N-dimensional invariant tori for the planar $(N+1)$-body problem *

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Abstract

For any $N \geq 2$ we prove the existence of quasi–periodic orbits lying on N–dimensional invariant elliptic tori for the planetary planar $(N+1)$–body problem. For concreteness, we focus on a caricature of the outer solar system.

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Contents

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1 Introduction and results

1.1 Quasi–periodic motions in the many–body problem

The existence of stable trajectories of the many body problem viewed as a model for the solar system has been the subject of researches of many distinguished scientists both in the past and in recent years. Despite eminent contributions given, among others, by Poincaré [Poi1905], Arnold [A63] and Herman [H95] and some numerical evidence of "stability" of the giant planets of the solar system (compare [L96]), the literature is still lacking in rigorous general proofs[1]. The main difficulty consists in the application of general tools (such as averaging and KAM theory) to particular cases of interest in celestial mechanics, where strong degeneracies appear. The scope of this paper is to make some progress in this direction, showing the existence of quasi–periodic orbits lying on N–dimensional invariant elliptic tori for the planar $(N+1)$–body problem. Though the method exposed here is quite general, for concreteness we will focus our attention on a caricature of the outer solar system.

More precisely, our model will be given by a Sun and N planets with relatively small masses (say, of order ε). All these $(N+1)$ bodies are considered as point masses in mutual gravitational interaction. Two planets (such as Jupiter and Saturn in the real world) will be assumed to have mass considerably bigger than the other planets. The bodies lie in a given plane and we assume that the initial configuration is far from collisions. We also assume, mimicking the case of the outer solar system, that the two big planets have an orbit which is internal with respect to the orbits of the small planets. We will establish, for a large set of semi–axes, the existence of quasi–periodic orbits with small eccentricities filling up N–dimensional invariant elliptic tori. Such orbits can be seen as continuations of "limiting" circular trajectories of the system obtained by neglecting the mutual interactions among the planets. A more precise statement is given in Theorem 1.1 below.

The above "outer model", which roughly mimics some traits of physically relevant cases, has also the nice feature of providing particulary simple expressions in the related perturbing functions, as we will see in Section 3 below. We stress, however, that many other situations (such as one large planet plus $N-1$ small planets; "inner" or "mixed" models, etc.) may be easily dealt with using the techniques and results presented in this paper.

The proof of our result is based on techniques developed in [BCV03] and on the explicit computation of the eigenvalues of the quadratic part of the so–called principal part of the perturbation for the planar many–body problem.

The first result on quasi–periodic orbits of interest in celestial mechanics goes back to [A63], where quasi–periodic orbits lying on 4–dimensional tori are shown to exist for the planar three–body problem (the general case was discussed there, but no complete proof was given). Related results were given in [JM66], which found linearly unstable quasi–periodic orbits lying on 2–dimensional tori for the non planar three–body problem. More recently, [LR95]–[R95] and [BCV03] proved the existence of quasi–periodic orbits for the non planar three–body problem, lying on 4–dimensional and linearly stable 2–dimensional tori, respectively. Periodic orbits of the non planar three–body problem winding around invariant tori have been constructed in [BBV03]. The result presented below, in the case $N=2$, has been obtained, with different methods, in [F02].

The paper is organized as follows. In Section 1.2, we give a more precise statement of our main result. In Section 2 we write down the $(N+1)$–body problem Hamiltonian in Delaunay–Poincaré variables. In Section 3 (which, in a sense, is the crucial part of the paper) we discuss degeneracies. In Section 4 we give the proof of the main result. The scheme of proof is similar to the one presented in [BCV03] (see also [BBV03]) in the three–body case and it is based on a "general" averaging theorem and on KAM theory for lower–dimensional tori ([P96], [BCV03], [BBV03]). For completeness, we include a classical (but not–easy–to–find) description of analytical properties of the Delaunay–Poincaré variables (see § 2

[1] See, however, [F03].

and Appendix A); in Appendix B we collect some simple linear algebra lemmata that are used in the arguments given in Section 3.

1.2 Statement of results

We denote the $N+1$ massive points ("bodies") by $P_0, \ldots, P_N$ and let $m_0, \ldots, m_N$ be their masses interacting through gravity (with constant of gravitation 1). Fix $m_0 > 0$ and assume that

$$m_i = \varepsilon \mu_i \ , \qquad i = 1, \ldots, N \ , \qquad 0 < \varepsilon < 1 \ . \tag{1.1}$$

Here, ε is regarded as a small parameter and μ_i is of order 1 in ε. The point P_0 represents the "Sun" and the points P_i, $i = 1, \ldots, N$, the "planets". We assume that all the bodies lie on a fixed plane, that will be identified with $\mathbf{R}^2$. The phase space of this dynamical system ("the planetary, planar $(N+1)$–body system") has dimension $4N$.

We will state the result in terms of orbital elements of the "osculating ellipses" of the two–body problems associated to (P_0, P_j). Let $u^{(0)}$ and $u^{(j)}$ denote the coordinates of P_0 and P_j (at a given time) and let $\dot{u}^{(0)}$ and $\dot{u}^{(j)}$ denote the corresponding velocities. By definition, the "osculating ellipse" is the ellipse described by the solution of the two–body problem (P_0, P_j) with initial data given by $(u^{(0)}, u^{(j)}, \dot{u}^{(0)}, \dot{u}^{(j)})$. Of course, such ellipses describe the motions of the full $(N+1)$–body problem only approximately; nevertheless, it provides a nice set of coordinates allowing, for example, to describe the true motions in terms of the eccentricities e_j and the major semi–axes a_j of the osculating ellipses. For further details and pictures of the orbital elements, we refer to [Ch88] and [BCV03].

In this paper we consider a planetary (planar) model with planets evolving from phase points corresponding to well separated nearly–circular ellipse ($e_i \ll 1$); here "well separated" means that

$$0 < a_i < \theta \ a_{i+1} \ , \qquad\qquad 1 \le i \le N-1 \ . \tag{1.2}$$

for a suitable constant $0 < \theta < 1$. For concreteness, we shall focus on a caricature of the outer solar system, i.e., we will assume that, for some $m_0 < \bar{\mu}_i < 4m_0$,

$$\begin{aligned} \mu_i &= \bar{\mu}_i && \text{for} \quad i = 1, 2 \, , \\ \mu_i &= \delta \bar{\mu}_i && \text{for} \quad i = 3, \ldots, N \, , \qquad 0 < \delta < 1 \, . \end{aligned} \tag{1.3}$$

In this setting, P_1 and P_2 imitate (in a very rough way, of course) the physical features of the giant planets Jupiter and Saturn, while P_3 and P_4 represent Uranus and Neptune[2].

Theorem 1.1 *Consider a planar, planetary $(N+1)$–body system satisfying (if $N \ge 3$) (1.3). Let $A \subset \mathbf{R}^N$ be a compact set of semi–axes where (1.2) hold. Then, there exists $\delta^\star > 0$ and for any $0 < \delta < \delta^\star$ there exists $\varepsilon^\star > 0$ so that the following holds. For any $0 < \varepsilon < \varepsilon^\star$, the planetary, planar $(N+1)$–body system possesses a family of N–dimensional elliptic invariant tori, travelled with Diophantine frequencies, provided that the osculating major semi–axes belong to a subset of A of density[3] $1 - C_1 \varepsilon^{c_1}$. These motions correspond to orbits with osculating eccentricities bounded by $C_2 \varepsilon^{c_2}$ and the variation in time of the osculating major semi–axes of these orbits is bounded by $C_3 \varepsilon^{c_3}$.*

A few comments:

- The numbers $\delta^\star$ and θ can be easily computed in the course of the proof and are not "very small"; in fact θ is a "universal" constant while $\delta^\star$ depends only on N and A. On the other hand $\varepsilon^\star$, which depend on N, A and δ, is related to a KAM smallness condition and rough estimates lead, as well known, to ridiculously small quantities (for somewhat more serious KAM estimates, we refer to [CC03]). Finally, the positive constants C_i's depend on N, A and δ, while the c_i's depend ony on N (and could, also, be easily calculated: see (4.93)).

[2] The Jupiter/Saturn mass ratio is approximately 3.34, while the Neptun/Uranus mass ratio is about 1.18.
[3] Here and in the sequel, the "density" is intended with respect to Lebesgue measure.

- The assumptions (1.2) and (1.3) in the theorem are used to check *explicitely* suitable “non degeneracy” conditions. However, giving up explicit constants and estimates, one can show that the thesis of the theorem holds, essentially, with no hypotheses on the semi–axes a_j and the rescaled masses μ_j (provided $a_i \neq a_j > 0$ and $\mu_j > 0$).

- The invariant tori found in Theorem 1.1 are *lower dimensional elliptic tori* meaning that the dimension of the tori are strictly smaller than (in fact, half of) the dimension of the Lagrangian (maximal) tori, which have dimension $2N$. “Elliptic” means that the tori are linearly stable. It is not difficult to show that such elliptic tori are surrounded by a set of positive measure of maximal tori.

- The proof given below is based on a well known elliptic KAM theorem, which works under “non degeneracy” (or Melnikov) conditions. To check these conditions one has to study the eigenvalues of the “secular” (or averaged) quadratic part of the Newtonian many–body interaction, which will be denoted $\overline{\mathcal{H}}_{1,2}$; “quadratic”, here, referes to the symplectic Cartesian variables measuring the eccentricity and the orientation of the osculating ellipses. The diagonalization of $\overline{\mathcal{H}}_{1,2}$ is trivial (under the only assumption that $a_i \neq a_j$), while conditions (1.2) and (1.3) will be used to check that the associated eigenvalues are non zero, simple and distinct so that Melnikov conditions are satisfied. The proof is non inductive on N.

- A more detailed and quantitative version of Theorem 1.1 is Theorem 4.2 below.

2 Poincaré Hamiltonian setting

The results described in this section are classical and go back to Delaunay and Poincaré; however since they are not so easy to find, for convenience of the reader, we reproduce them in Appendix A.

Consider $N+1$ bodies $P_0, \ldots, P_N$, in a fixed (ecliptic) plane, of masses $m_0, \ldots, m_N$ interacting through gravity (with constant of gravitation 1). We assume that the mass of P_0 (the “star”) is much larger that the mass of the other bodies (the “planets”), i.e., we assume (1.1). In heliocentric planar (suitably rescaled) variables, the dynamics of the planar $(N+1)$–body problem is governed (as explained in Appendix A) by the Hamiltonian

$$\mathcal{H}^{(N)}(X,x) := \mathcal{H}_0^{(N)}(X,x) + \varepsilon \mathcal{H}_1^{(N)}(X,x) \ , \tag{2.4}$$

where $X := (X^{(1)}, ..., X^{(N)}) \in \mathbf{R}^{2N}$ and $x := (x^{(1)}, ..., x^{(N)}) \in \mathbf{R}^{2N}$ are conjugated Cartesian symplectic variables and

$$\begin{aligned} \mathcal{H}_0^{(N)} &:= \sum_{i=1}^{N} \Big(\frac{1}{2\mathrm{m_i}} |X^{(i)}|^2 - \frac{\mathrm{m}_i \mathrm{M}_i}{|x^{(i)}|} \Big) \ , \\ \mathcal{H}_1^{(N)} &:= \sum_{1 \le i < j \le N} \Big(X^{(i)} \cdot X^{(j)} - \frac{\mu_i \mu_j}{m_0^2} \frac{1}{|x^{(i)} - x^{(j)}|} \Big) \ ; \end{aligned} \tag{2.5}$$

here we have introduced the dimensionless masses

$$\mathrm{M}_i := 1 + \varepsilon \frac{\mu_i}{m_0} \ , \qquad \mathrm{m}_i := \frac{\mu_i}{m_0 + \varepsilon \mu_i} = \frac{\mu_i}{m_0} \frac{1}{\mathrm{M}_i} \ . \tag{2.6}$$

The Hamiltonian $\mathcal{H}_0^{(N)}$ is simply the sum of N un–coupled planar Kepler problems (formed by the star and the i^{th} planet). Being interested in phase region where the un–coupled Kepler problem describe nearly–circular orbits, we introduce *planar Poincaré variables*, the construction of which are based on the following classical 4–dimensional symplectic map.

Lemma 2.1 (Planar Poincaré variables) *Let*

$$F_1(t) := \Big(1-\frac{t}{4}\Big)^{\frac12}\ , \qquad F_2(t) := \frac12\ \Big(1-\frac{t}{4}\Big)^{-1}\ , \qquad \Big(|t|<1/4\Big)\ ; \tag{2.7}$$

let $G_0(s,t) = t + st + \cdots$ be the function analytic in a neighborhood of $(0,0)$ implicitely defined by

$$G_0(0,0) = 0\ , \qquad G_0 = s\sin G_0 + t\cos G_0\ ; \tag{2.8}$$

define the following four functions of three variables $(\hat\eta,\hat\xi,\lambda)$ real–analytic in a neighborhood of the set $\{(\hat\eta,\hat\xi)=(0,0)\}\times \mathbf{T}$:

$$\begin{aligned}
G(\hat\eta,\hat\xi,\lambda) &:= G_0\Big((\hat\eta\cos\lambda-\hat\xi\sin\lambda)\ F_1(t)\ ,\ (\hat\xi\cos\lambda-\hat\eta\sin\lambda)\ F_1(t)\Big)\ ,\\
\mathcal{E}_{\rm s}(\hat\eta,\hat\xi,\lambda) &:= \Big(\hat\xi\cos(\lambda+G)+\hat\eta\sin(\lambda+G)\Big)\ F_1(t)\ ,\\
\mathcal{C}(\hat\eta,\hat\xi,\lambda) &:= \cos(\lambda+\mathcal{E}_{\rm s}) - \hat\eta\ F_1(t) - \hat\xi\,\mathcal{E}_{\rm s}\ F_1(t)\,F_2(t)\ ,\\
\mathcal{S}(\hat\eta,\hat\xi,\lambda) &:= \sin(\lambda+\mathcal{E}_{\rm s}) + \hat\xi\ F_1(t) - \hat\eta\,\mathcal{E}_{\rm s}\ F_1(t)\,F_2(t)\ ,
\end{aligned}\tag{2.9}$$

where t is short for $t=\hat\eta^2+\hat\xi^2$, G is short for $G(\hat\eta,\hat\xi,\lambda)$ and $\mathcal{E}_{\rm s}$ is short for $\mathcal{E}_{\rm s}(\hat\eta,\hat\xi,\lambda)$. Fix $\varepsilon,\mu,m_0>0$ and let

$$\begin{aligned}
&\mathrm{M} := 1+\varepsilon\frac{\mu}{m_0}\ , \qquad \mathrm{m} := \frac{\mu}{m_0}\ \frac{1}{\mathrm{M}}\ , \qquad \overline{\mathrm{m}} := \frac{\mu}{m_0}\ \frac{1}{\sqrt{\mathrm{M}}}\ ,\\
&\sigma := \Big(\frac{\mu}{m_0}\Big)^3\frac{1}{\mathrm{M}}\ , \qquad a = a(\Lambda;\mu,\varepsilon) := \frac{\Lambda^2}{\overline{\mathrm{m}}^2}\ .
\end{aligned}\tag{2.10}$$

Then, for any $\Lambda_+>\Lambda_->0$, there exists a ball B around the origin in $\mathbf{R}^2$ such that the 4–dimensional map

$$\Psi_{\rm P} : (\Lambda,\lambda,\eta,\xi)\in D := (\Lambda_-,\Lambda_+)\times\mathbf{T}\times B \to (X,x)\in\mathbf{R}^4$$

where

$$x_1 = x_1(\Lambda,\lambda,\eta,\xi;\mu,\varepsilon) := a(\Lambda;\mu,\varepsilon)\ \mathcal{C}\Big(\frac{\eta}{\sqrt\Lambda},\frac{\xi}{\sqrt\Lambda},\lambda\Big)\ , \tag{2.11}$$

$$x_2 = x_2(\Lambda,\lambda,\eta,\xi;\mu,\varepsilon) := a(\Lambda;\mu,\varepsilon)\ \mathcal{S}\Big(\frac{\eta}{\sqrt\Lambda},\frac{\xi}{\sqrt\Lambda},\lambda\Big)\ ,$$

$$X = X(\Lambda,\lambda,\eta,\xi;\mu,\varepsilon) := \frac{\overline{\mathrm{m}}^4}{\Lambda^3}\frac{\partial x}{\partial\lambda}(\Lambda,\lambda,\eta,\xi;\mu,\varepsilon) = \frac{\overline{\mathrm{m}}}{a(\Lambda;\mu,\varepsilon)^{3/2}}\ \frac{\partial x}{\partial\lambda}\ ,$$

is real-analytic in D and symplectic:

$$d\Lambda\wedge d\lambda + d\eta\wedge d\xi = dX_1\wedge dx_1 + dX_2\wedge dx_2\ .$$

Furthermore, if $\mathcal{H}_0^{(1)}$ denotes the two-body Hamiltonian

$$\mathcal{H}_0^{(1)}(X,x) := \frac{1}{2\mathrm{m}}|X|^2 - \frac{\mathrm{mM}}{|x|}\ ,$$

then, on the phase region of negative energies $\Big(\mathcal{H}_0^{(1)}\Big)^{-1}\Big(-\frac{\sigma}{2\Lambda_-^2},-\frac{\sigma}{2\Lambda_+^2}\Big)$, one has

$$\mathcal{H}_0^{(1)}\circ\Psi_{\rm P} = -\frac{\sigma}{2\Lambda^2}\ ;$$

in the planar coordinates $x\in\mathbf{R}^2$ the corresponding motion describes an ellipse of major semiaxis $a=a(\Lambda;\mu,\varepsilon)$ and eccentricity

$$e = \sqrt{\frac{\eta^2+\xi^2}{\Lambda}}\ F_1\Big(\frac{\eta^2+\xi^2}{\Lambda}\Big) = \sqrt{\frac{\eta^2+\xi^2}{\Lambda}\ \Big(1-\frac{\eta^2+\xi^2}{4\Lambda}\Big)}\ . \tag{2.12}$$

The proof of this lemma may be found in Appendix A. Note that

$$\mathcal{C}(0,0,\lambda) = \cos\lambda \ , \qquad \mathcal{S}(0,0,\lambda) = \sin\lambda \ , \tag{2.13}$$

so that the $(a,\lambda) \to x$ transformation is, for $\eta = \xi = 0$, just polar coordinates. Let, now, $\Psi_{\rm P}^{(N)}$ be the $4N$–dimensional map, parameterized by $(\mu_1, ..., \mu_N, \varepsilon)$, defined by

$$\Psi_{\rm P}^{(N)} : \Big((\Lambda_1, \lambda_1, \eta_1, \xi_1), ..., (\Lambda_N, \lambda_N, \eta_N, \xi_N) \Big) \in \Big((0,\infty) \times \mathbf{T} \times \mathbf{R}^2 \Big)^N \to (X, x) \tag{2.14}$$

with

$$\begin{aligned} &(X, x) = \Big((X^{(1)}, ..., X^{(N)}), (x^{(1)}, ..., x^{(N)}) \Big) \ , \\ &(X^{(i)}, x^{(i)}) = \Psi_{\rm P}(\Lambda_i, \lambda_i, \eta_i, \xi_i; \mu_i, \varepsilon) \ . \end{aligned} \tag{2.15}$$

Then, $\Psi_{\rm P}^{(N)}$ is symplectic and

$$\mathcal{H}_0^{(N)} \circ \Psi_{\rm P}^{(N)} = -\frac{1}{2} \sum_{i=1}^N \frac{\sigma_i}{\Lambda_i^2} =: \mathcal{H}_0(\Lambda) \ , \qquad \qquad \sigma_i := \Big(\frac{\mu_i}{m_0}\Big)^3 \frac{1}{{\rm M}_i} \ . \tag{2.16}$$

In such Poincaré variables the full planar $(N+1)$–body Hamiltonian $\mathcal{H}^{(N)}$ becomes

$$\mathcal{H}(\Lambda, \lambda, \eta, \xi) = \mathcal{H}_0(\Lambda) + \varepsilon \mathcal{H}_1(\Lambda, \lambda, \eta, \xi) \ , \qquad \mathcal{H}_1 := \mathcal{H}_1^{(N)} \circ \Psi_{\rm P}^{(N)} =: \mathcal{H}_1^{\rm compl} + \mathcal{H}_1^{\rm princ} \ , \tag{2.17}$$

where the so-called "complementary part" $\mathcal{H}_1^{\rm compl}$ and the "principal part" $\mathcal{H}_1^{\rm princ}$ of the perturbation are, respectively, the functions

$$\sum_{1 \le i < j \le N} X^{(i)} \cdot \ X^{(j)} \qquad \text{and} \qquad \sum_{1 \le i < j \le N} \frac{\mu_i \mu_j}{m_0^2} \frac{1}{|x^{(i)} - x^{(j)}|} \tag{2.18}$$

expressed in Poincaré variables[4]:

$$X^{(i)} = X(\Lambda_i, \lambda_i, \eta_i, \xi_i; \mu_i, \varepsilon) \qquad \text{and} \qquad x^{(i)} = x(\Lambda_i, \lambda_i, \eta_i, \xi_i; \mu_i, \varepsilon) \ .$$

Notice that, since $X^{(i)} = (\overline{\rm m}_i^4 / \Lambda_i^3) \partial_{\lambda_i} x^{(i)}$ the λ–average of $\mathcal{H}_1^{\rm compl}$ vanishes. Moreover, as it is well known, the λ–average of $\mathcal{H}_1$ is an even function of (η, ξ); see, also, Appendix A. Hence, we may split the perturbation function as

$$\mathcal{H}_1 = \overline{\mathcal{H}}_1 + \widetilde{\mathcal{H}}_1 \ , \tag{2.19}$$

with

$$\overline{\mathcal{H}}_1(\Lambda, \eta, \xi) := \int_{\mathbf{T}^N} \mathcal{H}_1 \frac{d\lambda}{(2\pi)^N} \ , \qquad \qquad \int_{\mathbf{T}^N} \widetilde{\mathcal{H}}_1 \, d\lambda = 0 \ . \tag{2.20}$$

Furthermore, $\overline{\mathcal{H}}_1$ may be written as

$$\overline{\mathcal{H}}_1(\Lambda, \eta, \xi) = \overline{\mathcal{H}}_{1,0}(\Lambda) + \overline{\mathcal{H}}_{1,2}(\Lambda, \eta, \xi) + \overline{\mathcal{H}}_{1,*}(\Lambda, \eta, \xi) \ , \tag{2.21}$$

where $\overline{\mathcal{H}}_{1,0} := \overline{\mathcal{H}}_1(\Lambda, 0, 0)$, $\overline{\mathcal{H}}_{1,2}$ is the (η, ξ)–quadratic part of $\overline{\mathcal{H}}_1$ while $\overline{\mathcal{H}}_{1,*}$ is the "remainder of order four":

$$|\overline{\mathcal{H}}_{1,*}(\Lambda, \eta, \xi)| \le \text{const} \, |(\eta, \xi)|^4 \, .$$

[4] $X = (X_1, X_2)$ and $x = (x_1, x_2)$ denote here the functions definded in (2.11).

3 The averaged quadratic potential $\overline{\mathcal{H}}_{1,2}$

In this section we analyze the function $\overline{\mathcal{H}}_{1,2}$ (i.e., the (η,ξ)–quadratic part of the λ–average of the perturbation) defined in (2.21), which may be written as

$$\overline{\mathcal{H}}_{1,2} = \frac{1}{2} \sum_{1\le i,j\le N} Q_{ij} \begin{pmatrix} \eta_j \\ \xi_j \end{pmatrix} \cdot \begin{pmatrix} \eta_i \\ \xi_i \end{pmatrix} \tag{3.22}$$

where Q_{ij} are (2×2)–matrices defined as

$$Q_{ij} := \begin{pmatrix} \partial^2_{\eta_i,\eta_j}\overline{\mathcal{H}}_{1,2} & \partial^2_{\eta_i,\xi_j}\overline{\mathcal{H}}_{1,2} \\ \partial^2_{\xi_i,\eta_j}\overline{\mathcal{H}}_{1,2} & \partial^2_{\xi_i,\xi_j}\overline{\mathcal{H}}_{1,2} \end{pmatrix}\Bigg|_{(\Lambda,0,0)} .$$

In view of the definition of the Poincaré variables, it is natural to look at the rescaled variables $(\hat\eta,\hat\xi)$ rather then (η,ξ). Therefore, we define

$$\begin{aligned} &\bar f_{ij}(\Lambda,\hat\eta,\hat\xi) := \\ &\frac{1}{(2\pi)^N}\int_{\mathbf{T}^N} \frac{d\lambda}{|x^{(i)}(\Lambda_i,\lambda_i,\sqrt{\Lambda_i}\hat\eta_i,\sqrt{\Lambda_i}\hat\xi_i;\mu_i,\varepsilon) - x^{(j)}(\Lambda_j,\lambda_j\sqrt{\Lambda_j}\hat\eta_j,\sqrt{\Lambda_j}\hat\xi_j;\mu_j,\varepsilon)|} . \end{aligned} \tag{3.23}$$

Thus, letting[5]

$$\begin{aligned} a_i &:= a(\Lambda_i;\mu_i,\varepsilon) , & c_{ij} &:= \frac{1}{m_0}\left(\frac{\mathrm{M}_i\mathrm{M}_j}{a_ia_j}\right)^{1/4} , \\ A_{ij} &:= \begin{pmatrix} \partial^2_{\hat\eta_i\hat\eta_j}\bar f_{ij} & \partial^2_{\hat\eta_i\hat\xi_j}\bar f_{ij} \\ \partial^2_{\hat\xi_i\hat\eta_j}\bar f_{ij} & \partial^2_{\hat\xi_i\hat\xi_j}\bar f_{ij} \end{pmatrix}\Bigg|_{\hat\eta=\hat\xi=0} , & B_{ij} &:= \begin{pmatrix} \partial^2_{\hat\eta_j\hat\eta_j}\bar f_{ij} & \partial^2_{\hat\eta_j\hat\xi_j}\bar f_{ij} \\ \partial^2_{\hat\xi_j\hat\eta_j}\bar f_{ij} & \partial^2_{\hat\xi_j\hat\xi_j}\bar f_{ij} \end{pmatrix}\Bigg|_{\hat\eta=\hat\xi=0} , \end{aligned} \tag{3.24}$$

we find

$$Q_{ij} = \begin{cases} \sqrt{\mu_i\mu_j}\, c_{ij}\; A_{ij} , & \text{if } i\neq j , \\ \displaystyle\sum_{k\neq j} \sqrt{\mu_k\mu_j}\, c_{kj}\; B_{kj} , & \text{if } i = j . \end{cases}$$

It is a remarkable fact that, *for the planar planetary* $(N+1)$*–body problem the matrices* A_{ij} *and* B_{ij} *are proportional to the* (2×2)*–identity matrix* $\mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and have simple integral representation. In fact, define, for $a\neq b$,

$$\begin{aligned} \mathcal{J}(a,b) &:= \frac{1}{2\pi}\int_0^{2\pi} \frac{-17ab\cos t + 8(a^2+b^2)\cos(2t) + ab\cos(3t)}{(a^2+b^2-2ab\cos t)^{5/2}}\, dt , \\ \mathcal{I}(a,b) &:= \frac{1}{2\pi}\int_0^{2\pi} \frac{-7ab + 4(a^2+b^2)\cos t - ab\cos(2t)}{(a^2+b^2-2ab\cos t)^{5/2}}\, dt , \end{aligned}$$

and denote, for $a_i\neq a_j$,

$$\alpha_{ij} := \frac{a_ia_j}{8}\mathcal{J}(a_i,a_j) , \qquad \beta_{ij} := \frac{a_ia_j}{4}\mathcal{I}(a_i,a_j) . \tag{3.25}$$

Then, the following "algebraic" result holds.

Proposition 3.1 *Assume $a_i\neq a_j$ for $i\neq j$. Then $A_{ij} = \alpha_{ij}\mathbf{1}_2$ and $B_{ij} = \beta_{ij}\mathbf{1}_2$.*

[5] Recall (2.10) and (2.6).

Remark 3.1 (i) An immediate corollary of this result is that, in the collisionless domain $\{a_i \neq a_j\}$, $\overline{\mathcal{H}}_{1,2}$ has the simple form

$$\overline{\mathcal{H}}_{1,2} = \frac{1}{2}\Big(M\eta \cdot \eta + M\xi \cdot \xi\Big) , \tag{3.26}$$

M being the real, symmetric $(N \times N)$–matrix with entries

$$M_{ij} = \begin{cases} \sqrt{\mu_i \mu_j}\, c_{ij}\ \alpha_{ij}\ , & \text{if } i \neq j\ , \\ \displaystyle\sum_{k \neq j} \sqrt{\mu_k \mu_j}\, c_{kj}\ \beta_{kj}\ , & \text{if } i = j\ . \end{cases} \tag{3.27}$$

The Hamiltonian (3.26) can be immediately put in symplectic normal form: if U is the real orthogonal matrix ($U^T = U^{-1}$) which diagonalize M ($U^T M U = \operatorname{diag}(\bar\Omega_1, ..., \bar\Omega_N)$), then the map $p = U^T \eta$, $q = U^T \xi$ is symplectic and, in such variables, the new Hamiltonian takes the form

$$\frac{1}{2} \sum_{i=1}^{N} \bar\Omega_i (p_i^2 + q_i^2)\ .$$

(ii) The functions $\mathcal{J}$ and $\mathcal{I}$ (which admit simple representations in terms of Gauss hypergeometric functions) are symmetric ($\mathcal{J}(a,b) = \mathcal{J}(b,a)$ and $\mathcal{I}(a,b) = \mathcal{I}(b,a)$) and satisfy

$$\mathcal{J}(a,b) = b^{-3} \mathcal{J}(a/b, 1)\ , \qquad \mathcal{I}(a,b) = b^{-3} \mathcal{I}(a/b, 1)\ , \qquad a < b\ .$$

The function of one real variables $s \in (-1,1) \to \mathcal{J}(s,1)$ and $s \in (-1,1) \to \mathcal{I}(s,1)$ are, respectively, even and odd in s, and satisfy, for small s, the following asymptotics

$$\mathcal{J}(s,1) = -\frac{15}{8} s^2 - \frac{105}{8} s^4 + O(s^6)\ , \qquad \mathcal{I}(s,1) = 3s + \frac{45}{8} s^3 + O(s^5)\,. \tag{3.28}$$

Proof (of Proposition 3.1) The computations we are going to perform are algebraic in character and it is therefore enough to consider real variables. Fix $i \neq j$ and define

$$\mathcal{R}_{ij}(\Lambda, \lambda, \hat\eta, \hat\xi) := \Big| x^{(i)}(\Lambda_i, \lambda_i, \sqrt{\Lambda_i}\hat\eta_i, \sqrt{\Lambda_i}\hat\xi_i; \mu_i, \varepsilon) - x^{(j)}(\Lambda_j, \lambda_j \sqrt{\Lambda_j}\hat\eta_j, \sqrt{\Lambda_j}\hat\xi_j; \mu_j, \varepsilon) \Big|^2\ , \tag{3.29}$$

so that (recall (3.23))

$$\bar f_{ij}(\Lambda, \hat\eta, \hat\xi) = \frac{1}{(2\pi)^N} \int_{\mathbf{T}^N} \frac{d\lambda}{\sqrt{\mathcal{R}_{ij}}}\ . \tag{3.30}$$

By (2.11) we find

$$\mathcal{R}_{ij} = a_i^2 \chi_i^2 + a_j^2 \chi_j^2 - 2 a_i a_j \Big(\mathcal{C}_i \mathcal{C}_j - \mathcal{S}_i \mathcal{S}_j \Big)\ , \tag{3.31}$$

where $\mathcal{C}_k$, $\mathcal{S}_k$ and χ_k are short for, respectively,

$$\mathcal{C}_k = \mathcal{C}(\hat\eta_k, \hat\xi_k, \lambda_k)\ , \quad \mathcal{S}_k = \mathcal{S}(\hat\eta_k, \hat\xi_k, \lambda_k) \quad \text{and} \quad \chi_k = \sqrt{\mathcal{C}_k^2 + \mathcal{S}_k^2}\ .$$

The proof will consist in computing explicitely λ–averages of quantities of the form

$$\rho_{\zeta_i, \zeta_j}(\lambda_i, \lambda_j) := \partial^2_{\zeta_i \zeta_j} \frac{1}{\sqrt{\mathcal{R}_{ij}}} \Bigg|_{\hat\eta = \hat\xi = 0} = \frac{3(\partial_{\zeta_i}\mathcal{R}_{ij})(\partial_{\zeta_j}\mathcal{R}_{ij}) - 2\mathcal{R}_{ij}(\partial_{\zeta_i \zeta_j}\mathcal{R}_{ij})}{4\mathcal{R}_{ij}^{5/2}} \Bigg|_{\hat\eta = \hat\xi = 0}\ , \tag{3.32}$$

where ζ_k denotes either of the variables $\hat\eta_k$ or $\hat\xi_k$. Thus, what we need to do is to compute suitable orders in the variables $(\hat\eta_k, \hat\xi_k)$ of the function $\mathcal{R}_{ij}$. For this purpose it will be useful the following

Lemma 3.1 *Define the following elementary functions*

$$
\begin{aligned}
&C_+(\lambda) := 1+\cos^2\lambda \;=\; \frac{3+\cos(2\lambda)}{2}\,,\\
&C_-(\lambda) := 1+\sin^2\lambda \;=\; \frac{3-\cos(2\lambda)}{2}\,,\\
&S_0(\lambda) := \cos\lambda\sin\lambda \;=\; \frac{1}{2}\sin(2\lambda)\,,\\
&\bar\chi(x,y,\lambda) := 1-2y\cos\lambda+2x\sin\lambda\\
&S(x,y,\lambda) := \sin\lambda + xC_+(\lambda)+yS_0(\lambda)\,,\\
&C(x,y,\lambda) := \cos\lambda - yC_-(\lambda)-xS_0(\lambda)\ ,
\end{aligned}
$$

and denote by $O_p(z_1,...,z_n)$ a function of the variables $(z_1,...,z_n)$ (depending possibly on other variables) analytic in a neighborhood of $(0,...,0)$ and starting with a homogeneous polynomial of degree p in $(z_1,...,z_n)$. Then,

$$
\begin{aligned}
&\chi_{_k}^2 = \mathcal{C}(\hat\eta_k,\hat\xi_k,\lambda_k)^2+\mathcal{S}(\hat\eta_k,\hat\xi_k,\lambda_k)^2 = \bar\chi(\hat\eta_k,\hat\xi_k,\lambda_k)+O_2(\hat\eta_k,\hat\xi_k)\,,\\
&\mathcal{C}_k = \mathcal{C}(\hat\eta_k,\hat\xi_k,\lambda_k) = C(\hat\xi_k,\hat\eta_k,\lambda_k)+O_2(\hat\eta_k,\hat\xi_k)\ ,\\
&\mathcal{S}_k = \mathcal{S}(\hat\eta_k,\hat\xi_k,\lambda_k) = S(\hat\xi_k,\hat\eta_k,\lambda_k)+O_2(\hat\eta_k,\hat\xi_k)\ .
\end{aligned}
\tag{3.33}
$$

The proof of this lemma follows at once from the explicit expressions for $\mathcal{C}$ and $\mathcal{S}$ given in Lemma 2.1 and is left to the reader.

We consider first the matrices A_{ij} (which allow to compute Q_{ij} for $i\neq j$) and then we turn to the matrices B_{ij} (which allow to compute Q_{jj}).

3.1 Computation of the matrices A_{ij}

First, observe that the two derivatives involved in the definition of A_{ij} are *always* mixed in the variables with indexes i and j. Thus, we can neglect the terms of third order in $(\hat\eta_i,\hat\xi_i,\hat\eta_j,\hat\xi_j)$ and the terms of *second order* of the type $O_2(\hat\eta_i,\hat\xi_i)$ and $O_2(\hat\eta_j,\hat\xi_j)$.

By Lemma 3.1, the function $\mathcal{R}_{ij}$ in (3.31) has the form

$$
\begin{aligned}
\mathcal{R}_{ij} \;=\;& a_i^2(1-2\hat\eta_i\cos\lambda_i+2\hat\xi_i\sin\lambda_i)+a_j^2(1-2\hat\eta_j\cos\lambda_j+2\hat\xi_j\sin\lambda_j)\\
&-2a_ia_j\Big[(\cos\lambda_i-\hat\xi_iS_0(\lambda_i)-\hat\eta_iC_-(\lambda_i))(\cos\lambda_j-\hat\xi_jS_0(\lambda_j)-\hat\eta_jC_-(\lambda_j))\\
&+(\sin\lambda_i+\hat\xi_iC_+(\lambda_i)+\hat\eta_iS_0(\lambda_i))(\sin\lambda_j+\hat\xi_jC_+(\lambda_j)+\hat\eta_jS_0(\lambda_j))\Big]\\
&+O_2(\hat\eta_i,\hat\xi_i)+O_2(\hat\eta_j,\hat\xi_j)+O_3(\hat\eta_i,\hat\xi_i,\hat\eta_j,\hat\xi_j)\,.
\end{aligned}
\tag{3.34}
$$

Therefore, letting $(\cdot)|_0$ be short for $(\cdot)|_{\hat\eta_i=\hat\xi_i=\hat\eta_j=\hat\xi_j=0}$, one finds

$$
\begin{aligned}
\mathcal{R}_{ij}\big|_0 \;&=\; a_i^2+a_j^2-2a_ia_j\cos(\lambda_i-\lambda_j)\,,\\
\partial_{\hat\eta_i}\mathcal{R}_{ij}\big|_0 \;&=\; -2a_i^2\cos\lambda_i-2a_ia_j[-C_-(\lambda_i)\cos\lambda_j+S_0(\lambda_i)\sin\lambda_j]\,,\\
\partial_{\hat\eta_j}\mathcal{R}_{ij}\big|_0 \;&=\; -2a_j^2\cos\lambda_j-2a_ia_j[-C_-(\lambda_j)\cos\lambda_i+S_0(\lambda_j)\sin\lambda_i]\,,\\
\partial_{\hat\xi_i}\mathcal{R}_{ij}\big|_0 \;&=\; 2a_i^2\sin\lambda_i-2a_ia_j[C_+(\lambda_i)\sin\lambda_j-S_0(\lambda_i)\cos\lambda_j]\,,\\
\partial_{\hat\xi_j}\mathcal{R}_{ij}\big|_0 \;&=\; 2a_j^2\sin\lambda_j-2a_ia_j[C_+(\lambda_j)\sin\lambda_i-S_0(\lambda_j)\cos\lambda_i]\,,\\
\partial^2_{\hat\eta_i\hat\eta_j}\mathcal{R}_{ij}\big|_0 \;&=\; -2a_ia_j[C_-(\lambda_i)C_-(\lambda_j)+S_0(\lambda_i)S_0(\lambda_j)]\,,\\
\partial^2_{\hat\xi_i\hat\xi_j}\mathcal{R}_{ij}\big|_0 \;&=\; -2a_ia_j[C_+(\lambda_i)C_+(\lambda_j)+S_0(\lambda_i)S_0(\lambda_j)]\,,
\end{aligned}
$$

$$
\begin{aligned}
\partial^2_{\hat\eta_i\hat\xi_j}\mathcal{R}_{ij}\big|_0 &= -2a_ia_j[C_-(\lambda_i)S_0(\lambda_j)+S_0(\lambda_i)C_+(\lambda_j)]\,,\\
\partial^2_{\hat\xi_i\hat\eta_j}\mathcal{R}_{ij}\big|_0 &= -2a_ia_j[S_0(\lambda_i)C_-(\lambda_j)+C_+(\lambda_i)S_0(\lambda_j)]\,.
\end{aligned}
\tag{3.35}
$$

In particular, $\mathcal{R}_{ij}$ and $\partial_{\hat\eta_i}\mathcal{R}_{ij}$ are even in $(\lambda_i,\lambda_j)\in\mathbf{T}^2$, while $\partial_{\hat\xi_j}\mathcal{R}_{ij}$ and $\partial^2_{\hat\eta_i\hat\xi_j}\mathcal{R}_{ij}$ are odd. Thus, recalling the definition of $\rho_{\varsigma_i,\varsigma_j}(\lambda_i,\lambda_j)$ in (3.32), we find that $\rho_{\hat\eta_i,\hat\xi_j}(\lambda_i,\lambda_j)$ is odd in (λ_i,λ_j) and it has therefore zero average. For the same reasons, also $\rho_{\hat\xi_i,\hat\eta_j}(\lambda_i,\lambda_j)$ has zero average. Hence, the off–diagonal terms of A_{ij} are zero. We now compute the diagonal terms of A_{ij}. We begin with $\rho_{\hat\eta_i,\hat\eta_j}(\lambda_i,\lambda_j)$. By (3.32) and the list in (3.35), we find

$$
\rho_{\hat\eta_i,\hat\eta_j}(\lambda_i,\lambda_j) := \partial^2_{\hat\eta_i\hat\eta_j}\frac{1}{\sqrt{\mathcal{R}_{ij}}}\bigg|_0 = \frac{\rho_1(\lambda_i,\lambda_j)}{\rho_2(\lambda_i,\lambda_j)}
\tag{3.36}
$$

with

$$
\begin{aligned}
\rho_1(\lambda_i,\lambda_j) &:= a_ia_j\cdot\Big[-24a_i^2\cos(2\lambda_i)-24a_j^2\cos(2\lambda_j)+8(a_i^2+a_j^2)\cos(2(\lambda_i-\lambda_j))\\
&\qquad -3a_ia_j\cos(\lambda_i-3\lambda_j)-17a_ia_j\cos(\lambda_i-\lambda_j)\\
&\qquad +a_ia_j\cos(3(\lambda_i-\lambda_j))-3a_ia_j\cos(3\lambda_i-\lambda_j)\\
&\qquad +54a_ia_j\cos(\lambda_i+\lambda_j)\Big]\,,\\
\rho_2(\lambda_i,\lambda_j) &:= 8\Big(a_i^2+a_j^2-2a_ia_j\cos(\lambda_i-\lambda_j)\Big)^{5/2}\,.
\end{aligned}
\tag{3.37}
$$

Thus, changing variable of integration, one finds

$$
\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{\mathbf{T}^2}\partial^2_{\hat\eta_i\hat\eta_j}\frac{1}{\sqrt{\mathcal{R}_{ij}}}\bigg|_0 d\lambda_i\,d\lambda_j\\
&\qquad = \frac{1}{2\pi}\int_{\mathbf{T}} a_ia_i\cdot\frac{-17a_ia_j\cos t+8(a_i^2+a_j^2)\cos(2t)+a_ia_j\cos(3t)}{8(a_i^2+a_j^2-2a_ia_j\cos t)^{5/2}}\,dt\\
&\qquad = \frac{a_ia_j}{8}\,\mathcal{J}(a_i,a_j) =: \alpha_{ij}\ .
\end{aligned}
$$

The case $\rho_{\hat\xi_i,\hat\xi_j}(\lambda_i,\lambda_j)$ is very similar (and will yield the same result). In place of (3.36) one finds

$$
\rho_{\hat\xi_i,\hat\xi_j}(\lambda_i,\lambda_j) := \partial^2_{\hat\xi_i\hat\xi_j}\frac{1}{\sqrt{\mathcal{R}_{ij}}}\bigg|_0 = \frac{\rho_3(\lambda_i,\lambda_j)}{\rho_2(\lambda_i,\lambda_j)}
\tag{3.38}
$$

with

$$
\begin{aligned}
\rho_3(\lambda_i,\lambda_j) &:= a_ia_j\cdot\Big[24a_i^2\cos(2\lambda_i)+24a_j^2\cos(2\lambda_j)+8(a_i^2+a_j^2)\cos(2(\lambda_i-\lambda_j))\\
&\qquad +3a_ia_j\cos(\lambda_i-3\lambda_j)-17a_ia_j\cos(\lambda_i-\lambda_j)\\
&\qquad +a_ia_j\cos(3(\lambda_i-\lambda_j))+3a_ia_j\cos(3\lambda_i-\lambda_j)\\
&\qquad -54a_ia_j\cos(\lambda_i+\lambda_j)\Big]\,.
\end{aligned}
\tag{3.39}
$$

Integrating one finds again

$$
\frac{1}{(2\pi)^2}\int_{\mathbf{T}^2}\partial^2_{\hat\xi_i\hat\xi_j}\frac{1}{\sqrt{\mathcal{R}_{ij}}}\bigg|_0 d\lambda_i\,d\lambda_j = \alpha_{ij}\ .
$$

This proves Proposition 3.1 in the case of Q_{ij}, with $i \neq j$.

3.2 Computation of the matrices B_{ij}

Observe that the derivatives involved in the definition of B_{ij} are two derivatives with the same index j. We can, therefore, neglect the third order terms and set $\hat\eta_i = \hat\xi_i = 0$.

Recalling (2.13) we see that $\chi_i|_{\hat\eta_i=\hat\xi_i=0} = 1$ and

$$\mathcal{R}_{ij}\Big|_{\hat\eta_i=\hat\xi_i=0} = a_i^2 + a_j^2\chi_j^2 - 2a_ia_j\chi_j\left((\cos\lambda_i)\,\frac{\mathcal{C}_j}{\chi_j} - (\sin\lambda_i)\,\frac{\mathcal{S}_j}{\chi_j}\right) . \tag{3.40}$$

Defining $\varphi_j = \varphi_j(\Lambda_j, \lambda_j, \hat\eta_j, \hat\xi_j)$ through the relation[6]

$$\cos\varphi_j = \frac{\mathcal{C}_j}{\chi_j} , \qquad \sin\varphi_j = \frac{\mathcal{S}_j}{\chi_j} , \tag{3.41}$$

we find

$$\mathcal{R}_{ij}\Big|_{\hat\eta_i=\hat\xi_i=0} = a_i^2 + a_j^2\chi_j^2 - 2a_ia_j\chi_j\cos(\varphi_j - \lambda_i) . \tag{3.42}$$

Denote by $\langle\, f\,\rangle_{\theta,\tau}$ the average of a function f over the angles θ and τ. Integrating *first* with respect to λ_i and changing variable of integration ($t = \lambda_i - \varphi_j$) one gets

$$\left\langle \frac{1}{\sqrt{\mathcal{R}_{ij}}}\Big|_{\hat\eta_i=\hat\xi_i=0}\right\rangle_{\lambda_i,\lambda_j} = \left\langle \frac{1}{\sqrt{\tilde{\mathcal{R}}_{ij}}}\right\rangle_{t,\lambda_j} \tag{3.43}$$

with

$$\tilde{\mathcal{R}}_{ij} := a_i^2 + a_j^2\chi_j^2 - 2a_ia_j\chi_j\cos t\,. \tag{3.44}$$

At this point, the argument is completely analogous to that used above. First, we observe that

$$\left\langle \partial^2_{\zeta_h\zeta_k}\frac{1}{\sqrt{\mathcal{R}_{ij}}}\Big|_0\right\rangle_{\lambda_i,\lambda_j} = \left\langle \frac{3(\partial_{\zeta_h}\tilde{\mathcal{R}}_{ij})(\partial_{\zeta_k}\tilde{\mathcal{R}}_{ij}) - 2\tilde{\mathcal{R}}_{ij}(\partial^2_{\zeta_h\zeta_k}\tilde{\mathcal{R}}_{ij})}{4\tilde{\mathcal{R}}_{ij}^{5/2}}\Big|_0\right\rangle_{t,\lambda_j} , \tag{3.45}$$

where ζ_ℓ denotes here any of the variables $\hat\eta_j, \hat\xi_j$. From Lemma 3.1 it follows that $\tilde{\mathcal{R}}_{ij}$ can be written as

$$\tilde{\mathcal{R}}_{ij} = f(t) + g(t)\Big(h_1 - h_2\Big) + a_j^2h_1^2 + O_3(\hat\eta_j, \hat\xi_j) ,$$

with

$$\begin{aligned} f(t) &:= a_i^2 + a_j^2 - 2a_ia_j\cos t\ , \qquad g(t) := -2a_j^2 + 2a_ia_j\cos t\ , \\ h_1 &:= \hat\eta_j\cos\lambda_j - \hat\xi_j\sin\lambda_j \qquad h_2 := \hat\xi_j^2\cos^2\lambda_j + \hat\eta_j^2\sin^2\lambda_j + \hat\eta_j\hat\xi_j\sin(2\lambda_j)\ . \end{aligned}$$

Thus, since h_1 is of order one in $(\hat\eta_j, \hat\xi_j)$ and h_2 is of order two in $(\hat\eta_j, \hat\xi_j)$,

$$\begin{aligned} \tilde{\mathcal{R}}_{ij}\Big|_0 &= f(t) \\ \partial_{\eta_j}\tilde{\mathcal{R}}_{ij}\Big|_0 &= g(t)\,\cos\lambda_j\,, \\ \partial_{\hat\xi_j}\tilde{\mathcal{R}}_{ij}\Big|_0 &= -g(t)\,\sin\lambda_j\,, \\ \partial^2_{\hat\eta_j\hat\eta_j}\tilde{\mathcal{R}}_{ij}\Big|_0 &= -2g(t)\,\sin^2\lambda_j + 2a_j^2\cos^2\lambda_j\,, \\ \partial^2_{\hat\eta_j\hat\xi_j}\tilde{\mathcal{R}}_{ij}\Big|_0 &= -(g(t)+a_j)^2\,\sin(2\lambda_j)\,, \\ \partial^2_{\hat\xi_j\hat\xi_j}\tilde{\mathcal{R}}_{ij}\Big|_0 &= -2g(t)\,\cos^2\lambda_j + 2a_j^2\sin^2\lambda_j\,. \end{aligned}$$

[6]Physically, φ_j coincide with $v_j + g_j$ where v_j and g_j are, respectively, the true anomaly and the argument of the perihelion of the osculating ellipse associated to the star and the j^{th} planet; compare Appendix A.

Therefore, using (3.45), one finds

$$\left\langle \partial^2_{\hat\eta_j \hat\xi_j} \frac{1}{\sqrt{\tilde{\mathcal{R}}_{ij}}}\Bigg|_0 \right\rangle_{\lambda_i,\lambda_j} = \left\langle \frac{-\left(\frac{3}{2}g^2 + 2(a_j^2 - g)f\right)\sin(2\lambda_j)}{4f^{5/2}}\Bigg|_0 \right\rangle_{t,\lambda_j} = 0$$

(since the integrand is odd in λ_j), showing that also B_{ij} is a diagonal matrix. To compute the diagonal elements we calculate

$$\partial^2_{\hat\eta_j \hat\eta_j} \frac{1}{\sqrt{\tilde{\mathcal{R}}_{ij}}}\Bigg|_0 = \frac{\tilde\rho_1(\lambda_j,t)}{\tilde\rho_2(\lambda_j,t)} \qquad \text{and} \qquad \partial^2_{\hat\xi_j \hat\xi_j} \frac{1}{\sqrt{\tilde{\mathcal{R}}_{ij}}}\Bigg|_0 = \frac{\tilde\rho_3(\lambda_j,t)}{\tilde\rho_2(\lambda_j,t)} \tag{3.46}$$

with

$$\begin{aligned}
\tilde\rho_1 &= -7\,a_i^2\,a_j^2 + (9\,a_i^2\,a_j^2 + 8\,{a_j}^4)\,\cos(2\,\lambda_j) + \frac{7}{2}\,a_i^2\,a_j^2\,\cos(2\,\lambda_j - 2\,t) \\
&\quad -(2\,{a_i}^3\,a_j + 10\,a_i\,{a_j}^3)\,\cos(2\,\lambda_j - t) + 4({a_i}^3\,a_j + a_i\,{a_j}^3)\,\cos(t) \\
&\quad -a_i^2\,a_j^2\,\cos(2\,t) - (2\,{a_i}^3\,a_j + 10\,a_i\,{a_j}^3)\,\cos(2\,\lambda_j + t) + \frac{7}{2}\,a_i^2\,a_j^2\,\cos(2\,\lambda_j + 2\,t)\ ; \\
\tilde\rho_2 &= 4(a_i^2 + a_j^2 - 2a_ia_j\cos t)^{5/2}\ ; \\
\tilde\rho_3 &= -7\,a_i^2\,a_j^2 - (9\,a_i^2\,a_j^2 + 8\,{a_j}^4)\,\cos(2\,\lambda_j) - \frac{7}{2}\,a_i^2\,a_j^2\,\cos(2\,\lambda_j - 2\,t) \\
&\quad +(2\,{a_i}^3\,a_j + 10\,a_i\,{a_j}^3)\,\cos(2\,\lambda_j - t) + 4({a_i}^3\,a_j + a_i\,{a_j}^3)\,\cos(t) \\
&\quad -a_i^2\,a_j^2\,\cos(2\,t) + (2\,{a_i}^3\,a_j + 10\,a_i\,{a_j}^3)\,\cos(2\,\lambda_j + t) - \frac{7}{2}\,a_i^2\,a_j^2\,\cos(2\,\lambda_j + 2\,t)\ ;
\end{aligned}$$

taking the λ_j–average, one finds immediately

$$\left\langle \partial^2_{\hat\eta_j \hat\eta_j} \frac{1}{\sqrt{\tilde{\mathcal{R}}_{ij}}}\Bigg|_0 \right\rangle_{\lambda_i,\lambda_j} = \left\langle \partial^2_{\hat\xi_j \hat\xi_j} \frac{1}{\sqrt{\tilde{\mathcal{R}}_{ij}}}\Bigg|_0 \right\rangle_{\lambda_i,\lambda_j} = \frac{a_ia_j}{4}\mathcal{I}(a_i,a_j) =: \beta_{ij}\ . \qquad \blacksquare$$

Next result shows that, for δ and ε small, generically the eigenvalues of M in (3.26)–(3.27) are non vanishing, simple and distinct. We formulate the result regarding the semi–axis a_j as *independent variables*. Recall the definitions of α_{ij} and β_{ij} in (3.25) and let (if $N \geq 3$)

$$\beta_j := \sum_{k=1,2} \frac{\sqrt{\bar\mu_k \bar\mu_j}}{m_0}\,\frac{1}{\sqrt[4]{a_k a_j}}\,\beta_{kj}\ , \qquad j \geq 3\ . \tag{3.47}$$

Proposition 3.2 *Assume that a_j and $\bar\mu_j$ verify*[7]

$$\alpha_{12} \neq 0 \quad \text{and} \quad \beta_{12} \neq \pm\alpha_{12}\ , \qquad \beta_i \neq 0 \quad \text{and} \quad \beta_i \neq \beta_j \ \text{ for } \ i \neq j\ . \tag{3.48}$$

Then, there exist $0 < \delta^\star < 1$ and $0 < \varepsilon_0 < 1$ such that for all $0 < \delta < \delta^\star$ and $0 \leq \varepsilon < \varepsilon_0$ the eigenvalues $\{\bar\Omega_1, ..., \bar\Omega_N\}$ of the matrix M are non vanishing, simple and distinct. Furthermore the following asymptotics holds[8]*:*

$$\bar\Omega_1 = \frac{\sqrt{\bar\mu_1\bar\mu_2}}{m_0}\ \frac{\beta_{12} + \alpha_{12}}{\sqrt[4]{a_1a_2}} + O(\sqrt{\delta}, \varepsilon)\ ,$$

[7] Clearly, if $N = 2$, the statements regarding the β_j and the eigenvalues $\bar\Omega_j$ for $j \geq 3$ have to be omitted.

[8] We use the standard notation: $a = O(\varepsilon) \iff \exists$ a constant $c > 0$ (independent of ε) and $0 < \varepsilon_0 < 1$ s.t. $|a| \leq c|\varepsilon|$ for all $|\varepsilon| \leq \varepsilon_0$; $O(\sigma, \varepsilon) = O(\sigma) + O(\varepsilon)$.

$$\bar\Omega_2 = \frac{\sqrt{\bar\mu_1\bar\mu_2}}{m_0}\ \frac{\beta_{12}-\alpha_{12}}{\sqrt[4]{a_1a_2}} + O(\sqrt\delta,\varepsilon)\ ,$$

$$\bar\Omega_j = \sqrt\delta \sum_{k=1,2} \frac{\sqrt{\bar\mu_k\bar\mu_j}}{m_0}\ \frac{\beta_{ij}}{\sqrt[4]{a_ka_j}} + \sqrt\delta\ O(\sqrt\delta,\varepsilon)\ , \qquad 3\le j\le N\,. \tag{3.49}$$

Proof First of all, from the definition of c_{ij} (see (3.24) and (2.6)) it follows that

$$c_{ij} = \frac{1}{m_0}\ \frac{1}{\sqrt[4]{a_ia_j}} + O(\varepsilon)\ . \tag{3.50}$$

Thus, by definition of M, by definition of β_j and α_{ij} and by the hypothesis on the masses μ_i (see (1.3)) we find the following asymptotics:

$$\begin{aligned}
&\begin{pmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix} = M^\star + O(\sqrt\delta,\varepsilon)\ , && \text{where}\quad M^\star := \frac{\sqrt{\bar\mu_1\bar\mu_2}}{m_0\sqrt[4]{a_1a_2}}\begin{pmatrix}\beta_{12} & \alpha_{12}\\ \alpha_{12} & \beta_{12}\end{pmatrix}\ ,\\
&M_{jj} = \sqrt\delta\beta_j + O(\delta,\varepsilon)\ , && \text{for } j\ge 3,\\
&M_{ij} = O(\sqrt\delta)\,, && \text{for } i=1,2 \text{ and } j\ge 3, \text{ or } j=1,2 \text{ and } i\ge 3,\\
&M_{ij} = O(\delta)\,, && \text{for } i,j\ge 3, \text{ with } i\ne j\,.
\end{aligned}$$

Therefore[9]

$$M = \begin{pmatrix} M^\star + O(\sqrt\delta,\varepsilon) & O(\sqrt\delta)\\ O(\sqrt\delta) & \sqrt\delta M_\star + O(\delta,\varepsilon)\end{pmatrix}\ ,$$

where

$$M_\star := \operatorname{diag}(\beta_3,\dots,\beta_N) \in \mathrm{Mat}((N-2)\times(N-2))\ .$$

The eigenvalues of $M^\star$ are

$$\frac{\sqrt{\bar\mu_1\bar\mu_2}}{m_0\sqrt[4]{a_1a_2}}\,(\beta_{12}+\alpha_{12}) \qquad \text{and} \qquad \frac{\sqrt{\bar\mu_1\bar\mu_2}}{m_0\sqrt[4]{a_1a_2}}\,(\beta_{12}-\alpha_{12})\,,$$

which by the first two requirements in (3.48), are non zero, simple and distinct. The matrix $M_\star$ is diagonal and its eigenvalues β_j are, also, non zero, simple and distinct by (3.48). The claim, now, follows by elementary linear algebra (compare,e.g., Lemma B.2 in Appendix B). ∎

Remark 3.2 (i) The hypotheses (3.48) of Proposition 3.2 are easily checked, for example, if a_j verifies (1.2) for a suitable $\theta > 0$. In fact the asymptotics for $\mathcal{J}(s,1)$ and $\mathcal{I}(s,1)$ (see (3.28)) yields immediately

$$\begin{aligned}
\alpha_{12} &= -\frac{15}{64}\ \frac{1}{a_2}\ \Big(\frac{a_1}{a_2}\Big)^3\ \Big[1+O\Big(\Big(\frac{a_1}{a_2}\Big)^2\Big)\Big]\,,\\
\beta_{12} &= \frac34\ \frac{1}{a_2}\ \Big(\frac{a_1}{a_2}\Big)^2\ \Big[1+O\Big(\Big(\frac{a_1}{a_2}\Big)^2\Big)\Big]\,,\\
\beta_j &= \frac34\ \frac{\sqrt{\bar\mu_j\bar\mu_2}}{m_0}\ \frac{1}{a_j^{3/2}}\ \Big(\frac{a_2}{a_j}\Big)^{7/4}\ \Big[1+O\Big(\Big(\frac{a_1}{a_2}\Big)^{7/4}\Big)+O\Big(\Big(\frac{a_2}{a_j}\Big)^2\Big)\Big]\ , \qquad j\ge 3\ ,\\
\beta_{12}\mp\alpha_{12} &= \frac34\ \frac{1}{a_2}\ \Big(\frac{a_1}{a_2}\Big)^2\ \Big[1+O\Big(\frac{a_1}{a_2}\Big)\Big]\,,\\
\beta_j-\beta_i &= \frac34\ \frac{\sqrt{\bar\mu_j\bar\mu_2}}{m_0}\ \frac{1}{a_j^{3/2}}\ \Big(\frac{a_2}{a_j}\Big)^{7/4}\ \times\\
&\qquad\times\Big[1+O\Big(\Big(\frac{a_1}{a_2}\Big)^{7/4}\Big)+O\Big(\Big(\frac{a_2}{a_j}\Big)^2\Big)+O\Big(\Big(\frac{a_i}{a_j}\Big)^{13/4}\Big)\Big]\ , \qquad i>j\ge 3\ .
\end{aligned}$$

[9]The $O(\sqrt\delta)$ in the upper–right part of M is a $(2\times(N-2))$ matrix while the $O(\sqrt\delta)$ in the lower–left part of M is a $((N-2)\times 2)$ matrix.

Thus, if θ is small enough and if (1.2) holds, one sees that

$$\begin{aligned}
&\alpha_{12} < 0 \ ,\\
&\beta_{12} \mp \alpha_{12} > 0 \ ,\\
&\beta_j > 0 \ , \qquad \forall\, j \geq 3 \ ,\\
&\beta_j - \beta_i > 0 \ , \qquad \forall\, i > j \geq 3 \ ,
\end{aligned}$$

and the hypotheses (3.48) are verified as claimed.

(ii) The $O(\cdot)$'s appearing in (3.49) (and in the proof of Proposition 3.2) depend on the a_j's (and on[10] m_0). Thus, it is important the order in fixing the various parameters. One way of proceeding is as follows. First determine θ as explained in the previous point (i). Then, let $\bar a_i$, $1 \leq i \leq N$, be positive numbers such that (1.2) holds, i.e., $\bar a_i/\bar a_{i+1} < \theta$ for any $1 \leq i \leq N-1$; (the $\bar a_i$ may be physically interpreted as observed mean major semi–axis). Now, consider a compact order–one neighborhood $A \subset \{0 < a_1 < \cdots < a_N\}$ of $(\bar a_1, ..., \bar a_N)$ for which (1.2) continues to be valid (such neighborhood exists simply by continuity). Finally, fix $\delta^\star$ and ε_0 so that Proposition 3.2 holds: such numbers will depend only on $\bar a_j$'s and the (order–one) size of the choosen neighborhood A.

4 Existence of N–dimensional elliptic invariant tori

In this Section we prove the existence of N–dimensional elliptic invariant tori for the $(N+1)$-body problem Hamiltonian $\mathcal{H}$ in (2.17) for any $N \geq 2$.

Let $m_0 < \bar\mu_j < 4m_0$, let θ, A, $\delta^\star$ and ε_0 be as in point (ii) of Remark 3.2 and fix $0 < \delta < \delta^\star$, which henceforth will be kept fixed. In the rest of the paper only ε is regarded a free parameter: at the moment, ε is assumed not to exceed ε_0 but later will be required to satisfy stronger smallness conditions. The semi–major axis map

$$\vec a : \Lambda = (\Lambda_1, ..., \Lambda_N) \mapsto \big(a(\Lambda_1; \mu_1, \varepsilon), ..., a(\Lambda_N; \mu_N, \varepsilon)\big) \tag{4.51}$$

is a real–analytic diffeomorphysm and we define

$$\mathfrak{I} = \vec a^{\,-1}(A) \ ,$$

then the Hamiltonian $\mathcal{H}$ is real–analytic (and bounded) on the domain $\mathfrak{I} \times \mathbf{T}^N \times B^{2N}_R$ for a suitable $R > 0$ (here B^n_r denotes the n–ball of radius r and center $0 \in \mathbf{R}^n$).

By Proposition 3.1, the quadratic part $\overline{\mathcal{H}}_{1,2}$ of the averaged Newtonian interaction $\mathcal{H}_1$ has the simple form (3.26), M being the symmetric matrix defined in (3.27). As already pointed out in Remark 3.1, the matrix M can be diagonalized with eigenvalues, which, thanks to our assumptions and to Proposition 3.2, have the form in (3.49) and, therefore, satisfy

$$\inf_{\mathfrak{I}} |\bar\Omega_j| > \bar c \ , \qquad \inf_{\mathfrak{I}} \left|\bar\Omega_i - \bar\Omega_j\right| > \bar c \, , \tag{4.52}$$

for any $i \neq j = 1, \ldots, N$ and for a suitable positive constant $\bar c$ independent of ε. If $U := U(\Lambda)$ is the symmetric matrix which diagonalize M, $U^T M U = \operatorname{diag}(\bar\Omega_1, ..., \bar\Omega_N)$, then the map

$$\Xi : (I, \varphi, p, q) \mapsto (\Lambda, \lambda, \eta, \xi) \ , \qquad \text{where} \qquad \begin{cases} p = U^T \eta \, , \qquad q = U^T \xi \, ,\\ I = \Lambda \, ,\\ \varphi = \lambda + \displaystyle\sum_{h,k,\ell} (\partial_\Lambda U_{k\ell}) U_{h\ell}\, \eta_k\, \xi_\ell \, , \end{cases} \tag{4.53}$$

[10]Recall that $m_0 < \bar\mu_j < 4m_0$; compare line before (1.3).

is symplectic (and real–analytic) and

$$\overline{\mathcal{H}}_{1,2} \circ \Xi = \frac{1}{2} \sum_{i=1}^{N} \bar{\Omega}_i(I)(p_i^2 + q_i^2) \, . \tag{4.54}$$

Thus, the $(N+1)$-body problem Hamiltonian $\mathcal{H}$ in (2.17), in the case we are considering, can be written as

$$\mathcal{H} \circ \Xi(I, \varphi, p, q; \varepsilon) = h(I) + f(I, \varphi, p, q; \varepsilon) \tag{4.55}$$

with

$$\begin{aligned} h &:= \mathcal{H}_0 \ , \qquad f := \varepsilon f_1(I, p, q; \varepsilon) + \varepsilon f_2(I, \varphi, p, q; \varepsilon) \ , \\ f_1 &:= f_{1,0}(I) + \frac{1}{2} \sum_{i=1}^{N} \bar{\Omega}_i(I)(p_i^2 + q_i^2) + \tilde{f}_1(I, p, q; \varepsilon) \ , \\ f_{1,0} &:= \overline{\mathcal{H}}_{1,0} \ , \qquad \tilde{f}_1 := \overline{\mathcal{H}}_{1,*} \circ \Xi \ , \qquad f_2 := \widetilde{\mathcal{H}}_1 \circ \Xi \, . \end{aligned}$$

Here h is uniformly strictly concave,

$$|\tilde{f}_1| \le \text{const}\, |(p,q)|^4 \ , \qquad \text{and} \qquad \int_{\mathbf{T}^N} f_2 \, d\varphi = 0 \, .$$

The construction of elliptic invariant tori for the Hamiltonian (4.55) is based on four steps, which we proceed to describe.

4.1 Averaging

Fix $\tau > N - 1 \ge 1$ and pick two numbers b_1, b_2 such that

$$0 < b_1 < \frac{1}{2} \ , \qquad 0 < b_2 < \Big(\frac{1}{2} - b_1\Big) \frac{1}{\tau + 1} \ . \tag{4.56}$$

Since the integrable Hamitlonian h depends only on the action I, the conjugated variable φ is a "fast" angle and, in "first approximation", the $(h + f)$-motions are governed by the averaged Hamiltonian $h + \varepsilon f_1$, which posses an elliptic equilibrium at $p = q = 0$. As we, now, proceed to describe, one may remove the φ-dependence of the perturbation function f up to high order in ε by using averaging theory; for detailed information on averaging theory in similar situations, see Proposition A.1 of [BCV03] or Proposition 7.1 of [BBV03].

Denote by D^n_R the complex n–ball of center zero and radius $R > 0$ and, for any $V \subset \mathbf{R}^N$, denote by V_R the complex neighborhood of radius $R > 0$ of the set V given by $V_R := \cup_{x \in V} D_R(x)$. Next, define the set $\hat{\mathfrak{I}}$ as the following "Diophantine subset" of $\mathfrak{I}$:

$$\hat{\mathfrak{I}} := \Big\{ I \in \mathfrak{I} \, : \, |\partial_I h(I) \cdot k| \ge \frac{\bar{\gamma}}{|k|^\tau} \, , \ \forall k \in \mathbf{Z}^N \setminus \{0\} \Big\} \quad \text{with} \quad \bar{\gamma} := \text{const}\ \varepsilon^{b_1} \ . \tag{4.57}$$

Notice that (as it is standard to prove)

$$\text{meas}\left(\mathfrak{I} \setminus \hat{\mathfrak{I}}\right) \le \text{const}\, \bar{\gamma} = \text{const}\, \varepsilon^{b_1} \ . \tag{4.58}$$

The Hamiltonian $h + f$ in (4.55) is real–analytic on the complex domain

$$\mathfrak{D}_{\mathrm{r},s,\rho} := \hat{\mathfrak{I}}_{\mathrm{r}} \times \mathbf{T}^N_s \times D^{2N}_\rho \subset \mathbf{C}^{4N} \ , \tag{4.59}$$

with

$$\mathrm{r} := \mathrm{const}\,\sqrt{\varepsilon}\ , \qquad s := \mathrm{const}\ , \qquad \rho := \mathrm{const}\,\varepsilon^{b_2}\ . \tag{4.60}$$

The definition of $\hat{\mathfrak{I}}$ is motivated by the necessity to have an estimate on small divisors. In fact, let $I \in \hat{\mathfrak{I}}_{\mathrm{r}}$ (and ε small enough) and let $I_0 \in \hat{\mathfrak{I}}$ be a point at distance less than r from I. Then, for any $k \in \mathbf{Z}^N \backslash \{0\}$ such that $|k| \le K :=\ \mathrm{const}\,\varepsilon^{-b_2}$, by the second relation in (4.56), by (4.57) and by Cauchy estimates, one finds

$$\begin{aligned} |\partial_I h(I)\cdot k| &\ge |\partial_I h(I_0)\cdot k| - |\partial_I h(I_0) - \partial_I h(I)|\ |k| \\ &\ge \frac{\bar{\gamma}}{K^\tau} - \max|\partial_I^2 h|\ r\ K \\ &\ge \frac{\bar{\gamma}}{2K^\tau} =: \alpha =\ \mathrm{const}\,\varepsilon^{b_1+\tau b_2}\ , \qquad \Big(0<|k|\le K := \mathrm{const}\,\varepsilon^{-b_2}\Big)\ . \end{aligned} \tag{4.61}$$

In order to apply averaging theory so as to remove the ε–dependence up to order $\exp(-\,\mathrm{const}\,K)$, one has to verify the following "smallness condition" (compare condition (A.2), p. 110 in [BCV03])

$$\|f\|_{\mathrm{r},s,\rho} \le \mathrm{const}\,\frac{\alpha\min\{\mathrm{r}s,\rho^2\}}{K}\ ,$$

where the norm $\|\cdot\|_{\mathrm{r},s,\rho}$ is defined as the standard "sup–Fourier norm"

$$\|f\|_{\mathrm{r},s,\rho} := \sum_{k\in\mathbf{Z}^N}\Big(\sup_{(I,p,q)\in\hat{\mathfrak{I}}_{\mathrm{r}}\times D^{2N}_\rho}|f_k(I,p,q)|\Big)e^{|k|s}\ , \tag{4.62}$$

($f_k(I,p,q)$ denotig Fourier coefficients of the multi–periodic, real–analytic function $\varphi\mapsto f(I,\varphi,p,q)$). Such condition, in view of (4.56), can be achieved by taking ε small enough since, by (4.61) and (4.60), one has that

$$\|f\|_{\mathrm{r},s,\rho} = O(\varepsilon) \qquad \text{and} \qquad \frac{\alpha\min\{\mathrm{r}s,\rho^2\}}{K} = O(\varepsilon^{b_1+(\tau+1)b_2+1/2})\ .$$

Hence, there exists a close–to–identity (real–analytic) symplectic change of variables $(I',\varphi',p',q')\mapsto (I,\varphi,p,q)$ verifying (compare formulae (2.16) and (A.7) of [BCV03])

$$|I'-I| \le\ \mathrm{const}\,\varepsilon^{\frac12+b_2}\ , \qquad \text{and} \qquad |p'-p|\,,|q'-q| \le \mathrm{const}\,\sqrt{\varepsilon}\ , \tag{4.63}$$

and such that the Hamiltonian expressed in the new symplectic variables becomes

$$h(I') + \hat{g}(I',p',q') + \hat{f}(I',\varphi',p',q')\,, \qquad \hat{g} := \varepsilon f_1(I',p',q') + \varepsilon\hat{f}_1(I',p',q') \tag{4.64}$$

with $\hat{f}_1$ and $\hat{f}$ real–analytic on the complex domain $\mathfrak{D}_{\mathrm{r}/2,s/6,\rho/2}$ and satisfying

$$\begin{aligned} &\|\hat{f}_1\|_{\mathrm{r}/2,s/6,\rho/2} \le \frac{\varepsilon}{\alpha\mathrm{r}} =\ \mathrm{const}\,\varepsilon^{b_2+b_3}\ , \quad \text{with} \quad b_3 := \frac12 - b_1 - (\tau+1)b_2 > 0\ , \\ &\|\hat{f}\|_{\mathrm{r}/2,s/6,\rho/2} \le\ \mathrm{const}\,e^{-\,\mathrm{const}\,K} \ll\ \mathrm{const}\,\varepsilon^3\ . \end{aligned} \tag{4.65}$$

4.2 New elliptic equilibrium

Due to the (small) term $\hat{f}_1$ in (4.64), zero is no longer an elliptic equilibrium for the "averaged" (i.e. φ-independent) Hamiltonian $h+\hat{g}$. Using the Implicit Function Theorem, we can find a new elliptic equilibrium for $h+\hat{g}$, which is $\varepsilon^{b_2+b_3}$ close to zero. Hence we construct a real–analytic symplectic transformation

$$(J',\psi',v',u')\mapsto(I',\varphi',p',q')\ , \quad \text{with} \quad I'=J' \quad \text{and } \varepsilon^{b_2+b_3}\text{-close-to-the-identity,} \tag{4.66}$$

such that in the new symplectic variables (J', ψ', v', u') the Hamiltonian takes the form

$$h(J') + \tilde{g}(J', v', u') + \tilde{f}(J', \psi', v', u'),$$

with $\tilde{g}$ having $v' = u' = 0$ as elliptic equilibrium; the functions $\tilde{g}$ and $\tilde{f}$ are real–analytic on a slighty smaller complex domain, say $\mathfrak{D}_{\mathrm{r}/7,s/7,\rho/7}$, where they satisfy bounds similar to those in (4.65). Furthermore, for $j = 1, \ldots, N$, the eigenvalues $\tilde{\Omega}_j(J')$ of the symplectic quadratic part of $\tilde{g}$ are purely imaginary and $\varepsilon^{1+b_2+b_3}$–close to $\varepsilon\bar{\Omega}_j(J')$.

4.3 Symplectic diagonalization of the quadratic term

Using a well known result on the symplectic diagonalization of quadratic Hamiltonians, we can find a real–analytic, symplectic transformation

$$(\tilde{J}, \tilde{\psi}, \tilde{v}, \tilde{u}) \mapsto (J', \psi', v', u'), \quad \text{with} \quad J' = \tilde{J} \quad \text{and } \varepsilon^{b_2+b_3}\text{-close-to-the-identity}, \tag{4.67}$$

such that the quadratic part of $\tilde{g}$ becomes, simply, $\sum_{i=1}^{N} \tilde{\Omega}_i(\tilde{J})\,(\tilde{u}_j^2 + \tilde{v}_j^2)$. Whence, the new Hamiltonian becomes (compare formula (2.22) of [BCV03])

$$\widetilde{\mathcal{H}} := h_0(\tilde{J}) + \sum_{i=1}^{N} \tilde{\Omega}_i(\tilde{J})\,(\tilde{u}_i^2 + \tilde{v}_i^2) + \tilde{g}_0(\tilde{J}, \tilde{v}, \tilde{u}) + \tilde{f}_0(\tilde{J}, \tilde{\psi}, \tilde{v}, \tilde{u}), \tag{4.68}$$

where

$$h_0(\tilde{J}) := h(\tilde{J}) + \varepsilon\tilde{g}(\tilde{J}, 0, 0), \tag{4.69}$$

$\tilde{g}_0$, $\tilde{f}_0$, $\tilde{\Omega}_j$ are real-analytic and

$$|\tilde{g}_0(\tilde{J}, \tilde{v}, \tilde{u})| \leq \text{ const } \varepsilon\, |(\tilde{v}, \tilde{u})|^3, \qquad |\tilde{\Omega}| \leq \text{ const}\, \varepsilon, \qquad \|\tilde{f}\|_{\mathrm{r}/8,s/8,\rho/8} \leq \text{ const}\, \varepsilon^3. \tag{4.70}$$

Finally, because of (4.52),

$$\inf |\tilde{\Omega}_i| \geq \text{ const } \varepsilon > 0, \qquad \inf |\tilde{\Omega}_2 - \tilde{\Omega}_1| \geq \text{ const } \varepsilon > 0. \tag{4.71}$$

4.4 Applying KAM Theory

We rewrite, now, the Hamiltonian $\widetilde{\mathcal{H}}$ in (4.68) in a form suitable for applying (elliptic) KAM theory. Introducing translated variables $y := \tilde{J} - \mathfrak{p}$ and complex variables $z, \bar{z}$, we define

$$H = \widetilde{\mathcal{H}}\Big(\mathfrak{p} + y, \psi, \frac{z + \bar{z}}{\sqrt{2}}, \frac{z - \bar{z}}{\mathrm{i}\sqrt{2}}\Big), \tag{4.72}$$

here $\mathfrak{p}$ is regarded as a parameter and the symplectic form is $\sum_{j=1}^{N} dy_j \wedge d\psi_j + \mathrm{i}\sum_{j=1}^{N} dz_j \wedge d\bar{z}_j$ with $\mathrm{i} := \sqrt{-1}$. The Hamiltonian H is then seen to have the form

$$H = \mathcal{N} + P$$

with

$$\mathcal{N} = e + \omega \cdot y + \sum_{j=1}^{N} \Omega_j z_j \bar{z}_j, \qquad e := h_0(\mathfrak{p}), \quad \omega := \partial_{\tilde{J}} h_0(\mathfrak{p}), \quad \Omega := \tilde{\Omega}(\mathfrak{p}) \tag{4.73}$$

and P a perturbation, which can be naturally be splitted in four terms:

$$P = \sum_{1 \le k \le 4} P_k \ ,$$

with

$$\begin{aligned} P_1 &= h_0(\mathfrak{p}+y) - h_0(\mathfrak{p}) - \partial_{\bar{J}} h_0(\mathfrak{p}) \cdot y \sim y^2 \,, \\ P_2 &= \sum_{j=1}^{n} \Big(\tilde{\Omega}_j(\mathfrak{p}+y) - \tilde{\Omega}_j(\mathfrak{p}) \Big) z_j \bar{z}_j \sim y|z||\bar{z}| \,, \\ P_3 &= \tilde{g}_0\Big(\mathfrak{p}+y, \frac{z+\bar{z}}{\sqrt{2}}, \frac{z-\bar{z}}{\mathrm{i}\sqrt{2}}\Big) \sim \varepsilon(|z|+|\bar{z}|)^3 \,, \qquad \Big(\text{by } \ (4.70)\Big) \ , \\ P_4 &= \tilde{f}_0\Big(\mathfrak{p}+y, \psi, \frac{z+\bar{z}}{\sqrt{2}}, \frac{z-\bar{z}}{\mathrm{i}\sqrt{2}}\Big) = O(\varepsilon^3) \,. \end{aligned} \tag{4.74}$$

The parameter $\mathfrak{p}$ runs over the Diophantine set $\hat{\mathfrak{J}}$ defined in (4.57). Notice that the integrable Hamiltonian $\mathcal{N}$ affords, for any given value of the parameter $\mathfrak{p}$, the $N-$*dimensional elliptic torus*

$$\{y=0\} \times \mathbf{T}^N \times \{z=\bar{z}=0\} \,, \tag{4.75}$$

which is invariant for the Hamiltonian flow generated by $\mathcal{N}$, the flow being, simply, the Diophantine translation $x \mapsto x + \omega t$, with ω as in (4.73).

Since det $\partial^2_{\bar{J}} h_0 \neq 0$, we can use the frequencies ω as parameters rather than the actions $\mathfrak{p}$. We, therefore, set

$$\mathcal{O} := \partial_{\bar{J}} h_0(\hat{\mathfrak{J}}) = \Big\{ \omega = \partial_{\bar{J}} h_0(\mathfrak{p}) \ : \ \mathfrak{p} \in \hat{\mathfrak{J}} \Big\} \,. \tag{4.76}$$

Notice that, by(4.57), we have

$$\mathrm{meas}\Big(\partial_{\bar{J}} h_0(\mathfrak{J}) \setminus \mathcal{O}\Big) \le \mathrm{const}\ \varepsilon^{b_1} \ . \tag{4.77}$$

Now, if we put $\mathfrak{p} = \mathfrak{p}(\omega) := (\partial_{\bar{J}} h_0)^{-1}(\omega)$ in (4.72), we can rewrite the $(N+1)$–body Hamiltonian in the form

$$H(y, \psi, z, \overline{z}; \omega) := \mathcal{N}(y, z, \bar{z}; \omega) + P(y, \psi, z, \bar{z}; \omega) \tag{4.78}$$

where

$$\mathcal{N}(y, z, \overline{z}; \omega) := e(\omega) + \omega \cdot y + \sum_{j=1}^{N} \Omega_j(\omega) z_j \bar{z}_j \ , \qquad e(\omega) := h_0(\mathfrak{p}(\omega)) \ , \qquad \Omega(\omega) := \widetilde{\Omega}(\mathfrak{p}(\omega)) \tag{4.79}$$

and the perturbation $P(y, \psi, z, \bar{z}; \omega)$ is obtained by replacing $\mathfrak{p}$ with $\mathfrak{p}(\omega)$ in (4.74). Recalling (4.60) the Hamiltonian H in (4.78) is real–analytic in

$$(y, \psi, z, \bar{z}; \omega) \in \mathfrak{D}_{\mathfrak{r}^2, \mathfrak{s}, \mathfrak{r}, \mathfrak{d}} := D^N_{\mathfrak{r}^2} \times \mathbf{T}^N_{\mathfrak{s}} \times D^{2N}_{\mathfrak{r}} \times \mathcal{O}^N_{\mathfrak{d}} \tag{4.80}$$

with

$$\mathfrak{r} := \mathrm{const}\, \varepsilon^{3/4} \ , \qquad \mathfrak{s} := \mathrm{const} \ , \qquad \mathfrak{d} := \mathrm{const}\, \sqrt{\varepsilon} \ . \tag{4.81}$$

We recall, now, a well known KAM result concerning the persistence of lower dimensional elliptic tori for nearly–integrable Hamiltonian systems ([M65], [E88], [K88]). The version we present here is, essentially, a reformulation of Pöschel's Theorem in [P89] (compare, also, with Theorem 5.1 of [BBV03]).

Theorem 4.1 *Let H have the form in* (4.78), (4.79) *and let it be real–analytic on a domain $\mathfrak{D}_{\mathfrak{r}^2, \mathfrak{s}, \mathfrak{r}, \mathfrak{d}}$ of the form* (4.80) *for some $\mathfrak{r}$, $\mathfrak{s}$ and $\mathfrak{d}$ positive. Assume that*

$$\sup_{\omega \in \mathcal{O}_{\mathfrak{d}}} |\partial_\omega \Omega(\omega)| \le \frac{1}{4} \tag{4.82}$$

and that the non-resonance (or Melnikov) condition

$$|\Omega(\omega)\cdot k| \geq \gamma_0 \ , \qquad \forall\ 1 \leq |k| \leq 2,\ k \in \mathbf{Z}^N, \qquad \forall\ \omega \in \mathcal{O}, \tag{4.83}$$

is satisfied for some $\gamma_0 > 0$. *Then, if* $\mathfrak{d} \geq \gamma_0$ *and* P *is sufficiently small, i.e.,*

$$|\!|\!| P |\!|\!|_{\mathfrak{r},\mathfrak{s},\mathfrak{d}} := \sup_{\omega\in\mathcal{O}_\mathfrak{d}} \|P(\,\cdot\,;\omega)\|_{\mathfrak{r}^2,\mathfrak{s},\mathfrak{r}} \leq \mathrm{const}\,\gamma_0\,\mathfrak{r}^2 \ , \tag{4.84}$$

then there exist a normal form $\mathcal{N}_* := e_*(\omega) + \omega\cdot y_* + \Omega_*(\omega) z_* \bar z_*$, *a Cantor set* $\mathcal{O}(\gamma_0) \subset \mathcal{O}$ *with*

$$\mathrm{meas}\Big(\mathcal{O}\setminus\mathcal{O}(\gamma_0)\Big) \leq \mathrm{const}\,\gamma_0 \ , \tag{4.85}$$

and a transformation

$$\begin{array}{rcl}\mathcal{F}:\ D^N_{\mathfrak{r}^2/4}\times\mathbf{T}^N_{\mathfrak{s}/2}\times D^{2N}_{\mathfrak{r}/2}\times\mathcal{O}(\gamma_0) & \longrightarrow & D^N_{\mathfrak{r}^2}\times\mathbf{T}^N_{\mathfrak{s}}\times D^{2N}_{\mathfrak{r}}\times\mathcal{O}_\mathfrak{d}\\ (y_*,\psi_*,z_*,\bar z_*;\omega) & \longmapsto & (y,\psi,z,\bar z;\omega)\end{array}$$

real–analytic and symplectic for each ω *and Whitney smooth in* ω, *such that*

$$H\circ\mathcal{F} = \mathcal{N}_* + R_* \qquad \textit{with} \quad \partial^j_{y_*}\partial^h_{z_*}\partial^k_{\bar z_*} R_* = 0 \ , \quad \textit{if} \quad 2|j| + |h+k| \leq 2 \ . \tag{4.86}$$

In particular, for each $\omega \in \mathcal{O}(\gamma_0)$, *the torus* $\{y_* = 0\}\times\mathbf{T}^N\times\{z_* = \bar z_* = 0\}$ *is an* N*–dimensional, linearly elliptic, invariant torus run by the flow* $\psi_* \to \psi_* + \omega t$. *Finally*

$$|y_* - y| \ , \quad \mathfrak{r}|z_* - z| \ , \quad \mathfrak{r}|\bar z_* - \bar z| \leq \ \mathrm{const}\,\frac{|\!|\!| P |\!|\!|_{\mathfrak{r},\mathfrak{s},\mathfrak{d}}}{\gamma_0} \ . \tag{4.87}$$

In this section we have shown that the many–body Hamiltonian (2.17) (under the hypotheses spelled out at the beginning the section) has indeed the form assumed in the KAM Theorem 4.1. Furthermore, by (4.71), the elliptic frequencies Ω_i verify the Melnikov conditions (4.83) with

$$\gamma_0 = \mathrm{const}\ \varepsilon \ , \tag{4.88}$$

and, by (4.74) and (4.81), the perturbation P verifies, for small ε, the KAM condition (4.84), since

$$|\!|\!| P |\!|\!|_{\mathfrak{r},\mathfrak{s},\mathfrak{d}} = O(\mathfrak{r}^4 + \varepsilon\mathfrak{r}^3 + \varepsilon^3) = O(\varepsilon^3) \leq \mathrm{const}\,\gamma_0\,\mathfrak{r}^2 = O(\varepsilon^{5/2}) \ . \tag{4.89}$$

Thus, the existence of the desired quasi-periodic orbits follows at once from Theorem 4.1. We may summarize the final result as follows.

Theorem 4.2 *Let* $N \geq 2$ *and let* $\mathcal{H}$ *be the* $(N+1)$*–body problem Hamiltonian in Poincaré variables defined in* (2.17). *Let* $m_0 < \bar\mu_j < 4m_0$, *let* θ, A, $\delta^\star$ *and* ε_0 *be as in point (ii) of Remark 3.2. Fix* $0 < \delta < \delta^\star$ *and let* $\mathfrak{I} = \vec a^{-1}(A)$ *where* $\vec a$ *is the semi–major axis map defined in* (4.51). *Let* $\tau > N-1$ *and pick* b_1, b_2 *as in* (4.56). *Finally, let* $0 < \varepsilon^\star < \varepsilon_0$ *be such that* (4.89) *holds for any* $\varepsilon \leq \varepsilon^\star$ *and such that all conditions on* ε *required for constructing the symplectic transformations introduced in § 4.1÷§ 4.3 are satisfied for* $\varepsilon < \varepsilon^\star$. *Then, for any* $\varepsilon < \varepsilon^\star$, *there exist a Cantor set* $\mathfrak{I}_* \subset \mathfrak{I}$, *with*

$$\mathrm{meas}\,(\mathfrak{I}\setminus\mathfrak{I}_*) \leq \mathrm{const}\,\varepsilon^{b_1} \ , \tag{4.90}$$

and a Lipschitz continuous family of tori embedding

$$\phi\ :\ (\vartheta,\mathfrak{p}) \in \mathbf{T}^N\times\mathfrak{I}_* \mapsto \Big(\Lambda(\vartheta;\mathfrak{p}),\ \lambda(\vartheta;\mathfrak{p}),\ \eta(\vartheta;\mathfrak{p}),\ \xi(\vartheta;\mathfrak{p})\Big) \in \mathfrak{I}\times\mathbf{T}^N\times B^{2N}_{\rho_*} \ ,$$

with $\rho_* := \mathrm{const}\,\varepsilon^{b_2}$ *such that, for any* $\mathfrak{p} \in \mathfrak{I}_*$, $\phi(\mathbf{T}^N;\mathfrak{p})$ *is a real-analytic elliptic* $\mathcal{H}$*-invariant torus, on which the* $\mathcal{H}$*–flow is analytically conjugated to the linear flow* $\vartheta \to \vartheta + \omega_* t$, ω_* *being* (γ,τ)*–Diophantine with* $\gamma = O(\varepsilon^{b_1})$. *Furthermore, the following bounds hold uniformly on* $\mathbf{T}^N\times\mathfrak{I}_*$:

$$\begin{aligned}|\Lambda(\vartheta;\mathfrak{p}) - \mathfrak{p}| &\leq \mathrm{const}\,\varepsilon^{\frac12 + b_2} \ , &(4.91)\\ |\eta(\vartheta;\mathfrak{p})| + |\xi(\vartheta;\mathfrak{p})| &\leq \mathrm{const}\,\varepsilon^{b_2} \ . &(4.92)\end{aligned}$$

Theorem 1.1 follows, now, by taking (recall the definitions of b_k in (4.56))

$$c_1 := b_1 \ , \qquad c_2 := b_2 \ , \qquad c_3 := b_2 + \frac{1}{2} \ . \tag{4.93}$$

In particular the statements on the density of the set of the osculating major semi–axes, on the bound on the osculating eccentricities and on the variation of the osculating major semi–axes, follows from (4.90), (2.12), (4.92), (2.10) and (4.91).

A Poincaré variables for the planar $(N+1)$–body problem

We briefly recall in this appendix the classical derivation of the Poincaré variables for the planar N–body problem, showing also the validity of Lemma 2.1. Though the methods exposed here essentially go back to [Poi1905], we believe that this sketchy presentation can be of use for pointing out some less known features of this set of variables that have been employed in this paper[11]. The question of the Poincaré variables for the N–body problem boils down to the two–body problem case. The set of coordinates thus obtained will then be extended to the one claimed in Lemma 2.1, just by taking the Poincaré variables given by the interaction between the sun and any of the planets.

A.1 Canonical variables for the two–body problem

Consider two bodies P_0, P_1 of masses $\mathfrak{m}_0$, $\mathfrak{m}_1$ and position $u^{(0)}$, $u^{(1)} \in \mathbf{R}^2$ (with respect to an inertial frame). We assume that P_0 and P_1 interact through gravity, with gravitational constant 1. By Newton's laws, the equations of motion for such two-body problem are

$$\begin{aligned} \ddot{u}^{(0)} &= \mathfrak{m}_1 \frac{(u^{(1)} - u^{(0)})}{|u^{(1)} - u^{(0)}|^3} \ , \\ \ddot{u}^{(1)} &= \mathfrak{m}_0 \frac{(u^{(0)} - u^{(1)})}{|u^{(0)} - u^{(1)}|^3} \ . \end{aligned}$$

Let

$$\mathfrak{M} := \mathfrak{m}_0 + \mathfrak{m}_1 \ , \quad \mathfrak{m} := \frac{\mathfrak{m}_0 \mathfrak{m}_1}{\mathfrak{M}} \ , \quad x := u^{(1)} - u^{(0)} \ , \quad X := \mathfrak{m}\dot{x} \ . \tag{A.94}$$

Then, the above equations of motion become

$$\ddot{x} = \frac{\mathfrak{M}x}{|x|^3} \ ,$$

and the motion of the two bodies is governed by the Hamiltonian

$$\mathcal{K}(X, x) = \frac{1}{2\mathfrak{m}}|X|^2 - \frac{\mathfrak{m}\mathfrak{M}}{|x|} \ , \tag{A.95}$$

with $(X, x) \in \mathbf{R}^2 \times \mathbf{R}^2$ conjugate variables, i.e., the equations of motion are $\dot{x} = \partial_X \mathcal{K}$, $\dot{X} = -\partial_x \mathcal{K}$.

As well known, such system is integrable and for $\mathcal{K} < 0$ the orbits are ellipses. More precisely, one has

[11] For a review of the Poincaré variables in the non-planar case, see, for instance, [Ch88] and [BCV03].

Proposition A.1 *Fix $\Lambda_- > 0 > \mathcal{K}_0$ and let $\Lambda_+ := \left(\frac{\mathfrak{m}^3\mathfrak{M}^2}{-2\mathcal{K}_0}\right)^{\frac{1}{2}} > \Lambda_-$. Then, there exist $\hat{\rho} > 0$ and a real–analytic symplectic transformation*

$$\Psi_{\rm DP} : \Big((\Lambda,\eta),(\lambda,\xi)\Big) \in \Big([\Lambda_-,\Lambda_+]\times[-\hat{\rho},\hat{\rho}]\Big)\times\Big(\mathbf{T}\times[-\hat{\rho},\hat{\rho}]\Big) \mapsto (X,x) \in \Big\{|x| \geq \frac{\hat{\rho}^2}{\mathfrak{m}^2\mathfrak{M}}\Big\}\,,$$

casting (A.95) *into the integrable Hamiltonian* $(-\mathfrak{m}^3\mathfrak{M}^2)/(2\Lambda^2)$.

This classical proposition is a planar version of the classical one, due to Poincaré (see [Poi1905], chapter III) and the variables $(\Lambda,\eta,\lambda,\xi)$ are, usually, called (planar) Poincaré variables. The proof of Proposition A.1 is particularly interesting from the physical point of view and rests upon the introduction of three different (famous) change of variables, which we, now, proceed to describe briefly.

Let ℓ and g denote, respectively, the mean anomaly and the argument of the perihelion.

Step 1. The system is set in *"symplectic" polar variables*: namely, we consider the symplectic map $\Psi_{\rm spc} : \Big((R,\Phi),(r,\varphi)\Big) \mapsto (X,x)$ (where $r > 0$ and $\varphi \in \mathbf{T}$) given by

$$\Psi_{\rm spc} : \begin{cases} x_1 = r\ \cos\varphi \\ \\ x_2 = r\ \sin\varphi \end{cases}, \qquad X = \begin{pmatrix} \cos\varphi & -\dfrac{\sin\varphi}{r} \\ \\ \sin\varphi & \dfrac{\cos\varphi}{r} \end{pmatrix}\begin{pmatrix} R \\ \\ \Phi \end{pmatrix} \tag{A.96}$$

and consider the new Hamiltonian $\mathcal{K}_{\rm spc} := \mathcal{K}\circ\Psi_{\rm spc}$.

Step 2. There is a symplectic map $\Psi_{\rm D} : \Big((L,G),(\ell,g)\Big) \mapsto \Big((R,\Phi),(r,\varphi)\Big)$ that integrates the system: $\Psi_{\rm D}$ is obtained via the generating function

$$S(L,G,r,\varphi) = \int\sqrt{-\frac{\mathfrak{m}^4\mathfrak{M}^2}{L^2} + \frac{2\mathfrak{m}^2\mathfrak{M}}{r} - \frac{G^2}{r^2}}\ dr\ +\ G\,\varphi\,. \tag{A.97}$$

The variables $\Big((L,G),(\ell,g)\Big)$ are known as (planar) *Delaunay variables*. In such variables, the new Hamiltonian becomes

$$\mathcal{K}_{\rm D} := \mathcal{K}_{\rm spc}\circ\Psi_{\rm D} = -\frac{\mathfrak{m}^3\mathfrak{M}^2}{2L^2}\,.$$

Also, if C is the angular momentum of the planet and a is the major semi–axis, by construction, one has that

$$G = |C|\,, \qquad \text{and} \qquad L = \mathfrak{m}\sqrt{\mathfrak{M}a}\,.$$

Step 3. We need now to remove singularities, which appear fro samll eccentricity. To this aim, we first introduce (planar) *Poincaré action–angle variables* by means of the linear symplectic transformation

$$\Psi_{\rm P_{aa}} : \Big((\Lambda,H),(\lambda,h)\Big) \mapsto \Big((L,G),(\ell,g)\Big)$$

given by

$$\Psi_{\rm P_{aa}} : \quad \begin{cases} \Lambda = L\,, \quad H = L - G\,, \\ \lambda = \ell + g\,, \quad h = -g\,. \end{cases} \tag{A.98}$$

Then, we let $\Psi_{\rm P} : \Big((\Lambda,\eta),(\lambda,\xi)\Big) \mapsto \Big((\Lambda,H),(\lambda,h)\Big)$ be the symplectic map defined by

$$\sqrt{2H}\ \cos h = \eta\,, \qquad \sqrt{2H}\ \sin h = \xi\ . \tag{A.99}$$

As Poincaré showed (see [Poi1905], [Ch88], [BCV03]), the symplectic map

$$\Psi_{\rm DP} : \Big((\Lambda,\eta),(\lambda,\xi)\Big) \mapsto (X,x)$$

with

$$\Psi_{\rm DP} := \Psi_{\rm spc} \circ \Psi_{\rm D} \circ \Psi_{\rm P_{aa}} \circ \Psi_{\rm P} \tag{A.100}$$

is real–analytic in a (complex) neighborhood of

$$\Lambda \in \Big[\Lambda_-, \Lambda_+\Big] , \qquad |\eta|, |\xi| \le {\rm const}\,\sqrt{\Lambda_-}, , \qquad \lambda \in {\bf T}\,.$$

Also, the two–body Hamiltonian, in Poincaré variables, is $\mathcal{K} \circ \Psi = -\frac{\mathfrak{m}^3\mathfrak{M}^2}{2\Lambda^2}$.

Remark A.1 (i) If we denote $(X, x) = \Phi_{\rm DP}\Big((\Lambda, \eta, {\rm p}), (\lambda, \xi, {\rm q})\Big)$ then,

$$X = \frac{\mathfrak{m}^4\mathfrak{M}^2}{\Lambda^3}\frac{\partial x}{\partial \lambda}\,. \tag{A.101}$$

Indeed, from the Hamilton equations one sees that : $\dot\lambda = \partial_\Lambda\Big(-\frac{\mathfrak{m}^3\mathfrak{M}^2}{2\Lambda^2}\Big) = \frac{\mathfrak{m}^3\mathfrak{M}^2}{\Lambda^3}$, and $\dot\Lambda = \dot\xi = \dot\eta = \dot{\rm p} = \dot{\rm q} = 0$. Thus, by the chain rule, $X = \mathfrak{m}\dot x = \mathfrak{m}(\partial_\lambda x)\,\dot\lambda = \frac{\mathfrak{m}^4\mathfrak{M}^2}{\Lambda^3}\frac{\partial x}{\partial\lambda}$, proving (A.101).

(ii) We collect some useful relations among the above introduced quantities. Let, as usual, e denote the eccentricity of the Keplerian ellipse and let a denote the major semi–axis. Then, by construction, one sees that

$$\Lambda = \mathfrak{m}\sqrt{\mathfrak{M}a}\,, \qquad \sqrt{\xi^2+\eta^2} = \sqrt{\Lambda}\,e\,(1+O(e^2))\,. \tag{A.102}$$

Also, if C is the angular momentum of the system, one infers that

$$|C| = \Lambda\,\sqrt{1-e^2} = \Lambda(1+O(e^2))\ . \tag{A.103}$$

The angular variables ℓ and g have the astronomical meaning of the mean anomaly and the longitude of the perihelion, respectively (see, for instance, [Ch88] and [BCV03] for pictures of the orbital elements and for some geometric insight).

(iii) A proof of the analyticity of Poincaré variables will also follow by directly inspecting the formulae given in Lemma 2.1, which is proved in the coming section.

A.2 Orbital elements

We now sketch a way to explicitly represent some quantities in terms of Poincaré variables. This will also lead to the proof of Lemma 2.1.

Let u and v denote the eccentric anomaly and the true anomaly, respectively. By geometric considerations,

$$u = \ell + e\sin u \tag{A.104}$$

and[12]

$$\cos v = \frac{\cos u - e}{1 - e\cos u}\,. \tag{A.105}$$

where

$$\ell = \lambda + h\,. \tag{A.106}$$

Also, by (A.99),

$$H = \frac{\eta^2+\xi^2}{2}\ . \tag{A.107}$$

[12]Such relations are classical and we refer to [Ch88] and [BCV03] for a geometric interpretation of these anomalies.

An explicit expression taking into account H, the eccentricity and the major semi–axis is given by

$$H = \Lambda\,(1 - \sqrt{1-e^2}) = \Lambda\,\frac{e^2}{2}(1 + O(e^2))\;, \tag{A.108}$$

$$e(H,\Lambda) = \sqrt{\frac{H}{\Lambda}\Big(2 - \frac{H}{\Lambda}\Big)}\,. \tag{A.109}$$

In light of (A.105),

$$\sin v = \frac{\sqrt{1-e^2}\sin u}{1 - e\cos u}\,. \tag{A.110}$$

By means of (A.104), we have

$$u - \ell = e\,\sin(u - \ell + \ell) = e\,\cos\ell\,\sin(u-\ell) + e\,\sin\ell\,\cos(u-\ell)\,.$$

Thus, in the notation of Lemma 2.1, if G_0 is implicitely defined by

$$G_0(x,y) = x\sin G_0(x,y) + y\cos G_0(x,y)\,,$$

with $G_0(0,0) = 0$, we have that G_0 is real–analytic, $G_0(x,y) = y + xy + O_3(x,y)$ and

$$u - \ell = G_0(e\cos\ell, e\sin\ell)\,. \tag{A.111}$$

Therefore, we deduce from (A.111) and (A.106) that

$$u = \lambda + h + G_0(e\cos h\cos\lambda - e\sin h\sin\lambda,\; e\sin h\cos\lambda + e\cos h\sin\lambda)\,. \tag{A.112}$$

Moreover, denoting

$$\hat\eta = \eta/\sqrt{\Lambda}\;, \qquad \hat\xi = \xi/\sqrt{\Lambda}\;, \tag{A.113}$$

we deduce from (A.99) and (A.109) that

$$e\sin h \;=\; \sqrt{\frac{2H}{\Lambda}}\cdot\sqrt{1 - \frac{H}{2\Lambda}}\sin h = \hat\xi\,F_1(\hat\eta^2 + \hat\xi^2)\,, \tag{A.114}$$

where $F_1(t) = \sqrt{1-(t/4)}$ is real–analytic for $|t| < 4$ (and agrees with the one introduced in Lemma 2.1). Analogously,

$$e\cos h = \hat\eta\,F_1(\hat\eta^2 + \hat\xi^2)\,. \tag{A.115}$$

Therefore, substituting (A.114) and (A.115) in (A.112), we can write G_0 as an analytic expression of $(\hat\eta, \hat\xi, \lambda)$: more formally, there exists a real–analytic $(\hat\eta, \hat\xi, \lambda) \mapsto G(\hat\eta, \hat\xi, \lambda)$ (which agrees with the one introduced in Lemma by (A.114) and (A.115)), so that

$$G_0(e\cos h\cos\lambda - e\sin h\sin\lambda,\; e\sin h\cos\lambda - e\cos h\sin\lambda) = G(\hat\eta, \hat\xi, \lambda)\,.$$

Hence, from of (A.112),

$$\begin{aligned} e\cos u &= e\cos h\cos(\lambda + G) - e\sin h\sin(\lambda + G)\,,\\ e\sin u &= e\sin h\cos(\lambda + G) + e\cos h\sin(\lambda + G)\,, \end{aligned} \tag{A.116}$$

with $G = G(\hat\eta, \hat\xi, \lambda)$. Notice also that, from the formulae in (A.109) and (A.107),

$$\frac{1 - \sqrt{1-e^2}}{e^2} = F_2(\hat\eta^2 + \hat\xi^2)\,,$$

for a suitable real–analytic function F_2 (actually, $F_2(t) = \frac{1}{2}\Big(1 - \frac{t}{4}\Big)^{-1}$, which agrees with the notation in Lemma 2.1). Thus, if we set $\varphi = \lambda + v - \ell = v - h$, recalling also (A.105) and (A.110), we have

$$\begin{aligned}
\sin\varphi &= \sin v \cos h - \cos v \sin h \\
&= \frac{1}{1 - e\cos u}[\sqrt{1-e^2}\sin u \cos h - \cos u \sin h + e \sin h] \\
&= \frac{1}{1 - e\cos u}[\sin(u-h) + e\sin h - F_2 \cdot (e\sin u)\cdot(e\cos h)] \\
&= \frac{1}{1 - e\cos u}[\sin(\lambda + e\sin u) + e\sin h - F_2 \cdot (e\sin u)\cdot(e\cos h)]\,, \qquad \text{(A.117)}
\end{aligned}$$

for $F_2 = F_2(\hat\eta^2 + \hat\xi^2)$ and analogously

$$\cos\varphi = \frac{1}{1 - e\cos u}[\cos(\lambda + e\sin u) - e\cos h - F_2 \cdot (e\sin u)\cdot(e\sin h)]\,. \qquad \text{(A.118)}$$

Hence, from (A.114), (A.115), (A.116), (A.117) and (A.118), it follows that $\sin\varphi$ and $\cos\varphi$ are real–analytic functions in $\lambda, \hat\eta, \hat\xi$, for $\lambda \in \mathbf{T}$ and small $\hat\xi, \hat\eta$. In particular, if $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{E}_{\rm s}$ are as defined in Lemma 2.1, we deduce from (A.116), (A.114) and (A.115) that

$$e\sin u = \mathcal{E}_{\rm s}\,, \qquad \text{(A.119)}$$

and then from (A.118) and (A.117) that

$$(1 - e\cos u)\cos\varphi = \mathcal{C} \qquad \text{and} \qquad (1 - e\cos u)\sin\varphi = \mathcal{S}\,. \qquad \text{(A.120)}$$

Finally, by geometric considerations, we have

$$r = a(1 - e\cos u) \qquad \text{(A.121)}$$

where r is the distance between the planet and the sun. Thus, the formulae in Lemma 2.1 follow at once by (A.119), (A.120), (A.96) and (A.101).

A.3 Hamiltonian setting for the planar many body problem

Consider $(N+1)$ bodies $P_0, \dots, P_N$ of mass $m_0, \dots, m_N$, all lying in the same plane, interacting through gravity (with constant of gravitation 1). Denote by $u^{(i)}$ the position of P_i in a given inertial frame of $\mathbf{R}^2$, with origin in the center of mass of the system. By Newton's laws, we have that

$$\ddot u^{(i)} = \sum_{0 \le j \ne i \le N} \frac{m_j(u^{(j)} - u(i))}{|u^{(j)} - u^{(i)}|^3}\,. \qquad \text{(A.122)}$$

Thus, if $U^{(i)} := m_i u^{(i)}$ denotes the momentum of P_i, we see that the equations of motion (A.122) come from the Hamiltonian

$$\sum_{i=0}^{N} \frac{1}{2m_i}|U^{(i)}|^2 - \sum_{0\le i<j\le N} \frac{m_i m_j}{|u^{(i)} - u^{(j)}|},$$

where $U = (U^{(0)}, \dots, U^{(N)}) \in \mathbf{R}^{2(N+1)}$ and $u = (u^{(0)}, \dots, u^{(N)}) \in \mathbf{R}^{2(N+1)}$ are conjugate symplectic variables.

We now consider P_0 as the "sun" and introduce canonical heliocentric variables via the linear symplectic transformation

$$\begin{aligned}
&u^{(0)} = r^{(0)}\ , \quad u^{(i)} = r^{(0)} + r^{(i)}\ , \\
&U^{(0)} = R^{(0)} - R^{(1)} - \dots - R^{(N)}\ , \quad U^{(i)} = R^{(i)}\ , \qquad \text{for} \quad i = 1, \dots, N\ . \qquad \text{(A.123)}
\end{aligned}$$

Notice that, by the conservation of the center of mass and our choice of coordinates, $R^{(0)} = 0$. Thus, the planar many body problem is governed by the $(2N)$-degree-of-freedom Hamiltonian

$$\sum_{i=1}^{N} \Big(\frac{m_0 + m_i}{2m_0 m_i} |R^{(i)}|^2 - \frac{m_0 m_i}{|r^{(i)}|} \Big) + \sum_{1 \le i < j \le N} \Big(\frac{R^{(i)} \cdot R^{(j)}}{m_0} - \frac{m_i m_j}{|r^{(i)} - r^{(j)}|} \Big) .$$

If $m_i = \varepsilon \mu_i$ for $i = 1, \ldots, N$, i.e., if the "planets" are way smaller than the "sun", the momenta $R^{(i)}$ are of order ε. Therefore, it is convenient to introduce the following rescaled symplectic variables:

$$X^{(i)} = \frac{R^{(i)}}{\varepsilon m_0^{5/3}} , \qquad x^{(i)} = \frac{r^{(i)}}{m_0^{2/3}} , \qquad i = 1, \ldots, N. \tag{A.124}$$

In such variables, after a time scale of factor $\varepsilon m_0^{7/3}$ we obtain the Hamiltonian in (2.4). Notice that, in that setting, the Hamiltonian $\mathcal{H}_0^{(N)}$ corresponds to the sum of N integrable Hamiltonians of the form (A.95), with $\mathfrak{m}$ and $\mathfrak{M}$ replaced by $\mathrm{m_i}$ and $\mathrm{M_i}$, respectively.

A.4 A parity property

We recall here the well known fact that the λ–average of $\mathcal{H}_1$ (as in (2.17)) is even in (η, ξ). The proof of this will be accomplished by a 180 degree rotation of the perihelia:

Proposition A.2 *Let*

$$f_1(\Lambda, \eta, \xi) := \frac{1}{\varepsilon} \int_{\mathbf{T}^2} \mathcal{H}_1(\Lambda, \eta, \lambda, \xi) \, d\lambda \, .$$

Then, $f_1(\Lambda, -\eta, -\xi) = f_1(\Lambda, \eta, \xi)$.

The rescaling by $\frac{1}{\varepsilon}$ is made so that f_1 is a (real–analytic) uniformly bounded (by an order–one constant) function.

Proof The eccentricity e_i, the semiaxis a_i and the mean anomaly ℓ_i of the osculating ellipse of P_i are invariant under the map $(\Lambda_i, \lambda_i, H_i, h_i) \mapsto (\Lambda_i, \lambda_i - \pi, H_i, h_i + \pi)$, while the argument of the perihelion g_i changes of π. Let us denote $\vec{\pi} := (\pi, \ldots, \pi) \in \mathbf{R}^N$. In light of the consideration above we have that the map

$$(\Lambda, \lambda, H, h) \mapsto (\Lambda, \lambda - \vec{\pi}, H, h + \vec{\pi}) \tag{A.125}$$

leaves $|r^{(i)} - r^{(j)}|$ invariant, for any $i, j = 1, \ldots, N$. The map in (A.125) corresponds to

$$(\Lambda, \lambda, \xi, \eta) \mapsto (\Lambda, \lambda - \pi, -\xi, -\eta) \, ,$$

usually referred to as "space inversion". ∎

B Simple eigenvalue perturbations

Lemma B.1 *Let* $M^\star \in \mathrm{Mat}(m \times m)$*,* $M_\star \in \mathrm{Mat}(k \times k)$*,* $M^\sharp \in \mathrm{Mat}(m \times k)$*. Then,*

$$\det \begin{pmatrix} M^\star & M^\sharp \\ \mathbf{0}_{k \times m} & M_\star \end{pmatrix} = \det(M^\star) \det(M_\star) \, .$$

Proof Induction over m. ∎

Lemma B.2 *Let $M^\star \in \mathrm{Mat}(m\times m)$, $M_\star \in \mathrm{Mat}(k\times k)$, $M^\sharp \in \mathrm{Mat}(m\times k)$, $M_\sharp \in \mathrm{Mat}(k\times m)$. Let*

$$M_\epsilon := \begin{pmatrix} M^\star + O(\epsilon) & \epsilon M^\sharp + O(\epsilon^2) \\ \\ \epsilon M_\sharp + O(\epsilon^2) & \epsilon M_\star + O(\epsilon^2) \end{pmatrix} .$$

Then:

- *if $\bar\lambda \neq 0$ is a simple eigenvalue of $M^\star$, then there exists $\bar\lambda_\epsilon = \bar\lambda + O(\epsilon)$ which is an eigenvalue of M_ϵ, provided $|\epsilon|$ is suitably small;*
- *if $\tilde\lambda$ is a simple eigenvalue of $M_\star$ and $\det(M^\star)\neq 0$, then there exists $\tilde\lambda_\epsilon = \tilde\lambda + O(\epsilon)$ so that $\epsilon\tilde\lambda_\epsilon$ is an eigenvalue of M_ϵ, provided $|\epsilon|$ is suitably small.*

Proof For the first claim, apply the Implicit Function Theorem to

$$\mathcal{F}_1(t,\epsilon) := \det(M_\epsilon - t\mathbf{1}_{(m+k)}) ,$$

noticing that $\mathcal{F}_1(t,0) = (-1)^k t^k \det(M^\star - t\mathbf{1}_m)$. For the second claim, apply the Implicit Function Theorem to

$$\mathcal{F}_2(t,\epsilon) := \det \begin{pmatrix} M^\star - \epsilon t \mathbf{1}_m & M^\sharp \\ \epsilon M_\sharp & M_\star - t\mathbf{1}_k \end{pmatrix} ,$$

noticing that $\epsilon^k \mathcal{F}_2(t,\epsilon) = \det(M_\epsilon - \epsilon t \mathbf{1}_{(m+k)})$ and that, by Lemma B.1, $\mathcal{F}_2(t,0) = \det(M^\star)\det(M_\star - t\mathbf{1}_k)$.
∎

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