On the injectivity of the circular Radon transform arising in thermoacoustic tomography

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Abstract

The circular Radon transform integrates a function over the set of all spheres with a given set of centers. The problem of injectivity of this transform (as well as inversion formulas, range descriptions, etc.) arises in many fields from approximation theory to integral geometry, to inverse problems for PDEs, and recently to newly developing types of tomography. A major breakthrough in the $2D$ case was made several years ago in a work by M. Agranovsky and E. T. Quinto. Their techniques involved intricate microlocal analysis and knowledge of geometry of zeros of harmonic polynomials in the plane, which are somewhat restrictive in more general circumstances. Since then there has been an active search for alternative methods, especially the ones based on simple PDE techniques. The article discusses known and provides new results that one can obtain by methods that essentially involve only the finite speed of propagation and domain dependence for the wave equation.

1 Introduction

Most tomographic methods of medical imaging (as well as industrial non-destructive evaluation, geological imaging, sonar, and radar) are based on the following procedure: one sends towards a non-transparent body some kind of a signal (acoustic or electromagnetic wave, X-ray or visual light photons, etc.) and measures the wave after it passes through the body. Then the problem becomes to use the measured information to recover the internal structure of the object of study. The common feature of all traditional methods of tomography is that the same kinds of physical signals are sent and are measured. Although the development of tomography in the last half of the century has brought many remarkable successes [26, 27], each of the methods has its own shortfalls. For instance, when imaging biological tissues, microwaves and optical imaging often provide good contrasts between different types of tissues, but are inferior in terms of resolution in comparison with ultrasound or X-rays. This, in particular, is responsible for practical impossibility of getting any good resolution in optical or electrical impedance tomography, unless one wants to image only skin-deep areas. On the other hand, ultrasound, while giving good resolution, does not do a good job in terms of contrast. It is amazing then that the idea of combining different types of radiation for triggering the signal and for the measured signal had to wait for such a long time to appear. By now, thermoacoustic tomography (TAT or TCT) and its sibling photoacoustic tomography (PAT) have already made significant advances (e.g., [21],[33]-[36]), while some others are still in a development stage. Since PAT in terms of the relevant mathematics is identical to TAT, we will describe briefly only the latter. In TAT, a short microwave or radiofrequency electromagnetic pulse is sent through the biological object. At each internal location x certain energy $H(x)$ will be absorbed. It is known (see the references above), that cancerous cells absorb several times more MW (or RF) energy than the normal ones, which means that significant increases of the values of $H(x)$ are expected at tumorous locations. It is believed that this contrast is due to the increased water and sodium content in tumors, which is partly due to extra blood vessel growth there. The absorbed energy, due to resulting heating, causes a thermoelastic expansion, which in turn creates a pressure wave. This wave can be detected by ultrasound transducers placed at the edges of the object. Since the pulses delivered are very short, the thermal diffusion during the experiment can be neglected. Now the former weakness of ultrasound (low contrast) becomes an advantage. Indeed, in many cases (e.g., for mammography) one can assume the sound speed to be constant. Hence, the sound waves detected at any moment t of time are coming from the locations at a constant distance (depending on time and sound speed) from the transducer. The strength of the signal coming from a location x reflects the energy absorption $H(x)$. Thus, one effectively measures the integrals of $H(x)$ over all spheres centered at the transducers' locations. In other words, one needs to invert a generalized Radon transform of H ("generalized," since integration is done over spheres). Exact implementation of this idea involves simple handling of the wave equation ([21],[33]-[36]).

This method amazingly combines advantages of two types of radiation used (contrast for microwaves and resolution for ultrasound), while avoiding their deficiencies.

It is clear from the dimension considerations that it should be sufficient to run the transducers along a curve in the case of a $2D$ problem or a surface in $3D$. The most popular geometries of these surfaces (curves) that have been implemented are spheres, planes, and cylinders [33]-[35].

Let us mention some of the central problems that arise in these studies:

- Uniqueness of reconstruction: is the information collected sufficient for the unique determination of the energy deposition function H?

- Reconstruction formulas and algorithms, and stability of the reconstruction.

- Description of the range of the transform: what conditions should ideal data satisfy?

- Incomplete data problems: what happens to the reconstruction if only a part of transducers' locations can be (or are) used?

All these questions have been essentially answered for the classical Radon transform that arises in X-ray CT, Positron Emission Tomography (PET), and Magnetic Resonance Imaging (MRI) [26, 27]. However, they are much more complex and not that well understood for the circular Radon transform that arises in TAT.

The aim of this paper is to address the uniqueness problem. We will survey the known results and describe some new approaches developed originally in [13] that significantly clarify this question. Then we show how one can make additional progress in this direction.

The results of this paper were presented at the special sessions on tomography at the AMS Meetings in Binghamton, NY in October 2003 and in Lawrenceville, NJ in April 2004 and at the Inverse problems workshop at IPAM in November 2003.

The next section contains the mathematical formulation of the problem, its history, and the description of the beautiful uniqueness result obtained by M. Agranovsky and E. T. Quinto in [3]. Alternative PDE methods developed in [2, 13] are also presented. The following section contains the main results of this paper. It is followed by sections containing further remarks and acknowledgements.

2 Formulation of the problem and known results

The discussion of the previous section motivates the study of the following "circular" Radon transform. Let $f(x)$ be a continuous function on $\mathbb{R}^n$, $n \geq 2$.

Definition 1 *The circular Radon transform of f is defined as*

$$Rf(p,r) = \int_{|y-p|=r} f(y) d\sigma(y),$$

where $d\sigma(y)$ is the surface area on the sphere $|y-p| = r$ centered at $p \in \mathbb{R}^n$.

In this definition we do not restrict the set of centers p or radii r. It is clear, however, that this mapping is overdetermined, since the dimension of pairs (p,r) is $n+1$, while the function f depends on n variables only. This (as well as the tomographic motivation of the previous section) suggests to restrict the set of centers to a set (hypersurface) $S \subset \mathbb{R}^n$, while not imposing any restrictions on the radii. This restricted transform will be denoted by R_S:

$$R_S f(p,r) = Rf(p,r)|_{p \in S}.$$

Definition 2 *The transform R is said to be injective on a set S (S is a* **set of injectivity***) if for any $f \in C_c(R^n)$ the condition $Rf(p,r) = 0$ for all $r \in \mathbb{R}$ and all $p \in S$ implies $f \equiv 0$.*

In other words, S is a set of injectivity, if the mapping R_S is injective on $C_c(\mathbb{R}^n)$.

One can wonder why we impose the condition of compactness of support on f, since it does not seem to be all that natural for the circular Radon transform . The answer is that the situation can be significantly different without compactness of support (or at least some decay) condition [2, 3]. Besides, the problem becomes significantly harder in this case. Fortunately, tomographic problems usually yield compactly supported functions (such as the energy deposition function H of the previous section).

One now arrives to the

Problem 3 *Describe all sets of injectivity for the circular Radon transform R on $C_c(\mathbb{R}^n)$.*

In other words, we are looking for a description of those sets of positions of transducers that enable one to recover uniquely the energy deposition function.

This problem has been around in different guises for quite a while. For instance, it is formulated (in a more general setting of integration over level sets of polynomials) in the recent book [11] by L. Ehrenpreis, preliminary drafts of which have been circulating among experts for years. The same question in its dual form was posed in terms of approximation theory and studied by V. Lin and A. Pincus [23, 24]:

Problem 4 *Describe sets $\Gamma \subset \mathbb{R}^n$, $n \geq 2$, such that the system of shifted radial functions (spherical waves)*

$$\psi(|x-a|),\ a \in \Gamma,\ \ \psi \textit{ is a function of one variable},$$

is complete in $C(\mathbb{R}^n)$ in the topology of uniform convergence on compacta.

As it will be clear a little bit later, this problem is equivalent to the description of possible nodal sets of oscillating infinite membranes. The paper [3] contains a survey of some other problems that lead to the injectivity question for R_S.

There might have been other sources of this problem, unknown to the authors.

As it turns out, the injectivity sets S are more common than the non-injectivity ones. This means that one should aim for a description of those "bad" non-injectivity sets (i.e. sets of transducers' positions from which one cannot recover the energy deposition function).

The first important observations concerning non-injectivity sets were made by V. Lin and A. Pincus [23, 24]. We will present these following the notations and formulations of [3].

First of all, convolution with radial mollifiers easily shows that one can assume that the function f to which the transform is applied is arbitrarily smooth (e.g., [3]). So, we will not bother with smoothness conditions, assuming that the function is at least continuous.

Let us associate with each $f \in C(\mathbb{R}^n)$ that decays at infinity faster than any power of $|x|$ the set

$$S[f] = \{x \in \mathbb{R}^n | \; Rf(x,r) = 0 \; \forall r \in \mathbb{R}_+\}.$$

We also introduce the infinite family of polynomials Q_k of degree $degQ_k \leq 2k$:

$$Q_k(x) = Q_k[f](x) = r^{2k} * f = \int_{\mathbb{R}^n} \|x-\xi\|^{2k} f(\xi) d\xi, \;\; r^2 = x_1^2 + \cdots + x_n^2.$$

For any polynomial Q with real coefficients, we denote by $V[Q]$ the real algebraic variety

$$V[Q] = \{x \in \mathbb{R}^n | \;\; Q(x) = 0\}.$$

Lemma 5 *(V. Lin and A. Pincus [23, 24])* $S[f] = \bigcap_{k=0}^{\infty} V[Q_k]$.

Lemma 6 *Let $f \in C_c(\mathbb{R}^n)$. Then $f \equiv 0$ if and only if $Q_k[f] \equiv 0$ for all $k = 0, 1, \ldots$. If f is not identically zero, and $P = Q_{k_0}[f]$ is the minimal degree nontrivial polynomial among Q_k, then P is harmonic.*

The harmonicity in Lemma 6 was discovered by N. Zobin [37].

These statements imply in particular that if R is not injective on S, then S is the zero set of a harmonic polynomial. Therefore we get a sufficient condition for injectivity:

Corollary 7 *Any set $S \subset \mathbb{R}^n$ of uniqueness for the harmonic polynomials is a set of injectivity for the transform R.*

In particular, this implies

Corollary 8 *If $U \subset \mathbb{R}^n$ is any bounded domain, then $S = \partial U$ is a injectivity set of R.*

We will see later a different proof of this fact that does not use harmonicity.

So, what are possible non-injectivity sets? It is rather obvious that any hyperplane S is such a set. Indeed, for any function f that is odd with respect to S one obviously gets $R_S f \equiv 0$. There are other options as well. In order to describe them in $2D$, let us first introduce the following definition.

Definition 9 *For any $N \in \mathbb{N}$ denote by Σ_N the Coxeter system of N lines $L_0, \ldots, L_{n-1}$:*

$$L_k = \{te^{i\pi k/n} | -\infty < t < \infty\}.$$

In other words, Σ_N is a "cross" of N lines passing through the origin and forming equal angles π/N. It is rather easy to construct a non-zero compactly supported function that is simultaneously odd with respect to all lines of a given Coxeter set (look at the Fourier series expansions with respect to the polar angle). Hence, Σ_N is a non-injectivity set as well. It is clear that applying any rigid motion ω one preserves non-injectivity property. Thus, all sets $\omega\Sigma_N$ are non-injectivity sets. It was conjectured by V. Lin and A. Pincus that these are the only non-injectivity sets for compactly supported functions on the plane. It was discovered in [3] that one can also add any finite set F preserving non-injectivity. And that is all that can happen. M. Agranovsky and E. Quinto presented the following complete solution in $2D$ for the (equivalent) Problems 3, 4 in [3]:

Theorem 10 *[3] The following condition is necessary and sufficient for a set $S \subset \mathbb{R}^2$ to be a set of injectivity for the circular Radon transform on $C_c(\mathbb{R}^2)$:*

S is not contained in any set of the form $\omega(\Sigma_N) \bigcup F$, where ω is a rigid motion in the plane and F is a finite set.

In other words, only Coxeter "crosses" plus finite sets are non-injectivity sets for the circular Radon transform R **considered on compactly supported functions**.

Before we go any further, we would like to state the (unproven) conjecture that describes non-injectivity sets in higher dimensions (still relative to the class of compactly supported functions).

Conjecture 11 *[3] The following condition is necessary and sufficient for S to be a set of injectivity for the circular Radon transform on $C_c(\mathbb{R}^n)$:*

S is not contained in any set of the form $\omega(\Sigma) \bigcup F$, where ω is a rigid motion of $\mathbb{R}^n$, Σ is the zero set of a **homogeneous** *harmonic polynomial, and F is an algebraic subset in $\mathbb{R}^n$ of co-dimension at least 2.*

The reader easily notices that for $n = 2$ this boils down to Theorem 10.

The beautiful proof of Theorem 10 by M. Agranovsky and E. Quinto is built upon three kinds of tools: Lemmas 5 and 6, microlocal analysis

(Fourier Integral Operators technique) that in particular guarantees existence of certain analytic wave front sets at the boundary of the support of a function located on one side of a smooth surface (8.5.6 in [19]), and known geometric structure of level sets of harmonic polynomials in $2D$ (e.g., [14]). These methods, unfortunately, restrict applicability of the proof. The microlocal tool used works at an edge of the support and hence is not applicable for non-compactly-supported functions, no matter how fast they decay. On the other hand, the geometry of level sets of harmonic polynomials does not work well in dimensions higher than 2, and definitely not on more general Riemannian manifolds (e.g., on the hyperbolic plane). Hence, the quest has been active for alternative approaches since [3] has appeared.

One can guess what such approaches could possibly be, by looking at alternative reformulations of the problem (which there are plenty [3]). We have already seen the one in terms of approximation theory, but unfortunately this has not lead to significant advances yet. There is, however, another revealing reformulation [3, 22] that stems from known relations between spherical integrals and the wave equation (e.g., [9, 20]). Namely, consider the initial value problem for the wave equation in $\mathbb{R}^n$:

$$u_{tt} - \triangle u = 0, \quad x \in \mathbb{R}^n, t \in \mathbb{R} \tag{1}$$

$$u|_{t=0} = 0, \quad u_t|_{t=0} = f. \tag{2}$$

Then

$$u(x,t) = \frac{1}{(n-2)!} \frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^t r(t^2 - r^2)^{(n-3)/2} (Rf)(x,r)dr, \quad t \geq 0.$$

Hence, it is not hard to show [3] that the original problem is equivalent to the problem of recovering $u_t(x,0)$ from the value of $u(x,t)$ on subsets of $S \times (-\infty, \infty)$.

Lemma 12 *[3, 22] A set S is a non-injectivity set for $C_c(\mathbb{R}^n)$ if and only if there exists a non-zero compactly supported continuous function f such that the solution $u(x,t)$ of the problem (1)-(2) vanishes for any $x \in S$ and any t.*

Hence, non-injectivity sets are exactly the nodal sets of oscillating free infinite membranes. In other words, injectivity sets are those that observing the motion of the membrane over S gives complete information about the motion of the whole membrane. Analogous relation holds also for solutions of the heat equation [3].

Lemma 12 suggests to try to understand the geometry of non-injectivity sets in terms of wave propagation. The first example of such a consideration was the original proof [22] of Corollary 8 that did not use harmonicity (which was not known at the time). The proof went essentially as follows (one can easily make what is written below precise). Let $S = \partial U$ be a non-injectivity (and hence nodal for wave equation) set, where U is a bounded domain. Then one can adopt two different points of view. On one hand, the membrane is free and hence the energy of the initial compactly supported perturbation must move away. Thus, its portion inside U should decay to zero. On the other hand, one can think that S is a fixed boundary (since it does not move anyway) and hence the energy inside must stay constant. This contradiction allows one to conclude that in fact $f = 0$. The same PDE idea, with many more technical details, was implemented in [2] to prove the following statement:

Theorem 13 *[2]If U is a bounded domain in $\mathbb{R}^n$, then $S = \partial U$ is an injectivity set for R in the space $L^q(\mathbb{R}^n)$ if $q \leq 2n/(n-1)$. This statement fails when $q > 2n/(n-1)$, in which case spheres fail to be injectivity sets.*

The rough idea of what is going on in this theorem can be explained as follows: when $q > 2n/(n-1)$, the function f in (1)-(2) has a long tail, and so there is enough energy at infinity to constantly come and replace the energy leaving the domain U. This can keep the set S nodal for all times. On the other hand, for $q \leq 2n/(n-1)$ there is not enough energy at infinity to keep the balance.

In spite of these limited results, it still remained unclear what distinguishes in terms of wave propagation the "bad" flat lines S in Theorem 10 that can be nodal for all times, from any truly curved S that according to this theorem cannot stay nodal. An approach to this question was found in the recent paper [13] by D. Finch, Rakesh, and S. Patch, where in particular some parts of the injectivity results due to [3] were re-proven by simple PDE means without using microlocal tools and harmonicity.

Theorem 14 *[13] Suppose D is a bounded, open, subset of $\mathbb{R}^n$, $n \geq 2$, with a strictly convex smooth boundary S. Let Γ be any relatively open subset of S. If f is a smooth function on $\mathbb{R}^n$, supported in $\bar{D}$, and $(Rf)(p,r) = 0$ for all $p \in \Gamma$ and all r, then $f = 0$.*

Equivalently, if u is the solution of the initial value problem (1), (2) and $u(p,t) = 0$ for all $p \in \Gamma$ and all t, then $f = 0$.

One can find out easily that the statement of this theorem follows immediately from a microlocal statement in [3][1] The significance of this theorem, however, lies not in its statement, but rather in the proof provided in [13] (that paper contains other important results as well, which we do not touch here).

The following two standard statements about the wave equation were the basis of the proof of the Theorem 14 in [13]. The first one concerns the unique continuation for the time-like Cauchy problem for the wave equation, while the second states finiteness of speed of propagation.

Proposition 15 *[13] Let $B_\epsilon(p) = \{x \in \mathbb{R}^n \mid |x-p| < \epsilon\}$. If u is a distribution and satisfies (1) and u is zero on $B_\epsilon(p) \times (-T, T)$ for some $\epsilon > 0$, and $p \in \mathbb{R}^n$, then u is zero on*

$$\{(x,t) : |x-p| + |t| < T\},$$

and in particular on

$$\{(x,0) : |x-p| < T\}.$$

Let now D be a bounded, open subset of $\mathbb{R}^n$ with a smooth boundary S. For points p, q outside D, let $d(p,q)$ denote the infimum of the lengths of all the piecewise C^1 paths in $\mathbb{R}^n \setminus D$ joining p to q. Then $d(p,q)$ is a metric on $\mathbb{R}^n \setminus D$. For any point p in $\mathbb{R}^n \setminus D$ and any positive number r, define $E_r(p)$ to be the ball of radius r and center at p in $\mathbb{R}^n \setminus D$ with respect to this metric, i.e.

$$E_r(p) = \{x \in \mathbb{R}^n \setminus D : d(x,p) < r\}.$$

The second statement is about the domain of dependence of solutions of the wave equation in an exterior domain. The result claims that the value of the solution of the wave equation in an exterior domain, at a point (x,s), affects the values of the solution at the point (y,t) only if $d(x,y) \le t - s$.

Proposition 16 *[13] Suppose D is a bounded, open, connected subset of $\mathbb{R}^n$, with a smooth boundary S. Let u be a smooth solution of the exterior problem*

$$u_{tt} - \triangle u = 0, \;\; x \in \mathbb{R}^n \setminus D, \;\; t \in R$$

$$u = h \;\; on \;\; S \times R.$$

[1] Results of [3] make the situation described in Theorem 14 impossible, since the support of f lies on one side of a tangent plane to Γ. See also Theorem 22

Suppose p is not in D, and $t_0 < t_1$ are real numbers. If $u(.,t_0)$ and $u_t(.,t_0)$ are zero on $E_{t_1-t_0}(p)$ and h is zero on

$$\{(x,t) : x \in S, t_0 \le t \le t_1, d(x,p) \le t_1 - t\},$$

then $u(p,t)$ and $u_t(p,t)$ are zero for all $t \in [t_0, t_1)$.

Now the proof of Theorem 14 roughly boils down to the following consideration: the signal from f can reach a point near Γ that is outside D at a time dictated by the "curvilinear" distance d, while on the other hand the membrane is free, so the signals can propagate along straight paths and so should have arrived earlier. This contradiction proves the theorem.

The last two propositions can be proven under less stringent conditions on the domains and functions, but as it will be seen in the next section, this is in fact not needed.

3 Further injectivity results by PDE means

We will show now how simple tools similar to the Propositions 15 and 16, namely finite speed of propagation and domain of dependence for the wave equation allow one to obtain stronger results concerning injectivity sets.

Let us start with some initial remarks that will narrow the cases we need to consider. First of all, according to Lemma 5, any non-injectivity set S is contained in an algebraic surface that is also a non-injectivity set. So, one can as well assume from the start algebraicity of S. Since it is known [3] that algebraic surfaces of co-dimension higher than 1 are automatically non-injectivity sets, we can restrict our attention to algebraic hypersurfaces in $\mathbb{R}^n$ only. The reader should understand that anything that is not algebraic (or rather does not fit into an algebraic surface) is automatically an injectivity set. So, confining ourselves to the case of algebraic hypersurfaces solely we do not lose any generality. One can also assume irreducibility of that surface, if this helps. When needed, one can also exclude the case of **closed hypersurfaces**, since according to Lemma 6 those are all injectivity sets, albeit this is mostly not needed in what follows. Notice that these reductions so far do not require us to confine ourselves to the class of compactly supported functions, since they work as well in the class of continuous functions decaying at infinity faster than any power of $|x|$. Thus, we do not impose at this stage compactness of support restriction, although it will appear in some of the results of this section.

Our goal now is to exclude many pairs (S, f), where S is an algebraic surface and f is a function as possible candidates for satisfying the non-injectivity condition $R_S f = 0$. Notice that Theorem 14 does exactly that when S contains an open part of the boundary of a strictly convex domain where f is supported. Theorem 10, on the other hand excludes all compactly supported f's in $\mathbb{R}^2$, unless $S = \omega\Sigma_N$. Similarly, Theorem 13 excludes boundaries S of bounded domains when f is in an appropriate space $L_p(\mathbb{R}^n)$.

Let S be an irreducible algebraic hypersurface that splits $\mathbb{R}^n$ into parts H^+ and H^- (for instance, if $P(x)$ is the irreducible polynomial defining S, we can take $H^+ = \{x \mid P(x) > 0\}$). One can define the interior metric in H^+ as follows:

$$d^+(p, q) = inf\{\text{length of}\,\gamma\}, \tag{3}$$

where the infimum is taken over all C^1-curves γ in H^+ joining points $p, q \in H^+$. This metric extends naturally to S. A similar metric d^- is defined on $H^- \cup S$.

Theorem 17 *Let S be as above and $f \in C(\mathbb{R}^n)$ be such that $R_S f = 0$. Let also $x \in H^+ \cup S$. Then*

$$\begin{aligned} dist(x, supp\ f \cap H^+) &= dist^+(x, supp\ f \cap H^+) \\ &\leq dist(x, supp\ f \cap H^-), \end{aligned} \tag{4}$$

where distances $dist^{\pm}$ are computed with respect to the metrics $d^{\pm}$, while dist is computed with respect to the Euclidean metric in $\mathbb{R}^n$.

A similar statement holds for $x \in H^- \cap S$.

In particular, for $x \in S$

$$\begin{aligned} &dist(x, supp\ f \cap H^+) = dist^+(x, supp\ f \cap H^+) \\ &= dist(x, supp\ f \cap H^-) = dist^-(x, supp\ f \cap H^-). \end{aligned} \tag{5}$$

Remark 18

- *Notice that the theorem does not require the function f to be compactly supported. It in fact does not require any control at infinity.*
- *The assumed algebraicity of S (although it comes for free when f decays faster than any power of $|x|$) is also not truly necessary. The reader can easily figure out from the proof that any piecewise-smooth (or even Lipschitz) hypersurface suffices.*

Proof of the theorem. Let us prove first the equality

$$dist(x, \mathrm{supp}\ f \cap H^+) = dist^+(x, \mathrm{supp}\ f \cap H^+), \tag{6}$$

which would also prove the similar one with d^- and H^-.

We need a simple auxiliary statement, where we use the notation $B_r(p)$ for the ball of radius r centered at p.

Lemma 19 *For any compact K, $\epsilon > 0$, and a point $p \in H^+ \cup S$ there exists a smooth function $d^+_\epsilon(q)$ in a neighborhood of $K \cap (H^+ \cup S)$ such that $|\nabla d^+_\epsilon(q)| \leq 1$ and $|d^+(p, q) - d^+_\epsilon(q)| < \epsilon$ for any $q \in K \cap (H^+ \cup S)$.*

This lemma removes the function theory difficulties related to Proposition 16 faced in [13].

Proof of the lemma. Consider for a small positive δ the enlarged set $H^+_\delta = \{x \mid P(x) > -\delta\} \supset H^+$. Introduce the distance D^+_δ in H^+_δ the same way d^+ was introduced in H^+. Then D^+_δ on H^+ does not exceed d^+ and for small δ can be made as close as necessary to d^+ on any compact $K \subset H^+ \cup S$. Mollifying D^+_δ with a mollifier of a small support, one obtains the required function d^+_ϵ. Indeed, closeness of the two functions is clear. Since D^+_δ has gradient of length not exceeding 1 a.e., we get the needed estimate on the gradient of d^+_ϵ on K. The lemma is proven.

Let us now return to the proof of (6). Since $d^+(p, q) \geq |p - q|$, it is sufficient to prove that the left hand side expression cannot be strictly smaller than the one on the right. Assume the opposite, that

$$dist(x, \mathrm{supp}\ f \cap H^+) = d_1 < d_2 = dist^+(x, \mathrm{supp}\ f \cap H^+). \tag{7}$$

Pick a smaller segment $[d_3, d_4] \subset (d_1, d_2)$. Then, by continuity, for any point p in a small ball $B \subset H^+$ near x (not necessarily containing x, which happens when $x \in S$) one has

$$dist(p, \mathrm{supp}\ f \cap H^+) \leq d_3 < d_4 \leq dist^+(p, \mathrm{supp}\ f \cap H^+). \tag{8}$$

For such a point p, let us choose $\epsilon < (d_4 - d_3)/2$, the compact K which is the ball with respect to d^+ centered at p and of radius $2d_4$, and a function $d^+_\epsilon(q)$ from the previous lemma. Then for any point $q \in K \cap \mathrm{supp}\ f \cap H^+$ we have

$$d^+_\epsilon(q) > (d_3 + d_4)/2 > d_3.$$

Consider the volume V in the space-time region $H^+ \times \mathbb{R}$ bounded by the space-like surfaces $t = 0$ and $t = (d_3 + d_4)/2 - d_\epsilon^+(p)$ and the "vertical" boundary $S \times \mathbb{R}$. Consider the solution $u(x, t)$ of the wave equation problem (1)-(2) with the initial velocity f. Then, by construction, this solution and its time derivative are equal to zero at the lower boundary $t = 0$ and on the lateral boundary S. Hence, the standard domain of dependence argument (see, e.g., Section 2.7, Ch. 1 in [8]) we conclude that $u = 0$ in V. In particular, $u(p, t) = 0$ for all $p \in B$ and $|t| \le (d_3 + d_4)/2 - \epsilon$. Notice that $(d_3 + d_4)/2 - \epsilon > d_3$. Now applying Proposition 15 to the wave equation in the whole space, we conclude that

$$dist(p, \text{supp } f) > d_3, \tag{9}$$

and hence

$$dist(p, \text{supp } f \cap H^+) > d_3, \tag{10}$$

which is a contradiction. This proves (6). It is now sufficient to prove

$$dist(x, \text{supp } f \cap H^+) \le dist(x, \text{supp } f \cap H^-). \tag{11}$$

This in fact is an immediate consequence of (9). Alternatively, we can repeat the same consideration as above in a simplified version. Namely, suppose that

$$dist(x, \text{supp } f \cap H^+) > d_2 > d_1 > dist(x, \text{supp } f \cap H^-) \tag{12}$$

for a point x, and hence for all points p in a small ball in H^+. Consider the volume V in the space-time region $H^+ \times \mathbb{R}$ bounded by the space-like surfaces $t = 0$ and $t = d_2 - |x - p|$ (p fixed in the small ball) and the boundary $S \times \mathbb{R}$. Consider the solution $u(x, t)$ of the wave equation problem (1)-(2) with the initial velocity f. Then, by construction, this solution and its time derivative are equal to zero at the lower boundary $t = 0$ and on the lateral boundary S. Hence, by the same standard domain of dependence argument (see, e.g., Section 2.7, Ch. 1 in [8]) we conclude that $u = 0$ in V. In particular, $u(p, t) = 0$ for all $p \in B$ and $|t| \le d_2$. Now applying Proposition 15 to the wave equation in the whole space, we conclude that

$$dist(p, \text{supp } f) > d_2,$$

and hence

$$dist(p, \text{supp } f \cap H^-) > d_2, \tag{13}$$

which is a contradiction. **Q.E.D.**

We will now show several corollaries that can be extracted from Theorem 17.

Let $S \subset \mathbb{R}^n$. For any points $p, q \in \mathbb{R}^n - S$ we define the distance $d_S(p, q)$ as the infimum of lengths of C^1 paths in $\mathbb{R}^n - S$ connecting these points. Clearly $d_S(p, q) \geq |p - q|$. Using this metric, we can define the corresponding distances $dist_S$ from points to sets.

Theorem 20 *Let a set $S \subset \mathbb{R}^n$ and a non-zero function $f \in C(\mathbb{R}^n)$ decaying at infinity faster than any power of $|x|$ be such that $R_S f = 0$. Then for any point $p \in \mathbb{R}^n - S$*

$$dist_S(p, suppf) = dist(p, suppf). \tag{14}$$

Proof. Assume that (14) does not hold, i.e.

$$dist_S(p, \mathrm{supp} f) > dist(p, \mathrm{supp} f).$$

As it has been mentioned before, under the conditions of the theorem, we can assume S to be a part of an algebraic surface Σ for which $R_\Sigma f = 0$. Since d_S does not react on presence of lower dimensional algebraic manifolds, we can assume S to be a part of an algebraic hypersurface Σ dividing the space into parts $H^\pm$. Then, in notations of the previous theorem, we have

$$dist^\pm(p, \mathrm{supp}\ f \cap H^\pm) \geq dist_S(p, \mathrm{supp} f) > dist(p, \mathrm{supp} f). \tag{15}$$

Then Theorem 17 shows that this is impossible. **Q.E.D.**

Corollary 21 *Let f be continuous and decaying faster than any power and $S \subset \mathbb{R}^n$ be an algebraic hypersurface such that $R_S f = 0$. Let L be any hyperplane such that $L \cap supp\ f \neq \emptyset$ and such that $supp\ f$ lies on one side of L. Let $x \in L \cap supp\ f$ and r_x be the open ray starting at x, perpendicular to L, and going into the direction opposite to the support of f. Then either $r_x \subset S$, or r_x does not intersect S (albeit tangency is allowed).*

Proof. Assuming otherwise, take any point p on r_x after the first intersection with S. The only point closest to p on supp f is x, and the segment connecting the two points intersects S. Then obviously $dist_S(p, \mathrm{supp}\ f) > |p - x| = dist(p, \mathrm{supp}\ f)$. Now the previous theorem finishes the job. **Q.E.D.**

Let us formulate another example of a geometric constraint on pairs S, f such that $R_S f = 0$.

Theorem 22 *Let $S \subset \mathbb{R}^n$ be a relatively open piece of a smooth hypersurface and f be a continuous function decaying faster than any power of $|x|$ such that $R_S f = 0$. If there is a point $p_0 \in S$ such that the support of f lies on one side of the tangent plane $T_{p_0}S$ to S at p_0, then $f = 0$.*[2]

Proof of the theorem . Let us denote by $K_p(\text{supp } f)$ the convex cone with the vertex p consisting of all the rays starting at p and passing through the convex hall of the support of f. Then $K_{p_0}(\text{supp } f)$, due to the condition of the theorem, lies on one side of $T_{p_0}S$. Let us pull the point p_0 to the other side of the tangent plane along the normal to a nearby position p. Then for p sufficiently close to p_0 all rays of the cone $K_p(\text{supp } f)$ will intersect S. This means in particular, that for this point p we have $dist_S(p, \text{supp } f) > dist(p, \text{supp } f)$. According to Theorem 20, this implies that $f = 0$. **Q.E.D.**

Corollary 23 *Let $S \subset \mathbb{R}^n$ be an algebraic hypersurface and f be a continuous function decaying at infinity faster than any power. If $R_S f = 0$, then every tangent plane to S intersects the convex hull of the support of f.*

The above results present significant restrictions on the geometry of the non-injectivity sets S and corresponding functions f in the kernel of R_S. One can draw from them more specific conclusions about these sets.

Proposition 24 *Let $S \subset \mathbb{R}^2$ be an algebraic curve such that $R_S f = 0$ for some non-zero compactly supported continuous function f. Then S has no compact components, and each its component has asymptotes at infinity.*

Proof. Corollary 8 excludes bounded components. So, we can think that S is an irreducible unbounded algebraic curve. Existence of its asymptotes can be shown as follows. Let us take a point $p \in S$ and send it to one of the infinite ends of S. According to Corollary 23, every tangent line T_pS intersects the convex hull of the support of f, which is a fixed compact in $\mathbb{R}^2$. This makes this set of lines on the plane compact. Hence, we can choose a sequence of points p_j such that the lines $T_{p_j}S$ converge to a line T in the natural topology of the space of lines (e.g., one can use normal coordinates of lines to introduce such topology). This line T is in fact the required asymptote. Indeed, let us choose the coordinate system where T is the x-axis. Then the slopes of the sequence $T_{p_j}S$ converge to zero. Due to algebraicity, for a tail

[2]This implies, in particular, Theorem 14.

of this sequence, the convergence is monotonic, and in particular holds for all $p \in S$ far in the tail of S. Let us for instance assume that these slopes are negative. Then the tail of S is the graph of a monotonically decreasing positive function. This means that S has a horizontal asymptote. This asymptote must be the x-axis T, otherwise the y-intercepts of $T_{p_j}S$ would not converge to zero, which would contradict the convergence of $T_{p_j}S$ to T. **Q.E.D.**

The next statement proves the Agranovsky-Quinto Theorem 10 in the particular case of functions of convex support.

Proposition 25 *Let $S \subset \mathbb{R}^2$ and $f \in C_c(\mathbb{R}^2)$ be such that the support of f contains the boundary of its convex hull (in particular, the support itself can be convex), $f \neq 0$, and $R_S f = 0$. Then $S \subset \omega\Sigma_N \cup F$ in notations of Theorem 10.*

Proof. First of all, up to a finite set, we can assume that S is an algebraic curve. Corollary 21 says that all normal exterior rays to the boundary of the support are either inside S, or do not intersect S. By the normal we mean here the normal to any supporting plane to the boundary (in particular, at non-smooth points of the boundary one might have the whole cone of such rays). This immediately implies that outside the support the set S consists of rays. Algebraicity and absence of bounded components means that S is the union of several lines L_j intersecting the support. It is known that any straight line L is a non-injectivity set, but the only functions annihilated by R_L are the ones odd with respect to L (e.g., [3, 9, 20]). Hence, f is odd with respect to all lines L_j. In particular, every of these lines passes through the center of mass of the support of f. Hence, lines L_j form a "cross"[3]. It remains now to show that the angles between the lines are commensurate with π. This can also be shown in several different ways. For instance, this follows immediately from existence of a **harmonic** polynomial vanishing on S. Another simple option is to notice that if this is not the case, there is no non-zero function that is odd simultaneously with respect to all the lines. **Q.E.D.**

[3]One can prove that all these lines pass through a joint point also in a different manner. Indeed, due to oddness of f, each line is a symmetry axis for the support of f. Then, considering the group generated by reflections through these lines, one can easily conclude that if they did not pass through a joint point, then the support of f must have been non-compact.

Exactly the same consideration as above shows that in higher dimensions at least the following statement is correct:

Proposition 26 *Let $S \subset \mathbb{R}^n$ and $f \in C_c(\mathbb{R}^n)$ be such that the support of f contains the boundary of its convex hull (e.g., the support itself is convex) , $f \neq 0$, and $R_S f = 0$. Then S is spanned by lines.*

Remark 27 *If we could also show that all these lines pass through the same point, then this would immediately imply, as in the previous proof, the validity of Conjecture 11 for this case of convex support. We, however, have not succeeded in proving this yet. As we were notified by M. Agranovsky, he and E. T. Quinto had made some recent further progress for this case, albeit using microlocal tools.*

4 Additional remarks

1. M. Agranovsky and E. T. Quinto have written several other papers besides [3] devoted to the problem considered here. They consider some partial cases (e.g., distributions f supported on a finite set) and variations of the problem (e.g., in bounded domains rather than the whole space). See [1, 4, 5, 6] for details.

2. One of our goals was to obtain the complete Theorem 10, the main result of [3] by simple PDE tools, avoiding using the geometry of zeros of harmonic polynomials and microlocal analysis (or at least one of those). Although we have not completely succeeded in this yet, the results presented (e.g., Proposition 24) are moving in this direction.

3. The PDE methods presented here in principle bear a potential for considering non-compactly-supported functions. In order to achieve this, one needs to have qualitative versions of statements like Proposition 16 and Theorem 20, where instead of just noticing whether a wave has come to certain point at a certain moment (which was our only tool), one observes the amount of energy it carries.

4. In this paper one of the motivations for studying the injectivity problem was the thermoacoustic tomography. One wonders then whether considerations of $2D$ problems bear any relevance for TAT. In fact, they do. If either the scanned sample is very thin, or the transducers

are collimated in such a way that they register the signals only coming parallel to a given plane, one arrives to a $2D$ problem.

5. Most of our results can be generalized to some Riemannian manifolds, in particular to the hyperbolic plane (where the natural analog of Theorem 10 has not been proven yet). We plan to address these issues elsewhere.

6. A closer inspection of the results of Section 3 shows that most results have their local versions, where it is not required that the whole transform R_S of a function is equal to zero, but rather only for radii up to a certain value. One can see an example of a local uniqueness theorem for the circular transform in [25]. We hope to address this issue elsewhere.

7. As J. Boman notified us during the April 2004 AMS meeting in Lawrenceville, he jointly with J. Sjostrand, being unaware of our work, had recently independently obtained some results analogous to some of those presented here (e.g., to Theorem 20).

8. We have not touched the problem of finding explicit inversion formulas for the circular transforms. Such formulas are known for the spherical, planar, and cylindrical sets of centers [7, 10, 12, 13, 28, 29, 33, 34, 35]. They come in two kinds: the ones involving expansions into special functions, and the ones of backprojection type. Exact backprojection type formulas are known for the planar geometry [10, 29] and recently for the spherical geometry in odd dimensions [13] if the function to be reconstructed is supported inside the sphere of transducers.

   Another problem deserving attention is finding the ranges of transforms R_S. Such knowledge could be used, for instance, to replenish missing data. Some necessary range conditions have been recently obtained in [30] for spherical location of transducers.

   An important problem in tomography of reconstruction with incomplete data was treated in [25, 36] based on an earlier work by E. T. Quinto in [32].

9. An important integral geometric technique of the so called κ-operator has been developed in I. Gelfand's school (e.g., [15, 16]). It has been applied recently to the problems of the circular Radon transform (see [17], the last chapter of [16], and references therein), albeit applicability

of this method to the problems of the kind we consider in this paper is not completely clear yet.

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