\,\epsilon^{N+5}\,[(N+\alpha)!]^{1/2},$$

and

$$\frac{\epsilon^{N+6}}{2}\,\|\mathcal{C}P_{i\leq 3(N+2)+\alpha-8}\|\,\|\hat{v}_{N+2}\| \;<\; g_{12}\,f_{12}^{N+6}\,\epsilon^{N+6}\,[(N+\alpha+1)!]^{1/2}.$$

The next term in (30) is estimated as follows.

$$\begin{aligned}
&\sum_{m=0}^{N} \epsilon^m \, \|\mathcal{D}^{[N-m+2]} A P_{i\le 3m+\alpha}\| \, \|\hat{w}_m\| \\
&< \; \epsilon^{N+3} \, \tilde{g}_{13} \, \tilde{f}_{13}^N \, \kappa^{3N} \, \sum_{l=0}^{N} \kappa^{-3l} \left[\frac{(3N-2l+\alpha+3)! \, (N-l+\alpha+1)!}{(3N-3l+\alpha+1)!} \right]^{1/2} \\
&\le \; \epsilon^{N+3} \, \tilde{g}_{13} \, \tilde{f}_{13}^N \, \kappa^{3N} \, \max_{0\le l\le N} \left[\frac{(3N-3l+\alpha+3)(3N-3l+\alpha+2)}{(N-l+\alpha+3)(N-l+\alpha+2)} \right]^{1/2} \\
&\quad \times \; \sum_{l=0}^{N} \kappa^{-3l} \left[\frac{(3N-2l+\alpha+3)! \, (N-l+\alpha+3)!}{(3N-3l+\alpha+3)!} \right]^{1/2} \\
&\le \; \epsilon^{N+3} \, 3 \, C_2 \, \tilde{g}_{13} \, \tilde{f}_{13}^N \, \kappa^{3N} \, [(N+\alpha+3)!]^{1/2} \\
&= \; g_{13} \, f_{13}^{N+3} \, \epsilon^{N+3} \, [(N+\alpha+3)!]^{1/2}. \qquad (33)
\end{aligned}$$

Finally, for the last term in inequality (30), we have

$$\begin{aligned}
&\sum_{m=N+3}^{2N+2} \epsilon^m \sum_{l=m-N}^{N+2} |E_l| \, \|\hat{w}_{m-l}\| \\
&< \; \sum_{m=N+3}^{2N+2} \epsilon^m \, b^{2(m-2)} \, \kappa^{3(m-2)} \sum_{l=m-N}^{N-2} [(l+\alpha-2)! \, (m-l+\alpha+1)!]^{1/2} \\
&< \; \sum_{m=N+3}^{2N+2} \epsilon^m \, b^{2(m-2)} \, \kappa^{3(m-2)} \, [(m+\alpha+1)!]^{1/2} \\
&\quad \times \; \sum_{l=0}^{m} \left[\frac{(l+\alpha+1)! \, (m-l+\alpha+1)!}{(m+\alpha+1)!} \right]^{1/2} \\
&\le \; \sum_{m=N+3}^{2N+2} g_{14} \, f_{14}^m \, \epsilon^m \, [(m+\alpha+1)!]^{1/2}. \qquad (34)
\end{aligned}$$

The upper bound for $\|S_N\|$ in the theorem now follows if we define $G_1 := \max\{\, g_1, \cdots, g_{14} \,\}$ and $F_1 := \max\{\, f_1, \cdots, f_{14} \,\}$.

The proof for the bound for $\|T_N\|$ involves the same kind of manipulations as done with $\|S_N\|$. We start with the following inequality:

$$\begin{aligned}
\|T_N\| \; \le \; & \frac{\epsilon^{N+3}}{2} \, \|\partial_x P_{i\le 3(N+1)+\alpha-7}\| \, \|\partial_x A\| \, \|\hat{v}_{N+1}\| \; + \; \frac{\epsilon^{N+4}}{2} \, \|\partial_x P_{i\le 3(N+2)+\alpha-7}\| \, \|\partial_x A\| \, \|\hat{v}_{N+2}\| \\
& + \sum_{m=0}^{N-1} \epsilon^{m+3} \, \|\mathcal{A}^{[N-m-1]} P_{i\le 3m+\alpha+1}\| \, \|\partial_x A\| \, \|\hat{w}_m\| \; + \; \epsilon^{N+3} \, \|\mathcal{A} P_{i\le 3N+\alpha+1}\| \, \|\partial_x A\| \, \|\hat{w}_N\|
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2} \sum_{m=3}^{N-2} \epsilon^{m+4} \, \|\mathcal{B}^{[N-m-2]} A P_{i \le 3m+\alpha-8}\| \, \|\hat{v}_m\| \\
&+ \frac{\epsilon^{N+3}}{2} \, \|\mathcal{B} P_{i \le 3(N-1)+\alpha-8}\| \, \|\hat{v}_{N-1}\| \; + \; \frac{\epsilon^{N+4}}{2} \, \|\mathcal{B} P_{i \le 3N+\alpha-8}\| \, \|\hat{v}_N\| \\
&+ \frac{\epsilon^{N+5}}{2} \, \|\mathcal{B} P_{i \le 3(N+1)+\alpha-8}\| \, \|\hat{v}_{N+1}\| \; + \; \frac{\epsilon^{N+6}}{2} \, \|\mathcal{B} P_{i \le 3(N+2)+\alpha-8}\| \, \|\hat{v}_{N+2}\| \\
&+ \frac{1}{2} \sum_{m=0}^{N-2} \epsilon^{m+4} \, \|\mathcal{C}^{[N-m-2]} A P_{i \le 3m+\alpha}\| \, \|\hat{w}_m\| \\
&+ \frac{\epsilon^{N+3}}{2} \, \|\mathcal{C} P_{i \le 3(N-1)+\alpha}\| \, \|\hat{w}_{N-1}\| \; + \; \frac{\epsilon^{N+4}}{2} \, \|\mathcal{C} P_{i \le 3N+\alpha}\| \, \|\hat{w}_N\| \\
&+ \sum_{m=3}^{N+2} \epsilon^{m} \, \|\mathcal{R}^{[N-m+2]} A P_{i \le 3m+\alpha-8}\| \, \|\hat{v}_m\| \\
&+ \sum_{m=N+3}^{2N+4} \epsilon^{m} \sum_{l=m-N-2}^{N+2} |E_l| \, \|\hat{v}_{m-l}\|. \qquad (35)
\end{aligned}
$$

Since $\|\partial_x P_{i \le n}\| = [(n+1)/2]^{1/2}$, we have

$$
\begin{aligned}
\|\partial_x A\| \, \|\hat{v}_{N+1}\| &< \tilde{a}_1 \, \tilde{b}_1^N \, \epsilon^{n+3} \, [(3N+\alpha-3)(N+\alpha)!]^{1/2} \\
&\le \tilde{a}_1 \, \tilde{b}_1^N \, \epsilon^{n+3} \max_{N \ge 3} \left[\frac{3N+\alpha-3}{N+\alpha+1} \right]^{1/2} [(N+\alpha+1)!]^{1/2} \\
&\le a_1 \, b_1^{N+3} \, \epsilon^{N+3} \, [(N+\alpha+1)!]^{1/2}.
\end{aligned}
$$

Similarly,

$$
\frac{\epsilon^{N+4}}{2} \, \|\partial_x P_{i \le 3(N+2)+\alpha-8}\| \, \|\partial_x A\| \, \|\hat{v}_{N+2}\| \; < \; a_2 \, b_2^{N+4} \, \epsilon^{N+4} \, [(N+\alpha+2)!]^{1/2}.
$$

The estimate for the third term in (35) can essentially be read out from (31). The result is

$$
\sum_{m=0}^{N-1} \epsilon^{m+3} \, \|\mathcal{A}^{[N-m-1]} P_{i \le 3m+\alpha+1}\| \, \|\partial_x A\| \, \|\hat{w}_m\| \; < \; a_3 \, b_3^{N+3} \, \epsilon^{N+3} \, [(N+\alpha+1)!]^{1/2}.
$$

For the next term we have

$$
\epsilon^{N+3} \, \|\mathcal{A} P_{i \le 3N+\alpha+1}\| \, \|\partial_x A\| \, \|\hat{w}_N\| \; < \; a_4 \, b_4^{N+3} \, \epsilon^{N+3} \, [(N+\alpha+1)!]^{1/2}.
$$

The fifth term in (35) is quite similar to the one in (30). As in (32), by resorting to part (ii) of Lemma 3 and part (ii) of Lemma 2, we get

$$
\frac{1}{2} \sum_{m=3}^{N-2} \epsilon^{m+4} \, \|\mathcal{B}^{[N-m-2]} A P_{i \le 3m+\alpha-8}\| \, \|\hat{v}_m\| \; < \; a_5 \, b_5^{N+3} \, \epsilon^{N+3} \, [(N+\alpha-1)!]^{1/2}.
$$

The bounds for the other terms that involve the function $\mathcal{B}$ do not deserve much comment. We have

$$\frac{\epsilon^{N+3}}{2}\,\|\mathcal{B}P_{i\leq 3(N-1)+\alpha-8}\|\,\|\hat{v}_{N-1}\| \;<\; a_6\, b_6^{N+3}\,\epsilon^{N+3}\,[(N+\alpha-2)!]^{1/2},$$

$$\frac{\epsilon^{N+4}}{2}\,\|\mathcal{B}P_{i\leq 3N+\alpha-8}\|\,\|\hat{v}_{N}\| \;<\; a_7\, b_7^{N+4}\,\epsilon^{N+4}\,[(N+\alpha-1)!]^{1/2},$$

$$\frac{\epsilon^{N+5}}{2}\,\|\mathcal{B}P_{i\leq 3(N+1)+\alpha-8}\|\,\|\hat{v}_{N+1}\| \;<\; a_8\, b_8^{N+5}\,\epsilon^{N+5}\,[(N+\alpha)!]^{1/2},$$

and

$$\frac{\epsilon^{N+6}}{2}\,\|\mathcal{B}P_{i\leq 3(N+2)+\alpha-8}\|\,\|\hat{v}_{N+2}\| \;<\; a_9\, b_9^{N+6}\,\epsilon^{N+6}\,[(N+\alpha+1)!]^{1/2}.$$

Similarly,

$$\frac{1}{2}\sum_{m=0}^{N-2}\epsilon^{m+4}\,\|\mathcal{C}^{[N-m-2]}AP_{i\leq 3m+\alpha}\|\,\|\hat{w}_m\| \;<\; a_{10}\, b_{10}^{N+3}\,\epsilon^{N+3}\,[(N+\alpha+1)!]^{1/2},$$

$$\frac{\epsilon^{N+3}}{2}\,\|\mathcal{C}P_{i\leq 3(N-1)+\alpha}\|\,\|\hat{w}_{N-1}\| \;<\; a_{11}\, b_{11}^{N+3}\,\epsilon^{N+3}\,[(N+\alpha)!]^{1/2},$$

and

$$\frac{\epsilon^{N+4}}{2}\,\|\mathcal{C}P_{i\leq 3N+\alpha}\|\,\|\hat{w}_{N}\| \;<\; a_{12}\, b_{12}^{N+4}\,\epsilon^{N+4}\,[(N+\alpha+1)!]^{1/2}.$$

We handle the last two terms in (35) in much the same way that we handled (33) and (34). Thus,

$$\sum_{m=3}^{N+2}\epsilon^{m}\,\|\mathcal{R}^{[N-m+2]}AP_{i\leq 3m+\alpha-8}\|\,\|\hat{v}_m\| \;<\; a_{13}\, b_{13}^{N+3}\,\epsilon^{N+3}\,[(N+\alpha+1)!]^{1/2},$$

and

$$\sum_{m=N+3}^{2N+4}\epsilon^{m}\sum_{l=m-N-2}^{N+2}|E_l|\,\|\hat{v}_{m-l}\| \;<\; \sum_{m=N+3}^{2N+4} a_{14}\, b_{14}^{m}\,\epsilon^{m}\,[(m+\alpha+1)!]^{1/2}.$$

Putting all these estimates together, we obtain the bound on $\|T_N\|$ in the theorem if we define $G_2 := \max\{\, a_1,\, \cdots,\, a_{14}\,\}$ and $F_2 := \max\{\, b_1,\, \cdots,\, b_{14}\,\}$.

The last statement of the theorem follows immediately. □

By using exactly the same ideas as in the proof of theorem 4 in [19], Theorem 3 implies the following result, which is a slightly more detailed statement of Theorem 1.

Theorem 4 *Let F be the constant defined in Theorem 3. For each $0 < g < F^{-2}$, there is $\epsilon_g > 0$ such that for $0 < \epsilon \leq \epsilon_g$, there exists $N(\epsilon)$ that behaves like g/ϵ^2, such that*

$$\left\| \left(H(\epsilon) - E^{N(\epsilon)}\right) \Psi^{N(\epsilon)} \right\| < \Lambda \exp\left(-\frac{\Gamma}{\epsilon^2}\right)$$

for some constants $\Lambda > 0$ and $\Gamma > 0$ independent of ϵ.

6 Appendix

In this appendix we derive formulas (23) and (24) which are the differences between the two sides of the equations when the truncated Rayleigh–Schrödinger series are substituted into equations (3) and (4).

We truncate the expansions for w, v, and E at orders N, $N+2$, and $N+2$, respectively. We then insert them into equation (3) and expand various functions that appear in their Taylor series of appropriate orders. The order to which we expand depends on the power of ϵ that it multiplies, as will be seen below. The results of our perturbation calculations guarantee that all terms of order ϵ^n with $n \leq N+2$ cancel. The quantity S_N is simply the sum of the terms that do not have to cancel. Therefore, we simply need to do the algebra to see that S_N can be put into the form given in (23).

To simplify some expressions, we define the following expressions, all of which are actually equal to $f(z)$.

$$\tilde{f}^{[n]}(z) \;=\; \begin{cases} \displaystyle\sum_{m=0}^{n} \frac{f^{(m)}}{m!}\, z^m \;+\; f^{[n]} & \text{if} \quad n \geq 0, \\[2ex] f^{[-1]}(z) & \text{if} \quad n \leq -1. \end{cases}$$

We also regard all empty sums, *e.g.*, $\displaystyle\sum_{n=4}^{2} \cdots$ as zero.

Before the making use of the cancellations we write S_N as

$$\begin{aligned} &(H_0 - E_2) \sum_{m=0}^{N} \epsilon^{n+2}\, w_n \;-\; \sum_{m=0}^{N+2} \epsilon^{m+3}\, \tilde{\mathcal{A}}^{[N-m-1]}\, \partial_x v_m + \frac{1}{2} \sum_{m=0}^{N} \epsilon^{n+4}\, \tilde{\mathcal{B}}^{[N-m-2]}\, w_m \\ &- \frac{1}{2} \sum_{m=0}^{N+2} \epsilon^{n+4}\, \tilde{\mathcal{C}}^{[N-m-2]}\, v_m \;+\; \sum_{m=0}^{N} \epsilon^{m}\, \tilde{\mathcal{D}}^{[N-m+2]}\, w_m \;-\; \sum_{k=3}^{N+2} \epsilon^{k}\, E_k \sum_{m=0}^{N} \epsilon^{m}\, w_m. \end{aligned} \tag{36}$$

Note that we have used $E_0 = E_1 = 0$, and that we have combined the E_2 term with the first term. We now rewrite each term in this expression in a way that makes the ϵ dependence clearer, and then drop all terms of orders ϵ^n with $n \leq N+2$.

The first term is simply dropped.

The second term in (36) equals

$$-\sum_{m=0}^{N} \epsilon^{m+3}\,\mathcal{A}^{[N-m-1]}\,\partial_x\,v_m \;-\; \epsilon^{N+4}\,\mathcal{A}^{[-1]}\,\partial_x\,v_{N+1} \;-\; \epsilon^{N+5}\,\mathcal{A}^{[-1]}\,\partial_x\,v_{N+2}$$

$$-\sum_{m=0}^{N}\sum_{k=0}^{N-m-1} \epsilon^{m+k+3}\,a_k\,x^k\,\partial_x\,v_m.$$

The first three terms in this expression appear in (23), and the last term is dropped since all of its contributions are $O(\epsilon^n)$ with $n \le N+2$.

Similarly, the third term in (36) can be written as

$$\frac{1}{2}\sum_{m=0}^{N-1} \epsilon^{m+4}\,\mathcal{B}^{[N-m-2]}\,w_m \;+\; \frac{\epsilon^{N+4}}{2}\,\mathcal{B}^{[-1]}\,w_N \;+\; \frac{1}{2}\sum_{m=0}^{N-1}\sum_{k=0}^{N-m-2} \epsilon^{m+k+4}\,b_k\,x^k\,w_m.$$

The first two terms here appear in (23), and the last term is dropped since all of its contributions are $O(\epsilon^n)$ with $n \le N+2$.

The fourth term is handled by the same technique.

The fifth term equals

$$\sum_{m=0}^{N} \epsilon^{m}\,\mathcal{D}^{[N-m+2]}\,w_m \;+\; \sum_{m=0}^{N+2}\sum_{k=0}^{N-m+2} \epsilon^{m+k}\,d_k\,x^k\,w_m.$$

The first term here is included in (23), and the second is dropped.

The final term in (36) equals

$$\begin{aligned}
&\sum_{k=3}^{N+2}\sum_{m=0}^{N} \epsilon^{k+m}\,E_k\,w_m \\
=\;& \sum_{l=3}^{2N+2} \epsilon^{l} \sum_{m=\max\{0,\,l-N-2\}}^{\min\{N,\,l-3\}} E_{l-m}\,w_m \\
=\;& \sum_{l=3}^{N+2} \epsilon^{l} \sum_{m=0}^{l-3} E_{l-m}\,w_m \;+\; \sum_{l=N+3}^{2N+2} \epsilon^{l} \sum_{m=l-N-2}^{N} E_{l-m}\,w_m.
\end{aligned}$$

The second term in the final expression here is contained in (23), and the first term is dropped.

This finishes the derivation, since the terms we have not dropped are precisely those given in (23).

The derivation of (24) follows exactly the same procedure, so we omit the details.

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