# Amenability of the Gauge Group.

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**Abstract**

Let $\mathcal{G}$ be one of the local gauge groups $C(X,\,U(n))$, $C^\infty(X,\,U(n))$, $C(X,\,SU(n))$ or $C^\infty(X,\,SU(n))$ where $X$ is a compact Riemannian manifold. We observe that $\mathcal{G}$ has a nontrivial group topology, coarser than its natural topology, w.r.t. which it is amenable, viz the relative weak topology of $C(X,\,M(n))$. This topology seems more useful than other known amenable topologies for $\mathcal{G}$. We construct a simple fermionic model containing an action of $\mathcal{G}$, continuous w.r.t. this amenable topology.

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# 1 Introduction

It is still one of the main open problems of mathematical physics to consistently "quantise" the Yang–Mills gauge field theory. In fact, at a more general level, we do not have a mathematically consistent quantum theory of a fermion interacting with a quantised local gauge potential (abelian or nonabelian) in four dimensional space–time,- an important component of the standard model of particle physics. The classical version of this theory has been constructed and studied for some time, as well as semiclassical models, e.g. of a quantized fermion in a classical gauge potential.

Nevertheless, whatever form the final version of local quantum gauge field theory takes, it is plausible to assume that there will be a unital C*-algebra $\mathcal{F}$ corresponding to bounded functions of the smeared quantum fields, on which there is a pointwise continuous action of a topological group $\mathcal{G}$ (the "local gauge group") and where the physically relevant states on $\mathcal{F}$ are the $\mathcal{G}$–invariant states, which must include a physically reasonable vacuum state. In order to ensure that such invariant states exist, the question arises as to whether there is a natural topology in which the local gauge group $\mathcal{G}$ is amenable, and w.r.t. which the action on $\mathcal{F}$ is continuous. We answer this question in the affirmative in this letter.

In classical gauge theories there is a range of local gauge groups $\mathcal{G}$. Initially $\mathcal{G}$ is taken to be the group of bundle isomorphisms of a principal fibre bundle $E$ which project down to the trivial automorphism on the base $M$. The base space $M$ is usually space–time $\mathbb{R}^4$, or its one–point compactification, and here we will always take the latter choice. The fibre group $G$ of $E$ is a compact Lie group, usually taken to be $SU(n)$ or a product of unitary groups. Now $\mathcal{G}$ has an equivalent expression as a set of continuous maps from $E$ to $G$ with pointwise multiplication, so the local gauge groups are usually considered as such mapping groups [3]. If $E$ is trivial, we can in fact consider $\mathcal{G}$ as a group of continuous maps from $M$ to $G$. We take $\mathcal{G} \subseteq C(X,\, SU(n))$ where $X$ is a fixed compact Riemannian manifold. Since the gauge transformations intertwine with the time evolutions of the classical fields it acts upon, and these are determined by differential equations (involving a gauge field), the gauge transformations must also be in some class of differentiable functions. The choices for $\mathcal{G}$ which occur in the literature are $C^k(X,\, SU(n))$, $C^\infty(X,\, SU(n))$ and $H^k(X,\, SU(n)) \equiv$ k–Sobolev maps for $k$ sufficiently high [1]. Now each of these gauge groups comes with its own natural topology, e.g. $C^\infty(X,\, SU(n))$ has the topology of uniform convergence w.r.t. the differential seminorms in any chart of $X$. So, ideally we would like to know whether these gauge groups with their natural topologies are amenable. Since they are not locally compact, this is a difficult question.

There is some flexibility in the quantum theory as to what topology one chooses for the local gauge group $\mathcal{G}$. This is because physics only requires that the expectation values of physical observables are continuous w.r.t. physical transformation groups. Since $\mathcal{G}$ is usually factored out, being a symmetry of non-observable fields, it would seem that its topology can be changed at will without affecting the physical theory.

# 2 Amenability results.

Let $\mathcal{G}$ be a topological group.

**2.1 Definition** *A bounded function* $f : \mathcal{G} \to \mathbb{C}$ *is* **right uniformly continuous** *if* $\lim\limits_{x\to e} \|f_x - f\| = 0$ *where* $f_x(y) := f(xy)$ *for all* $y \in \mathcal{G}$*. Denote the space of these by* $\mathrm{RUC}(\mathcal{G})$*, and note that* $f_x \in \mathrm{RUC}(\mathcal{G})$ *whenever* $f \in \mathrm{RUC}(\mathcal{G})$*. A* **left invariant mean** *is a functional* $m \in \mathrm{RUC}(\mathcal{G})^*$ *such that* $m(1) = 1 = \|m\|$ *and* $m(f_x) = m(f)$ *for all* $x \in \mathcal{G}$*,* $f \in \mathrm{RUC}(\mathcal{G})$*. We say* $\mathcal{G}$ *is* **amenable** *if it has a left invariant mean.*

Since $\mathrm{RUC}(\mathcal{G})$ is a C*–algebra, the definition of the left invariant mean implies that $m$ is a positive functional, hence a state.

Recall that if $\mathcal{G}$ is amenable w.r.t. some topology, then it is amenable w.r.t. all topologies which are coarser than this one (i.e. have fewer open sets). Let now $\mathcal{G}$ be one of the gauge groups above, i.e. $C^k(X, SU(n))$, $C^\infty(X, SU(n))$ or $H^k(X, SU(n))$. When $\mathcal{G}$ is abelian, i.e. $n = 1$, then it is amenable, since all discrete abelian groups are amenable. If $n > 1$, then $\mathcal{G}$ with the discrete topology is NOT amenable. This is because it contains the group of constant functions, which is of course isomorphic to $SU(n) \supset SU(2)$ which contains the free group of two generators, and this is nonamenable (cf. Pier, Corr 14.3 [12]). However, there is a nontrivial topology on $\mathcal{G}$ w.r.t. which it is amenable:-

**2.2 Theorem** *Let $\mathcal{G} = C(X, U(n))$ where $X$ is a compact topological space, equipped with pointwise multiplication. Equip $\mathcal{G} \subset C(X, M(n))$ with the relative weak topology of the C*–algebra $C(X, M(n))$, i.e. the topology defined by the seminorms $p_\omega$, $\omega \in C(X, M(n))^*$ where $p_\omega(g) := |\omega(g)|$. Then $\mathcal{G}$ is a topological group which is amenable.*

*Proof:* This is an easy application of Paterson's Theorem 2 in [9]. By compactness of $X$, the C*–algebra $C(X, M(n)) \cong C(X) \otimes M(n)$ is unital and as it is the (unique) tensor product of a commutative C*–algebra with a finite dimensional C*–algebra, it is nuclear. Thus we have fulfilled the hypotheses of Paterson's Theorem 2 [9], hence the unitary group of $C(X, M(n))$ with the relative weak topology is an amenable topological group. However, the unitary group of $C(X, M(n))$ is just $\mathcal{G} = C(X, U(n))$ and so we are done. ■

The relative weak topology is coarser than the norm topology (hence than the smooth topology on the subgroup $C^\infty(X, U(n)))$ so we do not know whether $\mathcal{G}$ with the norm topology is amenable.

**2.3 Theorem** *Let $\mathcal{G} = C(X, U(n))$ where $X$ is a compact $k$–manifold, and $\mathcal{G}$ has the relative weak topology as above. Then the following subgroups of $\mathcal{G}$ with the relative topology are amenable:*

(i) $C^\infty(X, U(n))$

(ii) $C(X, SU(n))$

(iii) $C^\infty(X, SU(n))$.

*Proof:* (i) Now $C^\infty(X, U(n)) \subset C(X, U(n)) = \mathcal{G}$ is dense with respect to the norm topology, i.e. for each $g \in \mathcal{G}$ there is a sequence $\{h_n\} \subset C^\infty(X, U(n))$ such that $h_n$ converges to $g$ in norm. But then $h_n$ also converges to $g$ in the weak topology because the norm topology is finer than the weak topology, i.e. $C^\infty(X, U(n))$ is also dense w.r.t. the weak topology. Since $\mathcal{G}$ is amenable, it follows from Paterson Proposition 1 (p720) [9] that $C^\infty(X, U(n))$ is amenable.
(ii) Note that $Z(U(n)) = \mathbb{T}\,\mathbb{1}$ and that $SU(n)$ is a non–simply connected cover of $U(n)/\mathbb{T}$. In fact by $SU(n) \subset U(n) \to U(n)/\mathbb{T}$ we get a continuous surjective homomorphism $SU(n) \to U(n)/\mathbb{T}$ with kernel $\{\exp(i2\pi k/n) \cdot \mathbb{1} \mid k = 0, \ldots, n-1\}$. Since the centre $Z(\mathcal{G}) = C(X, \mathbb{T})$ is a closed subgroup, the factoring $\mathcal{G} \to \mathcal{G}/Z(\mathcal{G})$ is a continuous homomorphism, and as $\mathcal{G}$ is amenable, so is $\mathcal{G}/Z(\mathcal{G}) = C\left(X, U(n)/\mathbb{T}\right) =: L$ by a direct adaptation of the argument in the proof of Prop. 13.1, p118 of Pier [12]. The factoring map restricts to the subgroup $K := C(X, SU(n)) \subset \mathcal{G}$, denoted $\xi : K \to C\left(X, U(n)/\mathbb{T}\right) = L$ and we already know that its image $L$ is amenable. Its kernel $H := Z(\mathcal{G}) \cap C(X, SU(n))$ is Abelian, hence amenable, so $\xi$ is a homomorphism with an amenable range and kernel. It now follows from Lemma 2.4 below that $K = C(X, SU(n))$ is amenable.
(iii) This follows from (ii) by the same argument as (i). ■

**2.4 Lemma** *If $K$ is a topological group, and $H$ is a closed normal subgroup of $K$ such that both $H$ and $L := K/H$ are amenable, then $K$ is amenable.*

*Proof:* For locally compact groups, this is in Prop. 13.4, p119 of Pier [12] and Prop 1.13 of Paterson [10]. These proofs do not use the local compactness, so are valid in our case, but as they are a bit condensed we do here an expanded version.
Let $m_1$ (resp. $m_2$) be a left invariant mean of $H$ (resp. $L$). Let $f \in \mathrm{RUC}(K)$, and for $k \in K$ define $f^k_{(H)} := (f_k)\restriction H$. Now

$$\begin{aligned}\left\|(f^k_{(H)})_x - f^k_{(H)}\right\| &= \sup_{y\in H}|f_k(xy) - f_k(y)| \\ &= \sup_{y\in H}|f(kxk^{-1}ky) - f(ky)| \\ &\leq \sup_{z\in K}|f(kxk^{-1}z) - f(z)| \\ &= \|f_{kxk^{-1}} - f\| \to 0\end{aligned}$$

as $x \to e$, since $f \in \mathrm{RUC}(K)$. Thus $f^k_{(H)} \in \mathrm{RUC}(H)$ and so we can define $\varphi(k) := m_1(f^k_{(H)})$. Then

$$\varphi(hx) = m_1(f^{hx}_{(H)}) = m_1((f^x_{(H)})_h) = m_1(f^x_{(H)}) = \varphi(x)$$

for all $h \in H$. Thus $\varphi$ is constant on cosets $Hx$ so we can identify it with a bounded function $\widehat{\varphi}$ on $K/H = L$. Let $\{x_\nu\} \subset L$ be a net converging to $e$, then since $L$ has the factor topology, there is a convergent net $\{\check{x}_\nu\} \subset K$, $\check{x}_\nu \to e$ such that $\xi(\check{x}_\nu) = x_\nu$. Now

$$\begin{aligned}\|\widehat{\varphi}_{x_\nu} - \widehat{\varphi}\| &= \sup_{z\in L}|\widehat{\varphi}(x_\nu z) - \widehat{\varphi}(z)| \leq \sup_{k\in K}|\varphi(\check{x}_\nu k) - \varphi(k)| \\ &= \sup_{k\in K}|m_1\left(f^{\check{x}_\nu k}_{(H)} - f^k_{(H)}\right)| \leq \sup_{k\in K}\left\|f^{\check{x}_\nu k}_{(H)} - f^k_{(H)}\right\| \\ &= \sup_{k\in K}\sup_{h\in H}|f_{\check{x}_\nu}(kh) - f(kh)| = \sup_{y\in K}|f_{\check{x}_\nu}(y) - f(y)| \\ &= \|f_{\check{x}_\nu} - f\| \to 0\end{aligned}$$

as $\check{x}_\nu \to e$. Thus $\widehat{\varphi} \in \mathrm{RUC}(L)$, and so we can define $M(f) := m_2(\widehat{\varphi})$, and by linearity and boundedness we have that $M$ is a functional on $\mathrm{RUC}(K)$. For invariance, observe first that $(f_k)^x_{(H)} = (f_{kx})\restriction H = f^{kx}_{(H)}$ for $k \in K$, hence $m_1((f_k)^x_{(H)}) = m_1(f^{kx}_{(H)}) = \varphi(kx) = \varphi_k(x)$. Now for $h \in H$ we have $\varphi_{hk}(x) = \varphi(hkx) = \varphi(kx) = \varphi_k(x)$, hence the map $(k,\, x) \to \varphi_k(x)$ in the first entry depends only on the coset $Hk$. Then it follows that

$$\varphi_k(hx) = \varphi(khx) = \varphi((h^{-1}kh)x) = \varphi_{h^{-1}kh}(x) = \varphi_k(x)\ ,$$

hence $\varphi_k(x)$ is also constant on the coset $Hx$ so it factors to a map on $L$ : $\widehat{(\varphi_k)} = (\widehat{\varphi})_{\xi(k)}$. Now we have:

$$M(f_k) = m_2(\widehat{(\varphi_k)}) = m_2((\widehat{\varphi})_{\xi(k)}) = m_2(\widehat{\varphi}) = M(f)\ .$$

So we obtain an invariant mean $M$ for $K$, so it is amenable. ∎

The real usefulness of Theorem 2.3, lies in the following well-known facts:

**2.5 Theorem** *Let $\alpha : \mathcal{G} \to \mathrm{Aut}\,\mathcal{F}$ be a pointwise continuous action of a topological group $\mathcal{G}$ on an unital $C^*$–algebra $\mathcal{F}$.*

*(i) If $\mathcal{G}$ is amenable, there is a $\mathcal{G}$–invariant state on $\mathcal{F}$.*

*(ii) If there is a $\mathcal{G}$–invariant state $\omega$ on $\mathcal{F}$, then in its GNS–representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$, there is a strong–operator continuous unitary representation $U : \mathcal{G} \to \mathcal{B}(\mathcal{H}_\omega)$ such that $U_g\pi_\omega(A)U_g^{-1} = \pi_\omega(\alpha_g(A))$ for all $A \in \mathcal{F}$, and $U_g\Omega_\omega = \Omega_\omega$ for all $g \in \mathcal{G}$, where $\Omega_\omega$ is the GNS–cyclic vector.*

*Proof:* For completeness, here are the proofs.
(i) Choose a state $\omega$ on $\mathcal{F}$ and define for each $A \in \mathcal{F}$ a function $f^A(g) := \omega(\alpha_{g^{-1}}(A))$, $g \in \mathcal{G}$. Then $f^A$ is bounded and

$$\begin{aligned} \left\| f^A_x - f^A \right\| &= \sup_{g\in G} |f^A(xg) - f^A(g)| = \sup_{g\in G} |\omega\left(\alpha_{g^{-1}}(\alpha_{x^{-1}}(A) - A)\right)| \\ &\leq \|\alpha_{x^{-1}}(A) - A\| \to 0 \end{aligned}$$

as $x \to e$. Hence $f^A \in \mathrm{RUC}(\mathcal{G})$, and so we can define $\varphi(A) := m(f^A)$ where $m$ is the left invariant mean of $\mathcal{G}$. Then $\varphi$ is linear, positive and normalised, i.e. a state. Observe that

$$f^{\alpha_g(A)}(x) = \omega(\alpha_{x^{-1}}(\alpha_g(A))) = \omega\left(\alpha_{(g^{-1}x)^{-1}}(A)\right) = f^A(g^{-1}x) = f^A_{g^{-1}}(x)$$

hence $\varphi(\alpha_g(A)) = m(f^A_{g^{-1}}) = m(f^A) = \varphi(A)$, i.e. $\varphi$ is an invariant state.
(ii) Recall that $\mathcal{H}_\omega$ is the closure of the factor space $\mathcal{F}/N_\omega$ w.r.t. the Hilbert norm $\|\xi(A)\|^2 := \omega(A^*A)$ where $\xi : \mathcal{F} \to \mathcal{F}/N_\omega$ denotes the factor map, and $N_\omega := \{\, A \in \mathcal{F} \;\mid\; \omega(A^*A) = 0\,\}$ is the left kernel of $\omega$. Then $\pi_\omega(A)\xi(B) = \xi(AB)$ and $\Omega_\omega = \xi(\mathbb{1})$. Now since $\omega$ is $\mathcal{G}$–invariant, $\alpha_g$ preserves $N_\omega$, hence $U_g\xi(A) := \xi(\alpha_g(A))$ is well-defined, and extends to a unitary on $\mathcal{H}_\omega$ by invariance of $\omega$. That $g \to U_g$ is a homomorphism is clear, covariance follows from

$$U_g\pi_\omega(A)U_g^{-1}\xi(B) = \xi\left(\alpha_g(A\alpha_{g^{-1}}(B))\right) = \xi(\alpha_g(A)B) = \pi_\omega(\alpha_g(A))\xi(B)$$

and $U_g\Omega_\omega = \xi(\alpha_g(\mathbb{1})) = \xi(\mathbb{1}) = \Omega_\omega$ is clear. As for strong–operator continuity, by the inequality

$$\begin{aligned} \|U_g\psi - \psi\| &= \|U_g(\psi - \varphi) + U_g\varphi - \varphi + \varphi - \psi\| \\ &\leq 2\|\psi - \varphi\| + \|U_g\varphi - \varphi\| \end{aligned}$$

for $\psi \in \mathcal{H}_\omega$, $\varphi \in \mathcal{F}/N_\omega$, it suffices to verify strong–operator continuity of $U_g$ on the dense space $\mathcal{F}/N_\omega$. This now follows from

$$\|U_g\xi(A) - \xi(A)\|^2 = \|\xi(\alpha_g(A) - A)\|^2 = \omega((\alpha_g(A) - A)^*(\alpha_g(A) - A)) \leq \|\alpha_g(A) - A\|^2$$

and the pointwise norm continuity of the action. ■

Below we will construct a simple model of a Fermion with a continuous action of the amenable group $C(X,\, U(n))$ on it.

Note that since the relative weak topology is coarser than the norm topology on $C(X,\, SU(n))$, and the smooth topology on $C^\infty(X,\, SU(n))$, any continuous map w.r.t. the relative weak topology is in fact also continuous w.r.t. these two topologies. In particular, any continuous action or unitary representation of the groups in Theorem 2.3 w.r.t. the relative weak topology is also continuous w.r.t. the usual topologies.

In physical models of gauge field theory, we have an action of the local gauge group $\mathcal{G} \subseteq C(X,\, SU(n))$ where $X$ is compactified space–time on a C*-algebra $\mathcal{F}$, as well as an action of the translation group $\mathbb{R}^4$ on $\mathcal{F}$ which intertwines with the natural action of translations on $\mathcal{G}$. Specifically there is a smooth action $\beta : \mathbb{R}^4 \to \mathrm{Diff}\, X$ of the translation group, leaving the point at infinity invariant which produces the natural action of $\mathbb{R}^4$ on $\mathcal{G}$ by

$$(\alpha_{\mathbf{a}}(g))(x) := g(\beta_{\mathbf{a}}(x))\,, \qquad x \in X,\ \mathbf{a} \in \mathbb{R}^4,\ g \in \mathcal{G}\,.$$

This action is the restriction of the natural action of $\mathbb{R}^4$ on $C(X, M(n))$, hence pointwise continuous w.r.t. the relative weak topology on $\mathcal{G}$. Thus if we combine the actions on $\mathcal{F}$, we have in fact an action of the semidirect product $K := \mathcal{G} \underset{\alpha}{\times} \mathbb{R}^4$ on $\mathcal{F}$. Now $\mathcal{G}$ is an amenable closed normal subgroup of $K$, and as $K/\mathcal{G} = \mathbb{R}^4$ is amenable, it follows from Lemma 2.4 that $K$ is amenable. Thus, providing that the action of $K$ on $\mathcal{F}$ is pointwise continuous, there will be $K$–invariant states, hence gauge invariant and translation invariant vacua. The relative weak topology on $\mathcal{G}$ seems to be easier for construction of models where the action of $K$ on $\mathcal{F}$ is continuous.

Concerning the freedom which we have to adjust the topology of the local gauge group, if $\mathcal{G}$ acts on a quantum field on $X$ in a local way, then in physical applications we must have covariant unitary representations $U : \mathbb{R}^4 \to \mathcal{U}(\mathcal{H})$, $V : \mathcal{G} \to \mathcal{U}(\mathcal{H})$ with $U_{\mathbf{a}} V_g U_{\mathbf{a}}^* = V_{\alpha_{\mathbf{a}}(g)}$ . If we require that $\mathbf{a} \to U_{\mathbf{a}}$ is strong operator continuous, then so is the map $\mathbf{a} \to V_{\alpha_{\mathbf{a}}(g)}$ , which suggests that we should at least equip $\mathcal{G}$ with the finest group topology which makes the maps $\mathbf{a} \in \mathbb{R}^4 \to \alpha_{\mathbf{a}}(g)$ continuous. However, there is no strong argument why we should require the strong operator continuity for $\mathbf{a} \to U_{\mathbf{a}}$ since at this point the the system still contains nonphysical degrees of freedom. Only for the final gauge invariant theory will the requirement of strong operator continuity for representations of the translations be justified.

## 3 Relation to other work.

It has been known for some time from work of Baez [2] that there is a Hausdorff group topology, weaker than the $C^\infty$-topology, in which the gauge group $C^\infty(X, G)$ is amenable where $G$ is locally compact, and $X$ is a compact Riemannian $k$–manifold. However, for the case where $G$ is compact, Baez's construction results in the amenability of a weaker group topology that is precompact. The easiest way to see it is to notice that the 'cylinder functions' appearing on p.3 in his ArXiv preprint [2] by their very definition, factor through continuous functions on finite products of copies of $G$ , and thus are almost periodic functions, continuous with regard to the Bohr topology on the gauge group (the finest precompact topology coarser than the original one). But it is a well-known classical result that the space of almost periodic functions on a topological group supports a unique bi-invariant mean. Moreover this topology is not obviously natural from the viewpoint of gauge groups acting by automorphisms of operator algebras.

There has been another known weaker amenable group topology on the group $C^\infty(X, G)$ , which is the topology of convergence in measure. Specifically, if the group $G$ is locally compact amenable, then the group of measurable maps from the standard Borel space $X$ , equipped with a non-atomic probability measure $\mu$, to $G$ is *extremely amenable,* that is, has a fixed point in every compact space on which it acts continuously cf. Theorem 2.2 in [11]. In the case where $G$ is compact, this is a result obtained a long time ago independently by Glasner (published much later in [6]) and Furstenberg and B. Weiss (unpublished). If $G$ is a compact Lie group and the measure $\mu$ has full support, the topology of convergence in measure on the group of maps coincides with the $L^p$ -topology for any $1 \leq p < \infty$ . Because of extreme amenability of $C(X, G)$ in this topology, this group has no non-trivial finite dimensional unitary representations. This is because every such a representation is a continuous homomorphism to some $U(n)$ and leads to an action of $C(X, G)$ on $U(n)$ without fixed points. Thus, it is minimally almost periodic, and incompatible with the Bohr topology.

The relative weak topology on the group $C(X, U(n))$ is more interesting, because it is actually finer than both the Bohr topology and the topology of convergence in measure. This can be seen by applying the fact (used, for instance, by Paterson in [9]) that the weak topology on a C*-algebra $\mathcal{A}$ coincides with the ultraweak topology coming from the universal enveloping von Neumann algebra of $\mathcal{A}$. It follows that every finite-dimensional unitary representation of $C(X, U(n))$ is relatively weakly continuous, and therefore the Bohr topology on the unitary group of $\mathcal{A}$ is weaker than the relative weak topology. Further, the von Neumann algebra $\mathcal{M} := L^\infty(X, M(n))$ is an enveloping von Neumann algebra for $C(X, M(n))$ (under the natural embedding, where a non-atomic measure $\mu$ on $X$ is presumed fixed), and therefore the restriction of the embedding to the unitary group of $C(X, M(n))$ , that is, $C(X, U(n))$ , is continuous with respect to the relative weak topology on the first group and the ultraweak (i.e. $\sigma(\mathcal{M}, \mathcal{M}^*)$) topology on the second. However, it can be proved that the ultraweak topology on the group of measurable maps from $X$ to $U(n)$ is the topology of convergence in measure.

In addition, the relative weak topology on $C(X, U(n))$ is not precompact. To see this, recall that it coincides with the ultraweak topology coming from the universal enveloping von Neumann algebra of $C(X, U(n))$, and the unitary group of a von Neumann algebra $\mathcal{M}$ with the ultraweak topology is precompact if and only if $\mathcal{M}$ is purely atomic. (For the proof of this, one looks at maximal abelian von Neumann subalgebras and uses the decomposition of commutative Von Neumann algebras in terms of basic types, as in Sect. 9.4 of [8]). However, this is not the case with the universal enveloping von Neumann algebra of $C(X) \otimes M(n)$. Therefore, the group $C(X, U(n))$ is not precompact

Giordano and Pestov have pointed out to us that one can prove a stronger fact about the unitary group $C(X, U(n))$ with the relative weak topology: it is strongly amenable in the sense of Glasner [7] that is, every proximal continuous action of this group on a compact space has a fixed point. This observation follows if instead of Paterson's theorem, used by us in the proof of Theorem 2.2, one applies the stronger Corollary 3.7 in [5]. They also independently noted the example in Theorem 2.2 but never published it because they did not find any applications of this fact.

# 4 Example.

We construct a model of a Fermion on space–time with a local gauge transformation.

Let $X$ be the one–point compactification of space–time, and let $\mu$ be any measure on $X$. Consider the Hilbert space $\mathcal{H} := L^2(X, \mu, \mathbb{C}^n) = L^2(X, \mu) \otimes \mathbb{C}^n$ where $\mathbb{C}^n$ has its usual inner product $\overline{\mathbf{z}} \cdot \mathbf{z}$. Then there is a representation $\rho : C(X,\, M(n)) \to \mathcal{B}(\mathcal{H})$ by $(\rho(A)\psi)(x) := A(x)\psi(x)$ for all $A \in C(X,\, M(n))$, $\psi \in \mathcal{H}$ and $x \in X$. Observe that $\rho$ restricts to a unitary representation of the group $\mathcal{G} = C(X,\, U(n))$ on $\mathcal{H}$ which is clearly continuous w.r.t. the weak topology.

Let $\mathcal{F}(\mathcal{H})$ be the C*–algebra of the canonical anticommutation relations over $\mathcal{H}$, i.e. the simple C*–algebra generated by the set $\{\, a(\psi) \mid \psi \in \mathcal{H}\,\}$ satisfying the relations

$$\{a(\psi),\, a(\xi)\} = 0 \qquad \{a(\psi),\, a(\xi)^*\} = (\psi, \xi)\, \mathbb{1} \quad \forall\, \psi,\, \xi \in \mathcal{H}$$

where $\{\cdot, \cdot\}$ denotes the anticommutator, and such that $\psi \to a(\psi)$ is an antilinear map, cf. [4]. Then we obtain an action $\gamma : \mathcal{G} \to \mathrm{Aut}\, \mathcal{F}(\mathcal{H})$ by $\gamma_g(a(\psi)) := a(\rho(g)\psi)$ for $\psi \in \mathcal{H}$, $g \in \mathcal{G}$. Now

$$\|\gamma_g(a(\psi)) - a(\psi)\| = \|a(\rho(g)\psi - \psi)\| = \|\rho(g)\psi - \psi\|$$

from which it follows that $\gamma : \mathcal{G} \to \mathrm{Aut}\, \mathcal{F}(\mathcal{H})$ is a pointwise continuous action w.r.t. the relative weak topology of $\mathcal{G}$. Thus by Theorem 2.5, $\mathcal{F}(\mathcal{H})$ has a $\gamma$–invariant state. (Of course since the Fock state is clearly $\gamma$–invariant, this is not new). Note that when $\mu$ is absolutely continuous w.r.t. the Lebesgue measure, there is a unitary representation of the translation group $\mathbb{R}^4$ on $\mathcal{H}$, and this produces an action of $\mathbb{R}^4$ on $\mathcal{F}(\mathcal{H})$ which intertwines with the action of $\mathbb{R}^4$ on $\mathcal{G}$.

A natural way to extend this trivial example, is to add a classical gauge field by tensoring on to $\mathcal{F}(\mathcal{H})$ the C*–algebra $C_b(\mathcal{C})$, i.e. the algebra of bounded continuous functions on the space of connections $\mathcal{C}$ of a trivial principal $SU(n)$–bundle over $X$, where $\mathcal{C}$ is equipped with its natural topology. At this point we encounter a problem, in that the action of the gauge group $\mathcal{G} := C^\infty(X, SU(n))$ on $\mathcal{C}$ is continuous w.r.t. the smooth topology, but not w.r.t. the relative weak topology. The way to solve this is to observe that only the topology of the orbit space $\mathcal{C}/\mathcal{G}$ has physical significance, so we can change the topology on the orbits of $\mathcal{G}$ on $\mathcal{C}$. In particular we can change the topology to make the maps $g \in \mathcal{G} \to g \cdot A \in \mathcal{C}$ continuous w.r.t. the relative weak topology on $\mathcal{G}$, where $A$ ranges over $\mathcal{C}$. Thus one obtains an action of $\mathcal{G}$ on $C_b(\mathcal{C})$ (note that $C_b(\mathcal{C})$ changes with the topology on $\mathcal{C}$) which is continuous w.r.t. the relative weak topology, and combines with the action $\gamma$ above to give a continuous action on the tensor product.

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