Stability of the magnetic Schrödinger operator in a waveguide

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Abstract

The spectrum of the Schrödinger operator in a quantum waveguide is known to be unstable in two and three dimensions. Any enlargement of the waveguide produces eigenvalues beneath the continuous spectrum [BGRS]. Also if the waveguide is bent eigenvalues will arise below the continuous spectrum [DE]. In this paper a magnetic field is added into the system. The spectrum of the magnetic Schrödinger operator is proved to be stable under small local deformations and also under small bending of the waveguide. The proof includes a magnetic Hardy-type inequality in the waveguide, which is interesting in its own.

1 Introduction

It has been known for a long time that an appropriate bending of a two dimensional quantum waveguide induces the existence of bound states, [EŠ], [GJ] and [DE]. From the mathematical point of view this means that the Dirichlet Laplacian on a smooth asymptotical straight planar waveguide has at least one isolated eigenvalue below the threshold of the essential spectrum. Similar results have been obtained for a locally deformed waveguide, which corresponds to adding a small "bump" to the straight waveguide, see [BGRS] and [BEGK]. In both cases an appropriate transformation is used to pass to a unitary equivalent operator on the straight waveguide with an additional potential, which is proved to be attractive. As a result at least one isolated eigenvalue appears below the essential spectrum for *any* nonzero curvature, satisfying certain regularity properties, respectively for an *arbitrarily small* "bump". The crucial point is that for low energy the Dirichlet Laplacian in a planar waveguide in $\mathbb{R}^2$ behaves effectively as a one dimensional system, in which the Schrödinger operators with attractive potentials have a negative discrete eigenvalue no matter how weak the potential is. This is related to the well known fact that the Hardy inequality fails to hold in dimensions one and two.

The purpose of this paper is to prove that in the presence of a suitable magnetic field some critical strength of the deformation is needed for these bound states to appear. The magnetic field is not supposed to affect the essential spectrum of the Dirichlet Laplacian. We will deal with two generic examples of magnetic field; a bounded differentiable field with

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compact support and an Aharonov-Bohm field. The crucial technical tool of the present work is a Hardy type inequality for magnetic Dirichlet forms in the waveguide.
For $d \geq 3$ the classical Hardy-inequality states that

$$\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\, dx \;\leq\; \frac{4}{(d-2)^2} \int_{\mathbb{R}^d} |\nabla u(x)|^2\, dx, \tag{1.1}$$

for all $u \in H^1(\mathbb{R}^d)$. Hence if $d \geq 3$ and $V \in C_0^\infty(\mathbb{R}^d), V \geq 0$, the operator $-\Delta - \varepsilon V$ does not have negative eigenvalues for small values of the parameter ε. However if $d = 1, 2$ then (1.1) fails to hold (see [BS2]) and hence the spectrum of $-\Delta - \varepsilon V$ contains some negative eigenvalues for any $\varepsilon > 0$. If $d = 2$ and a magnetic field is introduced a higher dimensional behavior appears. Let us consider the magnetic Schrödinger operator $(-\mathrm{i}\nabla + A)^2$, where $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a magnetic vector potential. In 1999 Laptev and Weidl proved a modified version of the inequality (1.1) in $\mathbb{R}^2$ for the quadratic form of a magnetic Schrödinger operator

$$\text{Const} \int_{\mathbb{R}^2} \frac{|u(x)|^2}{1+|x|^2}\, dx \;\leq\; \int_{\mathbb{R}^2} |(-\mathrm{i}\nabla + A)u(x)|^2\, dx, \tag{1.2}$$

see [LW], and gave a sharp result for the case of Aharonov-Bohm field. This was later extended in [B] to multiple Aharonov-Bohm magnetic potentials, see also [EL] and [BEL]. In our model the spectrum of $(-\mathrm{i}\nabla + A)^2$ starts from 1 and inequality (1.2) is not a good lower bound for functions in $H_0^1(\mathbb{R} \times (0, \pi))$. Our aim is therefore to prove that Hardy-inequality

$$\text{Const} \int_{\mathbb{R}\times(0,\pi)} \frac{|u(x)|^2}{1+x_1^2}\, dx \;\leq\; \int_{\mathbb{R}\times(0,\pi)} \left(|(-\mathrm{i}\nabla + A)u(x)|^2 - |u(x)|^2\right) dx, \tag{1.3}$$

holds true for all u in the Sobolev space $H_0^1(\mathbb{R} \times (0, \pi))$. Inequality (1.3) is then used to prove stability of the spectrum of the magnetic Schrödinger operator under local geometrical perturbations.

The text is organized in the following way. In Section 3 we prove inequality (1.3) for the magnetic Schrödinger operator with a bounded differentiable and compactly supported field, see Theorem 3.1. The main new ingredient of our result is that we subtract the threshold of essential spectrum. We also prove the asymptotical behavior of the corresponding constant in the Hardy inequality in the limit of weak fields.

In Section 4 we prove the stability of the essential spectrum of the operator $(-\mathrm{i}\nabla + A)^2$ in the deformed and curved waveguide for certain magnetic potentials, Theorem 4.1. The class of magnetic potentials for which the Theorem applies also includes the Aharonov-Bohm field.

In Section 5 we use (1.3) to prove that the spectrum of $(-\mathrm{i}\nabla + A)^2$ is stable under weak deformations of the boundary of the waveguide, Theorem 5.1. We also give an asymptotical estimate on the critical strength λ_0 of the deformation, for which the discrete spectrum $(-\mathrm{i}\nabla + A)^2$ will be empty. In particular, if the magnetic field equals αB, then λ_0 is proportional to α^2 as $\alpha \to 0$. Moreover, we prove by a trial function argument that the same behavior of λ, with another constant, is sufficient also for the presence of eigenvalues below the essential spectrum, Theorem 5.3. The latter shows that the order of α in our estimate is optimal.

Locally curved waveguides are studied in Section 6. We consider a waveguide with the curvature $\beta\gamma$, where β is a positive parameter and γ is some fixed smooth function with compact support. Similarly as in Section 5 we show in Theorem 6.1 that there exists a β_0, such that for all $\beta < \beta_0$ there will be no eigenvalues in the spectrum of $(-\mathrm{i}\nabla + A)^2$. The behavior of β_0 for in the limit of weak fields is at least proportional to α^2, as $\alpha \to 0$.

The Aharonov-Bohm field requires a bit different approach due to the technical difficulties coming from the fact that the corresponding magnetic potential has a singularity. However, all the results mentioned above can be extended also to this case. This is done in Section 7.

2 The main results

Here we formulate the main results of the paper without giving any explicit estimates on the involved constants. For more detailed formulations see the theorems in respectively sections.

We state the Hardy inequality for magnetic Dirichlet forms separately for the case of an Aharonov-Bohm field and for a bounded field.

Theorem 3.1. *Let $B \in C^1(\mathbb{R}^2)$ be a bounded, real-valued magnetic field which is non-trivial in Ω. Then there is a positive constant c such that*

$$c \int_{\mathbb{R}\times(0,\pi)} \frac{|u|^2}{1+x^2}\, dx\, dy \;\leq\; \int_{\mathbb{R}\times(0,\pi)} \left(|(-\mathrm{i}\nabla + A)u|^2 - |u|^2\right)\, dx\, dy, \tag{2.1}$$

for all $u \in H_0^1(\mathbb{R}\times(0,\pi))$, where A is a magnetic vector potential associated with B.

Theorem 7.1. *Let A be the magnetic vector potential*

$$A(x,y) = \Phi \cdot \left(\frac{-y+y_0}{x^2+(y-y_0)}, \frac{x}{x^2+(y-y_0)}\right), \tag{2.2}$$

where $\Phi \in \mathbb{R}\setminus\mathbb{Z}$ and $y_0 \in (0,\pi)$. Then there is a positive constant c such that

$$c \int_{\mathbb{R}\times(0,\pi)} \frac{|u|^2}{x^2+(y-y_0)^2}\, dx\, dy \;\leq\; \int_{\mathbb{R}\times(0,\pi)} \left(|(-\mathrm{i}\nabla + A)v|^2 - |v|^2\right)\, dx\, dy, \tag{2.3}$$

holds for all $u \in H_{0,A}^1(\mathbb{R}\times(0,\pi)\setminus\{(0,y_0)\})$.

As an application of Theorem 3.1. and Theorem 7.1. we prove stability results for the spectrum of the magnetic Schrödinger operator under geometrical perturbations. First we consider local deformations of a waveguide. Let f be a non-negative function in $C_0^1(\mathbb{R})$, $\lambda \geq 0$ and construct

$$\Omega_\lambda = \left\{(x,y) \in \mathbb{R}^2 : 0 < y < \pi + \lambda\pi f(x)\right\}. \tag{2.4}$$

Let M_d be the Friedrich's extension of the operator

$$\left(-\mathrm{i}\partial_x + a_1\right)^2 + \left(-\mathrm{i}\partial_y + a_2\right)^2, \tag{2.5}$$

defined on $C_0^\infty(\Omega_\lambda)$, where A is either the magnetic vector potential for the Aharonov-Bohm field inside the waveguide or a magnetic vector potential associated with a magnetic field $B \in C_0^1(\mathbb{R}^2)$, such that B is non-trivial in Ω_λ. Then the following statement holds:

Theorem 5.1. and 7.4. *There is a positive constant λ_0 such that for $\lambda \in (0,\lambda_0)$ the operator M_d has purely essential spectrum $[1,\infty)$.*

Assume that we replace the field B by αB, where $\alpha > 0$ then there are constants c_a and c_e such that if

$$\lambda < c_a\alpha^2 + \mathcal{O}(\alpha^4), \tag{2.6}$$

as $\alpha \to 0$, then the discrete spectrum of M_d is empty. But if

$$\alpha^2 < c_e\lambda + \mathcal{O}(\lambda^2), \tag{2.7}$$

as $\lambda \to 0$, then M_d has at least one eigenvalue.

If we now consider M_c being the same operator as M_d but in a curved waveguide Ω_β, where $\beta\gamma$ indicates the curvature of the boundary of the waveguide the results are similar.

Theorem 6.1. and 7.5. *There is a positive constant β_0 such that if $\beta \in (0,\beta_0)$ then the operator M_c has purely essential spectrum $[1,\infty)$.*

3 A Hardy-type inequality

In this section we will prove a Hardy inequality in the case of a general bounded, differentiable magnetic field.

Let $\Omega = \mathbb{R} \times (0, \pi)$ and let B be a bounded, real-valued magnetic field such that $B \in C^1(\mathbb{R}^2)$ and B is non-trivial in Ω. Choose a point $p \in \Omega$ such that there is a ball $\mathcal{B}_R(p) \subset \Omega$ with

$$\Phi(r) := \frac{1}{2\pi} \int_{\mathcal{B}_r(p)} B(x, y)\, dx\, dy \tag{3.1}$$

not identically zero for $r \in (0, R)$. For simplicity let $p = (0, y_0)$, for some $y_0 \in (0, \pi)$.

We can construct a magnetic vector potential for B as $A(x, y) = (a_1(x, y), a_2(x, y))$ defined on $\mathbb{R}^2$ in the following way

$$a_1(x, y) \;=\; -(y - y_0) \int_0^1 B(ux, u(y - y_0) + y_0)\, u\, du, \tag{3.2}$$

$$a_2(x, y) \;=\; x \int_0^1 B(ux, u(y - y_0) + y_0)\, u\, du. \tag{3.3}$$

Then $(\operatorname{curl} A)(x, y) = \partial_x a_2(x, y) - \partial_y a_1(x, y) = B(x, y)$ and the transversal gauge $A(x, y) \cdot (x, y - y_0) = 0$ for all $(x, y) \in \mathbb{R}^2$ is satisfied. Note that since $a_1, a_2 \in L^\infty(\mathbb{R}^2)$ we have $H^1_{0,A}(\Omega) = H^1_0(\Omega)$, where $H^1_{0,A}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in the norm

$$\|u\|^2_{H^1_{0,A}(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|(-\mathrm{i}\nabla + A)u\|^2_{L^2(\Omega)}. \tag{3.4}$$

Theorem 3.1. *Let $B \in C^1(\mathbb{R}^2)$ be a real-valued magnetic field such that $B \not\equiv 0$ in Ω. Then*

$$c_H \int_\Omega \frac{|u|^2}{1 + x^2}\, dx\, dy \leq \int_\Omega \left(|(-\mathrm{i}\nabla + A)u|^2 - |u|^2 \right) dx\, dy, \tag{3.5}$$

holds for all $u \in H^1_0(\Omega)$, where A is a magnetic vector potential associated with B and c_H is a positive constant, given in (3.31).

Proof. Due to gauge invariance of the inequality (3.5) we can without loss of generality assume that the components of A are given by (3.2) and (3.3). Let (r, θ) be polar coordinates centered at the point p. We will prove that the inequality

$$c \int_{\mathcal{B}_R(p)} |u|^2 r\, dr\, d\theta \leq \int_{\mathcal{B}_R(p)} \left(|u_r|^2 + r^{-2} |-\mathrm{i}u_\theta + r a(r, \theta) u|^2 \right) r\, dr\, d\theta, \tag{3.6}$$

holds for all $u \in H^1_0(\Omega)$, where $a(r, \theta) = A \cdot (-\sin\theta, \cos\theta)$ and c is a positive constant.

For fixed r we consider the operator $K_r = -\mathrm{i}\partial_\theta + r a(r, \theta)$ in $L^2(0, 2\pi)$, which was studied in [LW]. The operator K_r is self-adjoint on the domain $H^1(0, 2\pi)$ with periodic boundary conditions. The spectrum of K_r is discrete and the eigenvalues $\{\lambda_k\}_{k=-\infty}^{\infty}$ and the orthonormal set of eigenfunctions $\{\varphi_k\}_{k=-\infty}^{\infty}$ are given by

$$\lambda_k = \lambda_k(r) = k + \frac{r}{2\pi} \int_0^{2\pi} a(r, \theta)\, d\theta = k + \Phi(r), \tag{3.7}$$

and

$$\varphi_k(r, \theta) = \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}\lambda_k \theta - \mathrm{i}r \int_0^\theta a(r,s)\, ds}. \tag{3.8}$$

The quadratic form of K_r^2 satisfies the following inequality

$$\mu(r)^2 \int_0^{2\pi} |u|^2\, d\theta \leq \int_0^{2\pi} |-\mathrm{i}u_\theta + r a u|^2\, d\theta, \tag{3.9}$$

for all $u(r,\cdot) \in H^1(0,2\pi)$, where $\mu(r) = \mathrm{dist}(\Phi(r),\mathbb{Z})$. Thus

$$\int_{\mathcal{B}_R(p)} \frac{\mu^2}{r^2}|u|^2 r\,dr\,d\theta \leq \int_{\mathcal{B}_R(p)} r^{-2}|-\mathrm{i}u_\theta + rau|^2 r\,dr\,d\theta, \tag{3.10}$$

holds for all $u \in H^1(\Omega)$.

Define the function $\chi : [0,R] \to [0,1]$ by

$$\chi(r) = \frac{\mu_0^2\,\mu(r)^2}{r^2}, \qquad \text{where} \qquad \mu_0 = \left(\max_{r\in[0,R]} \frac{\mu(r)}{r}\right)^{-1}. \tag{3.11}$$

Since Φ is piecewise continuous differentiable and $\Phi(0) = 0$ it follows that χ is well defined. It is clear that $\chi(r) \in [0,1]$ and that there exists at least one $r_0 \in (0,R]$ such that $\chi(r_0) = 1$. Let $v \in H^1(0,R)$ such that $v(r_0) = 0$, then we have the following inequalities

$$\int_{r_0}^{R} |v(r)|^2 r\,dr \leq \frac{2R^3 - 3R^2 r_0 + r_0^3}{6r_0} \int_{r_0}^{R} |v'(r)|^2 r\,dr, \tag{3.12}$$

and

$$\int_0^{r_0} |v(r)|^2 r\,dr \leq \frac{r_0^2}{\nu_0^2} \int_0^{r_0} |v'(r)|^2 r\,dr, \tag{3.13}$$

where $\nu_0 \geq 2$ is the first zero of the Bessel function J_0. The latter comes from the lowest eigenvalue of $-\Delta$ in a circle with Dirichlet boundary conditions at the radius r_0. The first inequality follows by writing

$$|v(r)|^2 = \left|\int_{r_0}^{r} v'(t)\,dt\right|^2 \leq (r - r_0) \int_{r_0}^{R} |v'(r)|^2\,dr. \tag{3.14}$$

Using (3.12) and (3.13) we conclude that

$$\begin{aligned}
\int_{\mathcal{B}_R(p)} |u|^2 r\,dr\,d\theta &\leq 2\int_{\mathcal{B}_R(p)} \left(|\chi u|^2 + |(1-\chi)u|^2\right) r\,dr\,d\theta \\
&\leq 2\mu_0^2 \int_{\mathcal{B}_R(p)} r^{-2}|-\mathrm{i}u_\theta + rau|^2 r\,dr\,d\theta \\
&\quad + 2\int_0^{2\pi} \left(\frac{r_0^2}{\nu_0^2} \int_0^{r_0} |((1-\chi)u)'|^2 r\,dr \right. \\
&\quad \left. + \frac{2R^3 - 3R^2 r_0 + r_0^3}{6r_0} \int_{r_0}^{R} |((1-\chi)u)'|^2 r\,dr \right) d\theta \\
&\leq 2\mu_0^2 \int_{\mathcal{B}_R(p)} r^{-2}|-\mathrm{i}u_\theta + rau|^2 r\,dr\,d\theta \\
&\quad + c_0 \int_{\mathcal{B}_R(p)} \left(|\chi' u|^2 + |u_r|^2\right) r\,dr\,d\theta \\
&\leq c_1 \int_{\mathcal{B}_R(p)} \left(|u_r|^2 + r^{-2}|-\mathrm{i}u_\theta + rau|^2\right) r\,dr\,d\theta,
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
c_0 &= 4\max\left\{\nu_0^{-2} r_0^2, (6r_0)^{-1}(2R^3 - 3R^2 r_0 + r_0^3)\right\}, & (3.16)\\
c_1 &= \max\left\{2\mu_0^2 + 4c_0 c_2^2 \mu_0^4, c_0\right\}, & (3.17)\\
c_2 &= \max_{r\in[0,R]} |r^{-2}(r\mu'(r) - \mu(r))|. & (3.18)
\end{aligned}$$

The operator $-\frac{d^2}{dy^2} - 1$ on the domain $\left\{u \in H_0^2(0,\pi) : u(y_0) = 0\right\}$ is greater or equal to

$$c_3 := \pi^2 \min\left\{y_0^{-2}, (\pi - y_0)^{-2}\right\} - 1. \tag{3.19}$$

This can be easily verified by writing $-\frac{d^2}{dy^2}-1$ as the direct sum $\left(-\frac{d^2}{dy^2}-1\right)\oplus\left(-\frac{d^2}{dy^2}-1\right)$ on the set $H_0^2(0,y_0)\oplus H_0^2(y_0,\pi)$. In terms of quadratic forms this means that for v in $H^1(0,\pi)$ we have

$$\int_0^\pi |v(y)|^2\sin^2 y\,dy \le c_3^{-1}\int_0^\pi |v'(y)|^2\sin^2 y\,dy. \tag{3.20}$$

Let $u\in H^1(\Omega)$ and $\psi:(0,\pi)\to[0,1]$ be defined by

$$\psi(y)=\begin{cases}\frac{|y-y_0|}{\sqrt{R^2-x^2}} & \text{, if } h_-(x)<y<h_+(x),\\ 1 & \text{, otherwise.}\end{cases} \tag{3.21}$$

where $h_\pm(x)=y_0\pm\sqrt{R^2-x^2}$. We write $u=(1-\psi)u+\psi u$ and use (3.20) to obtain

$$\begin{aligned}\int_0^\pi |u|^2\sin^2 y\,dy \quad &\le \quad 2\int_{h_-(x)}^{h_+(x)}|(1-\psi)u|^2\sin^2 y\,dy \\ &+\frac{4}{c_3}\left(\int_0^\pi |u_y\psi|^2\sin^2 y\,dy+\int_{h_-(x)}^{h_+(x)}\frac{|u|^2\sin^2 y\,dy}{R^2-x^2}\right).\end{aligned} \tag{3.22}$$

Let $\Omega_R=(-R,R)\times(0,\pi)$, then by (3.15) and (3.22) we get

$$\begin{aligned}\int_{\Omega_R}(R^2-x^2)|u|^2\sin^2 y\,dy\,dx \quad &\le \quad \frac{c_1(2R^2c_3+4)}{c_3\cos^2(|y_0-\frac{\pi}{2}|+R)}\int_{\mathcal{B}_R(p)}|(-\mathrm{i}\nabla+A)u|^2\,\sin^2 y\,dx\,dy \\ &+\frac{4R^2}{c_3}\int_{\Omega_R}|u_y|^2\sin^2 y\,dy\,dx,\end{aligned} \tag{3.23}$$

for all $u\in H^1(\Omega)$. If $u=|v|$ where $v\in C^\infty(\overline{\Omega})$ then by the diamagnetic inequality (see for instance [K], [S], [AHS] and [HS]) saying that

$$|\nabla|v|(x,y)|\le|(-\mathrm{i}\nabla+A)v(x,y)| \tag{3.24}$$

holds almost everywhere, it follows that

$$\int_{\Omega_R}(R^2-x^2)|u|^2\sin^2 y\,dx\,dy\le c_4\int_{\Omega_R}|(-\mathrm{i}\nabla+A)u|^2\sin^2 y\,dx\,dy, \tag{3.25}$$

holds for all $u\in C^\infty(\overline{\Omega})$ with

$$c_4=\frac{2R^2c_1c_3+4c_1+4R^2}{c_3\,\cos^2(|y_0-\frac{\pi}{2}|+R)}\,. \tag{3.26}$$

We need the classical one-dimensional Hardy inequality saying that

$$\int_{-\infty}^{\infty}\frac{|v|^2}{t^2}\,dt\ \le\ 4\int_{-\infty}^{\infty}|v'|^2\,dt, \tag{3.27}$$

holds for any $v\in H^1(\mathbb{R})$, such that $v(0)=0$ (see [H]). Take $m=\frac{R}{\sqrt{2}}$ and let the mapping $\varphi:\mathbb{R}\to[0,1]$ be defined by

$$\varphi(x):=\begin{cases}1 & ,\ \text{ if } |x|>m,\\ \frac{|x|}{m} & ,\ \text{ if } |x|<m.\end{cases} \tag{3.28}$$

Let $u\in C^\infty(\overline{\Omega})\cap L^2(\Omega)$, by writing $u=u\varphi+u(1-\varphi)$ and using (3.24), (3.25) and (3.27) we obtain

$$\begin{aligned}\int_\Omega \frac{|u|^2\sin^2 y}{1+x^2}\,dx\,dy &\le 2\int_\Omega \frac{|u\varphi|^2+|u(1-\varphi)|^2}{1+x^2}\sin^2 y\,dx\,dy \\ &\le 16\int_\Omega \left(|u_x\varphi|^2+|u\varphi'|^2\right)\sin^2 y\,dx\,dy + 2\int_{\Omega_m}\frac{|u|^2\sin^2 y}{1+x^2}\,dx\,dy \\ &\le 16\int_\Omega |u_x|^2\sin^2 y\,dx\,dy + c_5\int_{\Omega_R}(R^2-x^2)|u|^2\sin^2 y\,dx\,dy \\ &\le c_6\int_\Omega |(-\mathrm{i}\nabla+A)u|^2\sin^2 y\,dx\,dy,\end{aligned} \tag{3.29}$$

where

$$c_5=\frac{64+4R^2}{R^4} \qquad \text{and} \qquad c_6=16+c_4c_5. \tag{3.30}$$

If we now substitute $v(x,y)=u(x,y)\sin y$ the statement of the theorem with

$$c_H=c_6^{-1} \tag{3.31}$$

will follow by continuity. □

Let us replace the field B by αB, where α is a positive constant. Let Φ_B be defined by (3.1) with the field B and define the following constants.

$$k_1 = \left(\max_{r\in[0,R]} r^{-1}\Phi_B(r)\right)^{-1}, \tag{3.32}$$

$$k_2 = \max_{r\in[0,R]} |r^{-2}(r\Phi_B'(r)-\Phi_B(r))|, \tag{3.33}$$

$$k_4 = \frac{(2R^2c_3+4)(2k_1^2+4c_0k_1^4k_2^2)}{c_3\cos^2(|y_0-\frac{\pi}{2}|+R)}. \tag{3.34}$$

Corollary 3.2. *If we replace B by αB in Theorem 3.1, then the constant c_H in (3.5) satisfies the following equality*

$$c_H \ge \frac{1}{k_4c_5}\,\alpha^2+\mathcal{O}(\alpha^4), \tag{3.35}$$

for $\alpha\to 0$.

Proof. We first note that the constants c_0, c_3 and c_5 are independent of α. As $\alpha\to 0$ the constant $c_1=(2k_1^2+4c_0k_1^4k_2^2)\alpha^{-2}$ and $c_2=k_2\alpha$. This implies that $c_4=k_4\alpha^{-2}+\mathcal{O}(1)$ and therefore (3.35) holds as $\alpha\to 0$. □

4 Stability of essential spectrum

Let Ω be a subset of $\mathbb{R}^2$ with $\partial\Omega$ being piecewise continuously differentiable and let us assume that there is a bounded set $\Omega_0\subset\mathbb{R}^2$ such that $\Omega\setminus\Omega_0$ consists up to translations and rotations of two half strips Ω_1 and Ω_2. By a half strip we denote the set $(0,\infty)\times(0,\pi)\setminus P$, where P is either empty or contains finite number of points in $\mathbb{R}^2$. Let M be the operator $(-\mathrm{i}\nabla+A)^2$ on $H^2_{0,A}(\Omega)$, for some magnetic vector potential A.

Theorem 4.1. *If the magnetic vector potential $A=(a_1,a_2)$ is such that for $j=1,2$ we have $a_j\in L^2_{\mathrm{loc}}(\Omega_1\cup\Omega_2)$, $a_j\in L^{2+\varepsilon}(\Omega_0)$ for some $\varepsilon>0$ and the functions $|A|$ and $\operatorname{div}A$ are for some $R>0$ in $L^2\left(\Omega_2\cap\left\{x\in\mathbb{R}^2:|x|>R\right\}\right)$, then*

$$\sigma_{\mathrm{ess}}(M)=[1,\infty). \tag{4.1}$$

Proof. We can without loss of generality assume that $\Omega_2 = (0, \infty) \times (0, \pi)$. To prove that $[1, \infty) \subset \sigma_{\text{ess}}(M)$ we construct Weyl sequences. Assume that λ is a non-negative real number. Let $\{h_n\}_{n=1}^{\infty}$ be a singular sequence of real-valued testfunctions for the operator $-\frac{d^2}{dx^2}$ in $L^2(\mathbb{R})$ at λ such that $\operatorname{supp} h_n \in (n, \infty)$ and such that $\|h_n\|_\infty$ and $\|h_n'\|_\infty$ are uniformly bounded in n. For instance let $\varphi \in C_0^\infty(\mathbb{R})$ be a non-negative function such that $\|\varphi\|_{L^2(\mathbb{R})} = 1$ and $\operatorname{supp} \varphi \subset (-1, 1)$. Let

$$\rho_n(x) = \begin{cases} 0 & \text{, if} \quad x < n \text{ or } x \geq n^2, \\ \frac{2x}{n(n-1)} - \frac{2}{n-1} & \text{, if} \quad n \leq x < \frac{n(n+1)}{2}, \\ \frac{-2x}{n(n-1)} + \frac{2n}{n-1} & \text{, if} \quad \frac{n(n+1)}{2} \leq x < n^2, \end{cases} \tag{4.2}$$

then h_n can be chosen as a subsequence of $(\rho_n * \varphi)(x) \cdot \cos(\sqrt{\alpha} x)$ such that the functions from the subsequence have disjoint support.

Construct the functions

$$g_n(x, y) = h_n(x) \sin y. \tag{4.3}$$

We will prove that g_n is a singular sequence for M at $1 + \lambda$. Clearly $g_n \in \mathcal{D}(M)$ for n large enough and

$$\|g_n\|_{L^2(\Omega)}^2 = \int_\Omega h_n(x)^2 \sin^2 y \, dx \, dy = \frac{\pi}{2} \|h_n\|_{L^2(\mathbb{R})}^2 > 0, \tag{4.4}$$

for every n.

Let u be any function in $L^2(\Omega)$, then

$$(u, g_n)_{L^2(\Omega)} = \int_0^\pi \sin y \int_n^\infty h_n(x) u(x, y) \, dx \, dy \to 0, \tag{4.5}$$

the latter follows since $u(\cdot, y)$ is in $L^2(\mathbb{R})$ for a.e. $y \in (0, \pi)$. Finally we must show that $(M - (\lambda + 1))g_n \to 0$, as $n \to \infty$. There is a constant c depending on $\|h_n\|_\infty$ and $\|h_n'\|_\infty$ such that

$$\|(M - (1 + \lambda))g_n\|_{L^2(\Omega)}^2 = c\Bigg(\int_n^\infty | - h_n' - \lambda h_n|^2 Dx \tag{4.6}$$

$$+ \int_0^\pi \int_n^\infty \left(|A|^2 + |\operatorname{div} A|^2 \right) \, dx \, dy \Bigg) \to 0, \tag{4.7}$$

as $n \to \infty$. We have proved that $1 + \lambda \in \sigma_{\text{ess}}(M)$ for all non-negative λ, i.e. $[1, \infty) \subset \sigma_{\text{ess}}(M)$.

To prove the reverse inclusion $\sigma_{\text{ess}}(M) \subset [1, \infty)$ it will be enough to prove that $\inf \sigma_{\text{ess}}(M) \geq 1$. We study the operator M_N being M with additional Neumann boundary condition at the intersections $\Omega_0 \cap \Omega_1$ and $\Omega_0 \cap \Omega_2$. Then M_N can be written as a direct sum of three operators $M_1 \oplus M_0 \oplus M_2$ on the domain $H_{0,A}^2(\Omega_1) \oplus H_{0,A}^2(\Omega_0) \oplus H_{0,A}^2(\Omega_2)$. Since the magnetic field is in $L^{2+\varepsilon}(\Omega_0)$ the norms in $H_A^1(\Omega_0)$ and $H_0^1(\Omega_0)$ are equivalent. This implies that the spectrum of M_0 is discrete. By the maximin principle we have

$$\inf \sigma_{\text{ess}}(M) \geq \inf \sigma_{\text{ess}}(M_N) = \inf \sigma_{\text{ess}}(M_2) \geq \inf \sigma(M_2). \tag{4.8}$$

By the diamagnetic inequality we get that

$$\inf \sigma(M_2) \geq \inf \sigma(-\Delta) = 1. \tag{4.9}$$

The last inequality follows since Dirichlet boundary conditions in the points contained in P don't affect the spectrum of $-\Delta$. Hence the proof is complete. $\square$

5 Locally deformed waveguides

Let f be a non-negative function in $C_0^1(\mathbb{R})$ and for $\lambda \geq 0$ we construct

$$\Omega_\lambda = \left\{ (s,t) \in \mathbb{R}^2 : 0 < t < \pi + \lambda \pi f(s) \right\}. \tag{5.1}$$

In [BGRS] it was proven that the Friedrich's extension of $-\Delta - 1$ defined on $C_0^\infty(\Omega_\lambda)$ had negative eigenvalues for all $\lambda > 0$. For small enough values of $\lambda > 0$ there is a unique simple negative eigenvalue $E(\lambda)$, the function $E(\lambda)$ is analytic at $\lambda = 0$ and

$$E(\lambda) = -\lambda^2 \left(\int_{\mathbb{R}} f(s)\, ds \right)^2 + \mathcal{O}(\lambda^3). \tag{5.2}$$

We will show that if we add a magnetic field to the Schrödinger operator it will prevent these negative eigenvalues to appear for small values of λ.

Assume that $B \in C_0^1(\mathbb{R}^2)$ such that B is not identically zero in Ω_λ. Let M_d be the Friedrich's extension of the symmetric, semi-bounded operator

$$\left(-\mathrm{i}\partial_s + a_1(s,t)\right)^2 + \left(-\mathrm{i}\partial_t + a_2(s,t)\right)^2, \tag{5.3}$$

defined on the domain $C_0^\infty(\Omega_\lambda)$, where $A(s,t) = (a_1(s,t), a_2(s,t))$ is a magnetic vector potential associated with B. Due to gauge invariance we can assume that A is defined by (3.2) and (3.3). Since B is bounded and of compact support, it follows from (3.2) and (3.3) that $a_1, a_2 \in L^\infty(\mathbb{R}^2)$ and for $r = |(s,t)| \to \infty$ we have

$$|a_j(s,t)| = \mathcal{O}(r^{-1}), \text{ for } j = 1,2. \tag{5.4}$$

This implies that the essential spectrum of M_d coincides by Theorem 4.1 with the half-line $[1,\infty)$.

Theorem 5.1. *There is a positive number λ_0 depending on $\|f\|_\infty$, $\|f'\|_\infty$, $\|a_1\|_\infty$ and $\|a_2\|_\infty$ such that for $\lambda \in (0,\lambda_0)$ the discrete spectrum of M_d is empty.*

Proof. We denote by $\mathfrak{q}_d$ the quadratic form associated with M_d, i.e.

$$\mathfrak{q}_d[\psi] = \int_{\Omega_\lambda} \left(|-\mathrm{i}\psi_s + a_1\psi|^2 + |-\mathrm{i}\psi_t + a_2\psi|^2 \right) ds\, dt, \tag{5.5}$$

with $\mathcal{D}(\mathfrak{q}_d) = H_0^1(\Omega_\lambda)$. Define

$$U_\lambda : L^2(\Omega_\lambda) \to L^2(\Omega_0) \tag{5.6}$$

to be the unitary operator given by

$$(U_\lambda\psi)(x,y) = \sqrt{1+\lambda f(x)}\,\psi(x,(1+\lambda f(x))y). \tag{5.7}$$

The operator M_d is unitary equivalent to the operator

$$M_\lambda := U_\lambda M_d U_\lambda^{-1}, \tag{5.8}$$

defined on the set $U_\lambda \mathcal{D}(M_d)$ in $L^2(\Omega_0)$. The form associated with M_λ is then given by

$$\mathfrak{q}_\lambda[\varphi] = \mathfrak{q}_d[U_\lambda^{-1}\varphi], \tag{5.9}$$

defined on the space $\mathcal{D}(\mathfrak{q}_\lambda) = U_\lambda \mathcal{D}(\mathfrak{q}_d)$. If we prove that $M_\lambda - 1$ is non-negative, then the theorem will follow from (5.8) and the fact that $\sigma_{\mathrm{ess}}(M_d) = [1,\infty)$.

For convenience let $g(s) = 1 + \lambda f(s)$, then

$$
\begin{aligned}
\mathfrak{q}_\lambda[\varphi] &= \mathfrak{q}_d[U_\lambda^{-1}\varphi] \\
&= \int_{\Omega_\lambda} \left(\left| (-\mathrm{i}\partial_s + a_1(s,t))(g(s)^{-\frac{1}{2}}\varphi(s, g(s)^{-1}t)) \right|^2 \right. \\
&\qquad \left. + \left| (-\mathrm{i}\partial_t + a_2(s,t))(g(s)^{-\frac{1}{2}}\varphi(s, g(s)^{-1}t)) \right|^2 \right) ds\, dt \\
&= \int_{\Omega_0} \left(\left| \frac{\mathrm{i}g'(x)}{2g(x)}\varphi(x,y) - \mathrm{i}\varphi_x(x,y) \right. \right. \\
&\qquad \left. + \frac{\mathrm{i}yg'(x)}{g(x)}\varphi_y(x,y) + \tilde{a}_1(x,y)\varphi(x,y) \right|^2 \\
&\qquad \left. + \left| -\frac{\mathrm{i}}{g(x)}\varphi_y(x,y) + \tilde{a}_2(x,y)\varphi(x,y) \right|^2 \right) dx\, dy,
\end{aligned} \tag{5.10}
$$

where

$$
\tilde{A}(x,y) = (\tilde{a}_1(x,y), \tilde{a}_2(x,y)) = A(x, g(x)y). \tag{5.11}
$$

Straightforward calculation gives

$$
\begin{aligned}
\mathfrak{q}_\lambda[\varphi] &= \int_{\Omega_0} \left(|-\mathrm{i}\varphi_x + \tilde{a}_1\varphi|^2 + |-\mathrm{i}\varphi_y + \tilde{a}_2\varphi|^2 - |\varphi_y|^2 \right. \\
&\qquad - \frac{g'}{2g}(\varphi\overline{\varphi_x} + \varphi_x\overline{\varphi}) - \frac{1}{4}\left(\frac{g'}{g}\right)^2 |\varphi|^2 - \frac{yg'}{g}(\varphi_x\overline{\varphi_y} + \varphi_y\overline{\varphi_x}) \\
&\qquad \left. + \frac{y^2(g')^2 + 1}{g^2}|\varphi_y|^2 + \mathrm{i}\frac{yg'\tilde{a}_1 + \lambda f\tilde{a}_2}{g}(\varphi_y\overline{\varphi} - \varphi\overline{\varphi_y}) \right) dx\, dy.
\end{aligned} \tag{5.12}
$$

Let $\mathfrak{q}$ be the quadratic form associated with the Schrödinger operator with the magnetic vector potential $\tilde{A}$ in the space $L^2(\Omega_0)$. We have

$$
\begin{aligned}
\mathfrak{q}_\lambda[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} &= \mathfrak{q}[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \\
&\quad + \int_{\Omega_0} \left(\frac{y^2\lambda^2(f')^2 - 2\lambda f - \lambda^2 f^2}{g^2}|\varphi_y|^2 - \frac{1}{4}\left(\frac{\lambda f'}{g}\right)^2 |\varphi|^2 \right. \\
&\qquad - \frac{y\lambda f'}{g}(\varphi_x\overline{\varphi_y} + \varphi_y\overline{\varphi_x}) - \frac{\lambda f'}{2g}(\varphi\overline{\varphi_x} + \varphi_x\overline{\varphi}) \\
&\qquad \left. + \mathrm{i}\lambda\frac{yf'\tilde{a}_1 + f\tilde{a}_2}{g}(\varphi_y\overline{\varphi} - \varphi\overline{\varphi_y}) \right) dx\, dy.
\end{aligned} \tag{5.13}
$$

Without loss of generality we can assume that $\lambda \leq 1$. Let χ be the characteristic function of the support of f. The following lower bound holds true,

$$
\mathfrak{q}_\lambda[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \geq \mathfrak{q}[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} - \lambda \int_{\Omega_0} \chi \cdot \left(c_7\left(|\varphi_x|^2 + |\varphi_y|^2\right) + c_8|\varphi|^2 \right) dx\, dy, \tag{5.14}
$$

where the constants are given by

$$
c_7 = \|f\|^2_\infty + (2 + \|a_2\|_\infty)\|f\|_\infty + (2^{-1} + \pi + \pi\|a_1\|_\infty)\|f'\|_\infty, \tag{5.15}
$$

$$
c_8 = 4^{-1}\|f'\|^2_\infty + 2^{-1}\|f'\|_\infty + \pi\|a_1\|_\infty\|f'\|_\infty + \|a_2\|_\infty\|f\|_\infty. \tag{5.16}
$$

By the pointwise inequality

$$
|\varphi_x|^2 + |\varphi_y|^2 \leq 2\left(|-\mathrm{i}\nabla\varphi + \tilde{A}\varphi|^2 + |\tilde{A}|^2|\varphi|^2 \right) \tag{5.17}
$$

and Theorem 3.1 we get

$$\begin{aligned}\mathfrak{q}_\lambda[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \quad \geq \quad & \left(\frac{1}{2} - 2\lambda c_7\right)\left(\mathfrak{q}[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)}\right) \\ & + \left(\frac{c_H}{2} - \lambda c_9(1+d^2)\right)\int_{\Omega_0} \frac{|\varphi|^2}{1+x^2}\,dx\,dy,\end{aligned} \tag{5.18}$$

where

$$d = \max \operatorname{supp} f \quad \text{and} \quad c_9 = 2(1 + \|a_1\|^2_\infty + \|a_2\|^2_\infty)c_7 + c_8 \tag{5.19}$$

and c_H is the constant from (3.5). Let

$$\lambda_0 = \frac{c_H}{2c_9(1+d^2)}, \tag{5.20}$$

then the right hand side of (5.18) is positive for all $\lambda \in (0, \lambda_0)$. $\square$

If we replace B by αB, A will be replaced αA. Let us define

$$k_9 := \lim_{\alpha\to 0} c_9 = \|f\|^2_\infty + 2\|f\|_\infty + 4^{-1}\|f'\|^2_\infty + (1+\pi)\|f'\|_\infty. \tag{5.21}$$

The following corollary is an immediate consequence of the previous Theorem and Corollary 3.2 and shows the asymptotical behavior of λ_0 for weak magnetic fields.

Corollary 5.2. *If we replace the magnetic field B by αB, where $\alpha \in \mathbb{R}$, then*

$$\lambda_0 \geq \frac{\alpha^2}{2k_4 k_9 c_5(1+d^2)} + \mathcal{O}(\alpha^4), \tag{5.22}$$

as $\alpha \to 0$, where the constants are given in (3.34), (3.30) *and* (5.19).

Without loss of generality we assume that Ω_λ includes a small triangle spanned by the points $(-s, 1)$, $(s, 1)$ and $(0, \pi(1+\beta\lambda))$ with s, $\beta > 0$.

Theorem 5.3. *Let the magnetic field B be replaced by αB, where $\alpha \in \mathbb{R}$ and assume that*

$$\alpha^2 \leq \frac{\pi s \beta}{4\|A\|^2}\lambda + \mathcal{O}(\lambda^2), \tag{5.23}$$

as $\lambda \to 0$, where A is any magnetic vector potential associated with B. Then the operator M_d has at least one eigenvalue below the essential spectrum.

Proof. Define the trial function φ introduced in [BGRS], as follows

$$\varphi(x,y) = \begin{cases} \sin y\, e^{-s\beta\lambda(|x|-s)} & , |x| \geq s,\ 0 < y < \pi, \\ \sin\left(\frac{y}{1+\beta\lambda\left(1-\frac{|x|}{s}\right)}\right) & , |x| < s,\ 0 < y < \pi\left(1 + \beta\lambda\left(1 - \frac{|x|}{s}\right)\right), \\ 0 & , \text{otherwise}. \end{cases} \tag{5.24}$$

Let $\|\cdot\| = \|\cdot\|_{L^2(\Omega_\lambda)}$. A simple calculation gives

$$\frac{\|\nabla\varphi\|^2}{\|\varphi\|^2} = 1 - \lambda^2\,\frac{s^2\beta^2}{2} + \mathcal{O}(\lambda^3), \tag{5.25}$$

for $\lambda \to 0$. In order to prove that the discrete spectrum of M_d is non-empty, it is enough to show that the inequality

$$\frac{\|(\mathrm{i}\nabla + \alpha A)\varphi\|^2}{\|\varphi\|^2} < 1 \tag{5.26}$$

is satisfied for certain values of λ and α. By (3.2) and (3.3) it follows that $|A| \in L^2(\Omega_\lambda)$. Since $\|\varphi\|_\infty = 1$, we have

$$\frac{\|(\mathrm{i}\nabla + \alpha A)\varphi\|^2}{\|\varphi\|^2} \leq \frac{\|\nabla\varphi\|^2}{\|\varphi\|^2} + \frac{\alpha^2\,\|A\|^2}{\|\varphi\|^2} = 1 - \lambda^2\,\frac{s^2\beta^2}{2} + \frac{\alpha^2\,\|A\|^2}{\|\varphi\|^2} + \mathcal{O}(\lambda^3), \tag{5.27}$$

Taking into account the fact that

$$\|\varphi\|^2 = \pi\left(\frac{1}{2s\beta\lambda} + s + \frac{\beta\lambda s}{2}\right) \tag{5.28}$$

we get

$$\alpha^2 \leq \frac{\pi s\beta}{4\|A\|^2}\,\lambda + \mathcal{O}(\lambda^2) \tag{5.29}$$

and the proof is complete. □

We remark that Corollary 5.2 together with Theorem 5.3 show that the order in the asymptotical behavior of the constant c_H given in Corollary 3.2 is sharp.

6 Locally curved waveguides

Let a and b be real-valued functions in $C^2(\mathbb{R})$. Define the set

$$\Omega_\gamma = \{(s,t) : s = a(x) - yb'(x),\ t = b(x) + ya'(x),\ \text{where } (x,y) \in \mathbb{R}\times(0,\pi)\}, \tag{6.1}$$

where γ is to be explained later. We assume that

$$a'(x)^2 + b'(x)^2 = 1, \tag{6.2}$$

for all $x \in \mathbb{R}$. The boundary of Ω_γ for which $y = 0$ is a curve $\Gamma \in \mathbb{R}^2$ given by

$$\Gamma = \{(a(x), b(x)) : x \in \mathbb{R}\}, \tag{6.3}$$

and the signed curvature $\gamma : \mathbb{R} \to \mathbb{R}$ of Γ is given by

$$\gamma(x) = b'(x)a''(x) - a'(x)b''(x). \tag{6.4}$$

Assume that $\gamma \in C_0^1(\mathbb{R})$ and let the natural condition

$$\gamma(x) > -\frac{1}{\pi}, \tag{6.5}$$

hold for all $x \in \mathbb{R}$. We prohibit Ω_γ to be self-intersecting.

We will formulate the theory and results in terms of the curvature γ and not in terms of the functions a and b. Those functions a and b can be constructed from γ uniquely up to rotations and translations from the identities

$$a(x) = a(0) + \int_0^x \cos\left(\int_0^{x_1} \gamma(x_2)\,dx_2\right)dx_1, \tag{6.6}$$

$$b(x) = b(0) + \int_0^x \sin\left(\int_0^{x_1} \gamma(x_2)\,dx_2\right)dx_1. \tag{6.7}$$

In 1994, Duclos and Exner [DE] gave a proof based on ideas from Goldstone and Jaffe [GJ] of existence of bound states below the essential spectrum for the Schrödinger operator $-\Delta$ in Ω_γ with Dirichlet boundary conditions, assuming that $\gamma \neq 0$. Our aim is to prove that if we introduce an appropriate magnetic field into the system it will make the threshold of the bottom of the essential spectrum stable if the curvature γ is weak enough.

To be able to study weak curvatures we replace γ by $\beta\gamma$, where β is a small positive real number. We will use the notation Ω_β for the set $\Omega_{\beta\gamma}$. Let $B \in C_0^1(\mathbb{R}^2)$ be a magnetic field such that B is not identically zero in Ω_β. Let the operator M_c be the Friedrich's extension of the symmetric, semi-bounded operator

$$(-\mathrm{i}\partial_s + a_1)^2 + (-\mathrm{i}\partial_t + a_2)^2 \tag{6.8}$$

on the domain $C_0^\infty(\Omega_\beta)$, where $A(s,t) = (a_1(s,t), a_2(s,t))$ is a magnetic vector potential associated with B. Without loss of generality we can assume that A is defined by the identities (3.2) and (3.3). By (5.4) and Theorem 4.1 we have $\sigma_{\mathrm{ess}}(M_c) = [1, \infty)$.

Theorem 6.1. *There exists positive number β_0 depending on $\|\gamma\|_\infty$, $\|\gamma'\|_\infty$, $\|a_1\|_\infty$ and $\|a_2\|_\infty$ such that for $\beta \in (0, \beta_0)$ the discrete spectrum of M_c is empty.*

Proof. The quadratic form $\mathfrak{q}_c$ associated with M_c is given by

$$\mathfrak{q}_c[\psi] = \int_{\Omega_\beta} \left(|-\mathrm{i}\psi_s + a_1\psi|^2 + |-\mathrm{i}\psi_t + a_2\psi|^2 \right) ds\, dt, \tag{6.9}$$

on $\mathcal{D}(\mathfrak{q}_c) = H_0^1(\Omega_\beta)$. Define the unitary operator

$$U_\beta : L^2(\Omega_\beta) \to L^2(\Omega_0) \tag{6.10}$$

as

$$(U_\beta\psi)(x,y) = \sqrt{1 + y\beta\gamma(x)}\ \psi(a(x) - yb'(x), b(x) + ya'(x)). \tag{6.11}$$

The operator M_c is unitary equivalent to the operator

$$M_\beta := U_\beta M_c U_\beta^{-1} \tag{6.12}$$

acting on the dense subspace $\mathcal{D}(M_\beta) = U_\beta \mathcal{D}(M_c)$ of the Hilbert space $L^2(\Omega_0)$. Our aim is to prove that the operator $M_\beta - 1$ is nonnegative. For this we calculate the quadratic form $\mathfrak{q}_\beta$ associated with M_β. Our change of variables gives us the Jacobian,

$$\frac{\partial\,(s,t)}{\partial\,(x,y)} = \begin{pmatrix} a'(x) - yb''(x) & b'(x) + ya''(x) \\ -b'(x) & a'(x) \end{pmatrix}. \tag{6.13}$$

Hence we have

$$\begin{cases} \partial_s &= (1 + y\beta\gamma)^{-1} \left(a'(x)\partial_x - (b'(x) + ya''(x))\partial_y\right) \\ \partial_t &= (1 + y\beta\gamma)^{-1} \left(b'(x)\partial_x - (a'(x) - yb''(x))\partial_y\right) \end{cases} \tag{6.14}$$

thus

$$\begin{aligned} \mathfrak{q}_\beta[\varphi] &= \mathfrak{q}_c[U_\beta^{-1}\varphi] \qquad (6.15)\\ &= \int_{\Omega_0} \left(\left| \left[\frac{-\mathrm{i}\,(a'(x)\partial_x - (b'(x) + ya''(x))\partial_y)}{1 + y\beta\gamma(x)} + \tilde{a}_1(x,y) \right] \left(\frac{\varphi(x,y)}{\sqrt{1 + y\beta\gamma(x)}} \right) \right|^2 \right. \\ &\quad \left. + \left| \left[\frac{-\mathrm{i}\,(b'(x)\partial_x + (a'(x) - yb''(x))\partial_y)}{1 + y\beta\gamma(x)} + \tilde{a}_2(x,y) \right] \left(\frac{\varphi(x,y)}{\sqrt{1 + y\beta\gamma(x)}} \right) \right|^2 \right) \\ &\quad (1 + y\beta\gamma(x))\, dx\, dy, \end{aligned}$$

where

$$\tilde{A}(x,y) = (\tilde{a}_1(x,y), \tilde{a}_2(x,y)) = A(a(x) - yb'(x), b(x) + ya'(x)). \tag{6.16}$$

We continue without writing arguments of the functions and use the identities $a'a'' + b'b'' = 0$ and $(a'')^2 + (b'')^2 = \beta^2\gamma^2$,

$$\begin{aligned} \mathfrak{q}_\beta[\varphi] \quad = \quad & \int_{\Omega_0} \Bigg(\frac{|\varphi_x|^2}{(1+y\beta\gamma)^2} - \frac{\mathrm{i}(a'\tilde{a}_1 + b'\tilde{a}_2)}{1+y\beta\gamma}(\varphi_x\overline{\varphi} - \varphi\overline{\varphi_x}) + |\varphi_y|^2 \\ & - \frac{\mathrm{i}\,(-(b'+ya'')\tilde{a}_1 + (a'-yb'')\tilde{a}_2)}{1+y\beta\gamma}(\varphi_y\overline{\varphi} - \varphi\overline{\varphi_y}) \\ & - \frac{y\beta\gamma'}{2(1+y\beta\gamma)^3}(\varphi\overline{\varphi_x} + \varphi_x\overline{\varphi}) - \frac{\beta\gamma}{2(1+y\beta\gamma)}(\varphi\overline{\varphi_y} + \varphi_y\overline{\varphi}) \\ & + \left(\frac{y^2\beta^2\,(\gamma')^2}{4(1+y\beta\gamma)^4} + \frac{\beta^2\gamma^2}{4(1+y\beta\gamma)^2} + \tilde{a}_1^2 + \tilde{a}_2^2 \right) |\varphi|^2 \Bigg)\, dx\, dy. \end{aligned} \tag{6.17}$$

We write the form q_β as a perturbation of the form

$$\mathfrak{q}[\varphi] := \int_{\Omega_0} |-\mathrm{i}\varphi_x + (a'\tilde{a}_1 + b'\tilde{a}_2)\varphi|^2 + |-\mathrm{i}\varphi_y + (-b'\tilde{a}_1 + a'\tilde{a}_2)\varphi|^2\, dx\, dy, \tag{6.18}$$

i.e.

$$\begin{aligned} \mathfrak{q}_\beta[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \quad = \quad & \mathfrak{q}[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \\ & - \int_{\Omega_0} \Bigg(\frac{2y\beta\gamma + y^2\beta^2\gamma^2}{1+y\beta\gamma}|\varphi_x|^2 - \mathrm{i}y\beta\gamma(a'\tilde{a}_1 + b'\tilde{a}_2)(\varphi_x\overline{\varphi} - \varphi\overline{\varphi_x}) \\ & \quad - \mathrm{i}y \left(-\beta\gamma b'\tilde{a}_1 + \beta\gamma a'\tilde{a}_2 + \frac{a''\tilde{a}_1 - b''\tilde{a}_2}{1+y\beta\gamma} \right) (\varphi_y\overline{\varphi} - \varphi\overline{\varphi_y}) \\ & \quad + \frac{y\beta\gamma'}{2(1+y\beta\gamma)^3}\,(\varphi\overline{\varphi_x} + \varphi_x\overline{\varphi}) + \frac{\beta\gamma}{2(1+y\beta\gamma)}\,(\varphi\overline{\varphi_y} + \varphi_y\overline{\varphi}) \\ & \quad - \left(\frac{y^2\beta^2\,(\gamma')^2}{4(1+y\beta\gamma)^4} + \frac{\beta^2\gamma^2}{4(1+y\beta\gamma)^2} \right) |\varphi|^2 \Bigg)\, dx\, dy. \end{aligned} \tag{6.19}$$

We can easily arrive at the following estimate

$$\begin{aligned} \mathfrak{q}_\beta[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \quad \geq \quad & \mathfrak{q}[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \\ & - \beta \int_{\Omega_0} \chi \left(c_{10} \left(|\varphi_x|^2 + |\varphi_y|^2 \right) + c_{11}|\varphi|^2 \right) dx\, dy, \end{aligned} \tag{6.20}$$

where χ is the characteristic function of the support of γ and

$$c_{10} \quad = \quad \pi^2\|\gamma\|^2_\infty + 2\pi\,(1 + \|a_1\|_\infty + \|a_2\|_\infty)\,\|\gamma\|_\infty + \frac{\pi}{2}\|\gamma'\|_\infty, \tag{6.21}$$

$$c_{11} \quad = \quad \left(\frac{1}{2} + 3\pi\|a_1\|_\infty + 3\pi\|a_2\|_\infty \right) \|\gamma\|_\infty + \frac{\pi}{2}\|\gamma'\|_\infty. \tag{6.22}$$

By using (3.5) and (5.17) we get

$$\begin{aligned} \mathfrak{q}_\beta[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \quad \geq \quad & \left(\frac{1}{2} - 2\beta c_{10} \right) \left(\mathfrak{q}[\varphi] - \|\varphi\|^2_{L^2(\Omega)} \right) \\ & + \left(\frac{c_H}{2} - \beta c_{12}(1+d^2) \right) \int_{\Omega_0} \frac{|\varphi|^2}{1+x^2}\, dx\, dy, \end{aligned} \tag{6.23}$$

where

$$d = \max \operatorname{supp} \gamma \quad \text{and} \quad c_{12} = 2\left(1 + \|a_1\|_\infty^2 + \|a_2\|_\infty^2\right) c_{10} + c_{11}. \tag{6.24}$$

The right hand side is positive if $\beta \in (0, \beta_0)$, with

$$\beta_0 := \frac{c_H}{2c_{12}(1+d^2)}. \tag{6.25}$$

Hence the operator M_d has empty discrete spectrum. $\square$

If we replace B by αB, A will be replaced αA. Let us define

$$k_{12} := \lim_{\alpha \to 0} c_{12} = 2\pi^2 \|\gamma\|_\infty^2 + (4\pi + 2^{-1})\|\gamma\|_\infty + \frac{3\pi}{2}\|\gamma'\|_\infty. \tag{6.26}$$

Corollary 6.2. *If we replace the magnetic field B by αB, where $\alpha \in \mathbb{R}$, then*

$$\beta_0 \geq \frac{\alpha^2}{2k_4 c_5 c_{12}(1+d^2)} + \mathcal{O}(\alpha^4), \tag{6.27}$$

as $\alpha \to 0$, where the constants are given in (3.34), (3.30) *and* (6.24).

7 Aharonov-Bohm field

In this last section we consider the Aharonov-Bohm field. The field is generated by a magnetic vector potential having a singularity in one point.

7.1 A Hardy-type inequality

Let p be the point $(0, y_0) \in \mathbb{R}^2$, where $y_0 \in (0, \pi)$ and define $A : \mathbb{R}^2 \to \mathbb{R}^2$ to be the vector field

$$A(x,y) = (a_1(x,y), a_2(x,y)) = \Phi \cdot \left(\frac{-y + y_0}{x^2 + (y-y_0)^2}, \frac{x}{x^2 + (y-y_0)^2} \right), \tag{7.1}$$

for $\Phi \in \mathbb{R}$. The vector field A is a magnetic vector potential for the Aharonov-Bohm magnetic field. The magnetic field $B : \mathbb{R}^2 \to \mathbb{R}$ is for $(x,y) \neq p$ given by

$$B(x,y) = \partial_x a_2 - \partial_y a_1 = 0 \tag{7.2}$$

and the constant $2\pi\Phi$ is the magnetic flux through the point p, i.e. let Γ be a closed simple curve containing p, then

$$\oint_\Gamma a_1\, dx + a_2\, dy = 2\pi\Phi. \tag{7.3}$$

Let $\Omega \subset \mathbb{R}^2$ be given by $\Omega = \mathbb{R} \times (0, \pi)$. The following Hardy-inequality holds true.

Theorem 7.1. *Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ be a given real-valued magnetic vector potential such that there exists a ball $\mathcal{B}_R(p) \subset \Omega$, for which $(x,y) \in \mathcal{B}_R(p)$ implies that*

$$A(x,y) = \Phi \cdot \left(\frac{-y + y_0}{x^2 + (y-y_0)^2}, \frac{x}{x^2 + (y-y_0)^2} \right), \tag{7.4}$$

where $\Phi \in \mathbb{R} \setminus \mathbb{Z}$. Then for all $v \in H^1_{0,A}(\Omega \setminus \{p\})$ the following inequality holds

$$c_{AB} \int_\Omega \frac{|v|^2\, dx\, dy}{x^2 + (y-y_0)^2} \quad \leq \quad \int_\Omega \left(|-\mathrm{i}\nabla v + Av|^2 - |v|^2 \right) dx\, dy, \tag{7.5}$$

where

$$c_{AB} = \frac{R^2\Psi^2\cos^2\left(\left|y_0 - \frac{\pi}{2}\right| + R\right)}{8\left(2R^2\Psi^2 + (2c_{13}\Psi^2 + 1 + 2c_{13})(9R^2 + 16\pi^2)\right)}, \tag{7.6}$$

$$\Psi = \min_{k\in\mathbb{Z}} |\Phi - k|, \tag{7.7}$$

$$c_{13} = \frac{4\pi^2}{\pi^2 - \max\left\{y_0^2, (\pi - y_0)^2\right\}}. \tag{7.8}$$

For the proof of the Theorem we need two lemmas.

Lemma 7.2. *Let R be chosen such that $\mathcal{B}_R(p) \subset \Omega$, then the inequality*

$$\int_{\mathcal{B}_R(p)} |-\mathrm{i}\nabla u + Au|^2 \sin^2 y\, dx\, dy \geq \Psi^2 \int_{\mathcal{B}_R(p)} \frac{\cos^2\left(|(x,y) - p|\right)|u|^2\sin^2 y\, dx\, dy}{x^2 + (y - y_0)^2} \tag{7.9}$$

holds true for all $u \in C^\infty(\overline{\mathcal{B}_R(p)})$ such that $u = 0$ in a neighborhood of p, where Ψ is given in (7.7)*.*

Proof. We follow ideas from [LW]. Let us introduce polar coordinates centered at the point p and let $D_n = \left\{(r,\theta) : (n-1)RN^{-1} < r < nRN^{-1}\right\}$, where N is a natural number. Let $u \in C^\infty(\overline{\mathcal{B}_R(p)})$ such that $u = 0$ in a neighborhood of p. In each D_n we have

$$\begin{aligned}\int_{D_n} |-\mathrm{i}\nabla u + Au|^2 \sin^2 y\, dx\, dy &= \int_{D_n} \left(|u_r|^2 + r^{-2}|-\mathrm{i}u_\theta + \Phi u|^2\cos^2(r\sin\theta)\right) r\, dr\, d\theta \\ &\geq \cos^2\left(\frac{nR}{N}\right)\int_{D_n} r^{-1}|-\mathrm{i}u_\theta + \Phi u|^2\, dr\, d\theta. \end{aligned}\tag{7.10}$$

To study the form (7.10) we make use of the one-dimensional self-adjoint operator K on $L^2(0,\pi)$ given by

$$K = -\mathrm{i}\partial_\theta + \Phi, \tag{7.11}$$

defined on the set

$$\mathcal{D}(K) = \left\{u \in H^1(0, 2\pi) : u(0) = u(2\pi)\right\}. \tag{7.12}$$

The spectrum of K is discrete and its eigenvalues $\{\lambda_k\}_{k\in\mathbb{Z}}$ and the complete orthonormal system of eigenfunctions $\{\varphi_k\}_{k\in\mathbb{Z}}$ are given by

$$\lambda_k = k + \Phi \tag{7.13}$$

and

$$\varphi_k(\theta) = \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}\theta(\lambda_k - \Phi)}. \tag{7.14}$$

We can write the function u in the Fourier expansion

$$u(r,\theta) = \sum_{k\in\mathbb{Z}} \omega_k(r)\varphi_k(\theta). \tag{7.15}$$

Then we have

$$\begin{aligned}\int_{D_n} r^{-1}|-\mathrm{i}u_\theta + \Phi u|^2\, dr\, d\theta &\geq \int_{D_n} r^{-1}\left|\sum_{k\in\mathbb{Z}}\omega_k\lambda_k\varphi_k\right|^2 d\theta\, dr \\ &\geq \int_{(n-1)RN^{-1}}^{nRN^{-1}} r^{-1}\sum_{k\in\mathbb{Z}}|\omega_k|^2\lambda_k^2\, dr \\ &\geq \Psi^2\int_{D_n} r^{-1}|u|^2\, dr\, d\theta. \end{aligned}\tag{7.16}$$

Finally we sum up the inequality over the rings. For any N we have

$$
\begin{aligned}
\int_{\mathcal{B}_R(p)} |-\mathrm{i}\nabla u + Au|^2 \sin^2 y \, dx\, dy &\geq \sum_{n=1}^{N} \cos^2\left(\frac{nR}{N}\right) \int_{D_n} r^{-1} |-\mathrm{i}u_\theta + \Phi u|^2 \, dr\, d\theta \\
&\geq \sum_{n=1}^{N} \cos^2\left(\frac{nR}{N}\right) \Psi^2 \int_{D_n} r^{-1} |u|^2 \, dr\, d\theta \qquad (7.17) \\
&\geq \Psi^2 \sum_{n=1}^{N} \int_{D_n} \cos^2\left(r + \frac{R}{N}\right) r^{-1} |u|^2 \, dr\, d\theta.
\end{aligned}
$$

Hence the desired result will follow as $N \to \infty$. □

Lemma 7.3. *The inequality*

$$\int_0^\pi \frac{|u(y)|^2 \sin^2 y \, dy}{(y-y_0)^2} \leq c_{13} \int_0^\pi |u'(y)|^2 \sin^2 y \, dy \tag{7.18}$$

holds true for all functions $u \in H^1(0,\pi)$ such that $u\left(\frac{\pi}{2}\right) = 0$, where c_{13} is given by (7.8).

Proof. It is clear that

$$\pi^2 \min\left\{y_0^{-2}, (\pi - y_0)^{-2}\right\} \leq -\frac{d^2}{dy^2}. \tag{7.19}$$

We will prove another estimate for $-\frac{d^2}{dy^2}$, namely the inequality

$$\frac{1}{4(y-y_0)^2} \leq -\frac{d^2}{dy^2}, \tag{7.20}$$

for the subspace of functions $v \in C_0^\infty(0,\pi)$ satisfying $v(y_0) = 0$. It will be enough to prove that

$$\frac{1}{4}\int_0^\beta \frac{|v(y)|^2 \, dy}{y^2} \leq \int_0^\beta |v'(y)|^2 \, dy, \tag{7.21}$$

for all functions $v \in C_0^\infty(0,\beta)$, where β is any positive number.

Let $v \in C_0^\infty(0,\beta)$ be a real-valued function, then

$$|v(y)|^2 = 2\int_0^y v(t) v'(t) dt. \tag{7.22}$$

Hence

$$
\begin{aligned}
\int_0^\beta \frac{|v(y)|^2 \, dy}{y^2} &= 2\int_0^\beta v(t) v'(t) \left(\frac{1}{t} - \frac{1}{\beta}\right) dt \qquad (7.23) \\
&\leq 2\left(\int_0^\beta |v(t)|^2 \left(\frac{1}{t} - \frac{1}{\beta}\right)^2 dt\right)^{\frac{1}{2}} \left(\int_0^\beta |v'(t)|^2 \, dt\right)^{\frac{1}{2}} \\
&\leq 2\left(\int_0^\beta \frac{|v(t)|^2 \, dt}{t^2}\right)^{\frac{1}{2}} \left(\int_0^\beta |v'(t)|^2 \, dt\right)^{\frac{1}{2}}
\end{aligned}
$$

from what (7.21) follows. The estimates (7.19) and (7.20) imply that

$$\frac{1}{(y-y_0)^2} \leq c_{13}\left(-\frac{d^2}{dy^2} - 1\right), \tag{7.24}$$

which in terms of the quadratic form means that

$$\int_0^\pi \frac{|v(y)|^2 \, dy}{(y-y_0)^2} \leq c_{13} \int_0^\pi |v'(y)|^2 - |v(y)|^2 \, dy, \tag{7.25}$$

holds for all $v \in H_0^1(0,\pi)$ such that $v(y_0) = 0$. The substitution $v(y) = u(y)\sin y$ implies that $u \in H^1(0,\pi)$ and that $u(y_0) = 0$. From (7.25) we get

$$\int_0^\pi \frac{|u(y)|^2 \sin^2 y\, dy}{(y-y_0)^2} \;\leq\; c_{13} \int_0^\pi |u'(y)|^2 \sin^2 y\, dy, \tag{7.26}$$

for functions $u \in H^1(0,\pi)$ such that $u(y_0) = 0$. □

Now we are in position to prove Theorem 7.1. Since the method used in the proof doesn't give a sharp constant we will not put an effort in using optimal inequalities with the risk of being lost in technicalities.

Proof of Theorem 7.1. If we substitute $v(x,y) = u(x,y)\sin y$ then inequality (7.5) becomes

$$c_{AB} \int_\Omega \frac{|u|^2 \sin^2 y\, dx\, dy}{x^2 + (y-y_0)^2} \;\leq\; \int_\Omega |-\mathrm{i}\nabla u + Au|^2 \sin^2 y\, dx\, dy. \tag{7.27}$$

We need to prove the inequality (7.27) for all $u \in C^\infty(\overline{\Omega}) \cap L^2(\Omega)$ such that $u = 0$ in a neighborhood of the point p.

Define for $R \in (0, \mathrm{dist}(y_0, \partial\Omega))$ the set $\Omega_R = (-R,R) \times (0,\pi)$ and let $h_\pm(x) = y_0 \pm \sqrt{R^2 - x^2}$. Assume $x \in (-R,R)$, $x \neq 0$, ψ is defined by (3.21) and let $u \in C^\infty(\overline{\Omega}) \cap L^2(\Omega)$ such that $u = 0$ in a neighborhood of the point p. Since $u(x,\cdot)\psi \in H^1(0,\pi)$ we have by Lemma 7.3 that

$$\begin{aligned}
\int_0^\pi \frac{|u|^2 \sin^2 y\, dy}{x^2 + (y-y_0)^2} \;&\leq\; 2c_{13} \int_0^\pi |u_y\psi + u\psi'|^2 \sin^2 y\, dy + 2\int_{h_-(x)}^{h_+(x)} \frac{|u|^2 \sin^2 y\, dy}{x^2 + (y-y_0)^2} \\
&\leq\; 4c_{13} \int_0^\pi |u_y|^2 \sin^2 y\, dy \\
&\quad + \left(2 + \frac{4c_{13}R^2}{R^2 - x^2}\right) \int_{h_-(x)}^{h_+(x)} \frac{|u|^2 \sin^2 y\, dy}{x^2 + (y-y_0)^2},
\end{aligned} \tag{7.28}$$

where c_{13} is given by (7.8). Thus the inequality

$$\begin{aligned}
\int_0^\pi \frac{|u|^2 (R^2 - x^2) \sin^2 y\, dy}{x^2 + (y-y_0)^2} \;&\leq\; 4c_{13}R^2 \int_0^\pi |u_y|^2 \sin^2 y\, dy \\
&\quad + 2R^2(1 + 2c_{13}) \int_{h_-(x)}^{h_+(x)} \frac{|u|^2 \sin^2 y\, dy}{x^2 + (y-y_0)^2},
\end{aligned} \tag{7.29}$$

holds. By continuity the inequality can be extended to $u(x,\cdot) \in H^1(0,\pi)$. We will make use of the diamagnetic inequality (3.24), for functions $v \in H^1_{0,A}(\Omega \setminus \{p\})$. Let $u(x,\cdot) = |w(x,\cdot)|$, where $w \in C^\infty(\overline{\Omega}) \cap L^2(\Omega)$ such that $w = 0$ in a neighborhood of the point p, then $u(x,\cdot) \in H^1(0,\pi)$ and by Lemma 7.2, (3.24) and (7.29) we have

$$\int_{\Omega_R} \frac{|w|^2 (R^2 - x^2) \sin^2 y\, dx\, dy}{x^2 + (y-y_0)^2} \;\leq\; c_{14} \int_{\Omega_R} |-\mathrm{i}\nabla w + Aw|^2 \sin^2 y\, dx\, dy, \tag{7.30}$$

where the constant

$$c_{14} = \frac{4R^2\Psi^2 c_{13} + 2R^2 + 4R^2 c_{13}}{\Psi^2 \cos^2(|y_0 - \frac{\pi}{2}| + R)}. \tag{7.31}$$

Let $m = \frac{R}{\sqrt{2}}$ and define φ by (3.28). For $u \in C^\infty(\overline{\Omega}) \cap L^2(\Omega)$ such that u vanishes in a neighborhood of the point p, $y \in (0,\pi)$, $y \neq y_0$, we write $u = u\varphi + u(1-\varphi)$ and use (3.27) to get

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{|u|^2\,dx}{x^2+(y-y_0)^2} &\leq 16\int_{-\infty}^{\infty}|u_x|^2\,dx + 16\int_{-m}^{m}|u\psi'|^2\,dx \\
&\quad + 2\int_{-m}^{m}\frac{|u|^2\,dx}{x^2+\left(y-\frac{\pi}{2}\right)^2} \qquad (7.32)\\
&= 16\int_{-\infty}^{\infty}|u_x|^2\,dx + c_{15}\int_{-m}^{m}\frac{|u|^2\,dx}{x^2+(y-y_0)^2},
\end{aligned}
$$

where $c_{15} = 18+\frac{32\pi^2}{R^2}$. Since $y \neq y_0$ the inequality can by continuity be extended to functions $u(\cdot,y) \in H_0^1(\mathbb{R})$. By using (3.24) one gets

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{|u|^2\,dx}{x^2+(y-y_0)^2} &\leq 16\int_{-\infty}^{\infty}|-\mathrm{i}\nabla u + Au|^2\,dx \qquad (7.33)\\
&\quad + c_{15}\int_{-m}^{m}\frac{|u|^2\,dx}{x^2+(y-y_0)^2},
\end{aligned}
$$

for all $u \in C^\infty(\overline{\Omega})\cap L^2(\Omega)$ such that $u=0$ in a neighborhood of p. Combining the inequalities (7.30) and (7.33) we have

$$
\int_\Omega \frac{|u|^2\sin^2 y\,dx\,dy}{x^2+(y-y_0)^2} \leq c_{16}\int_\Omega |-\mathrm{i}\nabla u + Au|^2\sin^2 y\;dx\,dy, \qquad (7.34)
$$

where the constant $c_{16} = 16+\frac{2c_{14}c_{15}}{R^2}$. This proves the inequality (7.27) with the constant $c_{AB} = c_{16}^{-1}$. □

7.2 Locally deformed waveguides

Let f be a non-negative function in $C_0^1(\mathbb{R})$ and for $\lambda \geq 0$ we define

$$
\Omega_\lambda = \big\{(x,y)\in\mathbb{R}^2 : 0<y<\pi+\lambda\pi f(x)\big\}\setminus\{p\}, \qquad (7.35)
$$

where $p=(0,y_0)$. Let M_d be the Friedrich's extension of the symmetric, semi-bounded operator

$$
(-\mathrm{i}\partial_s + a_1(s,t))^2 + (-\mathrm{i}\partial_t + a_2(s,t))^2\,, \qquad (7.36)
$$

on the domain $C_0^\infty(\Omega_\lambda)$, where the magnetic vector potential is for $\Phi\in\mathbb{R}\setminus\mathbb{Z}$ is defined by (7.1). For simplicity we assume that $\operatorname{supp} f \subset [\frac{\pi}{2},\infty)$. Since $\operatorname{div} A = 0$ and $|A| \in L^2((1,\infty)\times(0,\pi))$ we have by Theorem 4.1 that the essential spectrum of M_d equals $[1,\infty)$. The following Theorem says that the spectrum of M_d is stable under small deformations.

Theorem 7.4. *There exists a value λ_0 depending on $\|f\|_\infty$ and $\|f'\|_\infty$ such that for $\lambda\in(0,\lambda_0)$ the discrete spectrum of M_d is empty.*

Proof. Let the unitary mapping U_λ be given by (5.6) and (5.7). The operator M_d is unitary equivalent to

$$
M_\lambda := U_\lambda M_d U_\lambda^{-1}, \qquad (7.37)
$$

defined on the set $U_\lambda\mathcal{D}(M_d)$ in $L^2(\Omega)$. The quadratic form associated with M_d is

$$
\mathfrak{q}_d[\psi] = \int_{\Omega_\lambda} |-\mathrm{i}\psi_s + a_1\psi|^2 + |-\mathrm{i}\psi_t + a_2\psi|^2\,ds\,dt, \qquad (7.38)
$$

defined on $\mathcal{D}(\mathfrak{q}_d) = H_{0,A}^1(\Omega_\lambda)$. Hence the form associated with M_λ is

$$
\mathfrak{q}_\lambda[\varphi] = \mathfrak{q}_d[U_\lambda^{-1}\varphi] \qquad (7.39)
$$

defined on the space $\mathcal{D}(\mathfrak{q}_\lambda) = U_\lambda \mathcal{D}(\mathfrak{q}_d)$.

Since $\sigma_{\rm ess}(M_\lambda) = \sigma_{\rm ess}(M_d) = [1,\infty)$ it will be enough to prove that $M_\lambda - 1$ is non-negative. Let $g(s) = 1+\lambda f(s)$ and let $\mathfrak{q}$ be the quadratic form associated with the Schrödinger operator with the magnetic vector potential $\tilde{A}$ in the space $L^2(\Omega_0)$. Without loss of generality we assume that $\lambda \le 1$. It follows from (5.13) that

$$\begin{aligned}
\mathfrak{q}_\lambda[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} &= \mathfrak{q}[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \\
&\quad + \int_{\Omega_0} \Bigg(\frac{y^2\lambda^2 (f')^2 - 2\lambda f - \lambda^2 f^2}{g^2} |\varphi_y|^2 - \frac{1}{4}\left(\frac{\lambda f'}{g}\right)^2 |\varphi|^2 \\
&\quad - \frac{y\lambda f'}{g}(\varphi_x \overline{\varphi_y} + \varphi_y \overline{\varphi_x}) - \frac{\lambda f'}{2g}(\varphi \overline{\varphi_x} + \varphi_x \overline{\varphi}) \\
&\quad + \mathrm{i}\lambda \frac{y f' \tilde{a}_1 + f \tilde{a}_2}{g}(\varphi_y \overline{\varphi} - \varphi \overline{\varphi_y}) \Bigg)\, dx\, dy \\
&\ge \mathfrak{q}[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \\
&\quad - \lambda \int_{\Omega_0} \chi \cdot \left(c_{17}\left(|\varphi_x|^2 + |\varphi_y|^2\right) + \left(c_{18} + c_{19}(\tilde{a}_1^2 + \tilde{a}_2^2)\right)|\varphi|^2 \right) dx\, dy,
\end{aligned} \tag{7.40}$$

where $c_{17} = 2\pi\|f'\|_\infty + 3\|f\|_\infty + \|f\|_\infty^2$, $c_{18} = \frac{1}{4}\|f'\|^2_\infty + \frac{1}{2}\|f'\|_\infty$, $c_{19} = \pi\|f'\|_\infty + \|f\|_\infty$ and χ is the characteristic function of the support of f. From (5.17) we get

$$\begin{aligned}
\mathfrak{q}_\lambda[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} &\ge \mathfrak{q}[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \\
&\quad - \lambda \int_{\Omega_0} \chi \Bigg(2c_{17}\left(|-\mathrm{i}\nabla\varphi + \tilde{A}\varphi|^2 - |\varphi|^2 \right) \\
&\quad + \left(\frac{(2c_{17} + c_{18})(d^2+\pi^2)}{x^2 + (y-y_0)^2} + (2c_{17} + c_{19})(\tilde{a}_1^2 + \tilde{a}_2^2) \right) |\varphi|^2 \Bigg)\, dx\, dy,
\end{aligned} \tag{7.41}$$

where $d = \max \operatorname{supp} f$. We use the pointwise inequality

$$\chi(x) \cdot \left(\tilde{a}_1^2(x,y) + \tilde{a}_2^2(x,y)\right) \le \frac{4\Phi^2\left(d^2+\pi^2\right)}{\pi^2 (x^2 + (y-y_0))^2} \tag{7.42}$$

to get

$$\begin{aligned}
\mathfrak{q}_\lambda[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} &\ge \mathfrak{q}[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} - 2\lambda c_{17}\left(\mathfrak{q}[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)}\right) \\
&\quad - \lambda \int_{\Omega_0} \frac{c_{20} d^2 + c_{21}}{x^2 + (y-y_0)^2} |\varphi|^2\, dx\, dy,
\end{aligned} \tag{7.43}$$

where $c_{20} = 2c_{17} + c_{18} + 4\Phi^2\pi^{-2}(2c_{17} + c_{19})$ and $c_{21} = \pi^2(2c_{17} + c_{18}) + 4\Phi^2(2c_{17} + c_{19})$. From Theorem 7.1 we have

$$\begin{aligned}
\mathfrak{q}_\lambda[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} &\ge \left(\frac{1}{2} - 2\lambda c_{17}\right)\left(\mathfrak{q}[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)}\right) \\
&\quad + \left(\frac{c_{AB}}{2} - \lambda\left(c_{20} d^2 + c_{21}\right)\right) \int_{\Omega_0} \frac{|\varphi|^2}{x^2 + (y-y_0)^2}\, dx\, dy \\
&\ge 0,
\end{aligned}$$

for $\lambda \in (0, \lambda_0)$, where c_{AB} is the constant from (7.6) and

$$\lambda_0 = \frac{c_{AB}}{2(c_{20} d^2 + c_{21})}. \tag{7.44}$$

$\square$

7.3 Locally curved waveguides

Let A be given as in (7.1) and let Ω_γ be defined by (6.1) – (6.5) with the additional assumption that $a(x) = x$ and $b(x) = 0$ for $x \leq \frac{\pi}{2}$. To be able to study weak curvatures we replace γ by $\beta\gamma$ for arbitrary $\beta \geq 0$. We denote by Ω_β the set $\Omega_{\beta\gamma}$.

$$\mathfrak{q}_c[\psi] := \int_{\Omega_\beta} |-\mathrm{i}\psi_s + a_1\psi|^2 + |-\mathrm{i}\psi_t + a_2\psi|^2 \, ds\, dt, \tag{7.45}$$

be defined on $\mathcal{D}(\mathfrak{q}_c) = H^1_{0,A}(\Omega_\beta)$. Then $\mathfrak{q}_c$ is the quadratic form associated with the Friedrich's extension M_c of the the symmetric, semi-bounded operator

$$(-\mathrm{i}\partial_s + a_1(s,t))^2 + (-\mathrm{i}\partial_t + a_2(s,t))^2, \tag{7.46}$$

defined on $C_0^\infty(\Omega_\beta)$. For simplicity we assume that $\operatorname{supp}\gamma \subset [\frac{\pi}{2}, \infty)$. By Theorem 4.1 we get that the essential spectrum of M_c equals $[1, \infty)$.

Theorem 7.5. *There exists a positive number β_0 such that for $\beta \in (0, \beta_0)$ the discrete spectrum of M_c is empty.*

Proof. Denote by M_β the operator $U_\beta M_c U_\beta^{-1}$, where U_β is defined in (6.10) and (6.11). Let $\mathfrak{q}_\beta$ be the form associated with M_β defined on the domain $\mathcal{D}(\mathfrak{q}_\beta) = U_\beta \mathcal{D}(\mathfrak{q}_c)$. Following the calculations in (6.15) – (6.19) we get

$$\begin{aligned}
\mathfrak{q}_\beta[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \;=\;& \mathfrak{q}[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \\
& - \int_{\Omega_0} \left(\frac{2y\beta\gamma + y^2\beta^2\gamma^2}{1 + y\beta\gamma} |\varphi_x|^2 - \mathrm{i}y\beta\gamma(a'\tilde{a}_1 + b'\tilde{a}_2)(\varphi_x\overline{\varphi} - \varphi\overline{\varphi_x}) \right. \\
& -\mathrm{i}y \left(-\beta\gamma b'\tilde{a}_1 + \beta\gamma a'\tilde{a}_2 + \frac{a''\tilde{a}_1 - b''\tilde{a}_2}{1 + y\beta\gamma} \right) (\varphi_y\overline{\varphi} - \varphi\overline{\varphi_y}) \\
& + \frac{y\beta\gamma'}{2(1 + y\beta\gamma)^3} (\varphi\overline{\varphi_x} + \varphi_x\overline{\varphi}) + \frac{\beta\gamma}{2(1 + y\beta\gamma)} (\varphi\overline{\varphi_y} + \varphi_y\overline{\varphi}) \\
& \left. - \left(\frac{y^2\beta^2(\gamma')^2}{4(1 + y\beta\gamma)^4} + \frac{\beta^2\gamma^2}{4(1 + y\beta\gamma)^2} \right) |\varphi|^2 \right) dx\, dy.
\end{aligned} \tag{7.47}$$

Without loss of generality we can assume that $\beta \leq 1$, hence

$$\begin{aligned}
\mathfrak{q}_\beta[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \;\geq\;& \mathfrak{q}[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \\
& -\beta \int_{\Omega_0} \chi \left(c_{22} \left(|\varphi_x|^2 + |\varphi_y|^2 \right) + \left(c_{23} + c_{24}(\tilde{a}_1^2 + \tilde{a}_2^2) \right) |\varphi|^2 \right) dx\, dy,
\end{aligned} \tag{7.48}$$

where

$$c_{22} = 3\pi\|\gamma\|_\infty + \pi^2\|\gamma\|^2_\infty + 2^{-1}\pi\|\gamma'\|_\infty, \tag{7.49}$$

$$c_{23} = 2^{-1}(\|\gamma\|_\infty + \pi\|\gamma'\|_\infty), \tag{7.50}$$

$$c_{24} = \pi(1 + 2\|\gamma\|_\infty). \tag{7.51}$$

By the inequality (5.17), Theorem 7.1 and the fact that

$$\chi(x)(\tilde{a}_1^2(x,y) + \tilde{a}_2^2(x,y)) \leq \frac{d^2 + \pi^2}{(\mathrm{dist}(y_0, \partial\Omega_0))^2 \,(x^2 + (y - y_0)^2)}, \tag{7.52}$$

where $d = \max \operatorname{supp}\gamma$ we obtain

$$\begin{aligned}
\mathfrak{q}_\beta[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \;\geq\;& \left(\frac{1}{2} - 2\beta c_{22} \right) \left(\mathfrak{q}[\varphi] - \|\varphi\|^2_{L^2(\Omega_0)} \right) \\
& \left(\frac{c_{AB}}{2} - \beta c_{25} \right) \int_{\Omega_0} \frac{|\varphi|^2}{x^2 + (y - y_0)^2} \, dx\, dy,
\end{aligned} \tag{7.53}$$

where

$$c_{25} \quad = \quad (d^2+\pi^2)\left(2c_{22}+c_{23}+(\mathrm{dist}(y_0,\partial\Omega_0))^{-2}(2c_22+c_24)\right). \tag{7.54}$$

If we choose

$$\beta_0 = \frac{c_{AB}}{2c_{25}}, \tag{7.55}$$

it follows that the right hand side of 7.53 is positive. □

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