Provable first-order transitions for nonlinear vector and gauge models with continuous symmetries

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Abstract

We consider various sufficiently nonlinear vector models of ferromagnets, of nematic liquid crystals and of nonlinear lattice gauge theories with continuous symmetries. We show, employing the method of Reflection Positivity and Chessboard Estimates, that they all exhibit first-order transitions in the temperature, when the nonlinearity parameter is large enough. The results hold in dimension 2 or more for the ferromagnetic models and the RP^{N-1} liquid crystal models and in dimension 3 or more for the lattice gauge models. In the two-dimensional case our results clarify and solve a recent controversy about the possibility of such transitions. For lattice gauge models our methods provide the first proof of a first-order transition in a model with a continuous gauge symmetry.

1 Introduction

In this paper we prove a number of results showing that nearest neighbor models with a sufficiently nonlinear, rotation-invariant, nearest-neighbour interaction – sufficiently nonlinear meaning that the nearest neighbour interaction has the shape of a deep and narrow well – show a first-order transition in temperature. Part of our results have appeared in [15].

We remind the reader that the first-order transitions occur when the free energy density (or pressure) at some values of the thermodynamic parameters is non-differentiable as a function of one of the parameters in the Hamiltonian. In our examples this parameter will be the temperature. Equivalently, at these parameter values different infinite-volume Gibbs measures exist which have different expectation values for an observable dual to the nondifferentiability parameter. In our examples this observable will be the energy. For further aspects of Gibbs measure theory and the associated thermodynamic formalism we refer to [56, 20, 22].

An example of a model which has appeared in the literature and which can be treated by our methods is given by the ferromagnetic Hamiltonian

$$H = -J \sum_{<i,j>\in\mathbb{Z}^2} \left(\frac{1+\cos(\phi_i - \phi_j)}{2}\right)^p,$$

with p large.

Our results confirm earlier numerical work on this model [14, 8], which, however, has been contested by various authors. For some of this literature, see [25, 28, 57, 38, 27, 4].

Our analysis is not restricted to ferromagnets, but also applies to RP^{N-1} (liquid-crystal) models (such as were first introduced by Lasher and Lebwohl [35, 34]) and to lattice gauge models (which are invariant under local, as opposed to global, rotation symmetries).

We find that these nonlinear ferromagnetic, liquid-crystal and lattice gauge models, (with either abelian or non-abelian symmetries) all have 1st order transitions in the temperature.

The standard ferromagnetic N-vector models are either believed or sometimes rigorously known to have 2nd order transitions in $d = 3$ or higher, a "Kosterlitz-Thouless" transition in $d = 2, N = 2$, and no transition for $d = 2$ and higher N [1]. In the XY-model ($N = 2$) for either $d = 2$ or high d these results are rigorous, for the other models there is a consensus based on both numerics and heuristic arguments.

In contrast, for the standard versions of the liquid crystal and lattice gauge models, as well as for very non-linear ferromagnetic σ-models, both numerics and high temperature series suggested the existence of 1st order transitions, despite some theoretical and numerical analyses originally either suggesting 2nd order transitions, no transitions at all, or Kosterlitz-Thouless type transitions. Furthermore, the phase transition in the 3d liquid crystal models was observed to become more strongly first order when a nonlinear term was added. For some of this literature, see e.g. [32, 35, 34, 16, 2, 54, 10, 42, 48, 51, 53, 23, 47, 46, 36, 39, 18, 43, 44] and references therein. Moreover, in the limit where N approaches infinity (the spherical limit) 1st order transitions were found, in dimension 2 or more [31, 55, 53]. This spherical limit result also holds for our nonlinear interactions in the ferromagnetic case [9]. Whether such a first-order transition can also occur for finite N larger than 3 in $d = 2$, or whether it might be an artefact of the spherical limit has for a long time been a matter of controversy (see for example [53, 55]). In fact, Sokal and Starinets described the existence of such a first-order transition as a "pathology".

Our result finally settles this question: first-order phase transitions for models with continuous symmetry in $d = 2$ can occur, despite the conjecture to the contrary of [53]. Our results in $d = 2$ are thus essentially in agreement with the analysis of [55]. In contrast to what was suggested in most earlier analyses, the symmetry or the low-temperature properties of the model do not play a role of any great importance, and neither do the nature of the topological excitations or the spin-dimensionality. In fact, for our nonlinear choice of interaction the spin-dimensionality N does not need to be large and can be as small as 2. Also the lattice dependence of the phenomenon found in [55] seems somewhat of an artefact which disappears if one varies the nonlinearity parameter. The wide occurrence of first-order transitions in liquid-crystal and lattice gauge models indicates that a proof in these type of models may be of even more direct physical relevance than in the case of ferromagnets.

The main ingredient of our proofs is a similarity between such nonlinear models and high-q Potts models, which allows one to adapt proofs for Potts models, such as were first developed in [29], and based on [13], to prove first-order transitions in the temperature parameter. We remark that some similar results were found by L. Chayes [11], also making use of the Potts resemblance. See also [3], where high temperature uniqueness was proven, in almost the whole high-temperature region. The arguments for showing the existence of first-order transitions, such as have been used for Potts ferromagnets in d at least 2, apply to the ferromagnetic and liquid-crystal models, those developed for Potts lattice gauge models in d at least 3 [29], apply to the lattice gauge models.

The fact that our proofs are insensitive to the nature of the phases between which the transition takes place implies that one might have in the ferromagnetic or liquid-crystal models a transition between a disordered high-temperature phase and either a (ferromagnetically or nematically) ordered, a Kosterlitz-Touless or a disordered phase at low temperatures. Similarly one might find a transition either between a confining and a nonconfining – Coulomb-like – phase or between two confining phases in the lattice gauge models. Which one occurs in a particular case should depend on dimension and/or symmetry of the system, but our methods do not provide information on the low temperature regime, although in some cases known methods may apply. Another consequence of our methods is that we show examples where there are additional transitions between distinct ordered or distinct Kosterlitz-Thouless low-temperature phases.

In particular, we emphasize that our proofs are also insensitive as to whether the symmetry group of the lattice gauge model is abelian – in which case it is expected that in 4 dimensions a transition between a confined and a Coulomb-like phase occurs [24, 19] –, or nonabelian, in which case both states are expected to be confining, (this is also expected in general in $d = 3$). For a heterodox discussion on the difference between what is to be expected in abelian and nonabelian models, including some history of this problem cf [40].

2 Notation and Results

We consider a lattice $\mathbb{Z}^d$, and either spin models, in which the random variables σ_i live on the sites, or lattice gauge models, where the variables live on the bonds (or links) of the lattice. The parameters of our models are the spatial dimension d, the spin-dimension N, and the nonlinearity parameter p. The "standard" versions of the models are obtained by taking $p = 1$.

For ferromagnetic models the variables are N-component unit vectors, living on the sphere $\mathbb{S}^{N-1}$. We will present the argument in the 2-component case in detail, in the general case the proof is essentially the same. In the $N = 2$ case we also sometimes use angle variables ϕ_i to denote the spins.

The ferromagnetic models first considered in [14] and generalized to $N = 3$ in [8], were given by

$$H = -J \sum_{<i,j>\in\mathbb{Z}^2} \left(\frac{1 + \cos(\phi_i - \phi_j)}{2} \right)^p . \tag{1}$$

In fact the precise shape of the well-potential is not very important, as long as it is narrow enough. For convenience we first present the argument for a rectangular-well potential, and afterwards discuss the necessary adaptations to treat general wells.

The property we will always need is Reflection Positivity (RP) [47]. This will hold for all our examples, but, as is well known, it restricts us to nearest-neighbour-cube interactions (C-interactions, in the terminology of [21]) and in particular prevents us to extend our proof to quantum spin models.

We denote our spin variables by σ, and we consider interactions U which are nearest neighbor, and which contain only functions of inner products between neighboring spins.

Thus the general form of our model is:

$$H = - \sum_{<i,j>\in\mathbb{Z}^d} U(\sigma_i \cdot \sigma_j) \tag{2}$$

We will also use (by abuse of notation) the angle between neighboring spins as the argument of the function U, as we will consider U-s which are rotation invariant, and thus only depend on this angle.

When U has a maximum at 1 and is a decreasing function of the cosine of the angle ϕ between neighboring spins, the model is ferromagnetic.

We begin with the simplest case of a square-well potential in two-dimensions, with classical XY-spins. The parameter ε describing the width of the well will play the same role of a small parameter here as $\frac{1}{q}$ does in the q-state Potts model.

Theorem 1 *Let $U(\phi) = 1$ for $|\phi| \leq \varepsilon$, and $U = 0$ otherwise, $d = 2$ and $N = 2$. For ε small enough this model has a first-order phase transition in temperature. In particular, there exists a temperature where at least two different Gibbs measures with different energy densities coexist.*

Proof. We introduce projection P^o_b, which is the characteristic function of the event that the bond b is "ordered", that is the angle between the two spins at the ends of the bond b differ by less than ε, – and P_b^d, the indicator of the event "b is disordered", that is the spins at the ends of b differ by more than ε. It is immediate that the expectation of P_b^d at high temperatures is close to one, and that the expectation of P_b^o is close to one at low temperatures. We need to show that ordered and disordered bonds tend not to be neighbours at all temperatures. Thus we need to estimate the expectation of $P_b^d P_{b'}^o$, with b, b' two orthogonal bonds sharing the same site. Once we have it, the proof follows, see [50].

For this it is sufficient to apply a chessboard estimate, following the approach of [13, 29], going back to [17], and described in e.g. [50, 21]. We follow in particular [50]. Our two-dimensional model has RP property with respect to reflections in lines $\{x \pm y = k\}$, $k = ..., -1, 0, 1, ...$. We need to estimate the probability of the occurrence of a "universal contour", which in our case will be the set of configurations such that in a toroidal volume Λ consisting of L^2 sites, (where L is a multiple of 4) a quarter of the sites – those belonging to diagonals chosen periodically at distance 4 – are surrounded by ordered bonds, and another quarter of the sites, along the diagonals halfway, are surrounded by disordered bonds (compare [50], figure 6). We have thus alternating diagonal strips of ordered and disordered squares. Once we have obtained this estimate, we can apply a Peierls-type contour argument.

To estimate this universal-contour probability, we have to estimate the partition function over all configurations in which the universal contour occurs, from above. We do it by noticing that for three quarters of the sites one integrates over an interval of at most ε, and that half of the (2Λ) bonds contribute an energy β, from which we obtain

$$Z_{L,univcont}(\beta, \varepsilon) \leq Cst \; e^{\beta L^2} \varepsilon^{\frac{3}{4}L^2} e^{O(L)}.$$

We estimate the full partition function from below by

$$Z_L(\beta, \varepsilon) \geq max[1, Z_L^o].$$

The lower bound 1 follows from the positivity of the function U.

In the bound for the ordered partition function Z_L^o, we can restrict the integration over spin variables at each site, to the interval $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. This implies immediately that

$$Z_L^o \geq \varepsilon^{L^2} e^{2\beta L^2}$$

(which is larger than 1 for $\beta \geq \beta_0 = -\frac{ln\varepsilon}{2}$, which in its turn gives an approximate value for the transition temperature), so we obtain

$$\frac{Z_{L,univcont}(\beta,\varepsilon)}{Z_L(\beta,\varepsilon)} \leq \varepsilon^{O\left(\frac{L^2}{4}\right)}.$$

This is immediate for $\beta \geq \beta_0$, and for $\beta \leq \beta_0$ we use the observation that $e^{\beta} \leq e^{\beta_0} = \varepsilon^{-\frac{1}{2}}$. ■

Remark 1:

In 2-dimensional models, the Mermin-Wagner theorem [37] and its extension, the Dobrushin-Shlosman theorem [12] (a recent version of which also includes a non-continuous interaction such as the one under consideration here [26]), imply that all possible Gibbs measures are rotation-invariant.

Remark 2:

Of course, the above theorem can be extended to higher dimensions $d > 2$. The only difference in the proof would be that one can not use the RP in the 45° planes, so one uses instead RP in the coordinate planes and their integer shifts, i.e. in the planes $\{x_i = k\}$, $i = 1, 2, ..., d$, $k = ..., -1, 0, 1, ...$. That changes the definition of the universal contour. As a result, the corresponding estimates becomes somewhat weaker, but the conclusions of the theorem still hold. The generalization to $N > 2$ is immediate.

Our methods, combined with the technique of [41], also imply that more than one transition can occur, even infinitely many, at an infinite sequence of lower and lower temperatures, between all paramagnetic ($d = 2, n \geq 3$), all Kosterlitz-Thouless ($d = 2, n = 2$), or all ordered ($d \geq 3$) phases.

Choose e.g. for the potential function U a summable sum of characteristic functions on fastly decreasing intervals : $U(x) = \sum_n 2^{-n} 1_{\varepsilon_n}(x)$, with $\varepsilon_n (= \varepsilon_{n-1}^3) = \varepsilon^{3^{n-1}}$, with the first ε small enough.

Ground states for this interaction with wells in wells, which for obvious reasons we will call the Seuss-potential [49] (see fig. 1), are perfectly ordered, and the model is clearly ferromagnetic. At a sequence of increasing inverse temperatures β_n one has first-order transitions, where one has at β_n coexistence of Gibbs measures, one concentrated on configurations with most bonds in well n, evenly distributed, and another – on configurations with most bonds in well $n+1$. The widths and depths of the successive wells are chosen in such a way that the sequence of inverse transition temperatures is growing like $\frac{3}{2}^n$.

Theorem 2 *For ε small enough, the above Seuss-model has an infinite set of temperatures where first order transitions in the temperature occur, for all $N \geq 2, d \geq 2$.*

Our next step indicates how to generalize from rectangular wells to polynomial wells.

Theorem 3 *Consider the model ($N = 2, d = 2$) with nearest-neighbor Hamiltonian*

$$H = -J \sum_{<i,j>\in\mathbb{Z}^2} \left(\frac{1 + \cos(\phi_i - \phi_j)}{2} \right)^p . \tag{3}$$

For p large enough this model has a first-order transition.

Proof. We employ the fact that for small difference angles $cos(\phi_i - \phi_j)$ is approximately $1 - O([\phi_i - \phi_j]^2)$, and furthermore that $lim_{p\to\infty}(1 - \frac{1}{p})^p = \frac{1}{e}$. This suggests to choose $\varepsilon(p)$ to be $\frac{1}{\sqrt{p}}$. The difference with the rectangular-well model is that now the distinction between ordered and disordered bonds becomes somewhat arbitrary, and we make a slightly different choice, namely we call a bond ordered if

$$|\phi_i - \phi_j| \leq \varepsilon(p) \equiv \frac{C}{\sqrt{p}}.$$

We will choose C large, which implies that all disordered bonds have low energy (close to zero), which we will need in the estimate on the upper bound for the universal contour partition function.

We also introduce the notion of "strongly ordered" bonds, which have their energies close to maximal energy: a bond i, j is strongly ordered if

$$|\phi_i - \phi_j| \leq \frac{1}{C\sqrt{p}}.$$

We will use them in estimating the ordered partition function Z_L^o from below.

We then have, similarly to before, $Z_L(\beta, p) \geq max[1, Z_L^o]$. We bound the ordered partition function Z_L^o from below by integrating over the spin variables at each site within the strongly ordered interval $[-\frac{1}{C\sqrt{p}}, \frac{1}{C\sqrt{p}}]$. There the energy is close to its mininum, $-J$. Thus we obtain

$$Z_L(\beta, p) \geq \left(\frac{1}{C\sqrt{p}} \right)^{L^2} e^{2\beta J\left(1 - O(\frac{1}{C^2})L^2\right)}.$$

For the $Z_{L,univcont}$ we obtain similarly as before,

$$Z_{L,univcont}(\beta,p) \leq e^{\beta JL^2[1+O(e^{-C^2})]}\left(\frac{C}{\sqrt{p}}\right)^{\frac{3}{4}L^2+O(L)}.$$

The rest of the argument is identical to the one before, once we choose C large enough. For example a choice $C = p^\delta$ for some small positive δ will do. ■

Again, generalizations to higher N and d are immediate.

Remark 1:

The transition, and our proof for it, persists if one applies a small external field; thus it is immediately clear that no Lee-Yang circle theorem will hold, in contrast to the standard ferromagnetic XY-models.

Remark 2:

We know that at low temperatures percolation of ordered bonds holds [21]; it follows from our results that the associated percolation transition is also first order.

For liquid crystal RP^{N-1} models one either considers variables – usually denoted n_i – which live on the projective manifold, obtained by identifying a point on the N-sphere with its antipod, or equivalently one can consider ordinary spins on the N-sphere, and divide out this "local gauge symmetry" afterwards. The last approach is the route we will pursue, as it allows us to literally apply the identical proof in the ferromagnetic and the liquid crystalline cases.

Thus we consider Hamiltonians of the form

$$H = -J \sum_{<i,j>\in\mathbb{Z}^2} \left(\frac{1+\cos^2(\phi_i-\phi_j)}{2}\right)^p. \tag{4}$$

In the ferromagnetic case we called a bond "ordered" if the angle θ between two neighboring sites is small enough. Here we call it ordered if the angle $\theta \mathrm{mod} \pi$ is small enough. Then the argument goes through without any changes. There is a first-order phase transition for p chosen large enough (in general the values of p for which the proof works depend on N and d) between a high-temperature regime, in which most nearest neighbor bonds are disordered, and a low-temperature regime, in which most nearest neighbor bonds are ordered. This holds for each dimension at least 2, and whereas the Mermin-Wagner theorem excludes nematic long-range order in $d = 2$

[37], in $d = 3$ and higher long-range order it will occur [5]. Between the ordered and the disordered phase(s) free energy contours occur, whose probabilities are estimated to be uniformly small via a contour estimate valid over a whole temperature interval. In the contour estimate again use is made of the Reflection Positivity [47] of the model.

Theorem 4 *For any nonlinear RP^{N-1} model in dimension 2 or more and p high enough, there is a first order transition, that is, there exists a temperature at which the free energy is not differentiable in the temperature parameter. In particular, there exists a temperature where at least two different Gibbs measures with different energy densities coexist.*

For lattice gauge models the variables are elements of a unitary representation of a compact continuous gauge group, e.g. $U(1)$, $SU(n)$, or sums thereof [52]. Here we present the argument in the simplest case of a $U(1)$-invariant interaction in 3 dimensions:

$$H = -J \sum_{plaquettes\ P \in \mathbb{Z}^3} L\left(U_P\right), \tag{5}$$

with $L(U_P) = \left(\frac{1+\cos\left(\phi_{e_1}+\phi_{e_2}-\phi_{e_3}-\phi_{e_4}\right)}{2}\right)^p$. Here the e_i denote the 4 edges making up the plaquette P.

The effect of choosing the nonlinearity parameter p high is again that the potential, although it still has quadratic minima, becomes much steeper and narrower. In this way one constructs in a certain sense a "free energy barrier" between ordered and disordered phases.

The lattice gauge model proof becomes similar to the arguments from [29]. When the product over the link variables is sufficiently close to unity, we'll call the plaquette "ordered", "disordered" otherwise. This distinction corresponds to unfrustrated and frustrated plaquettes in the Potts case. We will sketch the argument for the toy model where the potential $L(U)$ is chosen to be 1 if the sum ϕ_P of the oriented angles along the plaquette P is between $-\frac{\varepsilon}{2}$ and $+\frac{\varepsilon}{2}$ and zero otherwise. The generalization to the non-rectangular-well potentials can then be done in the same way as before. The correspondence again is that ε is of order $O(\frac{1}{\sqrt{p}})$.

Our strategy is to find bounds for free energy contours between ordered phases, in which one has mainly cubes with 6 ordered plaquettes, and disordered phases, in which most cubes have 6 disordered plaquettes. We need thus to estimate the weights of contours consisting of cubes which are neither ordered nor disordered. The number of possibilities for such cubes includes the 7 possibilities given in [29], plus we have now the additional 8th possibility of having cubes with one disordered plaquette and five ordered ones.

For the partition function Z_L on a cube B_L of size L^3 we use the (quite rough) lower bound

$$Z_L \geq max(Z_L^d, Z_L^o), \tag{6}$$

where Z_L^d (resp., Z_L^o) is part of the partition function, calculated over all configurations which have all plaquettes disordered (resp., mostly ordered). For the disordered partition function Z_L^d we obtain the lower bound $(1-4\varepsilon)^{3L^3}$ (we take a normalized reference measure, giving a weight 1 to each link).

For the ordered partition function Z_L^o we proceed as follows: we first choose a set of bonds T_L in B_L, which is a tree, passing through every site. For example, we can put into T_L all vertical bonds – z-bonds – except these connecting sites with z-coordinates 0 and 1, plus all y-bonds in the plane $z = 0$, except these connecting the sites with y-coordinates 0 and 1, plus all x-bonds of the line $y = z = 0$, except the one between the sites $(0, 0, 0)$ and $(1, 0, 0)$. The site $(0, 0, 0)$ can be taken as a root of T_L. Note that the number of bonds in T_L is $L^3 - 1$. Therefore it is not surprising (and easy to see) that for every edge configuration $\phi = \{\phi_b, b \in T_L\}$ there exists a unique site configuration $\psi = \psi_\phi = \{\psi_x, x \in B_L\}$, such that the following holds:

1. Let $\mathfrak{g}^\psi$ denote the gauge transformation, defined by the configuration ψ. Then $\left(\mathfrak{g}^\psi \circ \phi\right)\Big|_b = 1$ for every bond $b \in T_L$;

2. $\psi_{(0,0,0)} = 1$.

For every family of bonds $S \subset B_L$ let us define a bigger family $C(S)$, by the rules:

1. $S \subset C(S)$,

2. for every four bonds $\{b_1, ..., b_4\}$, making a plaquette, such that three of them are in S, we have $\{b_1, ..., b_4\} \subset C(S)$.

Then we can consider also the sets $C^2(S) = C(C(S))$, $C^3(S)$, and so on. Define $\mathfrak{C}(S) = \cup_k C^k(S)$. Note that the number of plaquettes in $\mathfrak{C}(T_L)$ is

$3L^3 - O(L^2)$. We claim now that for every configuration $\phi_{T_L} = \{\phi_b, b \in T_L\}$ one can specify (in a continuous way) a collection of arcs $\left\{I_b = I_b(\phi_{T_L}) \subset S^1, b \in \mathfrak{C}(T_L) \setminus T_L, |I_b| = \frac{\varepsilon}{4}\right\}$, such that for every configuration ϕ on B_L, which coincides with ϕ_{T_L} on T_L, and for which the values ϕ_b on the bonds $b \in \mathfrak{C}(T_L) \setminus T_L$ belong to the above segments I_b, all the plaquettes that fall into $\mathfrak{C}(T_L)$ are non-frustrated. That would imply that

$$Z_L^o \geq \left(\frac{\varepsilon}{4}\right)^{2L^3} \exp\left\{3J\left(L^3 - O\left(L^2\right)\right)\right\}$$

by Fubini's theorem. To see the validity of our claim, consider first the case when the configuration $\phi_{T_L} \equiv \mathbf{1} \in S^1$ (here 1 is the neutral element). Then the choice of the segments I_b is easy: $I_b(\mathbf{1}) = \left[-\frac{\varepsilon}{8}, \frac{\varepsilon}{8}\right]$ for every $b \in \mathfrak{C}(T_L) \setminus T_L$. For a general ϕ_{T_L} let us take the corresponding gauge transformation $\mathfrak{g}^{\phi_{T_L}}$ (which is the identity for $\phi_{T_L} \equiv \mathbf{1}$), and we define our segments by

$$I_b(\phi_{T_L}) = \left(\mathfrak{g}^{\phi_{T_L}}\right)^{-1} I_b(\mathbf{1}).$$

This provides a lower bound

$$Z_L \geq \max\left[(1-4\varepsilon)^{3L^3}, \left(\frac{\varepsilon}{4}\right)^{2L^3} \exp\left\{3J\left(L^3 - O\left(L^2\right)\right)\right\}\right]. \tag{7}$$

This bound on the partition function as the maximum of the ordered and disordered term is similar to the argument in [15]. It plays the same role as the bound in terms of a fixed energy partition function given in [29].

To obtain our contour estimates, by Reflection Positivity we need to estimate the partition functions of configurations constrained to have a "universal contour". The estimates of the 7 types of universal contours mentioned in [29] are of a similar form as in that paper with the number of Potts states q replaced by $\frac{1}{\varepsilon}$, up to some constant. The universal contour due to the new case of cubes with one disordered plaquette consists of configurations in which the horizontal plaquettes in every other plane are disordered, and all the other ones are ordered. These configurations have a similar entropy contribution to the partition function as the ordered configurations, but the energy per cube is $\frac{5}{6}$ of that of a cube in the fully ordered situation. For ε small enough (which corresponds to p large enough) also such a contour is suppressed exponentially in the volume. The combinatorial factor in the contour estimate changes by some finite constant, which choosing ε small enough takes care of.

To summarize we have obtained the following result:

Theorem 5 *For lattice gauge models with plaquette action* $\left(\frac{1+L(U_P)}{2}\right)^p$, *(where* $L(U_P) = Tr(U_P + U_P^*)$*) in dimension* 3 *and more and* p *high enough, there is a first order transition, that is there exists a temperature at which the free energy is not differentiable in the temperature parameter. In particular, there exists a temperature where at least two different Gibbs measures with different energy densities coexist.*

Here U_P^* denotes the adjoint operator of U_P.

3 Summary and discussion

Our results provide a number of answers to questions which were raised before. As we discussed in the introduction, the nonlinear two-dimensional ferromagnetic models were studied numerically, and our results fully confirm what was found in [14, 8]. Our work provides to our knowledge the first case in which a first order transition for a lattice gauge model with a continuous gauge symmetry group is rigorously obtained. Whereas the example of the Potts lattice gauge model in $d = 3$ or higher is between a confining and a nonconfining phase [30, 33], in our theorem this is to be expected in $d = 4$, with $U(1)$ symmetry only. For $d = 3$ and also for $SU(n)$ in $d = 4$ we conjecture that confined phases exist on both sides of the phase transition.

Our proof only gives results for very high values of the nonlinearity parameter p. We will discuss some further aspects of what may actually be the p-values for which to expect first-order transitions, and what one might hope to prove. The recent work of Biskup and Chayes [6] shows that if a reflection positive model has a phase transition in mean field theory, then also at sufficiently high dimension a first-order transition occurs. They include in their discussion the RP^{N-1} model for $N = 3$, for which even for the standard choice $p = 1$ (so there is no strong nonlinearity in the interaction), a first-order transition is derived. The mean field analysis of [7] indicates that a similar result for the ferromagnetic case holds if $p = 3$, and here a sufficiently strong nonlinearity is indeed needed. For lattice gauge models on the other hand, also the standard (p=1) actions lead to first-order transitions in mean-field theory ([58], section 34.4), which indicates a first-order transition in sufficiently high dimension.

Similarly, if one believes that here the spherical (N to infinity) limit is not singular (which has been a matter of controversy itself), then for the square lattice, N large and p larger than 6 the sufficiently nonlinear ferromagnet might have a first-order transition, while for the RP^{N-1} case on the square or triangular lattice even for $p = 1$ a first order transition occurs, although for the hexagonal lattice one presumably needs a higher value of p [53, 55].

As mentioned before, numerical work suggests that in the standard ($p = 1$) Lebwohl-Lasher model with $N = 3$ in $d = 3$, as well as in the $U(1)$-lattice-gauge model in $d = 4$, a first order transition should occur; however, this seems far away from any provable result.

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