A Polymer Expasion for the Quantum Heisenberg Ferromagnet Wave Function

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Abstract

A polymer expansion is given for the Quantum Heisenberg Ferromagnet wave function. Working on a finite lattice, one is dealing entirely with algebraic identities; there is no question of convergence. The conjecture to be pursued in further work is that effects of large polymers are small. This is relevant to the question of the utility of the expansion and its possible extension to the infinite volume. In themselves the constructions of the present paper are neat and elegant and have surprising simplicity.

This paper assumes the fundamentals of the Heisenberg model but is basically self-contained; it arises from the work in [1], [2], [3], [4]. We hope to extend the program of this paper to questions relevant to a proof of the phase transition, via arguments in [3] and [4]. (Our paper [3] was circulated before we were aware of the work of Powers in [4], that contained all the same ideas.) In a less ambitious direction we intend to pursue the question of bounding the size of polymer contributions, including deriving the infinite volume limit.

We work with a finite rectangular lattice, V, in d-dimensions, $\mathcal{V}$ the set of its vertices. We consider a wave function of the Heisenberg ferromagnet, $\vec{\psi}(t)$, lying entirely in the N spin wave sector

$$\vec{\psi}(t) = \sum_{\mathcal{S}} C(\mathcal{S}, t) \bigotimes_{i \in \mathcal{S}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i \bigotimes_{j \notin \mathcal{S}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_j \tag{1}$$

where here the $\mathcal{S}$ are subsets of $\mathcal{V}$, with $\#(\mathcal{S}) = N$. We also write $C(\mathcal{S}, t) = c^N(\mathcal{S}, t)$ and define

$$c^r(\mathcal{S}, t) = 0 \quad \text{for} \quad r > N, \#(\mathcal{S}) = r. \tag{2}$$

and

$$c^r(\mathcal{S}, t) = \sum_{\mathcal{S}' \supset \mathcal{S}} C(\mathcal{S}', t) \;\; \text{for} \;\; N > r = \#(\mathcal{S}) \tag{3}$$

where the $\mathcal{S}'$ are summed over sets with $\#(\mathcal{S}') = N$. We have thus linear maps from the N spin wave state to lower spin wave states....that intertwine Hamiltonians! We define

$$c(\mathcal{S}, t) = c^r(\mathcal{S}, t) \quad \text{if} \quad \#(\mathcal{S}) = r. \tag{4}$$

We find it convenient to require the normalization

$$\sum_{\mathcal{S}} C(\mathcal{S}, t) = 1 \tag{5}$$

and this leads to, for $r \leq N$,

$$\sum_{\mathcal{S}} c^r(\mathcal{S}, t) = \frac{N!}{r!(N-r)!} \tag{6}$$

The sums in (5) and (6) are understood to be over sets of appropriate size.

$\vec{\psi}(t)$ satisfies the imaginary time Schroedinger equation

$$\vec{\psi}(t) = e^{-Ht}\vec{\psi}(0) \tag{7}$$

which more explicitly is

$$\frac{\partial}{\partial t}C(\mathcal{S},t) = \sum_{\mathcal{S}'\sim\mathcal{S}} \Big(C(\mathcal{S}',t) - C(\mathcal{S},t)\Big) \tag{8}$$

We write $\mathcal{S}' \sim \mathcal{S}$ if the two sets differ in a single element, by nearest neighbors:

$$\begin{aligned} \mathcal{S} &= \{i_1, ..., i_s\} \\ \mathcal{S}' &= \{i_1, ...i_{s-1}, i'_s\} \\ i_s &\sim i'_s. \end{aligned} \tag{9}$$

(i.e. i_s and i'_s nearest neighbors in V). It nicely follows that

$$\frac{\partial}{\partial t}c^r(\mathcal{S},t) = \sum_{\mathcal{S}'\sim\mathcal{S}} \Big(c^r(\mathcal{S}',t) - c^r(\mathcal{S},t)\Big). \tag{10}$$

Here $\#(\mathcal{S}) = \#(\mathcal{S})' = r$. This development is treated in Section II of [1]. A similar more complex parallel theory is given in [4] for random walks on the permutation group, instead of a lattice.

We start presenting the polymer expansion for $\vec{\psi}(t)$. We let $\mathcal{P}$ be a partition of $\mathcal{V}$. We write $\mathcal{S}_\alpha < \mathcal{P}$ for a subset $\mathcal{S}_\alpha$ of the partition $\mathcal{P}$. One has

$$S_\alpha \bigcap S_\beta = \emptyset, \quad \alpha \neq \beta \tag{11}$$

$$\bigcup_{\alpha\in I^P} \mathcal{S}_\alpha = \mathcal{V}. \tag{12}$$

We will have

$$\vec{\psi}(t) = \sum_{\mathcal{P}} \bigotimes_{\mathcal{S}_\alpha<\mathcal{P}} \vec{u}(\mathcal{S}_\alpha,t) \tag{13}$$

where

$$\vec{u}(\mathcal{S}_\alpha,t) = \begin{pmatrix} \phi_i(t) \\ 1-\phi_i(t) \end{pmatrix}_i \tag{14}$$

if $\mathcal{S}_\alpha = \{i\}$.

If $\#(\mathcal{S}_\alpha) = r > 1$

$$\vec{u}(\mathcal{S}_\alpha, t) = u^r(\mathcal{S}_\alpha, t) \bigotimes_{i \in \mathcal{S}_\alpha} \begin{pmatrix} 1 \\ -1 \end{pmatrix}_i \tag{15}$$

We also write

$$u(\mathcal{S}, t) = \begin{cases} \phi_i(t) & \text{if} \quad \mathcal{S} = \{i\} \\ u^r(\mathcal{S}, t) & \text{if} \quad \#(\mathcal{S}) = r > 1 \end{cases} . \tag{16}$$

Then we find that equation (13) is satisfied if the $u(\mathcal{S}, t)$ are chosen to satisfy:

$$c^r(\mathcal{S}, t) = u^r(\mathcal{S}, t) + \sum_{\mathcal{P}} \prod_{\mathcal{S}_\beta < \mathcal{P}} u(\mathcal{S}_\beta, t) \tag{17}$$

Where here $\mathcal{P}$ is a partition of $\mathcal{S}$ and $\#(\mathcal{S}) = r$. r will range from 1 to $\#(\mathcal{V})$. Equations (13) and (17) are prototype cluster-expansion/polymer-expansion equations. But the form of equation (15) is perhaps surprising? One checks the consistency of these equations by showing equation (1) and equation (13) have the same mapping onto each r spin wave sector under the mappings of equations (1) - (3).

References

[1] P. Federbush, “For the Quantum Heisenberg Ferromagnet, Some Conjectured Approximations”, math-ph/0101017.

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[3] P. Federbush, “For the Quantum Heisenberg Ferromagnet, Tao to the Proof of a Phase Transition”, math-ph/202044.

[4] Robert T. Powers, “Heisenberg Model and a Random Walk on the Permutation Group”, *Lett. in Math. Phys.* **1**, 125-130 (1976).