HALF-LINE SCHRÖDINGER OPERATORS WITH NO BOUND STATES

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ABSTRACT. We consider Schödinger operators on the half-line, both discrete and continuous, and show that the absence of bound states implies the absence of embedded singular spectrum. More precisely, in the discrete case we prove that if $\Delta + V$ has no spectrum outside of the interval $[-2, 2]$, then it has purely absolutely continuous spectrum. In the continuum case we show that if both $-\Delta + V$ and $-\Delta - V$ have no spectrum outside $[0, \infty)$, then both operators are purely absolutely continuous. These results extend to operators with finitely many bound states.

1. Introduction

We study half-line Schrödinger operators, both continuous and discrete, with a Dirichlet boundary condition at the origin. That is,

$$[h_V \psi](n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n) \tag{1}$$

acting in $\ell^2(\mathbb{Z}^+)$, $\mathbb{Z}^+ = \{1, 2, \dots\}$, where $\psi(0) = 0$; and, in the continuum case,

$$[H_V \psi](x) = -\psi''(x) + V(x)\psi(x) \tag{2}$$

acting in $L^2([0, \infty))$ with the boundary condition $\psi(0) = 0$. For convenience, we require that the potential, V, be uniformly locally square integrable. We write $\ell^\infty(L^2)$ for the Banach space of such functions.

The free operators, that is, when $V \equiv 0$, can be diagonalized by the Fourier transform. This shows that they have spectra $[-2, 2]$ and $[0, \infty)$, respectively, and that in both cases, the spectrum is purely absolutely continuous.

For a general discrete operator, the mere fact that the spectrum is contained in $[-2, 2]$ forces it to be purely absolutely continuous. This is our first main result:

Theorem 1. *A discrete half-line Schrödinger operator h_V with spectrum contained in $[-2, 2]$ has purely absolutely continuous spectrum.*

In fact, the proof shows that $[-2, 2]$ is the essential support of the absolutely continuous spectrum. That is, for every $S \subseteq [-2, 2]$ of positive Lebesgue measure, the spectral projection associated to S is non-zero.

For any V that is positive, the continuum operator H_V has spectrum contained in $[0, \infty)$. Consequently, one may conclude little from this requirement about the spectrum or its type: the spectrum may have gaps, as periodic potentials demonstrate, the spectral type may be pure point, such as occurs in random models [11, 15],

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or even purely singular continuous, as certain sparse potentials show [14, 18]. By treating V and $-V$ symmetrically, we obtain the continuum analogue of Theorem 1.

Theorem 2. *Suppose $V \in \ell^\infty(L^2)$. If the spectra of both H_V and H_{-V} are contained in $[0,\infty)$, then both operators have purely absolutely continuous spectrum. Moreover, $\sigma(H_V) = \sigma(H_{-V}) = [0,\infty)$.*

It follows from our proof that the essential support of the absolutely continuous spectrum is equal to $[0,\infty)$.

The reason that sign-definite potentials do not offer counterexamples to Theorem 1 is that, in the discrete case, the spectrum of the free operator has two sides. Positive potentials can produce spectrum above $+2$, and similarly, negative potentials can produce spectrum below -2. In fact, the operators h_{-V} and $-h_V$ are unitarily equivalent. The intertwining unitary operator is given by

$$[U\psi](n) = (-1)^n \psi(n). \tag{3}$$

Therefore, $\sigma(h_V) \subseteq [-2,2]$ is equivalent to $\sigma(h_V) \subseteq [-2,\infty)$ and $\sigma(h_{-V}) \subseteq [-2,\infty)$. In this way, we see that Theorem 2 is the natural analogue of Theorem 1.

It has been shown, in [13], that the free operator is the only discrete whole-line Schrödinger operator with spectrum contained in $[-2,2]$. A more transparent proof of this fact was given in [4]. This second proof is based on the construction, for $V \not\equiv 0$, of certain trial functions ψ such that

$$\langle \psi, (h_V - 2)\psi\rangle + \langle U\psi, (-h_V - 2)U\psi\rangle > 0$$

with U defined as in (3). (This inequality clearly implies that h_V must have spectrum outside $[-2,2]$.)

Similarly, on the whole space in two dimensions, only the free operator has spectrum contained in $[-4,4]$. The corresponding statement fails in three or more dimensions. (For proofs, see [4].) The validity of this result in one or two dimensions and its failure in three or more dimensions is intimately connected to certain well-known facts about Schrödinger operators in $\mathbb{R}^d$; see [3, 17, 20, 23]. For example, if $V \not\equiv 0$ is a non-positive, smooth, compactly supported potential on $\mathbb{R}^d$, then for $d = 1, 2$, $-\Delta + \lambda V$ has bound states (isolated eigenvalues) for any $\lambda > 0$, while for $d \geq 3$, $-\Delta + \lambda V$ has no bound states for small λ.

For operators on the half-line, however, there are non-zero potentials for which $\sigma(h_V) \subseteq [-2,2]$. The family of potentials $V(n) = \lambda(-1)^n/n$ was studied in [4]. It was shown that for $|\lambda| \leq 1$, h_V has spectrum $[-2,2]$, while for $|\lambda| > 1$, it has infinitely many eigenvalues outside $[-2,2]$.

On the other hand, absence of bound states is known to place fairly stringent restrictions on the potential. For example, it was shown in [13, Corollary 9.3] that the potential must be square summable. Moreover, by [6] (or [13]) this implies that the (essential support of the) absolutely continuous spectrum of the operator fills $[-2,2]$. In particular, it permits one to conclude that if $\sigma(h_V) \subseteq [-2,2]$, then actually $\sigma(h_V) = [-2,2]$. By the example given above, absence of bound states does not imply $V \in \ell^1$; however, Theorem 6 in Section 4 shows that V must be weak-ℓ^1 and so ℓ^p for every $p > 1$.

Further restrictions were derived in [4]. For example, by Theorem 5.2 of that paper, any potential V that does not produce bound states must satisfy the pointwise bound $|V(n)| \leq 2n^{-1/2}$. It was also shown that there exists a sequence of potentials V_m such that h_{V_m} has spectrum $[-2,2]$ for every m and $m^{1/2}|V_m(m)| \to 1$. Our

Proposition 4.3 shows that $|V(n)| \leq \sqrt{2/n}$ and that for each n, there is a potential that realizes this bound.

None of the estimates for V given above permits us to conclude that the spectrum on $[-2,2]$ is purely absolutely continuous (as is the case if $V \in \ell^1$, for example). Indeed, following Theorem 8, we exhibit, for any $\lambda > 1$, a potential of the form $V(n) = \lambda(-1)^n n^{-1} + O(n^{-2})$ for which zero is an eigenvalue. This example is essentially a discrete analogue of the classic Wigner–von Neumann construction [29]. As the potential $V(n) = (-1)^n n^{-1}$ has no bound states, we see that the thresholds for the appearance of eigenvalues inside and outside $[-2,2]$ are the same. For this reason, it is imperative that we obtain tight estimates at each step.

A second important realization is that the correct quantity to estimate is not the potential, V, but rather its "conditional integral", $\sum_{m=n}^{\infty} V(m)$. For example, for every $\varepsilon > 0$, there is a potential with $|V(n)| \leq \varepsilon/n$ and an embedded eigenvalue [8, 19]. However, Theorem 8 below shows that this is not the case if the conditional integral of V obeys such an estimate.

In the continuum case, absence of bound states does not imply that the potential goes to zero. Indeed, given any increasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$, there is a potential V such that $V(x_k) \geq h(x_k)$ for some sequence $x_k \to \infty$, and yet both H_V and H_{-V} have no bound states. This follows from Theorems 2.2 and A.1 in [5]. For example, if $h(x) = e^x$, one may choose

$$V(x) = \frac{d}{dx}\,\frac{\sin(e^{2x})}{4x}.$$

As a compromise between generality and simplicity, we have chosen to restrict our attention to potentials that are uniformly locally square integrable.

The methods we employ to prove Theorems 1 and 2 will allow us to prove the following stronger results:

Theorem 3. *If a discrete half-line Schrödinger operator has only finitely many eigenvalues outside* $[-2,2]$*, then it has purely absolutely continuous spectrum on* $[-2,2]$.

Theorem 4. *Suppose* $V \in \ell^\infty(L^2)$*. If both* H_V *and* H_{-V} *have only finitely many eigenvalues below energy zero, then both operators have purely absolutely continuous spectrum on the interval* $[0,\infty)$.

(Once again, the essential support of the absolutely continuous spectrum fills out the interval indicated.)

A Jacobi matrix is an operator of the form

$$[J\psi](n) = a_n\psi(n+1) + a_{n-1}\psi(n-1) + b_n\psi(n)$$

acting in $\ell^2(\mathbb{Z}^+)$. The first step in our analysis is to use the connection between such operators with spectrum contained in $[-2,2]$ and the theory of polynomials orthogonal on the unit circle, which seems to have first been made by Szegő (c.f. [27]). This is discussed in Section 2. In particular, it is proved that a Jacobi matrix has spectrum contained in $[-2,2]$ if and only if its parameters, a_n and b_n, can be represented in terms of a sequence of numbers $\gamma_n \in (-1,1)$ as described by equations (5) and (6). The coefficients γ_n occur in the continued fraction expansion of a certain function associated to the Jacobi matrix and, in the orthogonal polynomial context, are known as the Verblunsky coefficients.

Sturm oscillation theory gives an alternate criterion for a Jacobi matrix, J, to have $\sigma(J) \subseteq [-2,2]$ in terms of the behaviour of the generalized eigenfunctions at energies ± 2. We discuss this in Section 3 and, in particular, we determine the relation between these eigenfunctions and the Verblunsky coefficients. This is used to motivate the definition of the continuum analogue of the Verblunsky coefficients in Section 6 and also to prove that ± 2 are not eigenvalues. As the Verblunsky coefficients with even and odd indices play distinct roles in the discrete case, our continuum analogue consists of two functions: Γ_{e} and Γ_{o}.

A related but different continuum analogue of the Verblunsky coefficients was introduced by Kreĭn in his studies of the continuum analogue of polynomials orthogonal on the unit circle [16]. Specifically, his function A is given by our $\Gamma_{\mathrm{e}} - \Gamma_{\mathrm{o}}$. He did not consider the individual functions, nor any other combination of them.

Sections 4 and 6 are devoted to deriving estimates for the Verblunsky coefficients; they treat the discrete and continuum cases, respectively. It is also proved that there can be no eigenvalues at the edges of the spectrum.

As noted earlier, it is the conditional integral of the potential which proves to be the right object to study in order to prove the theorems presented above. In Sections 5 and 7, we prove that certain estimates on this conditional integral imply that the spectrum on $(-2,2)$ (resp., $(0,\infty)$) is purely absolutely continuous. This is the content of Theorems 8 and 10. As the conditional integral of the potential is given, to a good approximation, by the even Verblunsky coefficients, the estimates derived in Sections 4 and 6 provide the necessary input to these theorems.

In order to prove Theorems 8 and 10, we study solutions of the corresponding eigenfunction equations using Prüfer variables. On the one hand, we show that they may only grow or decay at a very restricted rate, and on the other, that they actually remain bounded except on a set of energies of zero Hausdorff dimension. By the Jitomirskaya-Last version [12] of subordinacy theory [10], the slow growth/decay of the solutions implies that the spectral measure assigns zero weight to sets of zero Hausdorff dimension. Moreover, the set of energies where all solutions are bounded supports no singular spectrum and, as just noted, the complement of this set has zero Hausdorff dimension. These two statements preclude the existence of embedded singular spectrum. A similar two-step procedure was used by Remling [19] to show that potentials which are $o(1/n)$ do not have embedded singular spectrum.

As outlined above, Theorem 1 follows from Theorem 6, which provides estimates for the conditional integral of the potential (see Section 4); Theorem 7, which precludes eigenvalues at ± 2 (see Section 4); and Theorem 8, which is a general criterion for the absence of singular spectrum embedded in $(-2,2)$ (see Section 5). Theorem 2 follows in a similar fashion from Theorem 9 of Section 6 and Theorem 10 of Section 7.

To obtain Theorems 3 and 4, which permit finitely many bound states, we use a truncation argument to show that the potential must obey estimates similar to those derived in the no-bound-state case; see Corollaries 4.6 and 6.5. These corollaries provide the input to Theorems 8 and 10 and also show that there are no eigenvalues at the edges of the spectrum.

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2. Verblunsky Coefficients for Jacobi Matrices

In this section we work in the more general setting of Jacobi matrices. Namely, we consider operators J acting in $\ell^2(\mathbb{Z}^+)$ by

$$[J\psi](n) = a_n\psi(n+1) + a_{n-1}\psi(n-1) + b_n\psi(n) \tag{4}$$

where $\psi(0)$ is to be regarded as zero. (Recall that $\mathbb{Z}^+ = \{1, 2, \dots\}$.) The coefficients a_n are positive and the b_n real. Both sequences are assumed to be bounded and so J defines a bounded self-adjoint operator.

The question we wish to address is the following: for which sequences of coefficients is the spectrum of J contained in $[-2, 2]$? (Note that in the case $a_n \equiv 1$, this is exactly the question of which discrete Schrödinger operators have no bound states.) While the criterion we prove in this section is by no means easy to check, it is the basis for almost all the analysis that follows.

Theorem 5. *A Jacobi matrix with coefficients a_n, b_n has spectrum $\sigma(J) \subseteq [-2, 2]$ if and only if there is a sequence $\gamma_n \in (-1, 1)$, $n \in \{0, 1, \dots\}$, that obeys*

$$b_{n+1} = (1 - \gamma_{2n-1})\gamma_{2n} - (1 + \gamma_{2n-1})\gamma_{2n-2} \tag{5}$$

$$a_{n+1}^2 = (1 - \gamma_{2n-1})(1 - \gamma_{2n}^2)(1 + \gamma_{2n+1}). \tag{6}$$

(Here $\gamma_{-1} = -1$ and the value of γ_{-2} is irrelevant since it is multiplied by zero.)

Most of this section is devoted to an exposition of the background material and the introduction of notation; the "Proof of Theorem 5" appears at the very end. Little that is said in this section is new save perhaps the style of presentation/derivation. In particular, Theorem 5 appears in Geronimus [9, §31], although not exactly in the form stated above.

The Geronimus proof of Theorem 5 employs the relation between orthogonal polynomials on the circle and on the interval $[-2, 2]$ derived by Szegő [27, Theorem 11.5]. Our proof is more closely related to continued fractions. The Schur algorithm provides a transformation on measures; it is therefore natural to ask what transformation it induces on Jacobi matrices. This is the content of Proposition 2.2, which seems to be new.

The proof of Theorem 5 presented below is short and self-contained; we feel our discussion would be incomplete without it.

It is not difficult to show that the vector $\delta_1 \in \ell^2(\mathbb{Z}^+)$ with entries $\delta_1(n) = \delta_{1,n}$ ($\delta_{n,m}$ denotes the Kronecker delta function) is cyclic for J. That is, $\{J^n\delta_1 : n = 0, 1, \dots\}$ spans the Hilbert space. Consequently, the spectrum of J is equal to the support of the spectral measure associated to δ_1, which we will denote by $d\mu$. In fact, cyclicity implies that J is unitarily equivalent to $g(x) \mapsto xg(x)$ in $L^2(d\mu)$. As $\ell^2(\mathbb{Z}^+)$ is infinite-dimensional, so must be $L^2(d\mu)$, which is equivalent to saying that $d\mu$ cannot be supported on a finite set. In fact, there is a one-to-one correspondence between compactly supported probability measures $d\mu$ on $\mathbb{R}$ that are not supported by a finite set and the set of Jacobi matrices with uniformly bounded $a_n > 0$, $b_n \in \mathbb{R}$. Given μ, the sequences a_n, b_n are exactly the coefficients of the recurrence relation obeyed by the polynomials orthonormal with respect to $d\mu$. (This is obvious once one realizes that the unitary mapping $\ell^2(\mathbb{Z}^+) \to L^2(d\mu)$ described above maps δ_n to the orthonormal polynomial of degree $n - 1$.)

We also wish to discuss the m-function associated to J, that is, the (1,1) entry of the Green function: $m_0(z) = \langle\delta_1|(J-z)^{-1}\delta_1\rangle$. Naturally, this can also be expressed in terms of the measure $d\mu$:

$$m_0(z) = \int \frac{1}{t-z}\, d\mu(t).$$

(The zero subscript is for consistency with what follows.)

If it happens that $\operatorname{supp}(d\mu) \subseteq [-2,2]$ (equivalently, $\sigma(J) \subseteq [-2,2]$), then it is possible to define a measure $d\rho$ on $\mathbb{S}^1 = \{\zeta : |\zeta| = 1\}$ which is symmetric with respect to complex conjugation and obeys

$$\int g(t)\, d\mu(t) = \int g(\zeta + \zeta^{-1}) d\rho(\zeta)$$

for any measurable function g. These conditions uniquely determine $d\rho$. In particular, note that $\rho(\mathbb{S}) = \mu([-2,2]) = 1$.

Associated to each measure on the circle is a Carathéodory function

$$F_0(\xi) = \int \frac{\zeta+\xi}{\zeta-\xi}\, d\rho(\zeta) = (\xi - \xi^{-1})\, m_0(\xi + \xi^{-1}),$$

defined and analytic for ξ in the unit disk. Notice that $F(0) = \rho(\mathbb{S}^1) = 1$ and that, because ρ is symmetric, $F_0 : (-1,1) \to \mathbb{R}$.

To each such Caratheodory function F_0 is associated a Schur function, an analytic mapping from the unit disk into itself, by

$$F_0(\xi) = \frac{1+\xi f_0(\xi)}{1-\xi f_0(\xi)} \quad \text{i.e.,} \quad f_0(\xi) = \frac{1}{\xi}\,\frac{F_0(\xi)-1}{F_0(\xi)+1}.$$

The analyticity of f_0 follows from the fact that $F_0(0) = 1$. As $F_0 : (-1,1) \to \mathbb{R}$, the same is true of f_0. Note also that f_0 cannot be a finite Blaschke product; if it were, then $d\rho$, and hence $d\mu$, would be supported on a finite set, namely $\{\zeta : f_0(\zeta) = 1\}$.

Recall that the Schur algorithm [21] gives a one-to-one correspondence between the set of Schur functions that are not finite Blaschke products and the set of complex sequences $\gamma : \{0,1,2,\dots\} \to \{z : |z| < 1\}$. It proceeds as follows:

$$f_{n+1}(\xi) = \frac{1}{\xi}\,\frac{f_n(\xi) - \gamma_n}{1 - \overline{\gamma}_n f_n(\xi)} \qquad \gamma_n = f_n(0).$$

The coefficients γ_n have many names; following [25], we term them the Verblunsky coefficients. (Other common names are the Schur, Szegő, Geronimus, or reflection coefficients.)

As the measure, $d\rho$, we consider is symmetric with respect to complex conjugation, so $f_0 : (-1,1) \to (-1,1)$. It is easy to verify inductively that this remains true for all f_n and consequently, that $\gamma_n \in (-1,1)$ for each $n \in \{0,1,2,\dots\}$.

Lemma 2.1. *The first two Verblunsky coefficients are*

$$\gamma_0 = \tfrac{1}{2} b_1 \quad \text{and} \quad \gamma_1 = -\frac{4 - b_1^2 - 2a_1^2}{4 - b_1^2}.$$

Equivalently, $b_1 = 2\gamma_0$ *and* $a_1^2 = 2(1-\gamma_0^2)(1+\gamma_1)$.

Proof. First,

$$\gamma_0 = f_0(0) = \tfrac{1}{2} F_0'(0) = \int \zeta^{-1}\, d\rho(\zeta) = \tfrac{1}{2}\int (\zeta + \zeta^{-1})\, d\rho(\zeta)$$

and so,

$$\gamma_0 = \tfrac{1}{2}\int t\, d\mu(t) = \tfrac{1}{2} b_1.$$

For the second coefficient,

$$\gamma_1 = f_1(0) = \frac{f_0'(0)}{1 - f_0(0)^2} = \frac{F_0''(0) - \big[F_0'(0)\big]^2}{4 - F_0'(0)}$$

and

$$F''(0) = 4\int \zeta^{-2}\, d\rho(\zeta) = \int 2t^2 - 4\, d\mu(t) = 2\big(b_1^2 + a_1^2\big) - 4$$

which implies that

$$\gamma_1 = -\frac{4 - b_1^2 - 2a_1^2}{4 - b_1^2}$$

as claimed. □

The process by which $d\rho$ determines the Schur function f_0 may be inverted and so each of the iterates f_n determines a measure on $\mathbb{S}^1$. The Carathéodory and m-functions of this new measure will be denoted by F_n and m_n, respectively; the Jacobi matrix, by J_n. It turns out that there is a simple relation between J_2, the Jacobi matrix resulting from two iterations of the Schur algorithm, and the original matrix J. Deriving this requires some computation. We begin by noting that

$$\begin{aligned} F_1(\xi) &= \frac{1 + \xi f_1(\xi)}{1 - \xi f_1(\xi)} = \frac{1-\gamma_0}{1+\gamma_0}\,\frac{1 + f_0(\xi)}{1 - f_0(\xi)} \\ &= \frac{1-\gamma_0}{1+\gamma_0}\,\frac{(\xi+1)F_0(\xi) + (\xi - 1)}{(\xi-1)F_0(\xi) + (\xi+1)} \end{aligned}$$

and that by iterating this,

$$F_2(\xi) = \frac{1-\gamma_1}{1+\gamma_1}\,\frac{(\xi^2 + 1 - 2\gamma_0\xi)F_0(\xi) + (\xi^2 - 1)}{(\xi^2 - 1)F_0(\xi) + (\xi^2 + 1 + 2\gamma_0\xi)}.$$

In this way, we obtain a relation between m_2 and m_0:

$$m_2(z) = \frac{1-\gamma_1}{1+\gamma_1}\,\frac{(z - 2\gamma_0)m_0(z) + 1}{(z^2-4)m_0(z) + (z + 2\gamma_0)} \tag{7}$$

$$= \frac{4 - b_1^2 - a_1^2}{a_1^2}\,\frac{(z - b_1)m_0(z) + 1}{(z^2-4)m_0(z) + (z + b_1)}, \tag{8}$$

where we used the expressions for γ_0 and γ_1 given in the lemma above.

Proposition 2.2. *If $\sigma(J) \subseteq [-2, 2]$, then the Jacobi matrix resulting from two iterations of the Schur algorithm is*

$$J_2 = \begin{bmatrix} b & a & 0 & 0 \\ a & b_3 & a_3 & 0 \\ 0 & a_3 & b_4 & \ddots \\ 0 & 0 & \ddots & \ddots \end{bmatrix} \tag{9}$$

where a and b are determined by

$$\begin{aligned} \kappa^2 &= \tfrac{4 - b_1^2}{4 - b_1^2 - a_1^2} = \tfrac{2}{1-\gamma_1} \\ a &= \kappa a_2 \\ b &= \kappa^2 b_2 + (\kappa^2 - 1)b_1 = \tfrac{2}{1-\gamma_1} b_2 + 2\tfrac{1+\gamma_1}{1-\gamma_1}\gamma_0. \end{aligned}$$

Throughout, a_n, b_n are the coefficients of the original Jacobi matrix.

Proof. Let $J^{(j)}$ denote the matrix resulting from J by the deletion of the first j rows and columns and let $m^{(j)}$ denote its m-function. For example,

$$J^{(2)} = \begin{bmatrix} b_3 & a_3 & 0 & 0 \\ a_3 & b_4 & a_4 & 0 \\ 0 & a_4 & b_5 & \ddots \\ 0 & 0 & \ddots & \ddots \end{bmatrix}.$$

If $\tilde{m}$ denotes the m-function for the matrix J_2 of (9) then, by Cramer's rule,

$$\tilde{m}(z) = \frac{1}{-z + b - a^2 m^{(2)}(z)}. \tag{10}$$

Similarly,

$$m^{(1)}(z) = \frac{1}{-z + b_2 - a_2^2 m^{(2)}(z)}$$

and so $a_2^2 m^{(2)}(z) = (b_2 - z) - [m^{(1)}(z)]^{-1}$. Substituting this into (10) gives

$$\tilde{m}(z) = \frac{m^{(1)}(z)}{(1-\kappa^2)(-z - b_1)m^{(1)}(z) + \kappa^2}.$$

We now use the fact that $a_1^2 m^{(1)}(z) = (b_1 - z) - [m_0(z)]^{-1}$ to obtain

$$\begin{aligned} \tilde{m}(z) &= \frac{(b_1 - z)m_0 - 1}{(1-\kappa^2)(-z - b_1)[(b_1 - z)m_0 - 1] + \kappa^2 a_1^2 m_0} \\ &= \frac{1}{\kappa^2 - 1} \, \frac{(z - b_1)m_0 + 1}{(z^2 - 4)m_0 + (z + b_1)}, \end{aligned}$$

where we also used $a_1^2\kappa^2 = (4 - b_1^2)(\kappa^2 - 1)$. From the definition of κ, this is exactly the same as the expression for m_2 in terms of m_0 given in (8). Therefore $\tilde{m} = m_2$ and m_2 is the m-function for J_2. Because a Jacobi matrix is uniquely determined by its m-function, this proves (9). ☐

Corollary 2.3. *If the Jacobi matrix J has $\sigma(J) \subseteq [-2, 2]$, then the corresponding Verblunsky coefficients $(\gamma_0, \gamma_1, \dots)$ are related to the Jacobi matrix coefficients, $(b_1, a_1, b_2, a_2, \dots)$, by*

$$\gamma_{2n} = \tfrac{1}{1-\gamma_{2n-1}} b_{n+1} + \tfrac{1+\gamma_{2n-1}}{1-\gamma_{2n-1}} \gamma_{2n-2} \tag{11}$$

$$\gamma_{2n+1} = (1 - \gamma_{2n-1})^{-1}(1 - \gamma_{2n}^2)^{-1} a_{n+1}^2 - 1 \tag{12}$$

or, what is equivalent, by

$$b_{n+1} = (1 - \gamma_{2n-1})\gamma_{2n} - (1 + \gamma_{2n-1})\gamma_{2n-2} \tag{13}$$

$$a_{n+1}^2 = (1 - \gamma_{2n-1})(1 - \gamma_{2n}^2)(1 + \gamma_{2n+1}). \tag{14}$$

In all formulae, $\gamma_{-1} = -1$.

Proof. By iterating the proposition above, one finds that m_{2n}, the m-function resulting from $2n$ iterations of the Schur algorithm, is associated to the Jacobi matrix

$$J_{2n} = \begin{bmatrix} b & a & 0 & 0 \\ a & b_{n+2} & a_{n+2} & 0 \\ 0 & a_{n+2} & b_{n+3} & \ddots \\ 0 & 0 & \ddots & \ddots \end{bmatrix}$$

where a and b are given by

$$a^2 = \frac{2a_{n+1}^2}{1-\gamma_{2n-1}} \qquad b = \frac{2}{1-\gamma_{2n-1}}\Big\{b_{n+1} + (1+\gamma_{2n-1})\gamma_{2n-2}\Big\}.$$

Hence by Lemma 2.1,

$$\gamma_{2n} = \tfrac{1}{2}b \quad \text{and} \quad \gamma_{2n+1} = \frac{2a^2}{4-b^2} - 1$$

from which (11) and (12) follow by substituting the formulae for a and b just given. ☐

Proof of Theorem 5. If $\sigma(J) \subseteq [-2,2]$, then the corollary above shows that the Verblunsky coefficients solve the equations (13) and (14), which are exactly the same as those stated in the theorem. This proves one direction.

Given J, suppose there is a sequence $(\gamma_0, \gamma_1, \dots)$ with values in $(-1,1)$ so that both (13) and (14) hold. Then there is a Schur function f that has these coefficients and it must obey $f(\bar\zeta) = \overline{f(\zeta)}$ because the coefficients are real. One may then define the corresponding F and so a probability measure $d\tilde\rho$ on $\mathbb{S}^1$ that is symmetric with respect to complex conjugation. This induces a probability measure $d\tilde\mu$ on $[-2,2]$ which gives rise to a Jacobi matrix, say $\tilde J$. But, the coefficients of $\tilde J$ are determined by the Verblunsky coefficients through (13)–(14) and so must equal the coefficients of J. This implies $d\mu = d\tilde\mu$ and so $d\mu$ is supported in $[-2,2]$, which shows that $\sigma(J) \subseteq [-2,2]$. ☐

3. Verblunsky Coefficients and Eigenfunctions

Let u and w denote the generalized eigenfunctions at energies 2 and -2, respectively, with the standard normalization. That is,

$$\begin{gathered} a_n u(n+1) + a_{n-1}u(n-1) + b_n u(n) = +2u(n) \\ u(0) = 0, \quad u(1) = 1 \end{gathered} \tag{15}$$

and

$$\begin{gathered} a_n w(n+1) + a_{n-1}w(n-1) + b_n w(n) = -2w(n) \\ w(0) = 0, \quad w(1) = 1. \end{gathered} \tag{16}$$

We also write $v(n)$ for $(-1)^{n-1}w(n)$, which obeys

$$\begin{gathered} a_n v(n+1) + a_{n-1}v(n-1) - b_n v(n) = +2v(n) \\ v(0) = 0, \quad v(1) = 1. \end{gathered} \tag{17}$$

Sturm oscillation theory for Jacobi matrices (see [28]) shows that the Jacobi matrix, J, with coefficients a_n and b_n (cf. (4)) has $\sigma(J) \subseteq [-2,2]$ if and only if $u(n)$ and $v(n)$ are positive for all $n \in \mathbb{Z}^+$. Hence we have an alternative to the criterion discussed in the previous section. However, we will not be using this alternate characterization.

This section is devoted to discussing the relation between the eigenfunctions u, w and the Verblunsky coefficients, and so provides a bridge between the two criteria. This serves two useful purposes: (1) it simplifies the demonstration that ± 2 cannot be eigenvalues of a discrete Schrödinger operator unless it has spectrum outside $[-2,2]$; and (2) it motivates the definition of the quantities that we regard as the continuum analogue of the Verblunsky coefficients.

Not surprisingly, the values of a_n and b_n for $1 \le n \le N$ can be recovered from the values of $u(n)$ and $w(n)$ for $2 \le n \le N+1$. The next lemma gives the precise formulae.

Lemma 3.1. *If $W(n) = u(n+1)w(n) - u(n)w(n+1)$, the Wronskian of u and w, and $\tilde{W}(n) = u(n+1)w(n) + u(n)w(n+1)$, then*

$$a_n = \frac{4}{W(n)} \sum_{k=1}^{n} u(k)w(k) \tag{18}$$

$$b_n = \frac{-2}{u(n)w(n)} \left\{ \frac{\tilde{W}(n)}{W(n)} \sum_{k=1}^{n} u(k)w(k) + \frac{\tilde{W}(n-1)}{W(n-1)} \sum_{k=1}^{n-1} u(k)w(k) \right\}. \tag{19}$$

Proof. Consider multiplying (15) by $w(n)$ and multiplying (16) by $u(n)$. Taking the difference gives

$$a_n W(n) - a_{n-1} W(n-1) = 4u(n)w(n) \tag{20}$$

while taking the sum gives

$$a_n \tilde{W}(n) + a_{n-1} \tilde{W}(n-1) + 2b_n u(n)w(n) = 0. \tag{21}$$

By summation, (20) implies

$$a_n W(n) = 4 \sum_{k=1}^{n} u(k)w(k)$$

which is equivalent to (18). Having found the formula for a_n we can now solve (21) for b_n. This gives (19). ◻

Similarly, one can write the Verblunsky coefficients in terms of u and w. The formulae actually look simpler than those for a_n and b_n:

Lemma 3.2. *With W and $\tilde{W}$ as in the previous lemma, we have*

$$\gamma_{2n} = -\frac{\tilde{W}(n+1)}{W(n+1)} \tag{22}$$

$$\gamma_{2n+1} = -1 - \frac{2}{u(n+2)w(n+2)} \sum_{k=1}^{n+1} u(k)w(k). \tag{23}$$

Proof. We proceed by induction on n. For $n = 0$, we have

$$\gamma_0 = \tfrac{1}{2} b_1 = -\frac{\tilde{W}(1)}{W(1)}$$

by Corollary 2.3 and (19), respectively. This implies that for $n = 0$,

$$\begin{aligned} 1 - \gamma_{2n}^2 &= \frac{W(n+1)^2 - \tilde{W}^2(n+1)}{W(n+1)^2} \\ &= -4 \frac{u(n+1)u(n+2)w(n+1)w(n+2)}{W(n+1)^2}. \end{aligned} \tag{24}$$

Using this together with Corollary 2.3 and (19) again,

$$1 + \gamma_1 = \tfrac{1}{2} a_1^2 (1 - \gamma_0^2)^{-1} = -\frac{2u(1)w(1)}{u(2)w(2)}.$$

Now for the inductive step. We assume that (22), (23), and (24) hold and will show that they remain true with n replaced by $n+1$.

Using Corollary 2.3 and then (22) and (23),

$$\begin{aligned}\gamma_{2n+2} &= \frac{1}{1-\gamma_{2n+1}}\Big\{b_{n+2} + (1+\gamma_{2n+1})\gamma_{2n}\Big\}\\ &= \frac{1}{1-\gamma_{2n+1}}\Big\{b_{n+2} + \frac{2\tilde{W}(n+1)}{u(n+2)w(n+2)W(n+1)}\sum_{k=1}^{n+1} u(k)w(k)\Big\}.\end{aligned}$$

Continuing using (19) and (23) gives

$$\begin{aligned}\gamma_{2n+2} &= \frac{1}{1-\gamma_{2n+1}}\ \frac{-2}{u(n+2)w(n+2)}\ \frac{\tilde{W}(n+2)}{W(n+2)}\sum_{k=1}^{n+2} u(k)w(k)\\ &= -\frac{\tilde{W}(n+2)}{W(n+2)}.\end{aligned}$$

Equation (24) with $n+1$ in lieu of n follows easily from this.

After first employing (18), (23) together with what we have just proved shows

$$\begin{aligned}1+\gamma_{2n+3} &= (1-\gamma_{2n+1})^{-1}(1-\gamma_{2n+2}^2)^{-1}a_{n+2}^2\\ &= \frac{-2}{u(n+3)w(n+3)}\sum_{k=1}^{n+2} u(k)w(k)\end{aligned}$$

just as is required to complete the proof. ◻

As we now demonstrate, the Verblunsky coefficients are intimately related to the logarithmic derivatives of u and v. In the case where $a_n \equiv 1$, to which we will turn our attention shortly, the odd and even coefficients are, to a good approximation, half their sum and half their difference, respectively.

Lemma 3.3. *Let $v(n) = (-1)^{n-1}w(n)$, as above, and write*

$$F(n) = 1 - \frac{u(n+1)}{u(n+2)}, \quad G(n) = 1 - \frac{v(n+1)}{v(n+2)} \tag{25}$$

for the logarithmic derivatives of u and v. We have

$$\gamma_{2n} = -\frac{F(n)-G(n)}{2-F(n)-G(n)}, \quad \gamma_{2n+1} = -a_{n+1}\frac{F(n)+G(n)}{2} + a_{n+1} - 1 \tag{26}$$

and

$$\begin{aligned}F(n) &= a_{n+1}^{-1}\big\{a_{n+1} - 1 - \gamma_{2n+1} - \gamma_{2n} - \gamma_{2n+1}\gamma_{2n}\big\},\\ G(n) &= a_{n+1}^{-1}\big\{a_{n+1} - 1 - \gamma_{2n+1} + \gamma_{2n} + \gamma_{2n+1}\gamma_{2n}\big\}.\end{aligned} \tag{27}$$

Proof. From (22) we have

$$\begin{aligned}\gamma_{2n} &= -\frac{u(n+2)w(n+1) + w(n+2)u(n+1)}{u(n+2)w(n+1) - w(n+2)u(n+1)}\\ &= -\frac{\Big(1-\frac{u(n+1)}{u(n+2)}\Big) - \Big(1-\frac{v(n+1)}{v(n+2)}\Big)}{2-\Big(1-\frac{u(n+1)}{u(n+2)}\Big) - \Big(1-\frac{v(n+1)}{v(n+2)}\Big)}.\end{aligned}$$

Combining (18) and (23) gives

$$\begin{aligned}\gamma_{2n+1} &= -1 - \frac{a_{n+1}}{2}\left\{\frac{w(n+1)}{w(n+2)} - \frac{u(n+1)}{u(n+2)}\right\}\\ &= -\frac{a_{n+1}}{2}\left\{\left(1-\frac{u(n+1)}{u(n+2)}\right) + \left(1-\frac{v(n+1)}{v(n+2)}\right)\right\} + a_{n+1} - 1.\end{aligned}$$

This proves (26). The identities in (27) are immediate consequences of (26). ☐

4. Estimates for the Verblunksy Coefficients and the Potential

Beginning with this section, we restrict ourselves to discrete Schrödinger operators $h = \Delta + V$. That is, Jacobi matrices with $a_n \equiv 1$ and where we rename $b_n = V(n)$.

From Theorem 5 we know that $\sigma(h) \subseteq [-2, 2]$ if and only if there exists a sequence γ_n with values in $(-1, 1)$ that solves

$$V(n+1) = (1 - \gamma_{2n-1})\gamma_{2n} - (1 + \gamma_{2n-1})\gamma_{2n-2} \tag{28}$$

$$1 = (1 - \gamma_{2n-1})(1 - \gamma_{2n}^2)(1 + \gamma_{2n+1}) \tag{29}$$

where $\gamma_{-1} = -1$ by definition. The proof shows that, when it exists, the solution of this system is given by the Verblunsky coefficients.

(We should also remind the reader that, as mentioned in the introduction, for discrete Schrödinger operators, $\sigma(h) \subseteq [-2, 2]$ is equivalent to $\sigma(h) = [-2, 2]$.)

The purpose of this section is to study the relations (28) and (29). We will show that γ_n must converge to zero fairly rapidly and consequently so must $V(n)$. We will also show, by means of examples, that the decay estimates derived in this section are optimal.

At the conclusion of the section, we show how estimates for the potential derived under the assumption that there are no bound states can be extended to the case where there are finitely many.

Lemma 4.1. *If $h = \Delta + V$ is a discrete Schrödinger operator with spectrum $[-2, 2]$, then the associated Verblunsky coefficients obey $\gamma_{2n-1} \le \gamma_{2n+1} \le 0$ for every $n \ge 1$. That is, the odd Verblunsky coefficients are increasing and non-positive.*

Proof. Assume that, for some $n \ge 1$, we have $\gamma_{2n-1} > 0$. Then, by (29),

$$1 + \gamma_{2n+1} = (1 - \gamma_{2n}^2)^{-1}(1 - \gamma_{2n-1})^{-1} \ge (1 - \gamma_{2n-1})^{-1} \ge 1 + \gamma_{2n-1}(1 + \gamma_{2n-1}).$$

Iterating this, we obtain

$$1 + \gamma_{2(n+m)+1} \ge 1 + \gamma_{2n-1}(1 + \gamma_{2n-1})^{m+1} \to \infty \text{ as } m \to \infty,$$

which is a contradiction. Therefore, $\gamma_{2n-1} \le 0$ and

$$1 + \gamma_{2n+1} \ge (1 - \gamma_{2n-1})^{-1} = 1 + \frac{\gamma_{2n-1}}{1 - \gamma_{2n-1}},$$

which yields $\gamma_{2n+1} \ge \gamma_{2n-1}$. ☐

Lemma 4.2. *If $h = \Delta + V$ is a discrete Schrödinger operator with no bound states and γ_n are the associated Verblunsky coefficients, then we have*

$$\gamma_{2n+1} \ge -\frac{1}{n+2} + \sum_{j=0}^{n} c_j^{(n)} \gamma_{2j}^2 \tag{30}$$

for every $n \geq 0$, where

$$c_j^{(n)} = \frac{(j+1)(j+2)}{(n+2)^2}.$$

In particular, $\gamma_{2n+1} \geq \frac{-1}{n+2}$.

Proof. We proceed by induction on n. The case $n = 0$ follows from

$$\gamma_1 = \tfrac{1}{2}(1-\gamma_0^2)^{-1} - 1 \geq \tfrac{1}{2}(1+\gamma_0^2) - 1 = -\tfrac{1}{2} + \tfrac{1}{2}\gamma_0^2.$$

For the induction step from $n-1$ to n, we note that

$$1 - \gamma_{2n-1} \leq 1 + \tfrac{1}{n+1} - \sum_{j=0}^{n-1} c_j^{(n-1)} \gamma_{2j}^2 = \tfrac{n+2}{n+1} - \sum_{j=0}^{n-1} c_j^{(n-1)} \gamma_{2j}^2,$$

and hence

$$\begin{aligned}
(1-\gamma_{2n-1})^{-1} &\geq \left(\tfrac{n+2}{n+1} - \sum_{j=0}^{n-1} c_j^{(n-1)} \gamma_{2j}^2 \right)^{-1} \\
&= \tfrac{n+1}{n+2} \left(1 - \sum_{j=0}^{n-1} \tfrac{n+1}{n+2}\, c_j^{(n-1)} \gamma_{2j}^2 \right)^{-1} \\
&\geq \tfrac{n+1}{n+2} \left(1 + \sum_{j=0}^{n-1} \tfrac{n+1}{n+2}\, c_j^{(n-1)} \gamma_{2j}^2 \right).
\end{aligned}$$

This yields

$$\begin{aligned}
\gamma_{2n+1} &\geq (1+\gamma_{2n}^2) \left(\tfrac{n+1}{n+2} + \sum_{j=0}^{n-1} \tfrac{(n+1)^2}{(n+2)^2}\, c_j^{(n-1)} \gamma_{2j}^2 \right) - 1 \\
&\geq -\frac{1}{n+2} + \sum_{j=0}^{n} c_j^{(n)} \gamma_{2j}^2
\end{aligned}$$

since, for $0 \leq j \leq n-1$, $c_j^{(n)} = c_j^{(n-1)}(n+1)^2/(n+2)^2$ and $c_n^{(n)} = (n+1)/(n+2)$. $\square$

The first main result in this section is the determination of the optimal pointwise estimate for potentials with no bound states. Weaker results were obtained in [4] by different methods.

Proposition 4.3. *If $\sigma(h_V) = [-2,2]$, then the potential obeys*

$$\big|V(n)\big| \leq \sqrt{\tfrac{2}{n}}. \tag{31}$$

Moreover, for each n, there is a potential V such that $\sigma(h_V) \subseteq [-2,2]$ and $V(n) = \sqrt{2/n}$.

Proof. The proof amounts to finding the sequence of $\gamma_j \in (-1,1)$ that maximizes

$$V(n+1) = (1-\gamma_{2n-1})\gamma_{2n} - (1+\gamma_{2n-1})\gamma_{2n-2} \tag{32}$$

subject to the constraint

$$1 = (1-\gamma_{2j-1})(1-\gamma_{2j}^2)(1+\gamma_{2j+1}) \quad \text{for all } j. \tag{33}$$

(The choice of $V(n+1)$ rather than $V(n)$ is to shorten the subscripts in the equations that follow.) The existence of an optimizer follows from the compactness of the set of $(\gamma_0, \gamma_1, \dots, \gamma_{2n+1})$ which obey (33) for $0 \leq j \leq n$ and have $\gamma_{2n+1} \leq 0$. Note that the final condition guarantees the possibility of extending this sequence so that (33) holds for all j. For example, one may choose $\gamma_j = 0$ for $j \geq 2n+3$; the value of γ_{2n+2} being determined by (33).

As the sign of γ_{2n} and of γ_{2n-2} can be changed without affecting the validity of (33), an optimizing sequence must have $\gamma_{2n} \geq 0$ and $\gamma_{2n-2} \leq 0$.

With γ_{2n-2} and γ_{2n-1} prescribed, $V(n+1)$ is maximized by making γ_{2n} as large as possible. Choosing $\gamma_j = 0$ for $j \geq 2n+1$ and substituting this into (33) shows that

$$\gamma_{2n}^2 = 1 - \frac{1}{1+\gamma_{2n-1}} = \frac{-\gamma_{2n-1}}{1-\gamma_{2n-1}} \tag{34}$$

is possible. In fact, since $\gamma_{2n+1} \leq 0$ by Lemma 4.1, this is also maximal.

Similarly, the optimizing γ_{2n-2} obeys

$$\gamma_{2n-2}^2 = 1 - \frac{1}{(1+\frac{1}{n})(1-\gamma_{2n-1})} = \frac{\frac{1}{n+1} + \gamma_{2n-1}}{1+\gamma_{2n-1}}. \tag{35}$$

In this case, we wished to make $-\gamma_{2n-3}$ as large as possible. By Lemma 4.2, $\gamma_{2n-3} \geq -\frac{1}{n}$. This bound can be achieved by choosing $\gamma_{2j} = 0$ for $0 \leq j \leq n-2$.

Combining the two preceding paragraphs, we see that we must find the value of $\gamma_{2n-1} \in [-\frac{1}{n+1}, 0]$ that optimizes

$$V(n+1) = \sqrt{-\gamma_{2n-1}(1-\gamma_{2n-1})} + \sqrt{(\tfrac{1}{n+1} + \gamma_{2n-1})(1+\gamma_{2n-1})}.$$

(This formula follows from substituting (34) and (35) into (32).) The resulting calculus exercise has solution $\gamma_{2n-1} = -\frac{1}{2n+1}$ which gives $V(n+1) = \sqrt{2/(n+1)}$. $\square$

In [4] it was shown that $|V(n)| \leq 2n^{-1/2}$ and examples were given showing that the power $\frac{1}{2}$ is optimal. As was also noticed in [4], this pointwise estimate does not tell the full story. The optimizing potential has only three non-zero values:

$$V_n(n-1) = \sqrt{\tfrac{n}{2(n+1)^2}}, \quad V_n(n) = \sqrt{\tfrac{2}{n}}, \quad V_n(n+1) = \sqrt{\tfrac{1}{2n}}.$$

Theorem 6 below shows that potentials without bound states must decay much more quickly in an averaged sense. First we give two propositions describing the decay of the Verblunsky coefficients.

Proposition 4.4. *Suppose $h = \Delta + V$ is a discrete Schrödinger operator with spectrum $[-2,2]$ and γ_n are the associated Verblunsky coefficients. For every $n \geq 0$,*

$$\sum_{j=0}^{n} (j+1)(j+2)\gamma_{2j}^2 \leq n+2. \tag{36}$$

This implies that for each $\varepsilon > 0$, $\sum (j+1)^{1-\varepsilon}\gamma_{2j}^2 < \infty$. It also implies that

$$\#\{j : |\gamma_{2j}| \geq \lambda\} \leq \frac{9}{\lambda} \tag{37}$$

and so (γ_{2j}) is weak-ℓ^1.

Proof. The bound (36) follows from (30) because, by Lemma 4.1, $\gamma_{2n+1} \leq 0$.

For the first implication, let $c_n = (n+2)^{-1-\varepsilon}$, which is summable, then

$$\sum_{n=0}^{\infty} c_n \geq \sum_{n=0}^{\infty} c_n \sum_{j=0}^{n} \tfrac{(j+1)(j+2)}{n+2}\gamma_{2j}^2 = \sum_{j=0}^{\infty} (j+1)(j+2)\gamma_{2j}^2 \sum_{n=j}^{\infty} (n+2)^{-2-\varepsilon}.$$

This proves the result because

$$(j+2)\sum_{n=j}^{\infty}(n+2)^{-2-\varepsilon} \geq (j+2)\int_{j+2}^{\infty} x^{-2-\varepsilon}\,dx \geq \tfrac{1}{1+\varepsilon}(j+2)^{-\varepsilon}.$$

To prove (37), let $N_\lambda = \#\{j : |\gamma_{2j}| \geq \lambda\}$. Note that the Verblunsky coefficients lie in $(-1,1)$, so we need only consider $\lambda < 1$. Clearly,

$$\#\{1 \leq j+1 < \lambda^{-1} : |\gamma_{2j}| \geq \lambda\} \leq \lambda^{-1}. \tag{38}$$

Moreover, for every $k \geq 0$, we infer from (36) that

$$\#\{2^k\lambda^{-1} \leq j+1 < 2^{k+1}\lambda^{-1} : |\gamma_{2j}| \geq \lambda\} \cdot 2^{2k}\lambda^{-2} \cdot \lambda^2 \leq 2 \cdot 2^{k+1}\lambda^{-1},$$

that is,

$$\#\{2^k\lambda^{-1} \leq j+1 < 2^{k+1}\lambda^{-1} : |\gamma_{2j}| \geq \lambda\} \leq 2^{2-k}\lambda^{-1}. \tag{39}$$

Combining (38) and (39), we obtain

$$N_\lambda \leq \lambda^{-1} + \sum_{k=0}^{\infty} 2^{2-k}\lambda^{-1} = 9\,\lambda^{-1}$$

which is exactly (37). ☐

Proposition 4.5. *Suppose $h = \Delta + V$ is a discrete Schrödinger operator with spectrum $[-2,2]$ and γ_n are the associated Verblunsky coefficients. There is a constant C such that, for every $n \geq 1$,*

$$\sum_{j=0}^{n}(j+1)\gamma_{2j}^2 \leq \tfrac{1}{4}\log(n) + C \tag{40}$$

and consequently,

$$\sum_{j=0}^{n}|\gamma_{2j}| \leq \tfrac{1}{2}\log(n) + C. \tag{41}$$

Proof. From (29) we have the identity

$$\left(\frac{1+\gamma_{2j-1}}{1-\gamma_{2j-1}}\right)\left(\frac{1-\gamma_{2j+1}}{1+\gamma_{2j+1}}\right) = (1-\gamma_{2j-1}^2)(1-\gamma_{2j}^2)^2(1-\gamma_{2j+1}^2).$$

Applying the function $x \mapsto -\frac{1}{2}\log(x)$ to both sides and expanding in Taylor series yields

$$(\gamma_{2j+1}-\gamma_{2j-1})\left\{1 + \sum_{l=1}^{\infty}\tfrac{1}{2l+1}\sum_{k=0}^{2l}\gamma_{2j+1}^{k}\gamma_{2j-1}^{2l-k}\right\} = \sum_{l=1}^{\infty}\tfrac{1}{2l}\left\{\gamma_{2j-1}^{2l} + 2\gamma_{2j}^{2l} + \gamma_{2j+1}^{2l}\right\}.$$

We now estimate the left-hand side from above using the following consequence of Lemma 4.2,

$$1 + \sum_{l=1}^{\infty}\tfrac{1}{2l+1}\sum_{k=0}^{2l}\gamma_{2j+1}^{k}\gamma_{2j-1}^{2l-k} \geq 1 + \sum_{l=1}^{\infty}(j+1)^{-2l} = \tfrac{(j+1)^2}{j(j+2)}$$

and estimate the right-hand side from below by neglecting all but the first term in the sum. This gives

$$\tfrac{(j+1)^2}{j(j+2)}(\gamma_{2j+1}-\gamma_{2j-1}) \geq \tfrac{1}{2}\gamma_{2j-1}^2 + \gamma_{2j}^2 + \tfrac{1}{2}\gamma_{2j+1}^2$$

or, what is equivalent,

$$\gamma_{2j+1} - \gamma_{2j-1} \geq \tfrac{j(j+2)}{2(j+1)^2}\big(\gamma_{2j-1}^2 + \gamma_{2j+1}^2\big) + Y_j \quad \text{with} \quad Y_j = \tfrac{j(j+2)}{(j+1)^2}\gamma_{2j}^2. \tag{42}$$

We now change variables according to

$$\gamma_{2j-1} = \frac{\alpha_j}{j+1}$$

so that $\alpha_0 = -1$ and, by Lemmas 4.1 and 4.2, $-1 \leq \alpha_j \leq 0$ for $j \geq 1$. This implies

$$-\tfrac{1}{4} \leq \alpha_j + \alpha_j^2 \leq 0 \quad \text{for all } j, \tag{43}$$

which we will use momentarily. In the new variables, (42) reads

$$\tfrac{(j+3/2)}{(j+1)(j+2)}[\alpha_{j+1} - \alpha_j] - \tfrac{1}{2(j+1)(j+2)}[\alpha_{j+1} + \alpha_j] \leq \tfrac{j(j+2)}{2(j+1)^2}\left[\tfrac{\alpha_{j+1}^2}{(j+2)^2} + \tfrac{\alpha_j^2}{(j+1)^2}\right] + Y_j.$$

So, by rearranging terms and then using $\alpha_j^2 \leq 1$ and $\alpha_{j+1}^2 \leq 1$,

$$\alpha_{j+1} - \alpha_j \geq \tfrac{1}{2(j+3/2)}\left\{\alpha_{j+1} + \tfrac{j}{j+1}\alpha_{j+1}^2 + \alpha_j + \tfrac{j(j+2)^2}{(j+1)^3}\alpha_j^2\right\} + \tfrac{(j+1)(j+2)}{(j+3/2)}Y_j \tag{44}$$

$$\geq \tfrac{1}{2(j+3/2)}\left\{\alpha_{j+1} + \alpha_{j+1}^2 + \alpha_j + \alpha_j^2\right\} - \tfrac{j+2}{(j+1)^3} + \tfrac{(j+1)(j+2)}{(j+3/2)}Y_j. \tag{45}$$

We now use (43) and sum both sides to obtain

$$\sum_{j=0}^{n} \tfrac{(j+1)(j+2)}{(j+3/2)}Y_j \leq \tfrac{1}{4}\log n + C,$$

from which (40) follows.

Equation (41) is an immediate consequence of (40) and the Cauchy–Schwarz inequality. □

We obtain the following corollaries for the potential $V(n)$.

Theorem 6. *If the discrete half-line Schödinger operator $\Delta + V$ has spectrum $[-2, 2]$, then*
(a) *the potential is weak-ℓ^1 and so belongs to all ℓ^p, $p > 1$;*
(b) *for all $\varepsilon > 0$, $\sum n^{1-\varepsilon}|V(n)|^2 < \infty$;*
(c) *there is a constant C such that for all $N \geq 1$,*

$$\sum_{n=1}^{N} |V(n)| \leq \log(N) + C;$$

(d) *it is possible to write $V(n) = W(n) - W(n-1) + Q(n)$ with $Q \in \ell^1$, $W \in \ell^2$, and*

$$\sum_{n=1}^{N} n|W(n)|^2 \leq \tfrac{1}{4}\log(N) + C.$$

Proof. As $V(n+1) = (1-\gamma_{2n-1})\gamma_{2n} - (1+\gamma_{2n-1})\gamma_{2n-2}$, parts (a)–(c) follow directly from the estimates on the Verblunsky coefficients proved above. For (d), we simply choose $W(n) = \gamma_{2n-2}$, $n \geq 1$, then $Q(n) = -\gamma_{2n-3}(\gamma_{2n-2} + \gamma_{2n-4})$ for $n \geq 2$ and $Q(1) = \gamma_0$ which is summable. □

Example. This example will show that all the statements in Theorem 6 are optimal. This in turn shows also that the estimates on the Verblunsky coefficients obtained above (e.g., Propositions 4.4 and 4.5) are optimal.

Consider the potential $V(n) = (-1)^n/n$. It was shown in [4, Proposition 5.9] that $\Delta + V$ has spectrum $[-2, 2]$. Consequently, weak-ℓ^1 in (a) cannot be replaced by ℓ^1, (b) cannot be improved to $\sum n|V(n)|^2 < \infty$, and the constant 1 in front of $\log(N)$ in (c) cannot be decreased.

In order to see that this example also shows that the constant 1/4 appearing in (d) is the smallest possible, we note the following:

$$\log(N) - C \leq \sum_{n=1}^{N} |V(n)| \leq \sum_{n=1}^{N} |Q(n)| + 2|W(n)|$$

and so, since $Q \in \ell^1$, $\sum_{n=1}^{N} |W(n)| \geq \frac{1}{2}\log(N) - C$. By applying the Cauchy–Schwarz inequality, we find $\sum_{n=1}^{N} n|W(n)|^2 \geq \frac{1}{4}\log(N) - C$.

Next we show that these estimates on the Verblunsky coefficients allow for a short proof that ± 2 are not eigenvalues.

Theorem 7. *If h is a discrete half-line Schrödinger operator with spectrum $[-2, 2]$, then ± 2 are not eigenvalues.*

Proof. Of course, $E = 2$ is an eigenvalue if and only if the generalized eigenfunction at this energy, which we denote by u (cf. (15)), is square integrable. (We will concentrate on $E = 2$; $E = -2$ can be dealt with in the same manner.)

Since we are studying the Schrödinger operator case, $a_n \equiv 1$, equations (25) and (27) give us the following relation between u and the Verblunsky coefficients:

$$\frac{u(n+1)}{u(n+2)} = 1 + \gamma_{2n+1} + \gamma_{2n} + \gamma_{2n+1}\gamma_{2n}.$$

Note that, by Sturm oscillation theory, $u(n) > 0$, $n \geq 1$, and that, by definition, $u(1) = 1$. Therefore, by neglecting the terms $\gamma_{2n+1} \leq 0$ and then using the summability of $\gamma_{2j+1}\gamma_{2j}$,

$$\begin{aligned}\log\big[u(n)\big] &= -\sum_{j=0}^{n-2} \log\big[1 + \gamma_{2j+1} + \gamma_{2j} + \gamma_{2j+1}\gamma_{2j}\big] \\ &\geq C - \sum_{j=0}^{n-2} \log\big[1 + \gamma_{2j}\big]\end{aligned}$$

for some constant C. But, $\log[1 + \gamma_{2j}] \leq |\gamma_{2j}|$ and so (41) gives

$$\big|u(n)\big| \geq cn^{-1/2},$$

which implies u is not square summable. $\square$

As intimated at the beginning of this section, the last two theorems can be extended to the case of finitely many bound states outside $[-2, 2]$.

Corollary 4.6. *If the spectrum of a discrete half-line Schrödinger operator $\Delta + V$ contains only finitely many points outside $[-2, 2]$, then*
(a) *the potential is weak-ℓ^1 and so belongs to all ℓ^p, $p > 1$;*
(b) *for all $\varepsilon > 0$, $\sum n^{1-\varepsilon}|V(n)|^2 < \infty$;*
(c) *there is a constant C such that for all $N \geq 1$,*

$$\sum_{n=1}^{N} |V(n)| \leq \log(N) + C;$$

(d) *it is possible to write* $V(n) = W(n) - W(n-1) + Q(n)$ *with* $Q \in \ell^1$, $W \in \ell^2$, *and*

$$\sum_{n=1}^{N} n|W(n)|^2 \leq \tfrac{1}{4}\log(N) + C;$$

(e) ± 2 *are not eigenvalues.*

Proof. As h_V has only finitely many eigenvalues outside $[-2,2]$, the solutions u and v, as defined in (15) and (17), pass through zero only finitely many times. (This follows from the discrete analogue of the classical Sturm theory [28].) So we may choose $k \in \mathbb{Z}^+$ such that $u(n)$ and $v(n)$ do not change sign for $n \geq k$.

Using Sturm theory again, we see that the operator with potential $V_1(n) = V(n+k)$ has no bound states. Thus, parts (a)–(d) are immediate consequences of Theorem 6.

To prove (e), we will simply show that u cannot be square summable. Similar arguments show that the same is true of v. This implies that ± 2 are not eigenvalues.

The sequence $\tilde{u}(n) = u(n+k)$ is the generalized eigenfunction at energy $+2$ for the operator with potential

$$V_2(n) = V(n+k) + \frac{u(k)}{u(k+1)}\delta_{n,1}.$$

As $\Delta + V_1$ has no eigenvalues outside $[-2,2]$ and $u(k)/u(k+1)$ is positive, $\Delta + V_2$ cannot have eigenvalues below -2. It also has no eigenvalues above $+2$; this is because $\tilde{u}$, the generalized eigenfunction at energy $+2$, does not pass through zero.

We have just seen that $\sigma(\Delta + V_2) \subseteq [-2,2]$; therefore, by Theorem 7, $\tilde{u}$ is not square summable. Consequently, u is not square summable either. ☐

5. Absence of Singular Spectrum: The Discrete Case

The purpose of this section is to complete the proofs of Theorems 1 and 3. We have already seen, in the previous section, that ± 2 are not eigenvalues, so it suffices to consider $(-2,2)$. Absence of singular spectrum in this interval is a consequence of the following general result whose applicability is guaranteed by part (d) of Theorem 6 or by the same part of Corollary 4.6.

Theorem 8. *A half-line Schrödinger operator whose potential admits the decomposition* $V(n) = W(n) - W(n-1) + Q(n)$ *with* $Q \in \ell^1$, $W \in \ell^2$, *and*

$$\sum_{n=1}^{N} n|W(n)|^2 \leq \tfrac{1}{4}\log(N) + C \tag{46}$$

has purely absolutely continuous spectrum on the interval $(-2,2)$.

Of course, (46) implies $W(n) \leq C' \log(n)/n$ for $n > 1$ and so the assumption that $W \in \ell^2$ is redundant.

Example. Define $\psi : \mathbb{Z}^+ \to \mathbb{R}$ as follows: the absolute value is given by $|\psi(n)| = n^{-\alpha}$ and the sign depends on the value of n mod 4 with the pattern $+,+,-,-,\ldots$. If $\alpha > 1/2$, then ψ is square summable and so a zero-energy eigenfunction for the potential

$$V(n) = -\frac{\psi(n+1) + \psi(n-1)}{\psi(n)}$$

for $n \geq 2$ and $V(1) = -\psi(2)/\psi(1)$. As $V(n) = -2\alpha(-1)^n/n + O(n^{-2})$, the argument from the example following Theorem 6 shows that any decomposition $V(n) = W(n) - W(n-1) + Q(n)$ with $Q \in \ell^1$ has

$$\sum_{n=1}^{N} n|W(n)|^2 \geq \alpha^2 \log(N) - C.$$

Consequently, the constant $1/4$ in (46) cannot be improved.

The proof of Theorem 8 will consume the remainder of this section. As this requires a number of technical ingredients, we first explain how the propositions that follow combine to establish the result.

Overview of Proof. The strategy we adopt to prove this theorem is inspired by Remling's proof of absence of embedded singular spectrum for $o(n^{-1})$ potentials [19]. The method consists of two steps, both combining the study of solutions to

$$\psi(n+1) + \psi(n-1) + V(n)\psi(n) = E\psi(n) \tag{47}$$

(with general initial conditions) with subordinacy theory.

First, we derive power-law estimates for all solutions of the Schrödinger equation. These results are contained in Proposition 5.2. It is shown that there are no non-zero ℓ^2 solutions for any $E \in (-2, 2)$ and so no embedded point spectrum. Further, it is shown that, for $E \in (-2, 0) \cup (0, 2)$, all non-zero solutions ψ obey

$$cn^{-3/5} \leq \big|\psi(n)\big|^2 + \big|\psi(n+1)\big|^2 \leq Cn^{3/5}.$$

By the Jitomirskaya–Last extension [12] of subordinacy theory [10], one may deduce that the restriction of the spectral measure $d\mu$ to $(-2, 2)$ gives zero weight to sets of Hausdorff dimension less than $2/5$. As noted a moment ago, $d\mu$ gives zero weight to single points—this is why we could write $(-2, 2)$ in the last sentence rather than just $(-2, 0) \cup (0, 2)$.

Second, we show that for all energies in $(-2, 2)$ that lie outside a set of zero Hausdorff dimension, all solutions of the Schrödinger equation are bounded. This is Proposition 5.5. By the most-used result of subordinacy theory, this implies that any embedded singular spectrum must be supported on this set of zero dimension and $[-2, 2]$ is contained in the essential support of the absolutely continuous spectrum (see, e.g., [24, 26]).

Combining the preceding paragraphs, we see that on $(-2, 2)$, the singular part of the spectral measure must be supported by a set of zero Hausdorff dimension, but also gives zero weight to sets of zero dimension. Of course, only the zero measure gives no weight to its support, so we may conclude that there is no embedded singular spectrum. □

As just described, we need to study solutions of

$$\psi(n+1) + \psi(n-1) + V(n)\psi(n) = 2\cos(k)\,\psi(n) \qquad n \geq 1 \tag{48}$$

where $\psi(0)$ is free to be anything—recall that the generalized eigenfunction vanishes at $n = 0$. The parametrization of energy $E \in (-2, 2)$ as $2\cos(k)$, $k \in (0, \pi)$, is standard and simplifies some of the formulae that follow.

Following [14, 19], we write $\psi(n)$ in terms of Prüfer variables R, θ:

$$\frac{1}{\sin k}\begin{pmatrix} \sin k & 0 \\ -\cos k & 1 \end{pmatrix}\begin{pmatrix} \psi(n-1) \\ \psi(n) \end{pmatrix} = R(n)\begin{pmatrix} \sin\big(\theta(n)/2\big) - k \\ \cos\big(\theta(n)/2\big) - k \end{pmatrix}.$$

These new variables obey the following equations:

$$\begin{aligned}\frac{R(n+1)^2}{R(n)^2} &= 1 - \frac{V(n)}{\sin k}\sin\theta(n) + \frac{V(n)^2}{\sin^2 k}\sin^2 \tfrac{1}{2}\theta(n),\\ \cot\big(\tfrac{1}{2}\theta(n+1) - k\big) &= \cot\big(\tfrac{1}{2}\theta(n)\big) - \frac{V(n)}{\sin k}.\end{aligned}$$

In the second equation, both sides being infinite is also permitted. From here, Taylor expansion yields

$$2\log\left[\frac{R(n+1)}{R(n)}\right] = -V(n)\frac{\sin\theta(n)}{\sin k} + O(V(n)^2) \tag{49}$$

$$\theta(n+1) - \theta(n) = 2k + \frac{V(n)}{\sin k}\,[1-\cos\theta(n)] + O(V(n)^2), \tag{50}$$

where the constants in the O-terms depend on k, but are independent of n.

As $V \in \ell^2$, (49) gives the following two-sided bound on solutions of (48):

$$\Big|\log\big[|\psi(N+1)|^2 + |\psi(N)|^2\big]\Big| \le \frac{1}{\sin k}\left|\sum_{n=1}^{N} V(n)\sin\theta(n)\right| + C. \tag{51}$$

(Note that by definition, $R(n)$ is comparable to the norm of the vector $[\psi(n-1), \psi(n)]$.) This shows that in order to control the behaviour of solutions, we must estimate $\sum V(n)\sin\theta(n)$. Naturally, the first step is to invoke the representation of V in terms of W and Q. Using

$$\begin{aligned}\sin\theta(n) - \sin\theta(n+1) &= 2\cos\left(\tfrac{\theta(n+1)+\theta(n)}{2}\right)\sin\left(\tfrac{\theta(n+1)-\theta(n)}{2}\right)\\ &= 2\cos\left(\tfrac{\theta(n+1)+\theta(n)}{2}\right)\sin\big(k + O(V(n))\big)\\ &= 2\sin(k)\cos\left(\tfrac{\theta(n+1)+\theta(n)}{2}\right) + O\big(V(n)\big)\end{aligned}$$

together with $Q \in \ell^1$ and $W \in \ell^2$ yields

$$\begin{aligned}\sum_{n=1}^{N} V(n)\sin\theta(n) &= \sum_{n=1}^{N} W(n)\big[\sin\theta(n) - \sin\theta(n+1)\big] + O(1)\\ &= 2\sin k\sum_{n=1}^{N} W(n)\cos\left(\tfrac{\theta(n+1)+\theta(n)}{2}\right) + O(1),\end{aligned} \tag{52}$$

where, as before, the implicit constants depend on $k \in (0,\pi)$.

Combining (51) and (52) shows that for each $E \in (-2,2)$,

$$\Big|\log\big[|\psi(N+1)|^2 + |\psi(N)|^2\big]\Big| \le 2\sum_{n=1}^{N} W(n)\cos\left(\tfrac{\theta(n+1)+\theta(n)}{2}\right) + C. \tag{53}$$

Note how the gain of a factor $\sin(k)$ is important; it cancels the $\frac{1}{\sin k}$ factor in front of the sum in (51). This is why estimates on W, the "indefinite integral" of V, control the behaviour of solutions uniformly in energy. Estimates of the form

$$\sum_{n=1}^{N} n|V(n)|^2 \le \alpha\log N + C$$

do not preclude embedded eigenvalues, no matter how small one chooses α; see [8, 19].

Lemma 5.1. *Given a sequence obeying $\phi(n+1)-\phi(n) = 2k+o(1)$ for some $k \in (0,\pi/2) \cup (\pi/2,\pi)$ and an $\varepsilon > 0$, there is a constant C so that*

$$\sum_{n=1}^{N} \frac{\cos^2\phi(n)}{n} \le \left[\tfrac{1}{2}+\varepsilon\right]\log(N)+C. \tag{54}$$

Proof. By writing $\cos^2\phi = \frac{1}{2}+\frac{1}{2}\cos 2\phi$ it suffices to show that $\sum \cos[2\phi(n)]/n \le \varepsilon \log(N)+C$.

Recall the following estimate for the Dirichlet kernel

$$\sup_{\delta}\left\{\sum_{j=0}^{\ell-1}\cos(4kj+\delta)\right\} = \left|\sum_{j=0}^{\ell-1} e^{4ikj}\right| = \left|\frac{\sin(2k\ell)}{\sin(2k)}\right| \le \frac{1}{|\sin 2k|}.$$

It follows that for fixed $\ell \ge 4\,|\varepsilon\sin(2k)|^{-1}$ and n sufficiently large, depending on k, ℓ, and ε,

$$\left|\sum_{j=0}^{\ell-1}\cos\bigl[\phi(n+j)\bigr]\right| \le \left|\sum_{j=0}^{\ell-1}\cos\bigl[2kj+\phi(n)\bigr]\right| + \sum_{j=0}^{\ell-1}\bigl|\phi(n+j)-\phi(n)-2kj\bigr| \le \tfrac{1}{2}\varepsilon\ell.$$

To finish the proof, note that for n sufficiently large,

$$\left|\sum_{j=0}^{\ell-1}\frac{\cos 2\phi(n+j)}{n+j}\right| \le \frac{1}{n}\left|\sum_{j=0}^{\ell-1}\cos[2\phi(n+j)]\right| + \sum_{j=0}^{\ell-1}\frac{j}{n(n+j)} \le \frac{\varepsilon\ell}{n}$$

so that (54) follows by summing over ℓ-sized blocks and absorbing the contribution from the initial segment, where n is not sufficiently large, into the constant C. $\square$

Proposition 5.2. *Suppose $V(n) = W(n)-W(n-1)+Q(n)$ with $Q \in \ell^1$ and $W \in \ell^2$ obeying (46). Then, for $E \in (-2,2)$, all solutions ψ of (47) that are not identically zero obey*

$$n^{-1} \lesssim \bigl|\psi(n)\bigr|^2 + \bigl|\psi(n+1)\bigr|^2 \lesssim n.$$

Moreover, for non-zero energies,

$$n^{-\eta} \lesssim \bigl|\psi(n)\bigr|^2 + \bigl|\psi(n+1)\bigr|^2 \lesssim n^{\eta}$$

for any $\eta > 1/\sqrt{2}$.

Proof. By (53) it suffices to show that

$$\left|\sum_{n=1}^{N} W(n)\cos\phi(n)\right| \le \alpha(k)\log(N)+O(1)$$

where $\phi(n) = \frac{1}{2}[\theta(n+1)+\theta(n)]$, $\alpha(\pi/2) = 1/2$, and $\alpha(k) = \eta$ when $k \ne \pi/2$. Applying Cauchy–Schwarz gives

$$\left|\sum_{n=1}^{N} W(n)\cos\phi(n)\right|^2 \le \sum_{n=1}^{N} n\bigl|W(n)\bigr|^2 \;\cdot\; \sum_{n=1}^{N}\frac{\cos^2\phi(n)}{n}.$$

By assumption, $\sum_1^N n|W(n)|^2 \le \frac{1}{4}\log(N)+C$ and so the $k=\pi/2$ case is an immediate consequence of $\cos^2\phi(n) \le 1$.

The $k \ne \pi/2$ case follows because, by (50),

$$\phi(n+1)-\phi(n) = \tfrac{1}{2}\bigl[\theta(n+2)-\theta(n)\bigr] = 2k+o(1) \tag{55}$$

so we can apply Lemma 5.1. $\square$

We now set about showing that the set of $E \in (-2,2)$ for which not all solutions of (47) are bounded is of zero Hausdorff dimension. We begin with a lemma modelled on Theorem 3.3 of [14].

Lemma 5.3. *Suppose $V(n) = W(n) - W(n-1) + Q(n)$ with W and Q as above and fix $k \in (0,\pi)$. If*

$$\widehat{W}(k;n) \equiv \lim_{M\to\infty} \sum_{m=n}^{M} W(m) e^{2ikm}$$

exists and obeys

$$\sum_{n=1}^{\infty} |W(n+j)\widehat{W}(k;n)| < \infty \qquad \forall j \in \{1,0,-1\}, \tag{56}$$

then all solutions of (48) *are bounded.*

Proof. By (53), it suffices to show that

$$\sum_{n=1}^{N} W(n) \exp\Big\{ \tfrac{i}{2}\big[\theta(n+1)+\theta(n)\big]\Big\}$$

is bounded for those k for which (56) holds. Writing $\phi(n) = \frac{1}{2}[\theta(n+1)+\theta(n)]$, we have

$$\begin{aligned}\sum_{n=1}^{N} W(n)e^{i\phi(n)} &= \sum_{n=1}^{N} \Big[\widehat{W}(k;n) - \widehat{W}(k;n+1)\Big] e^{i\phi(n)-2ikn} \\ &= \sum_{n=2}^{N} \widehat{W}(k;n)\Big[e^{i\phi(n)} - e^{i\phi(n-1)+2ik}\Big]e^{-2ikn} + O(1).\end{aligned}$$

But by (50), $|\phi(n)-\phi(n-1)-2k| \le 2[|W(n+1)|+2|W(n)|+|W(n-1)|]/\sin(k)+e_n$ where e_n is summable. The result now follows easily from the fact that $|e^{ix}-e^{iy}| \le |x-y|$. ☐

To control $\widehat{W}$ we use the following result from harmonic analysis. For a proof, see [31, §XIII.11] or [1, §V.5].

Lemma 5.4. *For each $\varepsilon \in (0,1)$ and every measurable function $m : [0,\pi] \to \mathbb{Z}$,*

$$\bigg\{ \int \bigg| \sum_{n=0}^{m(k)} c_n e^{-2ink} \bigg| \, d\nu(k) \bigg\}^2 \lesssim \mathcal{E}_\varepsilon(\nu) \sum_{n=0}^{\infty} n^{1-\varepsilon} |c_n|^2$$

where $\mathcal{E}_\varepsilon$ denotes the ε-energy of $d\nu$: $\mathcal{E}_\varepsilon(\nu) = \int\int |\sin(x-y)|^{-\varepsilon}\, d\nu(x)\, d\nu(y)$.

Combining these lemmas gives the following proposition, which completes the proof of Theorem 8 as described in the overview given above.

Proposition 5.5. *Suppose $V(n) = W(n) - W(n-1) + Q(n)$ with $Q \in \ell^1$ and $W \in \ell^2$ obeying* (46)*. There is a set $S \subseteq (-2,2)$ of zero Hausdorff dimension so that for all $E \in (-2,2) \setminus S$, all solutions ψ of* (47) *are bounded.*

Proof. By applying the Cauchy–Schwarz inequality to dyadic blocks, for example, we see that (46) implies $n^{-\varepsilon/4}W(n) \in \ell^1$ for all $\epsilon > 0$. Combining this with Lemma 5.3 shows that we need only prove that for all $\varepsilon > 0$, the set of k for which $n^{\varepsilon/4}\widehat{W}(k;n)$ is unbounded is of Hausdorff dimension no more than ε.

Let $m(k)$ be a measurable integer-valued function on $(0, \pi)$. Because of (46), Lemma 5.4 implies

$$\begin{aligned}\int \left| \sum_{n=m_l(k)}^{2^{l+1}-1} W(n) e^{2ikn} \right| d\nu(k) &= \int \left| \sum_{n=0}^{\tilde{m}_l(k)} W(2^{l+1}-1-n) e^{-2ikn} \right| d\nu(k) \\ &\lesssim \left\{ \sum_{n=2^l}^{2^{l+1}-1} n^{1-\varepsilon} \big|W(n)\big|^2 \right\}^{1/2} \sqrt{\mathcal{E}_\varepsilon(\nu)} \\ &\lesssim \sqrt{l}\, 2^{-\varepsilon l/2} \sqrt{\mathcal{E}_\varepsilon(\nu)}\end{aligned}$$

where $m_l(k) = \max\{m(k), 2^l\}$, $\tilde{m}_l(k) = \min\{2^l - 1, 2^{l+1} - 1 - m(k)\}$, and sums with lower index greater than their upper index are to be treated as zero. Multiplying both sides by $2^{\varepsilon l/4}$, summing this over l, and applying the triangle inequality on the left gives

$$\int \left| m(k)^{\varepsilon/4} \sum_{n=m(k)}^{\infty} W(n) e^{2ikn} \right| d\nu(k) \lesssim \sqrt{\mathcal{E}_\varepsilon(\nu)}.$$

That is, for any measurable integer-valued function $m(k)$,

$$\int m(k)^{\varepsilon/4} \big|\widehat{W}(k; m(k))\big|\, d\nu \lesssim \sqrt{\mathcal{E}_\varepsilon(\nu)}.$$

This implies that the set on which $n^{\varepsilon/4}\widehat{W}(k;n)$ is unbounded must be of zero ε-capacity (i.e., it does not support a measure of finite ε-energy).

As the Hausdorff dimension of sets of zero ε-capacity is less than or equal to ε (see [1, §IV.1]), this completes the proof. □

6. A Continuum Analogue of the Verblunsky Coefficients

As in the introduction, we write H_V for the Schrödinger operator associated to the potential V:

$$[H_V \psi](x) = -\psi''(x) + V(x)\psi(x).$$

We require a Dirichlet boundary condition at zero, $\psi(0) = 0$, and $V \in \ell^\infty(L^2)$, that is,

$$\sup_{n \geq 0} \int_n^{n+1} |V(t)|^2\, dt < \infty. \tag{57}$$

The purpose of this section is to identify the continuum analogue of the Verblunsky coefficients and to derive estimates for them.

It is well known that (57) ensures that for every energy E and every boundary condition α at zero, there exists a locally H^2 solution to

$$-\psi''(x) + V(x)\psi(x) = E\psi(x),\ \psi(0) = \sin\alpha,\ \psi'(0) = \cos\alpha. \tag{58}$$

See, for example, [30].

Let u and v denote the zero-energy normalized Dirichlet solutions of (58) with potential V and $-V$, respectively. That is,

$$\begin{aligned} -u'' + Vu = 0 \quad & u(0) = 0,\ u'(0) = 1 \\ -v'' - Vv = 0 \quad & v(0) = 0,\ v'(0) = 1. \end{aligned} \tag{59}$$

Notice that u and v play the same roles as they did in the discrete case; compare (15) and (17).

If both H_V and H_{-V} have no bound states, then it follows from oscillation theory (see, e.g., [2, 30]) that the functions u and v have no zeros in $(0, \infty)$.

From u and v we define the two functions, Γ_{e} and Γ_{o}, on $(0, \infty)$ in a manner inspired by (25) and (26):

$$\Gamma_{\mathrm{e}}(x) = \frac{1}{2}\left[\frac{u'(x)}{u(x)} - \frac{v'(x)}{v(x)}\right]$$
$$\Gamma_{\mathrm{o}}(x) = -\frac{1}{2}\left[\frac{u'(x)}{u(x)} + \frac{v'(x)}{v(x)}\right].$$

These two functions are the analogues of the Verblunsky coefficients in the discrete case with even and odd index, respectively. All the crucial properties of the γ_{2n}'s and the γ_{2n+1}'s carry over to the continuum case, as we will see. Lemma 6.1 below shows that they obey a pair of differential equations which are the analogues of the formulae (28) and (29).

That the Verblunsky coefficients are related to the logarithmic derivative of eigenfunctions in the discrete case appears in Geronimus' work on orthogonal polynomials [9, §31]. In Kreĭn's studies of a continuum analogue of polynomials orthogonal on the unit circle (see, e.g., [16]), he introduced a function A which plays the role of the Verblunsky coefficients. In the case where A is a real-valued function, it is given by the logarithmic derivative of the u associated with the potential $A' + A^2$. In this way, $A = \Gamma_{\mathrm{e}} - \Gamma_{\mathrm{o}}$. While the two approaches are related, the Kreĭn approach is not suited to our problem. For an example of how the Kreĭn approach may be employed in the study of Schrödinger operators, see [7].

Lemma 6.1. *The functions* $\Gamma_{\mathrm{e}}, \Gamma_{\mathrm{o}}$ *obey*

$$\Gamma_{\mathrm{e}}'(x) = V(x) + 2\Gamma_{\mathrm{e}}(x)\Gamma_{\mathrm{o}}(x) \tag{60}$$

$$\Gamma_{\mathrm{o}}'(x) = \Gamma_{\mathrm{o}}^2(x) + \Gamma_{\mathrm{e}}^2(x). \tag{61}$$

Proof. Write

$$F(x) = \frac{u'(x)}{u(x)} \text{ and } G(x) = \frac{v'(x)}{v(x)} \tag{62}$$

so that $\Gamma_{\mathrm{e}}(x) = [F(x) - G(x)]/2$ and $\Gamma_{\mathrm{o}}(x) = -[F(x) + G(x)]/2$. We infer from differential equations for u and w, (59), that

$$F'(x) = V(x) - F^2(x) \text{ and } G'(x) = -V(x) - G^2(x). \tag{63}$$

Subtraction gives

$$F'(x) - G'(x) = 2V(x) - F^2(x) + G^2(x)$$

and from this we get

$$V(x) = \tfrac{1}{2}(F'(x) - G'(x)) + \tfrac{1}{2}[(F(x) - G(x)][F(x) + G(x)],$$

which is (60). On the other hand, addition of the identities in (63) yields

$$-\tfrac{1}{2}(F'(x) + G'(x)) = \tfrac{1}{2}F^2(x) + \tfrac{1}{2}G^2(x) = \left(\frac{F(x) - G(x)}{2}\right)^2 + \left(-\frac{F(x) + G(x)}{2}\right)^2,$$

which is (61). ☐

We will now present three lemmas, which are the continuum analogues of results proved in Section 4. We begin with the counterpart to Lemmas 4.1 and 4.2.

Lemma 6.2. *For every $x > 0$, we have*

$$-\frac{1}{x} \leq \Gamma_{\mathrm{o}}(x) \leq 0.$$

Proof. Given $x_0 > 0$ and $y_0 \neq 0$, consider the initial value problem $\Gamma'(x) = \Gamma^2(x),\ \Gamma(x_0) = y_0$. Its solution is given by

$$\Gamma(x) = -\left(x - \frac{1 + x_0 y_0}{y_0}\right)^{-1}.$$

Notice that if $y_0 > 0$, then Γ blows up at finite $x > x_0$.

By (61), $\Gamma_{\mathrm{o}}'(x) \geq \Gamma_{\mathrm{o}}^2(x)$. Therefore,

$$\Gamma_{\mathrm{o}}(x) \geq -\left(x - \frac{1 + x_0 \Gamma_{\mathrm{o}}(x_0)}{\Gamma_{\mathrm{o}}(x_0)}\right)^{-1} \tag{64}$$

for $x > x_0$. As $\Gamma_{\mathrm{o}}(x)$ is regular, blow-up cannot occur and, by the remark made earlier, this implies that $\Gamma_{\mathrm{o}}(x) \leq 0$ for all $x \in (0, \infty)$.

By (64), $\Gamma_{\mathrm{o}}(x) \geq -1/x$ follows from

$$\lim_{x_0 \to 0} \frac{1 + x_0 \Gamma_{\mathrm{o}}(x_0)}{\Gamma_{\mathrm{o}}(x_0)} = 0,$$

which in turn follows from

$$\frac{1 + x_0 \Gamma_{\mathrm{o}}(x_0)}{\Gamma_{\mathrm{o}}(x_0)} = x_0 - 2\left(\frac{u'(x_0)}{u(x_0)} + \frac{v'(x_0)}{v(x_0)}\right)^{-1}$$

and the fact that u, u', v, and v' are continuous at the origin with the values given in (59). □

In place of Proposition 4.4 we have:

Lemma 6.3. *For every $x \geq 0$,*

$$\int_0^x t^2 \cdot \left[\Gamma_{\mathrm{e}}^2(t) + \left(\Gamma_{\mathrm{o}}(t) + \tfrac{1}{t}\right)^2\right] dt \leq x. \tag{65}$$

Moreover, $|\{x : |\Gamma_{\mathrm{e}}(x)| \geq \lambda\}| \leq 5\lambda^{-1}$ and so $\Gamma_{\mathrm{e}} \in L^1_w$.

Proof. Write

$$\Gamma_{\mathrm{o}}(t) = -\frac{1}{t} + h(t).$$

It follows from the definition of Γ_{o} and Lemma 6.2 that

$$0 \leq h(t) \leq \frac{1}{t} \text{ and } \lim_{t \to 0+} t^2 h(t) = 0. \tag{66}$$

Differentiating the definition of h gives

$$\Gamma_{\mathrm{o}}'(t) = \frac{1}{t^2} + h'(t),$$

while from (61) we have

$$\Gamma_{\mathrm{o}}'(t) = \Gamma_{\mathrm{o}}^2(t) + \Gamma_{\mathrm{e}}^2(t) = \frac{1}{t^2} - \frac{2h(t)}{t} + h^2(t) + \Gamma_{\mathrm{e}}^2(t).$$

Therefore,

$$h'(t) + \frac{2}{t} h(t) = h^2(t) + \Gamma_{\mathrm{e}}^2(t),$$

which in turn implies

$$(t^2 h(t))' = t^2 h^2(t) + t^2 \Gamma_{\mathrm{e}}^2(t).$$

The first inequality, (65), now follows by integrating this and applying (66).

To prove the second estimate, notice that for $k \geq 0$, (65) implies

$$|\{2^k\lambda^{-1} \leq x < 2^{k+1}\lambda^{-1} : |\Gamma_{\mathrm{e}}(x)| \geq \lambda\}| \cdot 2^{2k} \leq 2^{k+1}\lambda^{-1},$$

which yields

$$|\{x > 0 : |\Gamma_{\mathrm{e}}(x)| \geq \lambda\}| \leq \lambda^{-1} + \sum_{k=0}^{\infty} 2^{1-k}\lambda^{-1} = 5\lambda^{-1},$$

concluding the proof. □

Lastly, the continuum analogues of Proposition 4.5 and part (c) of Theorem 6 are given by the following:

Lemma 6.4. *The function Γ_{e} admits the following estimates: for all $x > y > 0$,*

$$\int_y^x t \cdot \Gamma_{\mathrm{e}}^2(t)\, dt \leq 1 + \tfrac{1}{4} \log\left(\tfrac{x}{y}\right)$$

and for $x > 1$,

$$\int_1^x |\Gamma_{\mathrm{e}}(t)|\, dt \leq \tfrac{1}{2}\log(x) + C. \tag{67}$$

Proof. Write

$$\Gamma_{\mathrm{o}}(t) = -\frac{\alpha(t)}{t}.$$

By Lemma 6.2, we have $0 \leq \alpha(t) \leq 1$ for every $t > 0$. From

$$\Gamma_{\mathrm{o}}'(t) = -\frac{\alpha'(t)}{t} + \frac{\alpha(t)}{t^2}$$

and (61) we obtain

$$\Gamma_{\mathrm{e}}^2(t) = \Gamma_{\mathrm{o}}'(t) - \Gamma_{\mathrm{o}}^2(t) = -\frac{\alpha'(t)}{t} + \frac{\alpha(t) - \alpha^2(t)}{t^2}.$$

Thus, because $0 \leq \alpha(t) \leq 1$ implies $0 \leq \alpha(t) - \alpha^2(t) \leq \frac{1}{4}$,

$$\int_y^x t \cdot \Gamma_{\mathrm{e}}^2(t)\, dt = \int_y^x -\alpha'(t) + \frac{\alpha(t) - \alpha^2(t)}{t}\, dt \leq 1 + \int_y^x \frac{1}{4t}\, dt,$$

from which the first estimate follows.

The second estimate follows from the first by applying the Cauchy–Schwarz inequality. □

The following theorem gives the input necessary to prove the absence of singular spectrum in the next section. Specifically, it shows that absence of bound states (for both H_V and H_{-V}) forces the potential to have a certain structure and so to be amenable to treatment by the general criterion given in Theorem 10 below.

Theorem 9. *If $V \in \ell^\infty(L^2)$ and the spectra of both H_V and H_{-V} are contained in $[0,\infty)$, then*

(a) *we can write $V = W' + Q$ with $Q \in L^1$, $W' \in \ell^\infty(L^2)$, and*

$$\int_1^x t[W(t)]^2 \leq \tfrac{1}{4}\log(x) + 1; \tag{68}$$

(b) *neither H_V nor H_{-V} has zero as an eigenvalue.*

Proof. (a) Let g be a C^∞ function on $\mathbb{R}^+$ with $g(x) = 0$ for $0 \leq x \leq 1/2$ and $g(x) = 1$ for $x \geq 1$. Let $W(x) = g(x)\Gamma_{\mathrm{e}}(x)$ and $Q = V - W'$. Thus, for $x \geq 1$, we have $W(x) = \Gamma_{\mathrm{e}}(x)$ and, by (60), $Q(x) = -2\Gamma_{\mathrm{e}}(x)\Gamma_{\mathrm{o}}(x)$. By (57), Q is absolutely integrable on $(0, 1)$, and by Lemmas 6.2 and 6.3, it is absolutely integrable on $(1, \infty)$. Moreover, $W' \in \ell^\infty(L^2)$ follows from (57), Lemma 6.2, and Lemma 6.3. Finally, the bound (68) follows from Lemma 6.4.

(b) From the definitions of Γ_{e} and Γ_{o},

$$\frac{u'(x)}{u(x)} = \Gamma_{\mathrm{e}}(x) - \Gamma_{\mathrm{o}}(x)$$

and hence, by Lemma 6.2,

$$\log\big[u(x)\big] \geq C - \int_1^x |\Gamma_{\mathrm{e}}(t)|\, dt$$

for $x > 1$. By using (67), we obtain

$$u(x) \gtrsim x^{-1/2}$$

for $x > 1$. Therefore, $u \notin L^2$ and so zero is not an eigenvalue of H_V. Similar reasoning shows that H_{-V} does not have zero as an eigenvalue. □

Corollary 6.5. *If $V \in \ell^\infty(L^2)$ and both H_V and H_{-V} have only finitely many eigenvalues below zero, then*

(a) *we can write $V = W' + Q$ with $Q \in L^1$, $W' \in \ell^\infty(L^2)$, and*

$$\int_1^x t[W(t)]^2 \leq \tfrac{1}{4}\log(x) + 1;$$

(b) *neither H_V nor H_{-V} has zero as an eigenvalue.*

Proof. As both H_V and H_{-V} have only finitely many eigenvalues below zero, the solutions u and v, as defined in (59), have only finitely many zeros. If we define $x_0 = \max\{x : u(x)v(x) = 0\}$, then $u(x)$ and $v(x)$ do not change sign for $x \geq x_0$. By symmetry, we may suppose that $u(x_0) = 0$.

Let $V_1(x) = V(x + x_0)$. As u and v do not change sign for $x > x_0$, both H_{V_1} and H_{-V_1} have spectrum contained in $[0, \infty)$. By the previous theorem, part (a) follows for V_1 and so also for V. It also shows that u cannot be square integrable.

To prove that w is not square integrable, we modify V_1 as follows. Consider $V_2 = V_1 + \lambda\chi_{[0,1]}$. As $\sigma(H_{V_1}) \subseteq [0, \infty)$, the same is true of H_{V_2} so long as $\lambda \geq 0$.

Choose λ to be the smallest eigenvalue of the following problem on $[0, 1]$:

$$-\frac{d^2\psi}{dx^2} - V_1\psi = \lambda\psi, \qquad \psi(0) = 0, \quad \psi'(1)v(x_0 + 1) - \psi(1)v'(x_0 + 1) = 0.$$

As $x \mapsto v(x_0 + x)$ does not have a zero in $[0, 1]$, λ cannot be negative. We denote the corresponding eigenfunction by ψ, normalized to have $\psi(1) = v(x_0 + 1)$.

For this value of λ, the function

$$v_2(x) = \begin{cases} \psi(x) & 0 \leq x \leq 1 \\ v(x_0 + x) & 1 \leq x < \infty \end{cases}$$

is the Dirichlet solution for the operator H_{-V_2} at energy zero and it does not change sign. This implies that $\sigma(H_{-V_2}) \subseteq [0, \infty)$.

We have just seen that for λ fixed as above, both H_{V_2} and H_{-V_2} have spectrum contained in $[0, \infty)$. By part (b) of Theorem 9, v_2 cannot be square integrable, which implies that v cannot be square integrable, either. □

7. Absence of Singular Spectrum: The Continuum Case

Our goal in this section is to prove Theorems 2 and 4; that is, to show that if the negative spectrum of both H_V and H_{-V} consists of only finitely many eigenvalues, then both operators have purely absolutely continuous spectrum on $[0,\infty)$. We have seen above that zero is not an eigenvalue, so it suffices to consider the open interval $(0,\infty)$. Absence of singular spectrum in this interval is a consequence of the following general result whose applicability is guaranteed by Theorem 9 or Corollary 6.5.

Theorem 10. *Let $H = -\Delta + V$ be a continuum half-line Schrödinger operator whose potential can be written as $V = W' + Q$ with $W' \in \ell^\infty(L^2)$, $Q \in L^1$, and*

$$\int_1^x t[W(t)]^2 \leq \tfrac{1}{4}\log(x) + C. \tag{69}$$

Then the essential support of the absolutely continuous spectrum of H is $(0,\infty)$ and the spectrum is purely absolutely continuous on this set.

The proof follows the same strategy as the proof of Theorem 8, that is, we prove estimates on the behaviour of generalized eigenfunctions and then use subordinacy theory.

Proposition 7.4 will show that the singular part of the spectral measure, restricted to $(0,\infty)$, does not assign any weight to sets of Hausdorff dimension zero.

Proposition 7.7 will show that for all energies in $(0,\infty)$, with the exception of a set of zero Hausdorff dimension, all solutions are bounded. This implies that $(0,\infty)$ is the essential support of the absolutely continuous spectrum and that any singular spectrum in $(0,\infty)$ must be supported on a set of zero Hausdorff dimension.

Notice that these two propositions preclude the existence of singular spectrum in $(0,\infty)$.

As a preliminary observation, we note the following:

Lemma 7.1. *If W is such that $W' \in \ell^\infty(L^2)$ and (69) is satisfied, then W is bounded, square integrable, and obeys the pointwise estimate*

$$|W(x)| \lesssim \left(\frac{\log x}{x}\right)^{1/4} \tag{70}$$

for x large enough. Moreover, $W^4 W' \in L^1$ and $W \in L^p$ for $p \geq 2$.

Proof. By (69), the integral of $|W|^2$ over the interval $[2^l, 2^{l+1}]$ is bounded by $Cl2^{-l}$. Summing this over l proves square integrability.

As $W' \in \ell^\infty(L^2)$, there is a constant C such that, for $|\delta| \leq 1$ and $x > 1$,

$$|W(x+\delta) - W(x)| = \left| \int_0^\delta W'(x+t)\,dt \right| \leq C|\delta|^{1/2}.$$

Thus,

$$|W(x+t)| \geq \tfrac{1}{2}|W(x)| \quad \text{for} \quad 0 \leq |t| \leq T_x = \min\left\{\tfrac{1}{4C}|W(x)|^2, 1\right\}.$$

Combining this with (69) gives

$$\min\left\{\tfrac{1}{8C}|W(x)|^4, \tfrac{1}{2}|W(x)|^2\right\} \leq \int_{-T_x}^{T_x} W(x+t)^2\,dt \leq \frac{\frac{1}{4}\log(x+1)+c}{x-1},$$

which implies that $W(x) \to 0$ as $x \to \infty$ and so (70). As this shows that $W \in L^\infty$ and we know $W \in L^2$, it follows that $W \in L^p$ for $p \geq 2$.

By the Cauchy–Schwarz inequality, $W' \in \ell^\infty(L^2)$, (69), and (70),

$$\begin{aligned}\int_n^{n+1} |W(x)^4 W'(x)|\,dx &\lesssim \left(\int_n^{n+1} |W(x)|^8\,dx\right)^{1/2} \\ &\leq \left(\sup_{n\leq x\leq n+1} |W(x)|^3\right)\left(\int_n^{n+1} |W(x)|^2\,dx\right)^{1/2} \\ &\lesssim \left(\frac{\log n}{n}\right)^{3/4}\left(\frac{\log n}{n}\right)^{1/2}.\end{aligned}$$

As this is summable, we find $W^4W' \in L^1$. □

As with its discrete analogue, Theorem 8, the proof of Theorem 10 rests on the study of solutions of the corresponding eigenfunction equation for all boundary conditions.

In order to study solutions of

$$-\psi''(x) + V(x)\psi(x) = k^2\psi(x), \tag{71}$$

we use the continuum Prüfer variables, $R(x)$ and $\theta(x)$. These are defined by

$$\psi(x) = R(x)\sin(\theta(x)/2), \quad \psi'(x) = kR(x)\cos(\theta(x)/2)$$

and the requirements that $R(x) > 0$ and θ be continuous (c.f. [14]). They obey the following differential equations:

$$\frac{d\log R(x)}{dx} = \frac{V(x)}{2k}\sin\theta(x) \tag{72}$$

$$\frac{d\theta(x)}{dx} = 2k - \frac{V(x)}{k}(1-\cos\theta(x)). \tag{73}$$

The following lemma isolates the main term in the asymptotics of the Prüfer amplitude $R(x)$.

Lemma 7.2. *Under the assumptions of Theorem 10,*

$$\log\left(\frac{R(x)}{R(0)}\right) = -\int_0^x W(t)\cos\theta(t)\,dt + O(1). \tag{74}$$

Proof. From (72) and $Q \in L^1$, we find

$$\log\left(\frac{R(x)}{R(0)}\right) = \tfrac{1}{2k}\int_0^x V(t)\sin\theta(t)\,dt = \tfrac{1}{2k}\int_0^x W'(t)\sin\theta(t)\,dt + O(1).$$

Integration by parts, Lemma 7.1, and (73) yield

$$\tfrac{1}{2k}\int_0^x W'(t)\sin\theta(t)\,dt = -\int_0^x W(t)\cos\theta(t)\left[1 - \frac{V(t)}{2k^2}[1-\cos\theta(t)]\right]dt + O(1),$$

so that (74) will follow once we show

$$\int_0^x W(t)W'(t)\cos\theta(t)[1-\cos\theta(t)]\,dt = O(1). \tag{75}$$

Note that $W(t)W'(t) = \frac{1}{2}(W(t)^2)'$. Integrating by parts, and reusing this idea, shows that (75) holds. Along the way we use the pointwise bound (70) to control the boundary terms, $W \in L^p$ for $p \geq 2$ to control integrals not containing W', and finally $W^4W' \in L^1$ to control the integral that contains this term. □

Lemma 7.3. *Assume that for all $L > 0$,*

$$\sup_{0\le t\le L} |\phi(x+t) - \phi(x) - 2kt| \to 0 \text{ as } x \to \infty. \tag{76}$$

Then, for every $\varepsilon > 0$, there is a constant C so that

$$\int_1^x \frac{\cos^2\phi(t)}{t}\,dt \le \left[\tfrac{1}{2} + \varepsilon\right]\log(x) + C.$$

Proof. Let $\varepsilon > 0$ be given. As $2\cos^2\phi = 1 + \cos 2\phi$, it suffices to show that

$$\int_1^x \frac{\cos 2\phi(t)}{t}\,dt \le \varepsilon\log(x) + C. \tag{77}$$

For L sufficiently large, say $L > 2/(\varepsilon k)$, we have

$$\sup_\delta \left| \int_0^L \cos(4kt+\delta)\,dt \right| \le \tfrac{1}{4}\varepsilon L.$$

For such an L and x large enough, we have

$$\left| \int_0^L \cos 2\phi(x+t)\,dt \right| \le \tfrac{1}{2}\varepsilon L.$$

Thus, again for x large enough,

$$\left| \int_0^L \frac{\cos 2\phi(x+t)}{x+t}\,dt \right| \le \frac{1}{x}\left| \int_0^L \cos 2\phi(x+t)\,dt \right| + \int_0^L \frac{t\,dt}{x(x+t)} \le \frac{\varepsilon L}{x}.$$

From this, (77) follows by breaking the integral over $[0,x]$ into L-sized blocks. $\square$

Proposition 7.4. *Suppose $V = W' + Q$ with $Q \in L^1$ and $W' \in \ell^\infty(L^2)$ obeying (69). Then, for $k > 0$, all solutions ψ of (71) that are not identically zero obey*

$$x^{-\eta} \lesssim |\psi(x)|^2 + |\psi'(x)|^2 \lesssim x^\eta$$

for any $\eta > 1/\sqrt{2}$ and $x \ge 1$. Consequently, the spectral measure gives zero weight to any subset of $(0,\infty)$ of Hausdorff dimension less than $1 - 2^{-1/2}$.

Proof. Fix $\eta > 1/\sqrt{2}$. By Lemma 7.2, it suffices to show

$$\left| \int_1^x W(t)\cos\theta(t)\,dt \right| \le \frac{\eta}{2}\log(x) + O(1).$$

By the Cauchy–Schwarz inequality,

$$\left| \int_1^x W(t)\cos\theta(t)\,dt \right|^2 \le \int_1^x tW(t)^2\,dt \cdot \int_1^x \frac{\cos^2\theta(t)}{t}\,dt.$$

Therefore, once we show that the function θ satisfies the condition (76), Lemma 7.3 and (69) allow us to conclude the proof. To this end, we note that

$$\theta(x+t) - \theta(x) - 2kt = -\tfrac{1}{k}\int_0^t V(x+s)[1-\cos\theta(x+s)]\,ds$$

and hence

$$\sup_{0\le t\le L} |\theta(x+t) - \theta(x) - 2kt| \le \sup_{0\le t\le L} \left| \tfrac{1}{k}\int_0^t W'(x+s)[1-\cos\theta(x+s)]\,ds \right| + \tfrac{2}{k}\int_0^L |Q(x+s)|\,ds.$$

As $Q \in L^1$, the second term goes to zero as $x \to \infty$. To show that

$$\sup_{0 \le t \le L} \left| \int_0^t W'(x+s)[1-\cos\theta(x+s)]\, ds \right| \to 0 \text{ as } x \to \infty,$$

we integrate by parts four times, as in the proof of Lemma 7.2, and then apply Lemma 7.1.

The statement about the spectral measure follows from the Jitomirskaya-Last version of subordinacy theory [12]. □

Our next goal is to show that the set of energies at which not all solutions of (71) are bounded is of zero Hausdorff dimension. First we prove a continuum analogue of Lemma 5.3; see [14, Theorem 3.2] for a related result.

Lemma 7.5. *Suppose $V = W' + Q$ with W and Q as above. Fix $k \in (0, \infty)$. If*

$$\widehat{W}(k;x) \equiv \lim_{M\to\infty} \int_x^M W(t) e^{2ikt}\, dt \tag{78}$$

exists and obeys

$$\widehat{W} W \in L^1, \tag{79}$$

then all solutions of (71) *are bounded.*

Proof. Let k be such that $\widehat{W}(k;x)$ exists and (79) holds. By Lemma 7.2, it suffices to show that $\int_0^x W(t)e^{i\theta(t)}\, dt$ is bounded. Notice that the existence of the limit in (78) implies that

$$\widehat{W}(k;x) \to 0 \text{ as } x \to \infty \tag{80}$$

and, by (70),

$$W(x) \to 0 \text{ as } x \to \infty. \tag{81}$$

Now we proceed as follows:

$$\begin{aligned} \int_0^x W(t)e^{i\theta(t)}\, dt &= \int_0^x \frac{\partial}{\partial t} \widehat{W}(k;t) e^{i\theta(t)-2ikt}\, dt \\ &= \tfrac{i}{k} \int_0^x \widehat{W}(k;t) V(t)[1-\cos\theta(t)] e^{i\theta(t)-2ikt}\, dt + O(1) \\ &= \tfrac{i}{k} \int_0^x \widehat{W}(k;t) W'(t) e^{-2ikt} P(\theta(t))\, dt + O(1), \end{aligned}$$

where we used (80) and (81) in the second step and $Q \in L^1$ in the last step. Here, $P(\cdot)$ denotes a trigonometric polynomial. Integrating by parts four times shows that this integral is bounded because $W^4 W' \in L^1$. To make this more explicit, one may use the following observation four times (with $l = 0$, 1, 2, and then 3): Given $l \ge 0$ and a trigonometric polynomial $P_1(t,\theta)$, there is a trigonometric polynomial $P_2(t,\theta)$ such that

$$\int_0^x \widehat{W}(k;t) W^l(t) W'(t) P_1(t,\theta(t))\, dt = \int_0^x \widehat{W}(k;t) W^{l+1}(t) W'(t) P_2(t,\theta(t))\, dt + O(1).$$

This is proved by integration by parts:

$$\begin{aligned}\int_0^x \widehat{W}(k;t)W^l(t)W'(t)P_1(t,\theta(t))\,dt &= \int_0^x \widehat{W}(k;t)\frac{(W^{l+1}(t))'}{l+1}P_1(t,\theta(t))\,dt\\ &= -\tfrac{1}{l+1}\int_0^x \widehat{W}(k;t)W^{l+1}(t)\left[\tfrac{\partial}{\partial t}P_1(t,\theta(t)) + \tfrac{\partial}{\partial\theta}P_1(t,\theta(t))\theta'(t)\right]\,dt\\ &\quad -\tfrac{1}{l+1}\int_0^x W^{l+2}(t)e^{2ikt}P_1(t,\theta(t))\,dt + O(1)\\ &= \int_0^x \widehat{W}(k;t)W^{l+1}(t)W'(t)P_2(t,\theta(t))\,dt + O(1).\end{aligned}$$

Here we used (80), (81), $W \in L^2$ (see Lemma 7.1), and the assumption (79), though only in the case $l = 0$. □

To use this lemma to show that the set of energies at which not all solutions of (71) are bounded is of zero Hausdorff dimension, we need to control $\widehat{W}$. For this, we use the following analogue of Lemma 5.4 whose proof is a straightforward adaptation of the arguments in [31, §XIII.11] or [1, §V.5]. The two ingredients will be combined in Proposition 7.7 below.

Lemma 7.6. *For each $\varepsilon \in (0,1)$, every measurable function $m : (0,\infty) \to \mathbb{R}$, and every measure ν, we have*

$$\left\{\int \left|\int_0^{m(k)} g(t)e^{2ikt}\,dt\right| d\nu(k)\right\}^2 \lesssim \mathcal{E}_\varepsilon(\nu)\int (1+t^2)^{\frac{1-\varepsilon}{2}}|g(t)|^2\,dt,$$

where $\mathcal{E}_\varepsilon(\nu) = \int\int (1+|x-y|^{-\varepsilon})\,d\nu(x)\,d\nu(y)$ denotes the ε-energy of $d\nu$.

Proposition 7.7. *Suppose $V = W' + Q$ with $Q \in L^1$ and $W' \in \ell^\infty(L^2)$ obeying (69). There is a set $S \subseteq (0,\infty)$ of zero Hausdorff dimension so that for all $E \in (0,\infty)\setminus S$, all solutions ψ of (71) are bounded. Consequently, the singular part of the spectral measure on $(0,\infty)$ is supported by a set of zero Hausdorff dimension.*

Proof. The proof is completely analogous to the proof of Proposition 5.5, so we just sketch the argument. Let $m(k)$ be a measurable function and for every $l \geq 0$, let $m_l(k) = \max\{2^l, m(k)\}$ and $\Omega_l = \{k : m(k) \leq 2^{l+1}\}$. Then, it follows from Lemma 7.6 that for every $\varepsilon \in (0,1)$,

$$\int_{\Omega_l}\left|\int_{m_l(k)}^{2^{l+1}} 2^{\varepsilon l/4}W(t)e^{2ikt}\,dt\right| d\nu(k) \lesssim \sqrt{\mathcal{E}_\varepsilon(\nu)}\,2^{-\varepsilon l/4}\sqrt{l}.$$

This shows that the set of k for which $x^{\varepsilon/4}\widehat{W}(k;x)$ is unbounded must be of zero ε-capacity, and hence of Hausdorff dimension no more than ε. Since $x^{-\varepsilon/4}W(x) \in L^1$, an application of Lemma 7.5 completes the proof of the proposition.

The last statement follows from the well-known fact that the spectral measure is purely absolutely continuous on the set of energies where all solutions are bounded [10, 24, 26]. □

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