Bifurcation of free vibrations for completely resonant wave equations

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Abstract: We prove existence of small amplitude, $2\pi/\omega$-periodic in time solutions of completely resonant nonlinear wave equations for any frequency ω belonging to a Cantor-like set of positive measure. The proof relies on a suitable Lyapunov-Schmidt decomposition and a variant of the Nash-Moser Implicit Function Theorem.

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1 Introduction and main result

We outline in this note recent results obtained in [4] on the existence of small amplitude, $2\pi/\omega$-periodic in time solutions of the *completely resonant* nonlinear wave equation

$$\begin{cases} u_{tt} - u_{xx} + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{1}$$

where the nonlinearity is analytic with respect to u for $|u|$ small. More precisely, we assume

(H) There is $\rho > 0$ such that $\forall (x,u) \in (0,\pi) \times (-\rho,\rho)$, $f(x,u) = \sum_{k=p}^{\infty} a_k(x) u^k$, $p \geq 2$, where $a_k \in H^1((0,\pi), \mathbf{R})$ and $\sum_{k=p}^{\infty} ||a_k||_{H^1} r^k < \infty$ for any $r \in (0,\rho)$.

We look for periodic solutions of (1) with frequency ω close to 1 in a set of *positive measure*.

Equation (1) is an infinite dimensional Hamiltonian system possessing an elliptic equilibrium at $u = 0$ with linear frequencies of small oscillations $\omega_j = j$, $\forall j = 1, 2, \ldots$ satisfying *infinitely many resonance* relations. Any solution $v = \sum_{j \geq 1} a_j \cos(jt + \theta_j) \sin(jx)$ of the linearized equation at $u = 0$,

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{2}$$

is 2π-periodic in time. For such reason equation (1) is called a *completely resonant* Hamiltonian PDE.

Existence of periodic solutions of *finite* dimensional Hamiltonian systems close to a completely resonant elliptic equilibrium has been proved by Weinstein, Moser and Fadell-Rabinowitz. The proofs are based on the classical Lyapunov-Schmidt decomposition which splits the problem in two equations: the so called *range equation*, solved through the standard Implicit Function Theorem, and the *bifurcation equation* solved via variational arguments.

For proving existence of small amplitude periodic solutions of completely resonant Hamiltonian PDEs like (1) two main difficulties must be overcome:

- (i) a "*small denominators*" problem which arises when solving the range equation;
- (ii) the presence of an *infinite dimensional* bifurcation equation: which solutions v of the linearized equation (2) can be continued to solutions of the nonlinear equation (1)?

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The appearance of the small denominators problem (i) is easily explained: the eigenvalues of the operator $\partial_{tt} - \partial_{xx}$ in the space of functions $u(t,x)$, $2\pi/\omega$-periodic in time and such that, say, $u(t,.) \in H_0^1(0,\pi)$ for all t, are $-\omega^2 l^2 + j^2$, $l \in \mathbf{Z}$, $j \geq 1$. Therefore, for almost every $\omega \in \mathbf{R}$, the eigenvalues accumulate to 0. As a consequence, for most ω, the inverse operator of $\partial_{tt} - \partial_{xx}$ is unbounded and the standard Implicit Function Theorem is not applicable.

The first existence results for small amplitude periodic solutions of (1) have been obtained in [8] for the specific nonlinearity $f(x,u) = u^3$ and periodic boundary conditions in x, and in [1] for $f(x,u) = u^3 + O(u^4)$, imposing a "strongly non-resonance" condition on the frequency ω satisfied in a *zero measure* set. For such ω's the spectrum of $\partial_{tt} - \partial_{xx}$ does not accumulate to 0 and so the small divisor problem (i) is bypassed. The bifurcation equation (problem (ii)) is solved proving that, for $f(x,u) = u^3$, the 0^{th}-order bifurcation equation possesses *non-degenerate* periodic solutions.

In [2]-[3], for the same set of strongly non-resonant frequencies, existence and multiplicity of periodic solutions has been proved for *any* nonlinearity $f(u)$. The novelty of [2]-[3] was to solve the bifurcation equation via a variational principle at fixed frequency which, jointly with min-max arguments, enables to find solutions of (1) as critical points of the Lagrangian action functional.

Unlike [1]-[2]-[3], a new feature of the results we present in this Note is that the set of frequencies ω for which we prove existence of $2\pi/\omega$-periodic in time solutions of (1) has positive measure.

Existence of periodic solutions for a positive measure set of frequencies has been proved in [5] in the case of periodic boundary conditions in x and for the specific nonlinearity $f(x,u) = u^3 + \sum_{4 \leq j \leq d} a_j(x) u^j$ where the $a_j(x)$ are trigonometric cosine polynomials in x. The nonlinear equation $u_{tt} - u_{xx} + u^3 = 0$ with periodic boundary conditions possesses a continuum of small amplitude, analytic and non-degenerate periodic solutions in the form of travelling waves $u(t,x) = \delta p_0(\omega t + x)$. With these properties at hand, the small divisors problem (i) is solved in [5] via a Nash-Moser Implicit function Theorem adapting the estimates of Craig-Wayne [6].

Recently, existence of periodic solutions of (1) for frequencies ω in a positive measure set has been proved in [7] using the Lindstedt series method for odd analytic nonlinearities $f(u) = au^3 + O(u^5)$ with $a \neq 0$. The need for the dominant term au^3 in the nonlinearity f relies, as in [1], in the way the infinite dimensional bifurcation equation is solved. The reason for which $f(u)$ must be odd is that the solutions are obtained as a sine-series in x, see the comments before Theorem 1.1.

In [4] we present a general method to prove existence of periodic solutions of the completely resonant wave equation (1) with Dirichlet boundary conditions, for not only a positive measure set of frequencies ω, but also for a *generic* nonlinearity $f(x,u)$ satisfying (H) (we underline we do not require the oddness assumption $f(-x,-u) = f(x,u)$), see *Theorem 1.1.*

Let's describe accurately our result. Normalizing the period to 2π, we look for solutions $u(t,x)$, 2π-periodic in time, of the equation

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{3}$$

in the real Hilbert space (which is actually a Banach algebra for $2s > 1$)

$$X_{\sigma,s} := \Big\{ u(t,x) = \sum_{l \in \mathbf{Z}} e^{ilt}\ u_l(x) \ \Big| \ u_l \in H_0^1((0,\pi),\mathbf{C}),\ \ \overline{u_l}(x) = u_{-l}(x)\ \forall l \in \mathbf{Z}, \text{ and } \ ||u||^2_{\sigma,s} := \sum_{l \in \mathbf{Z}} e^{2\sigma l}(l^{2s}+1)||u_l||^2_{H^1} < +\infty \Big\}.$$

For $\sigma > 0$ the space $X_{\sigma,s}$ is the space of all 2π-periodic in time functions with values in $H_0^1((0,\pi),\mathbf{R})$ which have a bounded analytic extension in the complex strip $|\mathrm{Im}\ t| < \sigma$ with trace function on $|\mathrm{Im}\ t| = \sigma$ belonging to $H^s(\mathbf{T}, H_0^1((0,\pi),\mathbf{C}))$

The space of the solutions of the linear equation $v_{tt} - v_{xx} = 0$ that belong to $X_{\sigma,s}$ is

$$V := \Big\{ v(t,x) = \sum_{l \geq 1} \Big(e^{ilt} u_l + e^{-ilt} \overline{u_l} \Big) \sin(lx) \ \Big| \ u_l \in \mathbf{C} \text{ and } ||v||^2_{\sigma,s} = \sum_{l \in \mathbf{Z}} e^{2\sigma l}(l^{2s}+1) l^2 |u_l|^2 < +\infty \Big\}.$$

Let $\varepsilon := \dfrac{\omega^2-1}{2}$. Instead of looking for solutions of (3) in a shrinking neighborhood of 0 it is a convenient devise to perform the rescaling $u \to \delta u$ with $\delta := |\varepsilon|^{1/p-1}$, obtaining

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + \varepsilon g(\delta, x, u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases}$$

where

$$g(\delta, x, u) := s^* \frac{f(x, \delta u)}{\delta^p} = s^* \Big(a_p(x) u^p + \delta a_{p+1}(x) u^{p+1} + \dots \Big)$$

with $s^* := \text{sign}(\varepsilon)$, namely $s^* = 1$ if $\omega \geq 1$ and $s^* = -1$ if $\omega < 1$. To fix the ideas, we shall consider here periodic solutions of frequency $\omega > 1$, so that $s^* = 1$ and $\omega = \sqrt{2\delta^{p-1}+1}$.

If we try to implement the usual Lyapunov-Schmidt reduction, i.e. to look for solutions $u = v + w$ with $v \in V$ and $w \in W := V^\perp$, we are led to solve the bifurcation equation (sometimes called the (Q)-equation) and the range equation (sometimes called the (P)-equation)

$$\begin{cases} -\Delta v = \Pi_V g(\delta, x, v + w) & (Q) \\ L_\omega w = \varepsilon \Pi_W g(\delta, x, v + w) & (P) \end{cases} \tag{4}$$

where

$$\Delta v := v_{xx} + v_{tt}, \qquad L_\omega := -\omega^2 \partial_{tt} + \partial_{xx}$$

and $\Pi_V : X_{\sigma,s} \to V$, $\Pi_W : X_{\sigma,s} \to W$ denote the projectors respectively on V and W.

Since V is infinite dimensional a difficulty arises in the application of the method of [6] in presence of small divisors : if $v \in V \cap X_{\sigma_0,s}$ then the solution $w(\delta, v)$ of the range equation, obtained with any Nash-Moser iteration scheme will have a lower regularity, e.g. $w(\delta, v) \in X_{\sigma_0/2,s}$. Therefore in solving next the bifurcation equation for $v \in V$, the best estimate we can obtain is $v \in V \cap X_{\sigma_0/2,s+2}$, which makes the scheme incoherent. Moreover we have to ensure that the 0^{th}-order bifurcation equation[2], i.e. the (Q)-equation for $\delta = 0$,

$$-\Delta v = \Pi_V \Big(a_p(x) v^p \Big) \tag{5}$$

has solutions $v \in V$ which are analytic, a necessary property to initiate an analytic Nash-Moser scheme (in [6] this problem does not arise since, dealing with *nonresonant* or *partially resonant* Hamiltonian PDEs like $u_{tt} - u_{xx} + a_1(x)u = f(x,u)$, the bifurcation equation is finite dimensional).

We overcome this difficulty thanks to a reduction to a *finite dimensional* bifurcation equation (on a subspace of V of dimension N independent of ω). This reduction can be implemented, in spite of the complete resonance of equation (1), thanks to the compactness of the operator $(-\Delta)^{-1}$.

We introduce a decomposition $V = V_1 \oplus V_2$ where

$$\begin{cases} V_1 := \Big\{ v \in V \mid v(t,x) = \sum_{l=1}^N \Big(e^{ilt} u_l + e^{-ilt} \overline{u_l} \Big) \sin(lx), \ u_l \in \mathbf{C} \Big\} \\ V_2 := \Big\{ v \in V \mid v(t,x) = \sum_{l \geq N+1} \Big(e^{ilt} u_l + e^{-ilt} \overline{u_l} \Big) \sin(lx), \ u_l \in \mathbf{C} \Big\} \end{cases}$$

Setting $v := v_1 + v_2$, with $v_1 \in V_1, v_2 \in V_2$, (4) is equivalent to

$$\begin{cases} -\Delta v_1 = \Pi_{V_1} g(\delta, x, v_1 + v_2 + w) & (Q_1) \\ -\Delta v_2 = \Pi_{V_2} g(\delta, x, v_1 + v_2 + w) & (Q_2) \\ L_\omega w = \varepsilon \Pi_W g(\delta, x, v_1 + v_2 + w) & (P) \end{cases} \tag{6}$$

where $\Pi_{V_i} : X_{\sigma,s} \to V_i$ $(i = 1,2)$, denote the orthogonal projectors on V_i $(i = 1,2)$.

Our strategy to find solutions of system (6) is the following. We solve first (*Step* 1) the (Q_2)-equation obtaining $v_2 = v_2(\delta, v_1, w) \in V_2 \cap X_{\sigma,s}$ by a standard Implicit Function Theorem provided we have chosen N large enough and σ small enough -depending on the nonlinearity f but *independent of* δ.

[2]We assume for simplicity of exposition that the right hand side $\Pi_V(a_p(x)v^p)$ is not identically equal to 0 in V. If not verified, the 0^{th}-order non-trivial bifurcation equation will involve the higher order terms of the nonlinearity, see [2].

Next (*Step* 2) we solve the (P)-equation obtaining $w = w(\delta, v_1) \in W \cap X_{\sigma/2,s}$ by means of a Nash-Moser Implicit Function Theorem for (δ, v_1) belonging to some Cantor-like set of parameters. A major role is played by the inversion of the *linearized operators*. Our approach -outlined in the next section- is much simpler than the ones usually employed and allows to deal nonlinearities which do NOT satisfy the oddness assumption $f(-x,-u) = -f(x,u)$. For this we develop $u(t,\cdot) \in H_0^1(0,\pi)$ in time-Fourier expansion only. Let us remark that $H_0^1(0,\pi)$ is the natural phase space to deal with Dirichlet boundary conditions instead of the usually employed spaces $\{u(x) = \sum_{j\geq 1} u_j \sin(jx) \mid \sum_j e^{2aj} j^{2\rho} |u_j|^2 < +\infty\}$, which force the nonlinearity f to be odd. We hope that the applicability of this technique can go far beyond the present results.

Finally (*Step* 3) we solve the *finite dimensional* (Q_1)-equation for a generic set of nonlinearities obtaining $v_1 = v_1(\delta) \in V_1$ for a set of δ's of positive measure.

In conclusion we prove:

Theorem 1.1 ([4]) *Consider the completely resonant nonlinear wave equation (1) where the nonlinearity* $f(x,u) = a_p(x)u^p + O(u^{p+1})$ *satisfies assumption* **(H)**.

There exists an open and dense set $\mathcal{A}_p$ *in* $H^1((0,\pi),\mathbf{R})$ *such that, for all* $a_p \in \mathcal{A}_p$*, there is* $\sigma > 0$ *and a smooth curve* $[0,\delta_0) \ni \delta \to u(\delta) \in X_{\sigma,s}$ *with the following properties:*

- (i) *There exists* $s^* \in \{-1, 1\}$ *and a Cantor set* $\mathcal{C}_{a_p} \subset [0,\delta_0)$ *satisfying*

$$\lim_{\eta\to 0^+} \frac{\mathrm{meas}(\mathcal{C}_{a_p} \cap (0,\eta))}{\eta} = 1 \tag{7}$$

  such that, for all $\delta \in \mathcal{C}_{a_p}$, $u(\delta)$ *is a* $2\pi/\omega$*-periodic in time solution of (1) with* $\omega = \sqrt{2s^*\delta^{p-1}+1}$*;*

- (ii) $||u(\delta) - \delta u_0||_{\sigma,s} = O(\delta^2)$ *for some* $u_0 \in V \backslash \{0\} \cap X_{\sigma,s}$.

The conclusions of the theorem hold true for any nonlinearity $f(x,u) = a_3 u^3 + \sum_{k\geq 4} a_k(x) u^k$, $a_3 \neq 0$, *with* $s^* = \mathrm{sign}(a_3)$.

2 Sketch of the proof

Step 1: solution of the (Q_2)-equation. The 0^{th}-order bifurcation equation (5) is the Euler-Lagrange equation of the functional $\Phi_0 : V \to \mathbf{R}$

$$\Phi_0(v) = \frac{||v||_{H_1}^2}{2} - \int_\Omega a_p(x) \frac{v^{p+1}}{p+1}\, dxdt, \qquad \Omega = (0,2\pi)\times(0,\pi). \tag{8}$$

Assume for definiteness there is $v \in V$ such that $\int_\Omega a_p(x) \frac{v^{p+1}}{p+1} > 0$ (if the integral is < 0 for some v we have to substitute $-a_p$ to a_p). Then Φ_0 possesses by the Mountain-pass Theorem a non-trivial critical set $K_0 := \{v \in V \mid \Phi_0'(v) = 0, \Phi_0(v) = c\}$ which is compact for the H_1-topology, see [2]. By a direct bootstrap argument any solution $v \in K_0$ of (5) belongs to $H^k(V)$, $\forall k \geq 0$ and therefore is C^∞. In particular the Mountain-Pass solutions of (5) satisfy the *a-priori estimate* $\sup_{v\in K_0} ||v||_{0,s+1} < R$ for some $0 < R < +\infty$.

Solutions of the (Q_2)-equation are the fixed points of the nonlinear operator $\mathcal{N}(\delta, v_1, w, \cdot) : V_2 \cap X_{\sigma,s} \to V_2 \cap X_{\sigma,s}$ defined by $\mathcal{N}(\delta, v_1, w, v_2) := (-\Delta)^{-1}\Pi_{V_2}\, g(\delta, x, v_1 + w + v_2)$. Using the *regularizing property* of $(-\Delta)^{-1}\Pi_2$ we can prove that $\mathcal{N}$ is a contraction and then solve the (Q_2)-equation in the space $V_2 \cap X_{\sigma,s}$ for N large enough and for $0 < \sigma < \overline{\sigma}$ (N and $\overline{\sigma}$ depend on R but *not on* δ).

Lemma 2.1 (Solution of the (Q_2)-equation) *There exist* $\overline{\sigma} > 0, N \in \mathbf{N}_+, \delta_0 > 0$ *such that,* $\forall 0 < \sigma < \overline{\sigma}$, $\forall ||v_1||_{0,s+1} \leq 2R$, $\forall ||w||_{\sigma,s} \leq 1$, $\forall |\delta| \leq \delta_0$, *there exists a unique* $v_2 = v_2(\delta, w, v_1) \in X_{\sigma,s}$ *with* $||v_2(\delta, w, v_1)||_{\sigma,s} \leq 1$ *which solves the* (Q_2)*-equation. Moreover* $v_2(\delta, w, v_1) \in X_{\sigma,s+2}$.

Lemma 2.1 implies, in particular, that any solution $v \in K_0$ of equation (5) is not only C^∞ but actually belongs to $X_{\sigma,s}$ and therefore is analytic in t (and hence in x).

Step 2: solution of the (P)-equation. By the previous step we are reduced to solve the (P)-equation with $v_2 = v_2(\delta, v_1, w)$, namely

$$L_\omega w = \varepsilon \Pi_W \Gamma(\delta, v_1, w) \tag{9}$$

where $\Gamma(\delta, v_1, w)(t,x) := g(\delta, x, v_1(t,x) + w(t,x) + v_2(\delta, v_1, w)(t,x))$.

The solution $w = w(\delta, v_1)$ of the (P)-equation (9) is obtained by means of a Nash-Moser Implicit Function Theorem for (δ, v_1) belonging to a Cantor-like set of parameters.

Consider the orthogonal splitting $W = W^{(p)} \oplus W^{(p)\perp}$ where $W^{(p)} = \{ w \in W \mid w = \sum_{l=0}^{L_p} e^{ilt} \ w_l(x) \}$, $W^{(p)\perp} = \{ w \in W \mid w = \sum_{l > L_p} e^{ilt} \ w_l(x) \}$ and $L_p = L_0 2^p$ for some large $L_0 \in \mathbf{N}$. We denote by $P_p : W \to W^{(p)}$, $P_p^\perp : W \to W^{(p)\perp}$ the orthogonal projectors onto $W^{(p)}$, $W^{(p)\perp}$. Define $\sigma_0 := \overline{\sigma}$, the "loss of analyticity at step p" $\gamma_p := \gamma_0/(p^2+1)$ and $\sigma_{p+1} = \sigma_p - \gamma_p$, $\forall \ p \geq 0$, with $\gamma_0 > 0$ small enough, such that the "total loss of analyticity" $\sum_{p \geq 0} \gamma_p = \gamma_0 \sum_{p \geq 0} 1/(p^2+1) \leq \overline{\sigma}/2$.

Proposition 2.1 (Nash-Moser iteration scheme) *Let $w_0 = 0$ and $A_0 := \{ (\delta, v_1) \mid |\delta| < \delta_0, ||v_1||_{0,s+1} \leq 2R \}$. There exist $\varepsilon_0, L_0 > 0$ such that $\forall |\varepsilon| < \varepsilon_0$, there exists a sequence $\{w_p\}_{p \geq 0}$, $w_p = w_p(\delta, v_1) \in W^{(p)}$, of solutions of*

$$(P_p) \qquad L_\omega w_p - \varepsilon P_p \Pi_W \Gamma(\delta, v_1, w_p) = 0,$$

defined for $(\delta, v_1) \in A_p \subseteq A_{p-1} \subseteq \ldots \subseteq A_1 \subseteq A_0$. For $(\delta, v_1) \in A_\infty := \cap_{p \geq 0} A_p$, $w_p(\delta, v_1)$ totally converges in $X_{\overline{\sigma}/2}$ to a solution $w(\delta, v_1)$ of the (P)-equation (9) with $||w(\delta, v_1)||_{\overline{\sigma}/2, s} = O(\varepsilon)$.

Moreover it is possible to define $w(\delta, v_1)$ in a smooth way on the whole A_0: there exists a function $\widetilde{w}(\delta, v_1) \in C^\infty(A_0, W)$ and a Cantor-like set $B_\infty \subset A_\infty$ such that, if $(\delta, v_1) \in B_\infty \subset A_\infty$ then $\widetilde{w}(\delta, v_1)$ solves the (P)-equation (9).

Of course, the above proposition does not mean very much if we do not specify A_∞ or B_∞. We refer to (12) for the definiton of A_p and just say that the set B_∞ is sufficiently large for our purpose.

The real core of the Nash-Moser convergence proof -and where the analysis of the small divisors enters into play- is the proof of the invertibility of the linearized operator

$$\begin{aligned} \mathcal{L}_p(\delta, v_1, w)[h] &:= L_\omega h - \varepsilon P_p \Pi_W D_w \Gamma(\delta, v_1, w)[h] \\ &= L_\omega h - \varepsilon P_p \Pi_W \Big(\partial_u g(\delta, x, v_1 + w + v_2(\delta, v_1, w)) \Big[h + \partial_w v_2(\delta, v_1, w)[h] \Big] \Big), \end{aligned}$$

where w is the approximate solution obtained at a given stage of the Nash-Moser iteration. We do not follow the approach of [6] which is based on the Fröhlich-Spencer techniques.

To invert $\mathcal{L}_p(\delta, v_1, w)$, we distinguish a "diagonal part" D. Let

$$\begin{cases} a(t,x) := \partial_u g(\delta, x, v_1(t,x) + w(t,x) + v_2(v_1, w)(t,x)) \\ a_0(x) := (1/2\pi) \int_0^{2\pi} a(t,x) \ dt \\ \overline{a}(t,x) := a(t,x) - a_0(x). \end{cases}$$

We can write

$$\mathcal{L}_p(\delta, v_1, w)[h] = Dh - M_1 h - M_2 h,$$

where D, M_1, $M_2 : W^{(p)} \to W^{(p)}$ are the linear operators

$$\begin{cases} Dh := L_\omega h - \varepsilon P_p \Pi_W (a_0 \ h) \\ M_1 h := \varepsilon P_p \Pi_W (\overline{a} \ h) \\ M_2 h := \varepsilon P_p \Pi_W (a \ \partial_w v_2[h]). \end{cases} \tag{10}$$

We next diagonalize the operator D using Sturm-Liouville spectral theory. We find out that the eigenvalues of D are $\omega^2 k^2 - \lambda_{k,j}$, $\forall |k| \leq L_p$, $j \geq 1$, $j \neq k$, and $\lambda_{k,j}$ satisfies the asympotic expansion

$$\lambda_{k,j} = \lambda_{k,j}(\delta, v_1, w) = j^2 + \varepsilon M(\delta, v_1, w) + O\Big(\frac{\varepsilon ||a_0||_{H^1}}{j} \Big) \qquad \text{as} \qquad j \to +\infty, \tag{11}$$

where $M(\delta, v_1, w) := (1/\pi) \int_0^\pi a_0(x)\ dx$.

Assuming, for some $\gamma > 0$ and $1 < \tau < 2$, the Diophantine condition (first order Melnikov condition)

$$(\delta, v_1) \in A_p \quad := \quad \Big\{ (\delta, v_1) \in A_{p-1} \ \Big| \ |\omega k - j| \geq \frac{\gamma}{(k+j)^\tau}, \ \Big|\omega k - j - \varepsilon \frac{M(\delta, v_1, w)}{2j}\Big| \geq \frac{\gamma}{(k+j)^\tau}, \\ \forall k \in \mathbf{N}, \ j \geq 1 \text{ s.t. } k \neq j, \ \frac{1}{3|\varepsilon|} < k, \ j \leq L_p \Big\} \subset A_{p-1}, \tag{12}$$

all the eigenvalues of D are *polynomially* bounded away from 0, since $\alpha_k := \min_{j \neq k, j \geq 1} |\omega^2 k^2 - \lambda_{k,j}| \geq \gamma / k^{\tau - 1}$, $\forall k$. Therefore D is invertible and D^{-1} has sufficiently good estimates for the convergence of the Nash-Moser iteration.

It remains to prove that the perturbative operators M_1, M_2 are small enough to get the invertibility of the whole $\mathcal{L}_p$. The smallness of M_2 is just a consequence of the regularizing property of $v_2 : X_{\sigma,s} \to X_{\sigma,s+2}$ stated in Lemma 2.1. The smallness of M_1 requires, on the contrary, an analysis of the "*small divisors*" α_k. For our method it is sufficient simply to prove that

$$\alpha_k \alpha_l \geq c|\varepsilon|^{\tau-1} > 0, \qquad \forall k \neq l \quad \text{with} \quad |k - l| \leq [\max\{k, l\}]^{2-\tau/\tau}.$$

We underline again that this approach works perfectly well for NOT odd nonlinearities f.

Step 3: solution of the (Q_1)-equation. Finally we have to solve the equation

$$(Q_1) \qquad\qquad -\Delta v_1 = \Pi_{V_1} \mathcal{G}(\delta, v_1)$$

where $\mathcal{G}(\delta, v_1)(t,x) := g(\delta, x, v_1(t,x) + \widetilde{w}(\delta, v_1)(t,x) + v_2(\delta, v_1, \widetilde{w}(\delta, v_1))(t,x))$ and to ensure that there are solutions $(\delta, v_1) \in B_\infty$ for δ in a set of positive measure (recall that if $(\delta, v_1) \in B_\infty \subset A_\infty$, then $\widetilde{w}(\delta, v_1)$ solves the (P)-equation (9)). Note that if $\omega = (1 + 2\delta^{p-1})^{1/2}$ belongs to the zero measure set of "strongly non-resonant" frequencies used in [2]-[3] then $(\delta, v_1) \in B_\infty$, $\forall v_1 \in V_1$ small enough.

The finite dimensional 0^{th}-order bifurcation equation, i.e. the (Q_1)-equation for $\delta = 0$,

$$-\Delta v_1 = \Pi_{V_1} \mathcal{G}(0, v_1) = \Pi_{V_1} \Big(a_p(x)(v_1 + v_2(0, v_1, 0))^p \Big),$$

is the Euler-Lagrange equation of the functional $\widetilde{\Phi}_0 : V_1 \to \mathbf{R}$ where $\widetilde{\Phi}_0 := \Phi_0(v_1 + v_2(0, v_1, 0))$ and $\Phi_0 : V \to \mathbf{R}$ is the functional defined in (8).

It can be proved that if a_p belongs to an *open* and *dense* subset $\mathcal{A}_p$ of $H^1((0,\pi), \mathbf{R})$, then $\widetilde{\Phi}_0 : V_1 \to \mathbf{R}$ (or the functional that one obtains when substituting $-a_p$ to a_p) possesses a non-trivial *non-degenerate* critical point $\overline{v}_1 \in V_1$ and so, by the Implicit function Theorem, there exists a smooth curve $v_1(\cdot) : (-\delta_0, \delta_0) \to V_1$ of solutions of the (Q_1)-equation with $v_1(0) = \overline{v}_1$.

The smoothness of $\delta \to v_1(\delta)$ then implies that $\{(\delta, v_1(\delta)); \delta > 0\}$ intersects B_∞ in a set whose projection on the δ coordinate is the Cantor set $\mathcal{C}_{a_p}$ of Theorem 1.1-(i), satisfying the measure estimate (7). Finally $u(\delta) = \delta u_0 + O(\delta^2)$ where $u_0 := \overline{v}_1 + v_2(0, \overline{v}_1, 0) \in V$ is a (non-degenerate, up to time translations) solution of the infinite dimensional bifurcation equation (5).

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