A New Approach to the Josephson Effect

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Abstract

We introduce a new approach to the Josephson effect in SIS tunnel junctions. The Josephson coupling energy is calculated from the overlap of real space Cooper pair wavefunctions in two superconductors through an insulating barrier. It is shown that the Josephson tunneling is limited by the size of the Cooper pair and its shrinking during the tunneling. Therefore, the Josephson coupling energy and the critical current become extremely small in high T_c superconductors, including MgB_2. This shrinking also causes the observed DC supercurrent in low T_c superconductors, such as Pb and Sn, to fall off much faster than $1/R_n$ for tunneling resistance R_n above several ohms. Consequently there is a material-dependent threshold resistance, above which the supercurrent decreases much faster with increasing resistance. The impurity-induced shrinking is also shown to limit the critical current. Furthermore, the (weak) temperature dependence of the Cooper pair size is found to contribute to the temperature dependence of the DC supercurrent.

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I. INTRODUCTION

The Josephson effect may be the most fascinating and intriguing property of superconductors.[1,2] In 1962 Josephson predicted that supercurrents can flow through the insulating barrier between two superconductors due to the Cooper pair tunneling.[1] The supercurrent depends on the relative phase of two superconductors, where the phase is associated with the order parameter of the superconducting condensed state. The prediction was confirmed in experiments quickly.[3,4] It is interesting that the experiment by Nicol et al.[5] showed already the possible DC supercurrent, although its reality was not noticed. This discovery led to the fabrication of the Josephson junction devices and the high precision measurement of fundamental constants.[6] Now it seems that our understanding of the Josephson effect is on firm ground.[6,7,8]

Josephson[1,2] used the tunneling Hamiltonian[9] to calculate tunneling currents between two superconductors and noticed that even at zero applied voltages, a DC supercurrent can occur. At finite voltages V, he found an AC supercurrent of the same amplitude as that of the DC supercurrent with frequency 2eV/h. The amplitude of the supercurrent was calculated by Anderson[7] and Ambegaokar and Baratoff.[10] At $T = 0K$ their result of the DC supercurrent, j, is

$$j = j_1 sin\phi, \tag{1}$$

with

$$j_1 = \frac{\pi}{2e}\frac{\Delta}{R_n}. \tag{2}$$

Here ϕ is the phase difference of the two superconductors with the same energy gap Δ. Note that the amplitude of the supercurrent is proportional to the energy gap and inversely proportional to the tunneling resistance.

However, there are still some fundamental experiments which remain puzzling.[11,12,13,14,15] For instance, the pair-quasi particle interference term was shown to be negative in experiments, whereas the theory predicts positive sign.[11] For AC Josephson effect, Likharev even claimed that *there are more questions than answers*.[12] Furthermore, high T_c cuprate Josephson junctions are mainly SNS type junctions, since SIS junctions do not show the Josephson effect.[13] This result is not consistent with the above theory, which predicts bigger Josephson current for higher T_c.[1,2,7,8,10] Recently, this behavior has also been found in MgB_2 SIS junctions.[14,15] In particular, the Josephson supercurrent was not observed for the big gap ($\sim 7meV$) of MgB_2. We stress that this behavior is closely related to the anomalous dependence of the maximum DC Josephson current on the tunneling resistance R_n, found in low T_c superconductors, such as Pb and Sn:[16,17,18,19] the Josephson current decreases much faster than $1/R_n$ above several Ω.

In this rapid communication we present Cooper pair wavefunction approach to the Josephson effect. It is shown that the Josephson coupling energy is determined by the overlap of the Cooper pair wavefunctions of two superconductors divided by a thin insulating layer. (In fact, this idea was suggested by Josephson in his original paper.[1,10] However, it was not pursued thoroughly.) The critical current is, then, calculated from the coupling energy E_J in the usual way.[2] Since the Cooper pair tunneling can be understood more easily from the Cooper pair wavefunction, this method is advantageous to deal with the Josephson effect in high T_c superconductors including MgB_2 and for the insulating barriers with high tunneling resistance R_n. We have found that the Cooper pair tunneling is strongly limited by the size of the Cooper pair and its reduction during the tunneling. Since the Cooper pair size is also reduced by the impurity potential scattering, ordinary impurity also limits the supercurrent. As a result, the Josephson current is much smaller than expected in higher T_c SIS junctions and falls off much faster than $1/R_n$ for thicker and/or higher insulating barriers, in good agreement with the experimental findings. The preliminary result was reported before.[20]

II. COOPER PAIR WAVEFUNCTION APPROACH TO JOSEPHSON TUNNELING : T=0K

A. Josephson Coupling Energy

The approximate expression of the supercurrent suggested by Josephson[1] is

$$j \cong \frac{1}{2} j_1 \psi_l^* \psi_r + \frac{1}{2} j_1 \psi_r^* \psi_l, \tag{3}$$

where ψ_l and ψ_r are the effective superconducting wavefunctions on the left and right sides, respectively.[21,22] Actually, it can be exact and it is better to write first the Josephson coupling energy E_J in terms of the Cooper pair wavefunctions and then calculate the supercurrent j from E_J:[2]

$$j = \frac{2e}{\hbar} \frac{\partial E_J}{\partial \phi}. \tag{4}$$

Figure 1 shows the real space Cooper pair wavefunctions of two superconductors with the same energy gap near the insulating barrier. The solid lines denote the Cooper pair wavefunctions in the absence of the barrier, whereas the thick lines denote the change of the Cooper pair wavefunctions due to the insulating barrier. The Josephson coupling energy is, therefore, determined by the overlap of the Cooper pair wavefunctions, connected by the phonon Green's function.[23,24] Previously, this idea was employed in some cases.[25,26] For instance, for a pure superconductor without a barrier with the Einstein phonons, the coupling energy or the pairing energy is given by

$$\begin{aligned} E_{int} &= V \int \int F^*(x,y) F(x,y) \delta(x-y) dx dy \\ &= V \sum_k \sum_{k'} u_k v_k u_{k'} v_{k'}, \end{aligned} \tag{5}$$

where the (effective) Cooper pair wavefunctions $F(x,y) (= \sum_k u_k v_k e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{y})})$ and $F^*(x,y)$ are coupled by the Einstein phonon Green's function, i.e., the Dirac delta function, $\delta(x-y)$. Here V is the phonon-mediated matrix element. In the same way, the Josephson coupling energy E_J is given by

$$E_J = V\int\int F_r^*(x,y)F_l(x,y)\delta(x-y)dxdy + V\int\int F_l^*(x,y)F_r(x,y)\delta(x-y)dxdy$$
$$= V\int F_r^*(x)F_l(x)dx + V\int F_l^*(x)F_r(x)dx, \tag{6}$$

with F_l and F_r being the effective Cooper pair wavefunctions in left and right sides. Observe that this expression has the same form as the approximate one, Eq. (3) except the integration. Figure 1 shows that it is essential to calculate the tail of the Cooper pair wavefunctions to determine the Josephson coupling energy.

It is necessary to emphasize that Eq. (6) leads to the familiar result in k-space. From the tunneling Hamiltonian[9]

$$H_T = \sum_{kk's}(T_{kk'}c^{r+}_{k's}c^l_{ks} + H.C.) \tag{7}$$

one finds the Josephson pair tunneling Hamiltonian, H_J,[27,28,29]

$$H_J = -\sum_{kq}(J_{kq}c^{r+}_{k\uparrow}c^{r+}_{-k\downarrow}c^l_{-q\downarrow}c^l_{q\uparrow} + H.C.). \tag{8}$$

Then, the coupling energy is

$$E_J = -J(\sum_k u^r_k v^{r*}_k \sum_q u^l_q v^l_q + \sum_k u^r_k v^r_k \sum_q u^l_q v^{l*}_q)$$
$$= -J(\sum_{kq} F^{r*}_k F^l_q + \sum_{kq} F^r_k F^{l*}_q) \tag{9}$$

with $J_{kq} \equiv J$. Comparing Eqs. (6) and (9), it is evident that J is proportional to the phonon-mediated matrix element V, i.e.,

$$J \propto V. \tag{10}$$

Accordingly, it is crucial to note that J_{kq} is the pair scattering matrix element across the barrier due to both the tunneling and the electron-phonon interaction. If we define

$$\Psi_r \equiv \sum_k F^r_k = |\Psi_r|e^{i\phi_r}$$
$$\Psi_l \equiv \sum_q F^l_q = |\Psi_l|e^{i\phi_l} \tag{11}$$

Eq. (9) is rewritten

$$E_J = -J(\Psi_l^*\Psi_r + \Psi_l\Psi_r^*) = -2J|\Psi_l||\Psi_r|cos(\phi_l - \phi_r). \tag{12}$$

It should be noticed that the result is formally the same as the internal Josephson coupling energy in two-band superconductors,[27,28] and believed to reproduce the Josephson coupling energy E_J in conventional SIS junctions.[27,28,29] However, the exact expression of $J_{kq} \equiv J$ is missing, although the (2nd order) tunneling-induced pair transfer term (J^T_{kq}),

$$J^T_{kq} = \frac{|T_{kq}|^2}{E^l_k + E^r_q}, \tag{13}$$

was confused with J.[27,29] Here E^l_k and E^r_q are quasi-particle energies. It would be dangerous indeed to suppose that $J_{kq} \equiv J$ (phonon-mediated pair-scattering matrix element across the barrier) and J^T_{kq} (tunneling-induced only pair-scattering matrix element) are the same when they are not. Note that J^T_{kq} is not proportional to V, unlike J, so Eq. (9) or Eq. (12) with J^T_{kq} does not reproduce the previous result, Eq. (2), which remains puzzling.[29]

B. k-space Approach

Now we need to compute Eq. (6) or Eq. (9) to find the Josephson coupling energy E_J and the Josephson current j. It is clear that both equations should lead to the same result. First, we consider Eq. (9), which requires the calculation of the pair-scattering matrix element J_{kq}. For simplicity, we assume two superconductors with the same gap, Δ, which will be generalized later. From Eq. (8) we obtain

$$J_{kq} = ^r< k\uparrow, -k\downarrow |V_{e-ph}| -q\downarrow, q\uparrow >^l \tag{14}$$

where V_{e-ph} is the phonon-mediated electron-electron interaction. Tunneling effect is included in the states of the left and right sides. It is noteworthy that this situation is much the same as the calculation of the matrix element between scattered-state pairs in Anderson's theory of dirty superconductors,[30,31] where impurity effect is included in the scattered states and the electron-phonon interaction couples the states.

Using the Tunneling Hamiltonian, Eq. (7), the one-particle wavefunctions are given by

$$\begin{aligned} \psi^l_q(r) &= N_q[\phi^l_q(r) + \sum_k \frac{T_{kq}}{\epsilon_q - \epsilon_k}\phi^r_k(r)] \\ \psi^r_k(r) &= N_k[\phi^r_k(r) + \sum_q \frac{T_{qk}}{\epsilon_k - \epsilon_q}\phi^l_q(r)] \end{aligned} \tag{15}$$

where ϕ^l_q and ϕ^r_k are the left- and right-hand states[32] with ϵ_q and ϵ_k being the electron energies. N_k and N_q are the normalization constants. Prange[32] proposed the *nonorthogonal almost confined states* for the left- and right-hand states, whereas Bardeen[33] introduced the WKB wavefunction,

$$\phi^l_m = Cp_x^{-1/2}e^{i(p_y y + p_z z)}sin(p_x x + \gamma), \qquad x < x_a \tag{16}$$

$$\phi^l_m = \frac{1}{2}C|p_x|^{-1/2}e^{i(p_y y + p_z z)}exp(-\int_{x_a}^{x}|p_x|dx), \quad x_a < x < x_b \tag{17}$$

where C is a normalization constant and $|p_x| = (2\mu U - p_y^2 - p_z^2)^{1/2}$ with $U(x)$ being the potential energy. μ is the electron mass. The barrier extends from x_a to x_b. Notice that the WKB wavefunction is a sinusoidal stationary wave (in x-direction). Thus, we have degenerate pairs $m = (p_x, p_y, p_z)$ and $\bar{m} = (p_x, -p_y, -p_z)$.

For the Einstein phonon model,[23,24,31] the pair-scattering matrix element is

$$\begin{aligned} J_{kq} &= V\int dr \psi^{r*}_k(r)\psi^{r*}_{-k}(r)\psi^l_{-q}(r)\psi^l_q(r) \\ &= V\int dr|\psi^r_k(r)|^2|\psi^l_q(r)|^2, \end{aligned} \tag{18}$$

which denotes the (density) correlation function between the eigenstates $\psi^l_q(r)$ and $\psi^r_k(r)$. Since the Bardeen's wavefunction is not much different from a plane wave, we may employ plane waves for ϕ^l_q and ϕ^r_k to obtain the matrix element,

$$J_{kq} \cong V\frac{1}{\Omega_r}\sum_k \frac{|T_{kq}|^2}{(\epsilon_q - \epsilon_k)^2} + V\frac{1}{\Omega_l}\sum_q \frac{|T_{qk}|^2}{(\epsilon_k - \epsilon_q)^2}, \tag{19}$$

where Ω_l and Ω_r are the volumes of the left and right sides. Unfortunately, the sums are divergent, which are common in the perturbation theory for continuous spectra. The remedy is to use the scattering theory with a cutoff (ξ_0), consistent with the bound state of Cooper pairs.[34] Since each sum denotes the relative probability contained in the virtual scattered wavelets (due to the barrier) within the BCS coherence length $\xi_0 = \hbar v_F/\pi\Delta$ (compared to the plane wave part),[34] we find

$$\frac{1}{\Omega_r}\sum_k \frac{|T_{kq}|^2}{(\epsilon_q-\epsilon_k)^2} \approx <|T_{kq}|^2> \frac{N_F\pi^2}{2\hbar v_F}\xi_0 =<|T_{kq}|^2> \frac{N_F\pi}{2\Delta}$$
$$\frac{1}{\Omega_l}\sum_q \frac{|T_{qk}|^2}{(\epsilon_k-\epsilon_q)^2} \approx <|T_{qk}|^2> \frac{N_F\pi^2}{2\hbar v_F}\xi_0 =<|T_{qk}|^2> \frac{N_F\pi}{2\Delta} \tag{20}$$

and

$$< J_{kq} >\equiv J \approx V < |T|^2 > \frac{N_F\pi}{\Delta}. \tag{21}$$

Here N_F is the density of states at the Fermi level and the angular brackets indicate the average value over the states.

Consequently, substituting Eq. (21) to Eq. (12) the Josephson coupling energy is written

$$E_J \approx -\frac{2\pi}{\lambda} < |T|^2 > N_F^2\Delta\ cos(\phi_l - \phi_r), \tag{22}$$

where $\lambda = N_F V$. Accordingly, the Josephson supercurrent is

$$j = \frac{1}{\lambda}\frac{\Delta}{eR_n}\ sin(\phi_l - \phi_r), \tag{23}$$

with

$$j_1 = \frac{1}{\lambda}\frac{\Delta}{eR_n}. \tag{24}$$

Although this equation for the maximum DC supercurrent is similar to the previous one, Eq. (2), physically it is significantly different. The factor $1/\lambda$ shows that the Josephson coupling energy and the supercurrent depend on the superconductor. (Notice that this factor also appears in the initial T_c decreases due to magnetic impurities[34] and weak localization.[31]) As a result, if we include the reduction of the Cooper pair size during the tunneling, which is important especially for superconductors with high T_c, the resulting supercurrent is not similar at all.

When the superconductors have different energy gaps, Δ_1 and Δ_2, the pair-scattering matrix elements are

$$< J_{kq} >\equiv J \approx V_1 < |T|^2 > \frac{N_{F,1}\pi}{2\Delta_2} + V_2 < |T|^2 > \frac{N_{F,2}\pi}{2\Delta_1} \tag{25}$$

where $N_{F,1}$ and $N_{F,2}$ are the density of states of left and right sides, respectively. Subsequently, one finds

$$E_J = -\pi < |T|^2 > N_{1,F} N_{F,2} \Delta_1 \Delta_2 (\frac{1}{\lambda_1 \Delta_1} + \frac{1}{\lambda_2 \Delta_2}) cos(\phi_l - \phi_r), \quad (26)$$

and

$$j = \frac{2\pi}{\hbar} e < |T|^2 > N_{F,1} N_{F,2} \Delta_1 \Delta_2 (\frac{1}{\lambda_1 \Delta_1} + \frac{1}{\lambda_2 \Delta_2}) sin(\phi_l - \phi_r), \quad (27)$$

with

$$j_1 = \frac{1}{2e} \frac{1}{R_n} \Delta_1 \Delta_2 (\frac{1}{\lambda_1 \Delta_1} + \frac{1}{\lambda_2 \Delta_2}). \quad (28)$$

Here λ_1 and λ_2 are the BCS coupling constants for two superconductors.

C. Cooper Pair Wavefunction Approach

We consider now Eq. (6), the Josephson coupling energy in terms of the Cooper pair wavefunctions. This method is more appropriate for high T_c superconductors, including MgB_2, which have tightly-bound Cooper pairs with small size. It is also desirable for the insulating barrier with high tunneling resistance R_n, since the Cooper pair size shrinks significantly during the tunneling in that case. A similar reduction of the Cooper pair size caused by the ordinary impurity scattering requires this approach, too.

The Cooper pair wavefunctions are expressed in terms of the one-particle wavefunctions, Eq. (15):

$$F_l(x) = \sum_q u_q v_q \psi_q^l(x) \psi_{-q}^l(x) \quad (29)$$

$$F_r(x) = \sum_k u_k v_k \psi_k^r(x) \psi_{-k}^r(x). \quad (30)$$

Substituting Eqs. (29) and (30) into Eq. (6), it is straightforward to show that

$$E_J = V(\frac{1}{\Omega_r} \sum_k \frac{|T_{kq}|^2}{(\epsilon_q - \epsilon_k)^2} + \frac{1}{\Omega_l} \sum_q \frac{|T_{qk}|^2}{(\epsilon_k - \epsilon_q)^2})(\sum_{kq} u_k^r v_k^{r*} u_q^l v_q^l + \sum_{kq} u_q^l v_q^{l*} u_k^r v_k^r) \quad (31)$$

which is indeed the same as that obtained by the k-space approach, i.e., Eq. (9) with Eq. (19). Whereas this real space approach allows us to take into account the change of the Cooper pair size during the tunneling through the Cooper pair wavefunctions.

If we insert the Bardeen's WKB wavefunction into Eq. (29), the left-side Cooper pair wavefunction shows the exponential decay in the barrier region:

$$F_l(x) = \frac{C^2}{4} (\sum_p u_p v_p) |p_x^0|^{-1} e^{-2\kappa(x - x_a)} \quad x_a < x < x_b \quad (32)$$

where $|p_x^0| = \hbar\kappa = (2\mu U_0 - p_y^2 - p_z^2)^{1/2}$ with U_0 being the potential energy. Beyond x_b, the tail of the Cooper pair wavefunction should be, after utilizing the one-particle wavefunction, Eq. (15),

$$F_l(x) = \sum_q u_q v_q [\sum_k \frac{|T_{kq}|^2}{(\epsilon_q - \epsilon_k)^2} \phi^r_k(x) \phi^r_{-k}(x)] \quad x > x_b. \tag{33}$$

We have used the fact that only (ϕ^r_k, ϕ^r_{-k}) pair contributes to the coupling energy, Eq. (31). (It is obvious that $|T_{kq}|^2 \propto e^{-2\kappa d}$, with the barrier thickness $d = x_b - x_a$.[35,36]) However, since ϕ^r_k and ϕ^r_{-k} are basically plane-wave-like, Eq. (33) shows not the presumed exponential tail but a constant amplitude. Of course, the divergence in the sum implies this behavior needs to be corrected according to the bounded Cooper pair wavefunction. Therefore, we assume

$$F_l(x) \approx < |T_{kq}|^2 > \sum_q u_q v_q \frac{N_F \pi^2}{2\hbar v_F} e^{-(x-x_b)/\xi_0} \quad x > x_b. \tag{34}$$

(In fact, x denotes the center of mass coordinate of the Cooper pair rather than the relative coordinate. Since the phonon-mediated attraction between the electrons in the left-hand side can propagate only up to the range of the Cooper pair size ξ_0 in the right-hand side, this assumption seems to be reasonable.) Inserting Eq. (34) into the first integral in Eq. (6) we obtain (in the right side)

$$\begin{aligned} V \int^r F^*_r(x) F_l(x) dx &= V < |T|^2 > \sum_k u^r_k v^{r*}_k \sum_q u^l_q v^l_q \frac{N_F \pi^2}{2\hbar v_F} \int_{x_b}^{\infty} e^{-(x-x_b)/\xi_0} dx \\ &= V < |T|^2 > \sum_k u^r_k v^{r*}_k \sum_q u^l_q v^l_q \frac{N_F \pi}{2\Delta} \end{aligned} \tag{35}$$

where $\int^r$ denotes the integration over the right side and $x_a = 0$. $F_r(x)$ will have the same exponential tail in the left side. As expected, the Cooper pair wavefunction method leads to the same result for the Josephson coupling energy and the supercurrent:

$$E_J = -\frac{2\pi}{\lambda} < |T|^2 > N_F^2 \Delta \; cos(\phi_l - \phi_r) \tag{36}$$

$$j = \frac{1}{\lambda} \frac{\Delta}{e R_n} \; sin(\phi_l - \phi_r), \tag{37}$$

where

$$j_1 = \frac{1}{\lambda} \frac{\Delta}{e R_n}. \tag{38}$$

D. Shrinking of the Cooper Pair Size Due to the Insulating Barrier

In the previous section we assumed that the tail of $F_l(x)$ in the right side, Eq. (34), is controlled by the same Cooper pair size, ξ_0, of the left side. In other words, we disregarded the effect of the insulating barrier potential on the Cooper pair size. However, it is well-known that the Cooper pair size is reduced due to the impurity potentials.[37,38] Therefore, it is obvious that this assumption is not appropriate especially for high T_c superconductors, including MgB_2, and for the insulating barriers with high tunneling resistance, R_n, since the Cooper pair will have so much difficulty in tunneling in these cases.

We estimate the reduction of the Cooper pair size due to the barrier potential. Equation (32) implies that the decay length of the Cooper pair wavefunction in the insulating barrier region is $\sim 1/2\kappa$. Whereas, after tunneling the Cooper pair (of size ξ_0) will have the memory of the phonon-mediated attraction up to $\xi_0 - d$. So we suppose that the Cooper pair have size $1/2\kappa$ with the weight d/ξ_0 and ξ_0 with the weight $(\xi_0 - d)/\xi_0$. Accordingly, the effective Cooper pair size ξ_{eff} may be written as

$$\frac{1}{\xi_{eff}} = \frac{1}{\xi_0}\frac{\xi_0 - d}{\xi_0} + 2\kappa\frac{d}{\xi_0}. \tag{39}$$

Since $d << \xi_0$ in most cases, we find the effective Cooper pair size in the right side,

$$\xi_{eff} \cong \frac{\xi_0}{1+2\kappa d}. \tag{40}$$

Then, the effective Cooper pair wavefunction of $F_\ell(x)$ in the right side is

$$F_l(x) \propto e^{-\frac{x}{\xi_{eff}}} = e^{-\frac{1+2\kappa d}{\xi_0}x} \qquad x \geq x_a = 0. \tag{41}$$

Now, it is important to notice that this reduction of the Cooper pair size is fully meaningful only when the exponential factor in Eq. (41) is $\sim e^{-1}$ just after tunneling, i.e.,

$$F_l(d) \propto e^{-\frac{d}{\xi_{eff}}} = e^{-\frac{1+2\kappa d}{\xi_0}d} \sim e^{-1}. \tag{42}$$

Thus, the Cooper pair size shrinks significantly when $d \sim \sqrt{\xi_0/2\kappa}$, i.e., for high T_c superconductors, such as MgB_2 and for the insulating barriers with high R_n. To put it another way, there is a (sample-dependent) threshold tunneling resistance, R_{th}, above which the supercurrent decreases much faster, although it is hard to determine the exact value of R_{th}.

The overlap of the Cooper pair wavefunctions is given by

$$\begin{aligned} V\int^r F_r^*(x)F_l(x)dx &\cong V < |T|^2 > \sum_k u_k^r v_k^{r*} \sum_q u_q^l v_q^l \frac{N_F\pi^2}{2\hbar v_F}\int_{x_b}^{\infty} e^{-\frac{1+2\kappa d}{\xi_0}x}dx \\ &\cong V < |T|^2 > \sum_k u_k^r v_k^{r*} \sum_q u_q^l v_q^l \frac{N_F\pi}{2\Delta}\frac{1}{1+2\kappa d}e^{-\frac{1+2\kappa d}{\xi_0}d}. \end{aligned} \tag{43}$$

Consequently, for $d \sim \sqrt{\xi_0/2\kappa}$ the Josephson coupling energy and the supercurrent are written as

$$E_J = -\frac{2\pi}{\lambda} < |T|^2 > N_F^2\Delta\frac{1}{1+2\kappa d}e^{-\frac{1+2\kappa d}{\xi_0}d}\ cos(\phi_l - \phi_r). \tag{44}$$

and

$$j = \frac{1}{\lambda}\frac{\Delta}{eR_n}\frac{1}{1+2\kappa d}e^{-\frac{1+2\kappa d}{\xi_0}d}\ sin(\phi_l - \phi_r), \tag{45}$$

with

$$j_1 = \frac{1}{\lambda}\frac{\Delta}{eR_n}\frac{1}{1+2\kappa d}e^{-\frac{1+2\kappa d}{\xi_0}d}. \tag{46}$$

Since the DC Josephson current is very small in this case, fabrication of MgB_2 SIS junctions requires very accurate nanometer scale manipulation of the insulating barriers. For instance, if $\xi_0 \sim 100\mathring{A}$, we need an insulating barrier of thickness d smaller than $\sim 10\mathring{A}$ (i.e., $R_{th} \sim 0.01 - 0.1\Omega$) to see the Josephson supercurrent, which explains why the previous experiments couldn't see the DC supercurrent corresponding to the big gap of MgB_2.[14,15]

E. R_n Dependence of the DC Supercurrent

We have found the maximum DC supercurrent:

$$j_1 = \frac{1}{\lambda}\frac{\Delta}{eR_n} \qquad \text{for low } \mathrm{R_n} \tag{47}$$

$$j_1 = \frac{1}{\lambda}\frac{\Delta}{eR_n}\frac{1}{1+2\kappa d}e^{-\frac{1+2\kappa d}{\xi_0}d} \qquad \text{for high } \mathrm{R_n}\ (\mathrm{d} \sim \sqrt{\xi_0/2\kappa}) \tag{48}$$

where

$$R_n^{-1} = \frac{4\pi e^2}{\hbar}N_F^2 < |T|^2 > . \tag{49}$$

For low R_n the DC supercurrent is inversely proportional to R_n, which may be called *linear region*, whereas for high R_n, corresponding to $d \sim \sqrt{\xi_0/2\kappa}$, the supercurrent decreases more quickly with increasing R_n, which may be called *steep region*, although the boundary is not that sharp. This behavior agrees with the experiments.[15–19] (Presumably in the *steep region* the supercurrent may drop to zero exponentially. This is the reason why we kept the exponential factor in the previous section.) One needs an interpolation formula that smoothly connects both limits, for low R_n and for high R_n. (Nevertheless, high R_n formula Eq. (48) is valid in a rather narrow range where $d \sim \sqrt{\xi_0/2\kappa}$ and the exponential factor $\sim e^{-1}$.) For that purpose, we may multiply $2\kappa d$ in Eq. (48) by the approximate function for the Heaviside unit step function,[39] i.e.,

$$S_n(\kappa d) = \frac{1}{2}[1 + tanh\ n(2\kappa d - 2\kappa d_c)] \tag{50}$$

where $2\kappa d_c$ may be chosen to give the exponential factor equal to e^{-1}, while n is selected to have the resistance spread of factor of 10, according to the experimental data. (In this case $2\kappa d_c$ gives much higher resistance than the threshold resistance R_{th}.)

In the presence of ordinary impurities, the Cooper pair size also decreases from ξ_0 to $\tilde{\xi}_0$, defined by[21,31,38]

$$\frac{1}{\tilde{\xi}_0} = \frac{1}{\xi_0} + \frac{1}{\ell}, \tag{51}$$

where ℓ is the mean free path. (It seems that Eq. (51) works better than the other expression, i.e., $\sqrt{\ell\xi_0}$.[37]) The effective Cooper pair size is then replaced by

$$\xi_{eff} \cong \frac{\ell\xi_0}{(1+2\kappa d)(\ell+\xi_0)}. \tag{52}$$

It is remarkable that ordinary impurities also decrease the supercurrent by reducing the Cooper pair size.

Figure 2 shows the comparison of our theoretical calculations with the experimental data for Pb-$\mathrm{PbO_x}$-Pb (PPP) at 4.2K by Schwidtal and Finnegan,[16] and Sn-SnO-Pb (SSP) at 1.4K by Tinkham's group (Danchi et. al.).[19] The coherence lengths ξ_0 for Pb and Sn are $\sim 820\mathring{A}$ and $\sim 1800\mathring{A}$, respectively.[40] Note that above $R_n \sim 40\Omega$ the supercurrent of

Sn-SnO-Pb junction becomes larger than that of Pb-PbO$_x$-Pb junction, due to the bigger Cooper pair size of Sn. For *linear region* $j_1 = 1.5/R_n(mA)$ for PPP and $j_1 = 1.2/R_n(mA)$ for SSP. McMillan and Rowell[41] observed that for a typical junction with a resistance of 30Ω, the exponential factor is roughly $exp(-20)$. So the tunneling resistance is assumed to be $R_n = 30\times exp(2\kappa d - 20)(\Omega)$. The threshold resistance for PPP is $R_{th} \sim 30\Omega$, (i.e., $d_{th} \sim 20\mathring{A}$), whereas for SSP, there are two threshold resistances for Sn and PB, leading to (average) value, $R_{th} \sim 50\Omega$. For PPP junction we used $Sn(\kappa d) = 1/2[1 + tanh\ 1.9(\kappa d - 21.25)]$, whereas for SSP junction we used $Sn(\kappa d) = 1/2[1 + tanh\ 1.0(\kappa d - 21.25)]$ for Pb and $Sn(\kappa d) = 1/2[1 + tanh\ 1.0(\kappa d - 22.75)]$ for Sn, respectively. Unfortunately, the mean free path ℓ is not available for those experimental data. However, since the thickness of the film is about $2,000\mathring{A}$,[16] we suppose $\ell \sim 1000\mathring{A}$. If we assume $\ell = 1000\mathring{A}$ for Pb-PbO$_x$-Pb junction, we obtain the impurity-limited coherence length $\tilde{\xi_0} = \ell\xi_0/(\ell + \xi_0) = 450.6\mathring{A}$, leading to $d_c \sim 21.23\mathring{A}$, in agreement with the above $R_n(\kappa d)$. For SSP junction, the coherence length of Sn, $\sim 1800\mathring{A}$ requires $\ell \sim 800\mathring{A}$ to get $d_c \cong 23.5\mathring{A}$, which is also consistent with the R_n used. Considering the uncertainty in ℓ, the perfect fit with the experimental data is rather fortuitous. Nevertheless, our approach explains clearly why the Cooper pair of Sn can tunnel through the barrier more easily than that of Pb.

III. JOSEPHSON EFFECT AT FINITE TEMPERATURE

Now we consider the Josephson coupling energy and the supercurrent at finite temperatures for low tunneling resistance R_n. The Bogoliubov-Valatin transformations are defined by[42]

$$\begin{aligned} c^r_{k\uparrow} &= u^r_k e_{k\uparrow} + v^r_k e^+_{-k\downarrow} \\ c^r_{k\downarrow} &= u^r_k e_{k\downarrow} - v^r_k e^+_{k\uparrow} \end{aligned} \tag{53}$$

$$\begin{aligned} c^l_{q\uparrow} &= u^l_q f_{q\uparrow} + v^l_q f^+_{-q\downarrow} \\ c^l_{q\downarrow} &= u^l_q f_{q\downarrow} - v^l_q f^+_{q\uparrow}. \end{aligned} \tag{54}$$

Consequently, the coupling energy is written

$$\begin{aligned} E_J &= -J\sum_{kq}[u^r_k v^{r*}_k(1-2f_k)u^l_q v^l_q(1-2f_q) + u^r_k v^r_k(1-2f_k)u^l_q v^l_{q*}(1-2f_q)] \\ &= -2J\sum_{kq}\frac{\Delta_1(T)\Delta_2(T)cos(\phi_l - \phi_r)}{2E^r_k \times 2E^l_q}(1-2f_k)(1-2f_q). \end{aligned} \tag{55}$$

At $T = 0K$ this expression leads to Eq. (12). After summations over k and q, we obtain

$$E_J = -2J\frac{\Delta_1(T)\Delta_2(T)}{V_1 V_2}cos(\phi_l - \phi_r). \tag{56}$$

Notice that the pair-scattering matrix element, J, has rather weak temperature dependence through the effective Cooper pair sizes $\xi_{eff,1}(T)$ and $\xi_{eff,2}(T)$:

$$J = V_1 < |T|^2 > \frac{N_{F,1}\pi^2}{2\hbar v_F}\xi_{eff,2}(T) + V_2 < |T|^2 > \frac{N_{F,2}\pi^2}{2\hbar v_F}\xi_{eff,1}(T). \tag{57}$$

It is well-known that the pair-correlation amplitude, or the Cooper pair wavefunction falls exponentially:[21,26]

$$F(r) \propto exp(-\frac{r}{\pi\xi_0}) \qquad T = 0K \tag{58}$$

$$F(r) \propto exp(-\frac{2r}{3.5\xi_0}) \qquad T \to T_c \tag{59}$$

leading to the decrease of the effective Cooper pair size $\xi_{eff}(T)$ from $\pi\xi_0/2$ at $T = 0K$ to $3.5\xi_0/4$ near T_c. (In the previous sections we used ξ_0 instead of $\pi\xi_0/2$ at $T = 0K$.) The limiting behavior of the effective Cooper pair size can be shown to be[43]

$$\xi_{eff}(T) \cong \frac{\pi\xi_0}{2}(1 - \frac{\pi^2 T^2}{2\Delta^2}) \qquad \text{near T} = 0\text{K} \tag{60}$$

$$\xi_{eff}(T) \cong \frac{1}{2}\frac{\hbar v_F}{\pi T}(1 - \frac{\Delta^2}{2\pi^2 T^2}) \qquad \text{near } \mathrm{T_c}, \tag{61}$$

where $\xi_{eff}(T) \cong \frac{\pi\Delta_0\xi_0}{2\sqrt{\pi^2T^2+\Delta^2}}$ at any temperature. Unlike $\Delta(T)$, near $T = 0K$, $\xi_{eff}(T)$ decreases with increasing temperature in a parabolic manner.

Accordingly, for a symmetric junction the Josephson coupling energy and the supercurrent are written as

$$E_J = -2|T|^2\frac{\pi^2}{\hbar v_F}\frac{\xi_{eff}(T)}{\lambda}N_F^2\Delta^2(T)cos\phi \tag{62}$$

$$j = \frac{1}{\lambda}\frac{\pi\xi_{eff}(T)}{\hbar v_F}\frac{\Delta^2(T)}{eR_n}sin\phi, \tag{63}$$

with

$$j_1 = \frac{1}{\lambda}\frac{\pi\xi_{eff}(T)}{\hbar v_F}\frac{\Delta^2(T)}{eR_n}. \tag{64}$$

The reduced dc Josephson current is, then, given by

$$\frac{j_1(T)}{j_1(0K)} = \frac{\xi_{eff}(T)}{\frac{\pi}{2}\xi_0}\frac{\Delta^2(T)}{\Delta_0^2} \tag{65}$$

$$= \frac{\Delta^2(T)}{\Delta_0\sqrt{\pi^2T^2 + \Delta^2(T)}}. \tag{66}$$

For an asymmetric junction, one finds the coupling energy and the supercurrent:

$$E_J = -\pi < |T|^2 > N_{F,1}N_{F,2}\Delta_1(T)\Delta_2(T)(\frac{\pi\xi_{eff,1}(T)}{\lambda_1\hbar v_{F,1}} + \frac{\pi\xi_{eff,2}(T)}{\lambda_2\hbar v_{F,2}})cos(\phi_l - \phi_r), \tag{67}$$

and

$$j = \frac{2\pi}{\hbar}e < |T|^2 > N_{F,1}N_{F,2}\Delta_1(T)\Delta_2(T)(\frac{\pi\xi_{eff,1}(T)}{\lambda_1\hbar v_{F,1}} + \frac{\pi\xi_{eff,2}(T)}{\lambda_2\hbar v_{F,2}})sin(\phi_l - \phi_r), \tag{68}$$

with

$$j_1 = \frac{1}{2e}\frac{1}{R_n}\Delta_1(T)\Delta_2(T)(\frac{\pi\xi_{eff,1}(T)}{\lambda_1\hbar v_{F,1}} + \frac{\pi\xi_{eff,2}(T)}{\lambda_2\hbar v_{F,2}}). \tag{69}$$

Therefore, the reduced DC Josephson current for an asymmetrical junction is

$$\frac{j_1(T)}{j_1(0K)} = \frac{\Delta_1(T)\Delta_2(T)}{\Delta_1(0)\Delta_2(0)}\frac{[\frac{\xi_{eff,1}(T)}{\lambda_1} + \frac{\xi_{eff,2}(T)}{\lambda_2}]}{\frac{\pi}{2}[\frac{\xi_{0,1}}{\lambda_1} + \frac{\xi_{0,2}}{\lambda_2}]}. \tag{70}$$

Figure 3 shows the temperature dependence of the maximum DC supercurrent $j_1(T)$ for a Sn/Sn and a Pb/Sn junction compared with our theoretical calculation from Eqs. (64) and (68). Data are from Fiske.[44] As in Figure 2, we used $\ell = 800\mathring{A}$ and $\xi_0 = 820\mathring{A}$ for Pb and $\xi_0 = 1800\mathring{A}$ for Sn, respectively. The approximate temperature dependence of $\xi_{eff}(T)$ is given by $\xi_{eff}(T) = \frac{\pi\Delta_0\xi_0}{2\sqrt{\pi^2T^2+\Delta^2}}$, which is almost identical to the accurate numerical calculation of $\xi_{eff}(T)$.[43] As can be seen, the agreement between theory and experiment is fairly good.

IV. DISCUSSION

It is clear that more study is needed to understand the intriguing properties of the Josephson effect. In particular, the sign of pair-quasi particle interference term, generalization of this approach to weak links and SNS junctions, and explanation of many unsolved problems in AC Josephson effect may be interesting problems. For AC Josephson effect an investigation based on this approach will be published separately.[45]

The Josephson effect in MgB_2 requires more careful study. It is highly desirable to fabricate clear-cut SIS junction with very low tunneling resistance ($R_n \sim 0.01 - 0.1\Omega$) for the detection of the Josephson current for the big gap. The insulating layer should have thickness not larger than $10\mathring{A}$ and a small band gap.

Very recently, a similar problem has been found in the flux quantization of superconducting cylinders.[46] Whereas most previous approaches focused on the phase of the effective Cooper pair wavefunction in the presence of the magnetic flux, flux quantization turned out to be due to the flux dependence of the pairing energy.[46]

V. CONCLUSION

We have introduced Cooper pair wavefunction approach to the Josephson effect in SIS tunnel junctions. We have found that the Josephson tunneling depends on the size of the Cooper pair and its shrinking during the tunneling. Accordingly there is a material-dependent threshold of tunneling resistance above which the DC Josephson current decreases much faster with increasing the tunneling resistance. High T_c superconductors, including MgB_2, have tightly-bound Cooper pairs with small size, which can not tunnel through the insulating barrier easily, leading to extremely small critical current, in agreement with experimental findings. This understanding also explains why the observed DC supercurrent of low T_c superconductors, such as Sn and Pb, decreases much faster than $1/R_n$ above the tunneling resistances R_n in excess of several ohms. It is of interest that ordinary impurities limit the supercurrent by reducing the Cooper pair size, too. We have also shown that the (weak) temperature dependence of the Cooper pair size contribute to the temperature dependence of DC supercurrent.

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FIGURES

FIG. 1. Cooper pair wavefunctions near the insulating barrier. The solid lines denote the Cooper pair wavefunctions without the barrier, while the thick lines show the change of the Cooper pair wavefunctions due to the barrier.

FIG. 2. The maximum DC supercurrent vs the tunneling resistance R_n for Pb-PbO$_\mathrm{x}$-Pb junction (at 4.2K) and for Sn-SnO-Pb junction (at 1.4K). Data are from Schwidtal and Finnegan, Ref. 16 and Danchi et. al., Ref. 19. Notice the crossing of the supercurrents near $R_n \sim 40\Omega$. The solid lines are our theoretical calculation from Eqs. (47) and (48).

FIG. 3. Temperature dependence of the DC supercurrent of a Sn/Sn and a Pb/Sn junction in comparison with our theoretical calculation. Data are from Fiske, Ref. 44.