Quantum electrodynamics of relativistic bound states with cutoffs.

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Abstract

We consider an Hamiltonian with ultraviolet and infrared cutoffs, describing the interaction of relativistic electrons and positrons in a Coulomb potential with photons in Coulomb gauge. The interaction includes both interaction of the current-density with transversal photons and the Wick ordering corrected Coulomb interaction of charge-density with itself.

We prove that the Hamiltonian is self-adjoint and has a ground state for sufficiently small coupling constants.

1 Introduction

In [15] an Hamiltonian with cutoffs describing relativistic electrons and positrons in a Coulomb potential interacting with transversal photons in Coulomb gauge is considered.

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In that note [15], results concerning the self-adjointness of the Hamiltonian and the existence of a ground state for it are announced.

In this article, we consider the full model of QED by adding the Wick ordering corrected Coulomb interaction of the charge density with itself to the Hamiltonian described in [15]. We thus study an Hamiltonian with ultraviolet and infrared cutoff functions with respect to the momenta of photons, but also with respect to the momenta of electrons and positrons. The total Hamiltonian in the Fock space of electrons, positrons and photons is then well defined in the Furry picture.

In this paper, we prove results concerning the self-adjointness of the Hamiltonian and the existence of a ground state when the coupling constant is sufficiently small.

In [7], Bach, Fröhlich and Sigal proved the existence of a ground state for the Pauli-Fierz Hamiltonian with an ultraviolet cutoff for photons, and for sufficiently small values of the fine structure constant, without introducing an infrared cutoff. Their result has been extended by Griesemer, Lieb and Loss [20] under the binding condition. For related results see [4, 5, 6, 17, 36].

No-pair Hamiltonians for relativistic electrons in QED have been recently considered in [10, 16, 28, 31]. In [3], Arai has analyzed the hamiltonian of a Dirac particle interacting with the quantum radiation field. In [35], an Hamiltonian without conservation of the particle number is studied. In [1], the scattering theory for the spin fermion model is studied.

The case of relativistic electrons in classical magnetic fields was studied earlier in [30] and [21]. There, it was proven instability for the Brown and Ravenhall model in the free picture. In [21] it is even deduced from this result that instability also holds in QED context, i.e. for the Brown and Ravenhall model in the free picture, coupled to the quantized radiation field with or without cutoff.

In our case, working in the Furry picture and imposing both electronic and photonic cutoffs prevent from instability.

Our methods follow those of [2, 7, 8, 9, 18], in which the spectral theory of the spin-boson and Pauli-Fierz Hamiltonians is studied.

The infrared conditions on the cutoff functions with respect to the momenta of photons are stronger than those appearing in [7] and [20]. The authors plan to weaken these infrared conditions by using the gauge invariance of the model.

Some of the results obtained in this paper have been announced in [15] and [11].

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2 The quantization of the Dirac-Coulomb field

The Dirac electron of mass m_0 in a Coulomb potential is described by the following Hamiltonian:

$$H_D := H_D(e) = c\boldsymbol{\alpha} \cdot \frac{1}{i}\nabla + \beta m_0 c^2 - \frac{e^2 Z}{|x|},$$

acting in the Hilbert space $L^2(\mathbb{R}^3; \mathbb{C}^4)$ with domain $\mathfrak{D}(H_D) = H^1(\mathbb{R}^3; \mathbb{C}^4)$, the Sobolev space of order 1. Here $\hbar = 1$. We refer to [37] for a discussion of the Dirac operator (see also [33]).

Here e is the negative charge of the electron, $-Ze$ is the positive charge of the nucleus and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, β are the Dirac matrices in the standard form:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad i = 1, 2, 3,$$

where σ_i are the usual Pauli matrices.
It is well known that $H_D(e)$ is self-adjoint for $Z \leq 118$ ([37, Theorem 4.4]). The eigenstates of the Dirac-Coulomb operator H_D are labelled by the angular momentum quantum numbers $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$, $m_j = -j, -j+1, \dots, j-1, j$, by the spin-orbit quantum number $k_j = \pm(j + \frac{1}{2}) = \pm 1, \pm 2, \dots$ and by the quantum number $n = 0, 1, 2, \dots$, counting the non-degenerate eigenvalues of the Dirac radial operator associated with k_j. From now on we set $\gamma = (j, m_j, k_j)$.

Let $\psi_{\gamma,n}$ denote the eigenstates of the Hamiltonian H_D. We have

$$H_D(e)\psi_{\gamma,n} = E_{\gamma,n}\psi_{\gamma,n},$$

with

$$E_{\gamma,n} = m_0 c^2 \left(1 + \frac{(Ze^2)^2/c^2}{(n + \sqrt{k_j^2 - (Ze^2)^2/c^2})^2}\right)^{-1/2}$$

Each eigenvalue is degenerate. The infimum of the discrete spectrum is an isolated eigenvalue of multiplicity two. Together with the bound state energy levels the continuum energy levels are given by

$$\pm\omega(p), \qquad \omega(p) = (m_0^2 c^4 + p^2)^{\frac{1}{2}}, \qquad p = |\mathbf{p}| \tag{1}$$

Here $\mathbf{p}$ is the momentum of the electron.

Let $\psi_{\gamma,\pm}(p, x)$ denote the continuum eigenstates of $H_D(e)$. We then have

$$H_D\psi_{\gamma,\pm}(p, x) = \pm\omega(p)\psi_{\gamma,\pm}(p, x) \ .$$

Here, $\psi_{\gamma,n}$ and $\psi_{\gamma,\pm}(p,x)$ are normalized in such a way that

$$\begin{aligned}
\int_{\mathbb{R}^3} \psi^\dagger_{\gamma,n}(x)\psi_{\gamma',n'}(x)\mathrm{d}^3x &= \delta_{nn'}\delta_{\gamma\gamma'}, \\
\int_{\mathbb{R}^3} \psi^\dagger_{\gamma,\pm}(p,x)\psi_{\gamma',\pm}(p',x)\mathrm{d}^3x &= \delta_{\gamma\gamma'}\delta(p-p'), \\
\int_{\mathbb{R}^3} \psi^\dagger_{\gamma,\pm}(p,x)\psi_{\gamma',\mp}(p',x)\mathrm{d}^3x &= 0, \\
\int_{\mathbb{R}^3} \psi^\dagger_{\gamma,n}(x)\psi_{\gamma',\pm}(p,x)\mathrm{d}^3x &= 0.
\end{aligned}$$

Here $\psi^\dagger_{\gamma,\pm}(p,x)$ (resp. $\psi^\dagger_{\gamma,n}(x)$) is the adjoint spinor of $\psi_{\gamma,\pm}(p,x)$ (resp. $\psi_{\gamma,n}(x)$). The spinors $(\psi_{\gamma,n})_{\gamma,n}$ and $(\psi_{\gamma,\pm})_\gamma$ generate a spectral representation of $H_D(e)$.

According to the hole theory [12, 25, 26, 33, 34, 37, 38], the absence in the Dirac theory of an electron with energy $E<0$ and charge e is equivalent to the presence of a positron with energy $-E>0$ and charge $-e$. Indeed, by the charge conjugation $C\psi = i\beta\alpha_2\overline{\psi}$, one obtains

$$CH_D(e)C^{-1} = -\left(c\alpha\cdot\frac{1}{i}\nabla + \beta m_0c^2 + \frac{e^2Z}{|x|}\right).$$

Let us split the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^3;\mathbb{C}^4)$ into

$$\mathfrak{H}_{c^-} = P_{(-\infty,0]}(H_D)\mathfrak{H}, \qquad \mathfrak{H}_d = P_{(0,m_0c^2)}(H_D)\mathfrak{H}, \qquad \mathfrak{H}_{c^+} = P_{[m_0c^2,+\infty)}(H_D)\mathfrak{H}.$$

Here $P_I(H_D)$ denotes the spectral projection of H_D corresponding to the interval I. Since $(\psi_{n,\gamma})_{\gamma,n}$ is a basis in $\mathfrak{H}_d$, every $\psi\in\mathfrak{H}_d$ can be written as

$$\psi(x) = \mathrm{L.i.m}\sum_{\gamma,n} b_{\gamma,n}\,\psi_{n,\gamma}(x)\ ,$$

with

$$\sum_{\gamma,n}|b_{\gamma,n}|^2<\infty.$$

Hence we can identify $\mathfrak{H}_d$ with the Hilbert space $\tilde{\mathfrak{H}}_d = \oplus_\gamma F_\gamma$, where F_γ is a closed subspace of $L^2(\mathbb{R}_+)$. Precisely, there exists a unitary operator U_d from $\mathfrak{H}_d$ onto $\tilde{\mathfrak{H}}_d$ given by

$$U_d\psi = \left(\sum_n b_{\gamma,n}\,\mathbb{1}_{[n,n+1)}(p)\right)_\gamma, \qquad \psi\in\mathfrak{H}_d$$

where $p\in\mathbb{R}_+$.

Similarly, $\mathfrak{H}_{c\pm}$ can be identified with $\tilde{\mathfrak{H}}_{c\pm} := \oplus_\gamma L^2(\mathbb{R}_+)$ by using the unitary operators $U_{c\pm}$:

$$U_{c\pm}\psi = \left(\mathrm{L.i.m}\int \psi^\dagger_{\gamma,\pm}(p,x)\,\psi(x)\mathrm{d}x\right)_\gamma,$$

where $\psi\in\mathfrak{H}_{c\pm}$.

2.1 The Fock space for electrons and positrons

Let $\mathfrak{G}$ be any separable Hilbert space. Let $\otimes_a^n \mathfrak{G}$ denotes the antisymmetric n-th tensor power of $\mathfrak{G}$ appropriate to Fermi-Dirac statistics. We define the Fermi-Fock space over $\mathfrak{G}$, denoted by $\mathfrak{F}_a(\mathfrak{G})$, to be the direct sum

$$\mathfrak{F}_a(\mathfrak{G}) = \mathop{\oplus}_{n=0}^{\infty} \otimes_a^n \mathfrak{G} \ ,$$

where $\otimes_a^0 \mathfrak{G} := \mathbb{C}$. The state Ω will denote the vacuum vector, i.e., the vector $(1, 0, 0, \cdots)$.

Let $\mathfrak{F}_{a,d}$, $\mathfrak{F}_{a,+}$ and $\mathfrak{F}_{a,-}$ be the Fermi-Fock spaces over $\tilde{\mathfrak{H}}_d$, $\tilde{\mathfrak{H}}_{c+}$ and $\tilde{\mathfrak{H}}_{c-}$ respectively. Ω_d, Ω_+ and Ω_- denote the associated vacua.

In the Furry picture, the Fermi-Fock space for electrons and positrons, denoted by $\mathfrak{F}_D$, is then the following Hilbert space

$$\mathfrak{F}_D = \mathfrak{F}_{a,d} \otimes \mathfrak{F}_{a,+} \otimes \mathfrak{F}_{a,-} \tag{2}$$

The vector $\Omega_d \otimes \Omega_+ \otimes \Omega_-$ is the vacuum of electrons and positrons.

One has

$$\mathfrak{F}_D = \mathop{\oplus}_{q,r,s=0}^{\infty} \mathfrak{F}_a^{(q,r,s)}$$

where $\mathfrak{F}_d^{(q,r,s)} = (\otimes_a^q \tilde{\mathfrak{H}}_d) \otimes (\otimes_a^r \tilde{\mathfrak{H}}_{c+}) \otimes (\otimes_a^s \tilde{\mathfrak{H}}_{c-})$.

Let us remark that $\mathfrak{F}_D$ is unitarily equivalent to $\mathfrak{F}_a(\tilde{\mathfrak{H}}_d \oplus \tilde{\mathfrak{H}}_{c+} \oplus \tilde{\mathfrak{H}}_{c-})$. (See [13] and [14]).

2.2 Creation and annihilation operators

For every $\varphi \in \mathfrak{H}$ we define in $\mathfrak{F}_a(\mathfrak{H})$ the annihilation operator, denoted by $b(\varphi)$, which maps $\otimes_a^{n+1} \mathfrak{H}$ into $\otimes_a^n \mathfrak{H}$:

$$b(\varphi)\left(A_{n+1}(\varphi_1 \otimes \ldots \otimes \varphi_{n+1})\right)$$

$$= \frac{\sqrt{n+1}}{(n+1)!} \sum_\sigma sgn(\sigma) \left(\varphi, \varphi_{\sigma(1)}\right) \varphi_{\sigma(2)} \otimes \ldots \otimes \varphi_{\sigma(n+1)}$$

where $\varphi_i \in \mathfrak{H}$.

The creation operator, denoted by $b^*(\varphi)$, is the adjoint of $b(\varphi)$. The operators $b^*(\varphi)$ and $b(\varphi)$ are bounded in $\mathfrak{F}_a(\mathfrak{H})$ and $\|b(\varphi)\| = \|b^*(\varphi)\| = \|\varphi\|$.

We now define the annihilation and creation operators in $\mathfrak{F}_D$.

2.2.1 Bound states

For every (γ, n), we define in $\mathfrak{F}_D$ the annihilation operator, denoted by $b_{\gamma,n}$, which maps $\mathfrak{F}^{(q+1,r,s)}$ into $\mathfrak{F}^{(q,r,s)}$:

$$b_{\gamma,n}\left(A_{q+1}(f_1 \otimes \ldots \otimes f_{q+1}) \otimes A_r(g_1 \otimes \ldots \otimes g_r) \otimes A_s(h_1 \otimes \ldots \otimes h_s)\right)$$

$$= [b(U_d\, \psi_{\gamma,n}) A_{q+1}(f_1 \otimes \ldots \otimes f_{r+1})] \otimes A_r(g_1 \otimes \ldots \otimes g_r) \otimes A_s(h_1 \otimes \ldots \otimes h_s)$$

where $f_i \in \tilde{\mathfrak{H}}_d$, $g_j \in \tilde{\mathfrak{H}}_{c+}$ and $h_k \in \tilde{\mathfrak{H}}_{c-}$.

The creation operator $b^*_{\gamma,n}$ is the adjoint of $b_{\gamma,n}$.

$b^*_{\gamma,n}$ and $b_{\gamma,n}$ are bounded operators in $\mathfrak{F}_D$.

2.2.2 Electrons in the continuum

For every $g \in \tilde{\mathfrak{H}}_{c+}$, we define in $\mathfrak{F}_D$ the annihilation operator, denoted by $b_+(g)$, which maps $\mathfrak{F}^{(q,r+1,s)}$ into $\mathfrak{F}^{(q,r,s)}$ as follows

$$b_+(g)\left(A_q(f_1 \otimes \ldots \otimes f_q) \otimes A_{r+1}(g_1 \otimes \ldots \otimes g_{r+1}) \otimes A_s(h_1 \otimes \ldots \otimes h_s)\right)$$

$$= A_q(f_1 \otimes \ldots \otimes f_q) \otimes [(-1)^q\, b(g) A_{r+1}(g_1 \otimes \ldots \otimes g_{r+1})] \otimes A_s(h_1 \otimes \ldots \otimes h_s)$$

The creation operator $b^*_+(g)$ is the adjoint of $b_+(g)$. $b^*_+(g)$ and $b_+(g)$ are bounded operators in $\mathfrak{F}_D$.

We set, for every $\psi \in \tilde{\mathfrak{H}}_{c+}$,

$$\begin{aligned} b_{\gamma,+}(\psi) &= b_+(P^+_\gamma \psi) \\ b^*_{\gamma,+}(\psi) &= b^*_+(P^+_\gamma \psi) \end{aligned}$$

where P^+_γ is the projection of $\tilde{\mathfrak{H}}_{c+}$ onto the γ-th component.

2.2.3 Positrons

For every $h \in \tilde{\mathfrak{H}}_{c-}$, we define in $\mathfrak{F}_D$ the annihilation operator, denoted by $b_-(h)$, which maps $\mathfrak{F}^{(q,r,s+1)}$ into $\mathfrak{F}^{(q,r,s)}$:

$$b_-(h)(A_q(f_1 \otimes \ldots \otimes f_q) \otimes A_r(g_1 \otimes \ldots \otimes g_r) \otimes A_{s+1}(h_1 \otimes \ldots \otimes h_{s+1}))$$

$$= A_q(h_1 \otimes \ldots \otimes f_q) \otimes A_r(g_1 \otimes \ldots \otimes g_r) \otimes [(-1)^{q+r} b(h) A_{s+1}(h_1 \otimes \ldots \otimes h_{s+1})]$$

The creation operator $b^*_-(h)$ is the adjoint of $b_-(h)$. $b^*_-(h)$ and $b_-(h)$ are bounded operators in $\mathfrak{F}_D$.

We set for every $\psi \in \tilde{\mathfrak{H}}_{c-}$,

$$\begin{aligned} b_{\gamma,-}(\psi) &= b_-(P^-_\gamma \psi) \\ b^*_{\gamma,-}(\psi) &= b^*_-(P^-_\gamma \psi) \end{aligned}$$

where P^-_γ is the projection of $\tilde{\mathfrak{H}}_{c-}$ onto the γ-th component.

A simple computation shows that the following anti-commutation relations hold

$$\{b^*_{\gamma,n}, b_{\beta,m}\} = \delta_{\gamma,\beta}\, \delta_{n,m}$$

$$\{b^*_{\gamma,n}, b^*_{\beta,m}\} = 0 = \{b_{\gamma,n}, b_{\beta,m}\}$$

$$\{b_{\gamma,+}(\psi_1), b^*_{\beta,+}(\psi_2)\} = \delta_{\gamma,\beta}(P^+_\gamma \psi_1, P^+_\gamma \psi_2)_{L^2(\mathbb{R}_+)}, \qquad \psi_i \in \tilde{\mathfrak{H}}_{c+}$$

$$\{b_{\gamma,-}(\psi_1), b^*_{\beta,-}(\psi_2)\} = \delta_{\gamma,\beta}(P^-_\gamma \psi_1, P^-_\gamma \psi_2)_{L^2(\mathbb{R}_+)}, \qquad \psi_i \in \tilde{\mathfrak{H}}_{c-}$$

Furthermore

$$\{b^{\#}_{\gamma,\pm}(\psi), b^{\#}_{\beta,n}\} = 0.$$

Here $b^{\#}$ is b or b^*.

$$\{b_{\gamma,+}(\psi_1), b_{\beta,-}(\psi_2)\} = \{b_{\gamma,+}(\psi_1), b^*_{\beta,-}(\psi_2)\} = 0,$$

$$\{b^*_{\gamma,+}(\psi_1), b_{\beta,-}(\psi_2)\} = \{b^*_{\gamma,+}(\psi_1), b^*_{\beta,-}(\psi_2)\} = 0,$$

where $\psi_1 \in \tilde{\mathfrak{H}}_{c+}$ and $\psi_2 \in \tilde{\mathfrak{H}}_{c-}$. One should remark that, in contrast to [37], the charge conjugation operator C is not included in the definition of the annihilation and creation operators for the positrons.

Our definition is close to the hole theory and is the one occurring in many text books in Quantum Field Theory as in [25], [26], [34] and [38]. The other method for the quantization of the Dirac field is close to the so-called symmetric theory of charge. Both approaches are described in [26].

In the first approach the Fock space for electrons and positrons is built upon the subspaces $\tilde{\mathfrak{H}}_d$, $\tilde{\mathfrak{H}}_{c+}$ and $\tilde{\mathfrak{H}}_{c-}$ of the one-particle Hilbert space and, in the second case, upon the subspaces $\tilde{\mathfrak{H}}_d$, $\tilde{\mathfrak{H}}_{c+}$ and $C'\tilde{\mathfrak{H}}_{c-}$ where $C' = U_{c-}\, C\, U_{c-}^{-1}$.

As in [32, chapter X], we introduce operator-valued distributions $b_{\gamma,\pm}(p)$ and $b^*_{\gamma,\pm}(p)$ such that we write

$$\begin{aligned} b_{\gamma,\pm}(\psi) &= \int_{\mathbb{R}^+} \mathrm{d}p\; b_{\gamma,\pm}(p)\,\overline{(P^{\pm}_{\gamma}\psi)\,(p)} \\ b^*_{\gamma,\pm}(\psi) &= \int_{\mathbb{R}^+} \mathrm{d}p\; b^*_{\gamma,\pm}(p)\,(P^{\pm}_{\gamma}\psi)\,(p) \end{aligned}$$

where $\psi \in \tilde{\mathfrak{H}}_{c+} \oplus \tilde{\mathfrak{H}}_{c-}$.

We now give a representation of $b_{\gamma,\pm}(p)$ and $b^*_{\gamma,\pm}(p)$. Let $\mathfrak{D}_D$ denote the set of smooth vectors $\Phi \in \mathfrak{F}_D$ for which $\Phi^{(q,r,s)}$ has a compact support and $\Phi^{(q,r,s)} = 0$ for all but finitely many (q,r,s).

For every (p,γ), $b_{\gamma,+}(p)$ maps $\mathfrak{F}_a^{(q,r+1,s)} \cap \mathfrak{D}_D$ into $\mathfrak{F}_a^{(q,r,s)} \cap \mathfrak{D}_D$ and we have

$$\begin{aligned} &(b_{\gamma,+}(p)\Phi)^{(q,r,s)}(p_1,\gamma_1,\ldots,p_q,\gamma_q;p_1,\gamma_1,\ldots,p_r,\gamma_r;p_1,\gamma_1,\ldots,p_s,\gamma_s) \\ &= \sqrt{r+1}(-1)^q\Phi^{(q,r+1,s)}(p_1,\gamma_1,\ldots,p_q,\gamma_q;p,\gamma,p_1,\gamma_1,\ldots,p_r,\gamma_r;p_1,\gamma_1,\ldots,p_s,\gamma_s) \end{aligned}$$

$b^*_{\gamma,+}(p)$ is then given by:

$$\begin{aligned} &(b^*_{\gamma,+}(p)\Phi)^{(q,r+1,s)}(p_1,\gamma_1,\ldots,p_q,\gamma_q;p_1,\gamma_1,\ldots,p_{r+1},\gamma_{r+1};p_1,\gamma_1,\ldots,p_s,\gamma_s) = \\ &\frac{(-1)^q}{\sqrt{r+1}}\sum_{i=1}^{r+1}(-1)^{i+1}\delta_{\gamma_i\gamma}\delta(p-p_i) \\ &\Phi^{(q,r,s)}(p_1,\gamma_1,\ldots,p_q,\gamma_q;p_1,\gamma_1,\ldots,\widehat{p_i,\gamma_i},\ldots,p_{r+1},\gamma_{r+1};p_1,\gamma_1,\ldots,p_s,\gamma_s) \end{aligned}$$

where $\widehat{\cdot}$ denotes that the i-th variable has to be omitted.

Similarly $b_{\gamma,-}(p)$ maps $\mathfrak{F}_a^{(q,r,s+1)} \cap \mathfrak{D}_D$ into $\mathfrak{F}_a^{(q,r,s)} \cap \mathfrak{D}_D$ such that

$$\begin{aligned}&(b_{\gamma,-}(p)\Phi)^{(q,r,s)}(p_1,\gamma_1,\dots,p_q,\gamma_q;p_1,\gamma_1,\dots,p_r,\gamma_r;p_1,\gamma_1,\dots,p_s,\gamma_s) = \\ &\sqrt{s+1}(-1)^{q+r}\Phi^{(q,r,s+1)}(p_1,\gamma_1,\dots,p_q,\gamma_q;p_1,\gamma_1,\dots,p_r,\gamma_r;p,\gamma,p_1,\gamma_1,\dots,p_s,\gamma_s)\end{aligned}$$

$b^*_{\gamma,-}(p)$ is then given by

$$\begin{aligned}&(b^*_{\gamma,-}(p)\Phi)^{(q,r,s+1)}(p_1,\gamma_1,\dots,p_q,\gamma_q;p_1,\gamma_1,\dots,p_r,\gamma_r;p_1,\gamma_1,\dots,p_{s+1},\gamma_{s+1}) = \\ &\frac{1}{\sqrt{s+1}}(-1)^{q+r}\sum_{i=1}^{s+1}(-1)^{i+1}\delta_{\gamma,\gamma_i}\delta(p-p_i) \\ &\Phi^{(q,r,s)}(p_1,\gamma_1,\dots,p_q,\gamma_q;p_1,\gamma_1,\dots,p_r,\gamma_r;p_1,\gamma_1,\dots,\widehat{p_i,\gamma_i},\dots,p_{s+1},\gamma_{s+1})\end{aligned}$$

Let us recall that $\Phi^{(q,r,s)}$ is antisymmetric in the bound states, the electron in the continuum and the positron variables separately. We have

$$\{b_{\gamma,+}(p), b^*_{\gamma',+}(p')\} = \{b_{\gamma,-}(p), b^*_{\gamma',-}(p')\} = \delta_{\gamma,\gamma'}\delta(p-p')$$

Any other anti-commutation relation is equal to zero. In particular we have

$$\{b_{\gamma,\pm}(p), b^\#_{\gamma',n}\} = \{b^*_{\gamma,\pm}(p), b^\#_{\gamma',n}\} = 0$$

where $b^\#$ is b or b^*.

2.2.4 The Hamiltonian for the quantized Dirac-Coulomb field

The quantization of the Dirac-Coulomb Hamiltonian H_D, denoted by $d\Gamma(H_D)$, is now given by

$$\begin{aligned}d\Gamma(H_D) \;&=\; \sum_{\gamma,n} E_{\gamma,n}\, b^*_{\gamma,n}\, b_{\gamma,n} + \sum_\gamma \int \mathrm{d}p\; \omega(p)\, b^*_{\gamma,+}(p)\, b_{\gamma,+}(p) \\ &\quad + \sum_\gamma \int \mathrm{d}p\; \omega(p)\, b^*_{\gamma,-}(p)\, b_{\gamma,-}(p)\ ,\end{aligned}$$

with $\omega(p)$ given in (1). The operator $d\Gamma(H_D)$ is the Hamiltonian of the quantized Dirac-Coulomb field. It is well defined on the dense subset $\mathfrak{D}_D$ and it is essentially self-adjoint on $\mathfrak{D}_D$. The self-adjoint extension will be also denoted by $d\Gamma(H_D)$ with domain $\mathfrak{D}(d\Gamma(H_D))$. Moreover the operator number of electrons and positrons, denoted by N_D, is given by

$$N_D = \sum_{\gamma,n} b^*_{\gamma,n} b_{\gamma,n} + \sum_\gamma \int \mathrm{d}p\; b^*_{\gamma,+}(p) b_{\gamma,+}(p) + \sum_\gamma \int \mathrm{d}p\; b^*_{\gamma,-}(p) b_{\gamma,-}(p)\ .$$

The operator N_D is essentially self-adjoint on $\mathfrak{D}_D$. The self-adjoint extension will be also denoted by N_D with domain $\mathfrak{D}(N_D)$.
Let Δ denote the set of all eigenvalues of the operator

$$d\Gamma(H_D)_d := \sum_{\gamma,n} E_{\gamma,n} b^*_{\gamma,n} b_{\gamma,n}$$

Each $E_{\gamma,n}$ is in Δ and we have

$$0 < E_0 = \inf_{\gamma,n} E_{\gamma,n}, \quad \text{and} \quad \inf_{E\in\Delta} E = 0.$$

0 is the eigenvalue associated with Ω_D. Furthermore

$$\sigma(d\Gamma(H_D)) = \Delta \cup [m_0 c^2, \infty) \ .$$

The set $[m_0c^2,\infty)$ is the absolutely continuous spectrum of $d\Gamma(H_D)$. Let us remark that, for $E\in\Delta$ with $E > m_0c^2$, E is an eigenvalue embedded in the continuous spectrum.

Let $P_D^\perp$ be the orthogonal projection on the orthogonal subspace to Ω_D in $\mathfrak{F}_D$. Then E_0 is the ground state of the self-adjoint operator $P_D^\perp d\Gamma(H_D) P_D^\perp$ in $P_D^\perp \mathfrak{F}_D$.

2.3 The Fock space for transversal photons

The one photon Hilbert space is given by $L^2(\mathbb{R}^3,\mathbb{C}^2)$. For every f in $L^2(\mathbb{R}^3,\mathbb{C}^2)$, we shall write $f(k,\mu)$ where $k\in\mathbb{R}^3$ is the momentum variable of the photon and $\mu = 1,2$ is its polarization index associated with two given independent real transversal polarizations $\varepsilon_\mu(k)$ of the photon in the Coulomb gauge such that $\varepsilon_\mu(k).\varepsilon_{\mu'}(k) = \delta_{\mu\mu'}$ and $\varepsilon_\mu(k).k = 0$.

Let $\mathfrak{F}_{ph}$ denote the Fock space for transversal photons :

$$\mathfrak{F}_{ph} = \oplus_{n=0}^\infty L^2(\mathbb{R}^3,\mathbb{C}^2)^{\otimes_s^n}$$

where $L^2(\mathbb{R}^3,\mathbb{C}^2)^{\otimes_s^0} = \mathbb{C}$. Here $L^2(\mathbb{R}^3,\mathbb{C}^2)^{\otimes_s^n}$ is the symmetric n-tensor power of $L^2(\mathbb{R}^3,\mathbb{C}^2)$ appropriate for Bose-Einstein statistics.

For $f\in L^2(\mathbb{R}^3,\mathbb{C}^2)$, the annihilation and creation operators, denoted by $a(f)$ and $a^*(f)$ respectively, are now given by:

$$\begin{aligned} a(f) &= \sum_{\mu=1,2} a_\mu(f(\cdot,\mu)) \\ a^*(f) &= \sum_{\mu=1,2} a^*_\mu(f(\cdot,\mu)) \end{aligned}$$

where

$$(a_\mu(f(\cdot,\mu))\Psi)^{(n)}(k_1,\mu_1,k_2,\mu_2,\ldots,k_n,\mu_n) =$$

$$\sqrt{n+1}\int \mathrm{d}^3k\, \overline{f(k,\mu)}\, \Psi^{(n+1)}(k,\mu,k_1,\mu_1,k_2,\mu_2,\ldots,k_n,\mu_n)$$

$$(a_\mu^*(f(\cdot,\mu))\Psi)^{(n)}(k_1,\mu_1,\dots,k_n,\mu_n) =$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} f(k_i,\mu_i)\,\Psi^{(n-1)}(k_1,\mu_1,\dots,\widehat{k_i,\mu_i},\dots,k_n,\mu_n)$$

where $\widehat{\cdot}$ denotes that the i-th variable has to be omitted. Note that $a_\mu^*(f(\cdot,\mu))$ and $a_\mu(f(\cdot,\mu))$ are linear and anti-linear with respect to f respectively, so that we can introduce operator valued distributions, i.e, fields $a_\mu(k)$ and $a_\mu^*(k)$ such that

$$a_\mu(f(\cdot,\mu)) = \int \mathrm{d}^3k\ \overline{f(k,\mu)}\,a_\mu(k)$$

and

$$a_\mu^*(f(\cdot,\mu)) = \int \mathrm{d}^3k\ f(k,\mu)\,a_\mu^*(k)$$

where $f(\cdot,\mu)\in L^2(\mathbb{R}^3)$.

Let $\mathfrak{D}_{ph}$ denote the set of smooth vectors $\Psi\in\mathfrak{F}_{ph}$ for which $\Psi^{(n)}$ has a compact support and $\Psi^{(n)}=0$ for all but finitely many n. Then, for any $\Psi\in\mathfrak{D}_{ph}$, the action of $a_\mu(k)$ and $a_\mu^*(k)$ is given by

$$\begin{aligned}&(a_\mu(k)\Psi)^{(n)}(k_1,\mu_1,k_2,\mu_2,\dots,k_n,\mu_n)\\ &\quad= \sqrt{n+1}\Psi^{(n+1)}(k,\mu,k_1,\mu_1,\dots,k_n,\mu_n)\end{aligned}$$

$$\begin{aligned}&(a_\mu^*(k)\Psi)^{(n+1)}(k_1,\mu_1;k_2,\mu_2,\dots,k_{n+1},\mu_{n+1})\\ &= \frac{1}{\sqrt{n+1}}\sum_{i=1}^{n+1}\delta_{\mu_i,\mu}\,\delta(k_i-k)\,\Psi^{(n)}(k_1,\mu_1,\dots,\widehat{k_i,\mu_i},\dots,k_{n+1},\mu_{n+1})\end{aligned}$$

We have the canonical commutation relations

$$[a_\mu(k),a_{\mu'}^*(k')] = \delta_{\mu\mu'}\delta(k-k')$$

Any other commutation relation is equal to zero.

The Hamiltonian of the quantized electromagnetic field, denoted by H_{ph}, is

$$H_{ph} = \sum_{\mu=1,2}\int \mathrm{d}^3k\ \omega(k)\,a_\mu^*(k)\,a_\mu(k)$$

where $\omega(k)=c|k|$. H_{ph} is essentially self-adjoint on $\mathfrak{D}_{ph}$.

The state $(1,0,\dots)\in\mathfrak{F}_{ph}$ is the vacuum of photons and will be denoted Ω_{ph}.

The spectrum of H_{ph} consists of an absolutely continuous spectrum covering $[0,+\infty)$ and a simple eigenvalue, equal to zero, whose the corresponding eigenvector is the vacuum state $\Omega\in\mathfrak{F}_{ph}$.

2.4 The total Hamiltonian and the main results

The Fock space for electrons, positrons and photons is the following Hilbert space:

$$\mathfrak{F} = \mathfrak{F}_D \otimes \mathfrak{F}_{ph}.$$

$\Omega = \Omega_D \otimes \Omega_{ph}$ is the vacuum of $\mathfrak{F}$.

The free Hamiltonian for electrons, positrons and photons, denoted by H_0, is the following operator in $\mathfrak{F}$:

$$H_0 = d\Gamma(H_D) \otimes \mathbb{1}_{ph} + \mathbb{1}_D \otimes H_{ph}$$

H_0 is a self-adjoint operator with domain

$$\mathfrak{D}\left(d\Gamma(H_D) \otimes \mathbb{1}_{ph}\right) \cap \mathfrak{D}\left(\mathbb{1}_D \otimes H_{ph}\right) \; .$$

The operator H_0 has the same point spectrum as $d\Gamma(H_D)$ and its continuous spectrum covers the half axis $[0, \infty)$. Hence the point spectrum of $d\Gamma(H_D)$ is embedded in the continuous spectrum of H_0. The eigenfunctions of H_0 corresponding to the eigenvalues E have the form $\varphi \otimes \Omega_{ph}$, where φ are the eigenfunctions of $d\Gamma(H_D)$ corresponding to the eigenvalues E.

Let us now describe the interaction between electrons, positrons and photons in Coulomb gauge that we consider.

Let us first recall the physical interaction in Q. E. D. with the Coulomb gauge.

The quantized Dirac-Coulomb field is given by

$$\begin{aligned}\psi(x) \;&=\; \sum_{\gamma,n} b_{\gamma,n}\psi_{\gamma,n}(x)\\ &+\sum_{\gamma}\int_{\mathbb{R}^+}\mathrm{d}p\; b_{\gamma,+}(p)\psi_{\gamma,+}(p,x) + \sum_{\gamma}\int_{\mathbb{R}^+}\mathrm{d}p\; b^*_{\gamma,-}(p)\psi_{\gamma,-}(p,x)\end{aligned}$$

together with

$$\begin{aligned}\psi^\dagger(x) \;&=\; \sum_{\gamma,n} b^*_{\gamma,n}\psi^\dagger_{\gamma,n}(x) + \sum_{\gamma}\int_{\mathbb{R}^+}\mathrm{d}p\; b^*_{\gamma,+}(p)\psi^\dagger_{\gamma,n}(p,x)\\ &+\sum_{\gamma}\int_{\mathbb{R}^+}\mathrm{d}p\; b_{\gamma,-}(p)\psi^\dagger_{\gamma,-}(p,x)\end{aligned}$$

for the quantized adjoint spinor.

Here $b_{\gamma,n}$, $b^*_{\gamma,n}$, $b_{\gamma,\pm}(p)$ and $b^*_{\gamma,\pm}(p)$ are the annihilation and creation operators defined above.

The density of charge is then given by

$$\rho(x) = e : \psi^\dagger(x)\, \psi(x) :$$

and the density of current by

$$j(x) = ec : \psi^\dagger(x)\, \alpha\, \psi(x) :$$

Here : : is the normal ordering and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$.

The interaction of electrons and positrons with photons is given by two terms. The first one is

$$\frac{e^2}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mathrm{d}^3x\, \mathrm{d}^3x' \; \frac{\rho(x)\, \rho(x')}{|x - x'|} \; , \tag{3}$$

and describes the Coulomb interaction of densities of charges. The second one,

$$-e \int_{\mathbb{R}^3} \mathrm{d}^3x \; j(x) \cdot A(x) \; ,$$

describes the interaction between the current and the transversal photons. Here $A(x)$ is the quantized electromagnetic field in Coulomb gauge

$$A(x) = \sum_{\mu=1,2} \left(\frac{\hbar}{2\varepsilon_0 (2\pi)^3} \right)^{1/2} \int \frac{\mathrm{d}^3 k}{\sqrt{2\omega_{ph}(k)}} \left(\varepsilon_\mu(k) e^{ik\cdot x} a_\mu(k) + \varepsilon_\mu(k) e^{-ik\cdot x} a^*_\mu(k) \right)$$

(See [12] and [19]), $a^*_\mu(k)$ and $a_\mu(k)$ are the creation and annihilation operators on $\mathfrak{F}_{ph}$.

It is a known fact that in order to get a well-defined total Hamiltonian, one first needs to rewrite (3) with a Wick ordering correction, i.e., in (3), the term $e^2 : \psi^\dagger(x)\psi(x) :: \psi^\dagger(x')\psi(x') :$ is replaced by $e^2 : \psi^\dagger(x)\psi(x)\, \psi^\dagger(x')\psi(x') :$. Furthermore, we have to introduce several cutoff functions in the Dirac-Coulomb field and the electromagnetic vector potential in order to get a well defined total Hamiltonian in the Fock space (see [29]).

Then, by a formal computation, the interaction term can be expressed in terms of the annihilation and creation operators $b_{\gamma,n}$, $b^*_{\gamma,n}$, $b_{\gamma,\pm}(p)$, $b^*_{\gamma,\pm}(p)$, $a_\mu(k)$ and $a^*_\mu(k)$.

The interaction between electrons, positrons and photons consists in two terms. The first term in the interaction, denoted by $H_I^{(1)}$, is given by

$$
\begin{aligned}
&H_I^{(1)} = \\
&\sum_{\gamma,\gamma',n,\ell}\sum_{\mu=1,2}\int \mathrm{d}^3k\left(G^{\mu}_{d,\gamma,\gamma',n,\ell}(k)b^*_{\gamma,n}b_{\gamma',\ell}a^*_{\mu}(k)+\text{h.c.}\right)\\
&+\sum_{\epsilon=+,-}\sum_{\gamma,\gamma',n}\sum_{\mu=1,2}\int \mathrm{d}^3k\,\mathrm{d}p\left(G^{\mu}_{d,\epsilon,\gamma,\gamma',n}(p;k)\Big(b^*_{\gamma,n}b_{\gamma',\epsilon}(p)\right.\\
&\qquad\qquad\left.+b^*_{\gamma,\epsilon}(p)b_{\gamma',n}\Big)a^*_{\mu}(k)+\text{h.c.}\right)\\
&+\sum_{\gamma,\gamma'}\sum_{\mu=1,2}\int \mathrm{d}^3k\,\mathrm{d}p\,\mathrm{d}p'\left(G^{\mu}_{+,-,\gamma,\gamma'}(p,p';k)\Big(b^*_{\gamma,+}(p)b^*_{\gamma',-}(p')\right.\\
&\qquad\qquad\left.+b_{\gamma,-}(p)b_{\gamma',+}(p')\Big)a^*_{\mu}(k)+\text{h.c.}\right)\\
&+\sum_{\epsilon=+,-}\sum_{\gamma,\gamma'}\sum_{\mu=1,2}\int \mathrm{d}^3k\,\mathrm{d}p\,\mathrm{d}p'\left(G^{\mu}_{\epsilon,\epsilon,\gamma,\gamma'}(p,p';k)b^*_{\gamma,\epsilon}(p)b_{\gamma',\epsilon}(p')a^*_{\mu}(k)+\text{h.c.}\right)
\end{aligned}
$$

For the second term let us introduce some notations. In the case of electrons, ξ will be equal to (γ,p) and (γ,n) with $\int \mathrm{d}\xi=\sum_{\gamma}\int \mathrm{d}p+\sum_{\gamma,n}$. In the case of positrons, ξ will be equal to (γ,p) with $\int \mathrm{d}\xi=\sum_{\gamma}\int \mathrm{d}p$. The second term of the interaction, denoted by $H_I^{(2)}$, is an operator in $\mathfrak{F}_D$ given by

$$
\begin{aligned}
&H_I^{(2)} = \\
&\int \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\xi_4\,F^{(1)}(\xi_1,\xi_2,\xi_3,\xi_4)b^*_+(\xi_1)b^*_-(\xi_2)b_+(\xi_3)b_-(\xi_4)\\
&+\sum_{\epsilon=+,-}\int \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\xi_4\,F^{(2)}_{\epsilon}(\xi_1,\xi_2,\xi_3,\xi_4)b^*_{\epsilon}(\xi_1)b^*_{\epsilon}(\xi_2)b_{\epsilon}(\xi_3)b_{\epsilon}(\xi_4)\\
&+\sum_{\substack{\epsilon,\epsilon'=+,-,\\ \epsilon+\epsilon'=0}}\int \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\xi_4\left(F^{(3)}_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4)b^*_{\epsilon}(\xi_1)b_{\epsilon}(\xi_2)b_{\epsilon'}(\xi_3)b_{\epsilon}(\xi_4)\right.\\
&\qquad\qquad\left.-\overline{F^{(3)}_{\epsilon,\epsilon'}(\xi_4,\xi_2,\xi_3,\xi_1)}b^*_{\epsilon}(\xi_1)b^*_{\epsilon}(\xi_2)b^*_{\epsilon'}(\xi_3)b_{\epsilon}(\xi_4)\right)\\
&+\sum_{\substack{\epsilon,\epsilon'=+,-,\\ \epsilon+\epsilon'=0}}\int \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\xi_4\left(F^{(4)}_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4)b_{\epsilon}(\xi_1)b_{\epsilon}(\xi_2)b_{\epsilon'}(\xi_3)b_{\epsilon'}(\xi_4)\right.\\
&\qquad\qquad\left.+\overline{F^{(4)}_{\epsilon,\epsilon'}(\xi_4,\xi_2,\xi_3,\xi_1)}b^*_{\epsilon}(\xi_1)b^*_{\epsilon}(\xi_2)b^*_{\epsilon'}(\xi_3)b^*_{\epsilon'}(\xi_4)\right)\,.
\end{aligned}
$$

Furthermore, we suppose that

$$F^{(1)}(\xi_1,\xi_2,\xi_3,\xi_4)=\overline{F^{(1)}(\xi_3,\xi_4,\xi_1,\xi_2)}$$

and

$$F^{(2)}_\epsilon(\xi_1,\xi_2,\xi_3,\xi_4) = \overline{F^{(2)}_\epsilon(\xi_4,\xi_3,\xi_2,\xi_1)}\ .$$

We set

$$F^{(2),a}_\epsilon(\xi_1,\xi_2,\xi_3,\xi_4) = F^{(2)}_\epsilon(\xi_1,\xi_2,\xi_3,\xi_4) - F^{(2)}_\epsilon(\xi_1,\xi_2,\xi_4,\xi_3)$$

Definition 2.1. *The Hamiltonian for relativistic electrons and positrons in a Coulomb potential interacting with photons in Coulomb gauge that we consider is given by*

$$H(g_1,\,g_2) := H_0 + g_1\,H_I^{(1)} + g_2 H_I^{(2)} \otimes \mathbb{1}\ ,$$

where g_i, $i = 1,\,2$, are real coupling constants.

It is easy to show that $H(g)$ is a symmetric operator in $\mathcal{F}$ as soon as the kernels $F^{(i)}$'s and $G^{(i)}$'s are square integrable. With stronger conditions on the $F^{(i)}$'s and $G^{(i)}$'s, we recover a self-adjoint operator with a ground state, as stated in Theorem 2.2 and 2.3 below.

Let, for $\beta = 0, 1, 2$,

$$C_\beta = \sum_{\mu=1,2}\left(\sum_{\gamma,\gamma',n,\ell}\int_{\mathbb{R}^3}|G^\mu_{d,\gamma,\gamma',n,\ell}(k)|^2\omega(k)^{-\beta}\mathrm{d}^3k\right)^{1/2}$$

$$+\sum_{\varepsilon=+,-}\sum_{\mu=1,2}\left(\sum_{\gamma,\gamma',n}\int_{\mathbb{R}^3\times\mathbb{R}^+}|G^\mu_{d,\varepsilon,\gamma,\gamma',n}(p;k)|^2\omega(k)^{-\beta}\mathrm{d}p\,\mathrm{d}^3k\right)^{1/2}$$

$$+\sum_{r}\sum_{\mu=1,2}\left(\sum_{\gamma,\gamma'}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^+}|G^\mu_{r,\gamma,\gamma'}(p,p';k)|^2\omega(k)^{-\beta}\mathrm{d}p\,\mathrm{d}p'\,\mathrm{d}^3k\right)^{1/2}$$

where $r = \{+,+\}, \{+,-\}, \{-,-\}$.

We now state our main results.

Theorem 2.2. *We assume that every $F^{(j)} \in L^2$ ($j = 1, 2, 3, 4$). Furthermore, we suppose that $C_0 < \infty$, $C_1 < \infty$ and*

$$\frac{|g_1|}{\sqrt{E_0}}C_1 + \frac{|g_2|}{E_0}\left(\frac{1}{\sqrt{2}}\|F^{(1)}\| + \|F^{(2),a}_+\| + \|F^{(2),a}_-\|\right) < 1 \tag{4}$$

Then, $H(g_1, g_2)$ is self adjoint on the domain $\mathfrak{D}(H_0)$.

Theorem 2.3. *We assume that every $F^{(j)} \in L^2$ ($j = 1, 2, 3, 4$), and that $C_0 < \infty$, $C_1 < \infty$ and $C_2 < \infty$. Furthermore, we suppose that* (4) *holds true. Then there exists $g_0 > 0$ such that for $|g_1| + |g_2| \le g_0$, the self-adjoint operator $H(g_1, g_2)$ has a ground state.*

Remark 2.4. *The same proof shows that the operator $(P_D^\perp \otimes \mathbb{1})H(g)(P_D^\perp \otimes \mathbb{1})$, with domain $(P_D^\perp \otimes \mathbb{1})\mathfrak{F}$, has a ground state under the same conditions. This ground state is a perturbation of the one associated with the ground state energy E_0 of $(P_D^\perp \otimes \mathbb{1})d\Gamma(H_D)(P_D^\perp \otimes \mathbb{1})$ in $(P_D^\perp \otimes \mathbb{1})\mathfrak{F}$. (Recall that E_0 is also the ground state energy of the Dirac-Coulomb operator H_D.)*

3 Proofs of Theorems 2.2 and 2.3

Let

$$\begin{aligned}\mathcal{D} &= \{\gamma = (j, m_j, k_j); j = \frac{1}{2}, \frac{3}{2}, \dots, m_j = -j+1, \dots, j, k_j = \pm 1, \pm 2, \dots, \}\\ \mathcal{D}_d &= \{(\gamma, n); \gamma \in \mathcal{D} \text{ and } n = 0, 1, 2, \dots\}\end{aligned}$$

We set

$$\begin{aligned}\Sigma_+ &= \Sigma_- = \mathbb{R}^+ \times \mathcal{D}\\ \Sigma_d &= \mathbb{R}^+ \times \mathcal{D}_d\end{aligned}$$

We will write $\xi = (p, \gamma)$ where (p, γ) belongs to Σ_+ and Σ_- respectively and $\xi_d = (p, \gamma, n)$ where (p, γ, n) belongs to Σ_d. Then

$$L^2(\Sigma_\pm) = \left\{ f : \Sigma_\pm \to \mathbb{C};\ \int_{\Sigma_\pm} |f(\xi)|^2 \mathrm{d}\xi := \Sigma_{\gamma\in\mathcal{D}} \int_{\mathbb{R}^+} |f(p,\gamma)|^2 \mathrm{d}p < \infty \right\} .$$

Thus the map

$$(f_\gamma(\cdot))_{\gamma\in\mathcal{D}} \in \bigoplus_{\gamma\in\mathcal{D}} L^2(\mathbb{R}^+) \longrightarrow f(\xi) = f(p,\gamma) \in L^2(\Sigma_\pm) ,$$

is unitary and we will identify the space $\tilde{\mathfrak{H}}_{c_+}$ (resp. $\tilde{\mathfrak{H}}_{c_-}$) introduced in Section 2 with $L^2(\Sigma_+)$(resp. $L^2(\Sigma_-)$).

Similarly we identify the space $\tilde{\mathfrak{H}}_d$ with a closed subspace, denoted by F, of $L^2(\Sigma_d)$. According to Section 2, F is the closure of the following subspace

$$\left\{ \sum_{(\gamma,n)\in\mathcal{D}_d} c_{\gamma,n}\, \mathbb{1}_{[n,n+1]}(p) \ ; \sum_{(\gamma,n)\in\mathcal{D}_d} |c_{\gamma,n}|^2 < \infty \right\}$$

Hence the Fermi-Fock space $\mathfrak{F}_D$ will be identified with

$$\mathfrak{F}_a(F) \otimes \mathfrak{F}_a(L^2(\Sigma_+)) \otimes \mathfrak{F}_a(L^2(\Sigma_-))$$

Let us now define the new annihilation and creation operators. Recall that (See [37])

$$\begin{aligned}\left\| \int_{\mathbb{R}^+} \mathrm{d}p\, \overline{h_\pm(p,\gamma)}\, b_{\gamma\pm}(p) \right\| &= \left\| \int \mathrm{d}p\, h_\pm(p,\gamma)\, b^*_{\gamma,\pm}(p) \right\|\\ &= \|h_\pm(\cdot,\gamma)\|_{L^2(\mathbb{R}^+)}\end{aligned}$$

Thus the series $\displaystyle\sum_{\gamma\in\mathcal{D}} \int_{\mathbb{R}^+} \mathrm{d}p\, \overline{h_\pm(p,\gamma)} b_{\gamma,\pm}(p)$ and $\displaystyle\sum_{\gamma\in\mathcal{D}} \int_{\mathbb{R}^+} \mathrm{d}p\, h_\pm(p,\gamma) b^*_{\gamma,\pm}(p)$ are normally convergent in $\ell^2(\mathcal{D},\ B(\mathfrak{F}_a(\tilde{\mathfrak{H}}_{c\pm}))$ respectively, for every $h_\pm$ in $L^2(\Sigma_\pm)$.

Here $B(\mathfrak{F}_a(\tilde{\mathfrak{H}}_{c\pm}))$ is the set of bounded operators in $\mathfrak{F}_a(\tilde{\mathfrak{H}}_{c\pm})$ respectively.

We then set, for every $h_\pm$ in $L^2(\Sigma_\pm)$ respectively,

$$
\begin{aligned}
b_\pm(h_\pm) &:= \int \mathrm{d}\xi\, \overline{h_\pm(\xi)}\, b_\pm(\xi) = \sum_{\gamma\in\mathcal{D}} \int_{\mathbb{R}^+} \mathrm{d}p\, \overline{h_\pm(p,\gamma)}\, b_{\gamma,\pm}(p) \\
b^*_\pm(h_\pm) &:= \int \mathrm{d}\xi\, h_\pm(\xi)\, b^*_\pm(\xi) = \sum_{\gamma\in\mathcal{D}} \int_{\mathbb{R}^+} \mathrm{d}p\, h_\pm(p,\gamma)\, b^*_{\gamma,\pm}(p)
\end{aligned}
$$

where $b_{\gamma,\pm}(p)$ and $b^*_{\gamma,\pm}(p)$ are defined in Section 2.

Similarly the series $\sum_{(\gamma,n)\in\mathcal{D}_d} \overline{c_{\gamma,n}}\, b_{\gamma,n}$ and $\sum_{(\gamma,n)\in\mathcal{D}_d} c_{\gamma,n}\, b^*_{\gamma,n}$, where $\sum_{(\gamma,n)\in\mathcal{D}_d} |c_{\gamma,n}|^2 < \infty$, are normally convergent in $\ell^2(\mathcal{D}_d, B(\mathfrak{F}_a(\tilde{\mathfrak{H}}_d)))$.

Thus, for any $h_d(\xi_d) = \sum_{(\gamma,n)\in\mathcal{D}_d} c_{\gamma,n}\, \mathbb{1}_{[n,n+1]}(p) \in F \subset L^2(\Sigma_d)$, we set

$$
\begin{aligned}
b_d(h_d) &= \sum_{(\gamma,n)\in\mathcal{D}_d} \overline{c_{\gamma,n}}\, b_{\gamma,n}\ , \\
b^*_d(h_d) &= \sum_{(\gamma,n)\in\mathcal{D}_d} c_{\gamma,n}\, b^*_{\gamma,n}\ ,
\end{aligned}
$$

where $b_{\gamma,n}$ and $b^*_{\gamma,n}$ are defined in Section 2.2.1.

The new canonical anti-commutation relations are the following ones

$$\{b_\pm(h^1_\pm),\ b^*_\pm(h^2_\pm)\} = \langle h^1_\pm, h^2_\pm\rangle_{L^2(\Sigma_\pm)} \tag{5}$$

$$\{b^\#_\pm(h^1_\pm),\ b^\#_\pm(h^2_\pm)\} = 0 \tag{6}$$

$$\{b^\#_\pm(h_\pm),\ b^\#_d(h_d)\} = 0 \tag{7}$$

$$\{b_d(h^1_d),\ b^*_d(h^2_d)\} = (h^1_d, h^2_d)_{L^2(\Sigma_d)} \tag{8}$$

Here $b^\#$ is b or b^*.

We now have the following lemmas whose proofs are well-known.

Lemma 3.1. *Let f be in $L^2(\mathbb{R}^3, \mathbb{C}^2; \omega(k)^{-\beta}\mathrm{d}^3k)$ for $\beta = 0, 1$. We have*

$$
\begin{aligned}
\|a(f)\Psi\|^2_{\mathfrak{F}_{ph}} &\le \left(\sum_{\mu=1,2} \int \frac{|f(k,\mu)|^2}{\omega(k)}\, \mathrm{d}^3k\right) \|H^{1/2}_{ph}\Psi\|^2_{\mathfrak{F}_{ph}} \\
\|a^*(f)\Psi\|^2_{\mathfrak{F}_{ph}} &\le \left(\sum_{\mu=1,2} \int \frac{|f(k,\mu)|^2}{\omega(k)}\, \mathrm{d}^3k\right) \|H^{1/2}_{ph}\Psi\|^2_{\mathfrak{F}_{ph}} \\
&\quad + \left(\sum_{\mu=1,2} \int |f(k,\mu)|^2\, \mathrm{d}^3k\right) \|\Psi\|^2_{\mathfrak{F}_{ph}}
\end{aligned}
$$

for every $\Psi \in \mathfrak{D}(H_{ph})$ *(See [8]).*

Lemma 3.2. *For* $h_\pm \in L^2(\Sigma_\pm)$*,* $b_\pm(h_\pm)$ *and* $b^*_\pm(h_\pm)$ *are bounded operators and we have*

$$\|b_\pm(h)\Psi\|^2 + \|b^*_\pm(h)\Psi\|^2 = \|\Psi\|^2\|h\|^2_{L^2(\Sigma_\pm)}.$$

In particular,

$$\|b_\pm(h)\| = \|b^*_\pm(h)\| = \|h\|_{L^2(\Sigma_\pm)}$$

for every $\Psi \in \mathfrak{F}_a(L^2(\Sigma_\pm))$ *respectively (See [37]).*

Lemma 3.3. *For every* $h_d \in F$*,* $b^\#_d(h_d)$ *is a bounded operator and we have*

$$\|b_d(h_d)\Psi\|^2 + \|b^*_d(h_d)\Psi\|^2 = \|h_d\|^2_{L^2(\Sigma_d)}\|\Psi\|^2$$

for all $\Psi \in \mathfrak{F}_a(L^2(\Sigma_d))$*.*

Let $\{f_i, i = 1, 2, \dots\}$ (resp. $\{g_j, j = 1, 2, \dots\}$, $\{h_k, k = 1, 2, \dots\}$) be an orthonormal basis of $L^2(F)$ (resp. $L^2(\Sigma_+)$, $L^2(\Sigma_-)$). We suppose that the g_j's and the h_k's are smooth functions in the Schwartz space with respect to p.

We will now consider vectors in $\mathfrak{F}_D$ of the following form:

$$\Phi^{(\ell,m,n)} := b^*_d(f_{i_1})\dots b^*_d(f_{i_\ell})b^*_+(g_{j_1})\dots b^*_+(g_{j_m})b^*_-(h_{k_1})\dots b^*_-(h_{k_n})\,\Omega\ , \tag{9}$$

where l, m, n are positive integers and $\Omega = \Omega_d \otimes \Omega_{c+} \otimes \Omega_{c-}$. The indexes will be assumed ordered such that $i_1 < \dots < i_l$, $j_1 < \dots < j_m$ and $k_1 < \dots < k_n$. It is known that the set $\{\Phi^{(\ell,m,n)}; \ell, m, n = 0, 1, 2, \dots\}$ is an orthonormal basis of $\mathfrak{F}_D$ (See [37]). The set

$$\mathfrak{F}_{0,D} = \{\Psi \in \mathfrak{F}_D;\ \Psi \text{ is a finite linear combination of basis vectors of the form (9)}\}$$

is dense in $\mathfrak{F}_D$.

In the following propositions we investigate several operators in $\mathfrak{F}_D$ built from product of creation operators or annihilation operators only.

For $G \in L^2(\Sigma_\pm \times \Sigma_\pm)$ the formal operators

$$\int_{\Sigma_\pm\times\Sigma_\pm} \mathrm{d}\xi_1\,\mathrm{d}\xi_2\overline{G(\xi_1,\xi_2)}b_\pm(\xi_1)b_\pm(\xi_2)$$

are defined as quadratic forms on $\mathfrak{F}_{0,D} \times \mathfrak{F}_{0,D}$:

$$\int_{\Sigma_\pm\times\Sigma_\pm} \mathrm{d}\xi_1\,\mathrm{d}\xi_2\langle\Psi,\,\overline{G(\xi_1,\xi_2)}b_\pm(\xi_1)b_\pm(\xi_2)\Phi\rangle\ ,$$

where $\Psi, \Phi \in \mathfrak{F}_{0,D}$. Mimicking the proof of [32, Theorem X.44], we get two operators, denoted by $A_\pm$, associated with these two forms such that $A_\pm$ are the unique operators in $\mathfrak{F}_D$ so that $\mathfrak{F}_{0,D} \subset \mathfrak{D}(A_\pm)$ is a core for $A_\pm$ and

$$A_\pm = \int_{\Sigma_\pm\times\Sigma_\pm} \mathrm{d}\xi_1\,\mathrm{d}\xi_2\overline{G(\xi_1,\xi_2)}b_\pm(\xi_1)b_\pm(\xi_2)$$

as quadratic forms on $\mathfrak{F}_{0,D} \times \mathfrak{F}_{0,D}$.

From now on we denote A_ϵ the operators $A_\pm$ where $\epsilon = +, -$, associated with the kernels $G_\epsilon(\xi_1, \xi_2)$. In the same way, we define as quadratic forms on $\mathfrak{F}_{0,D} \times \mathfrak{F}_{0,D}$ the operators $A_{\epsilon\epsilon'}$, $A_{\epsilon\epsilon'\epsilon}$ and $A_{\epsilon\epsilon\epsilon'\epsilon'}$, where $\epsilon, \epsilon' = +, -$ and $\epsilon \neq \epsilon'$ as follows

$$A_{\epsilon\epsilon'} = \int_{\Sigma_\epsilon \times \Sigma_{\epsilon'}} \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \overline{G_{\epsilon\epsilon'}(\xi_1, \xi_2)} b_\epsilon(\xi_1) b_{\epsilon'}(\xi_2) \ ,$$

$$A_{\epsilon\epsilon'\epsilon} = \int_{\Sigma_\epsilon \times \Sigma_{\epsilon'} \times \Sigma_\epsilon} \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \, \mathrm{d}\xi_3 \overline{G_{\epsilon\epsilon'\epsilon}(\xi_1, \xi_2, \xi_3)} b_\epsilon(\xi_1) b_{\epsilon'}(\xi_2) b_\epsilon(\xi_3) \ ,$$

$$A_{\epsilon\epsilon\epsilon'\epsilon'} = \int_{\Sigma_\epsilon \times \Sigma_\epsilon \times \Sigma_{\epsilon'} \times \Sigma_{\epsilon'}} \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \, \mathrm{d}\xi_3 \, \mathrm{d}\xi_4 \overline{G_{\epsilon\epsilon\epsilon'\epsilon'}(\xi_1, \xi_2, \xi_3, \xi_4)} b_\epsilon(\xi_1) b_\epsilon(\xi_2) b_{\epsilon'}(\xi_3) b_{\epsilon'}(\xi_4) \ ,$$

with $G_{\epsilon\epsilon'}$, $G_{\epsilon\epsilon'\epsilon}$ and $G_{\epsilon\epsilon\epsilon'\epsilon'}$ in L^2.

Let $\Phi^{(\ell,m,n)}$ be a vector of the form (9). In addition, for simplicity, assume that $\{i_1, \ldots, i_\ell\} = \{1, \ldots, \ell\}$, $\{j_1, \ldots, j_m\} = \{1, \ldots, m\}$ and $\{k_1, \ldots, k_n\} = \{1, \ldots, n\}$, i.e.

$$\Phi^{(\ell,m,n)} = \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{j=1}^{m} b_+^*(g_j) \otimes \prod_{k=1}^{n} b_-^*(h_k)\Omega \ .$$

We claim that

$$A_+\Phi^{(\ell,m,n)} = \sum_{1 \le \alpha < \beta \le m} (-1)^{\alpha+\beta} \langle G_+^a, \, g_\alpha \otimes g_\beta \rangle \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \ldots, m\} \\ j \neq \alpha, j \neq \beta}} b_+^*(g_j) \otimes \prod_{k=1}^{n} b_+^*(h_k) \tag{10}$$

where

$$G_+^a(\xi_1, \xi_2) = G_+(\xi_1, \xi_2) - G_+(\xi_2, \xi_1)$$

and

$$\langle G_+^a, g_\alpha \otimes g_\beta \rangle = \int \overline{G_+^a(\xi_1, \xi_2)} g_\alpha(\xi_1) g_\beta(\xi_2) \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \ .$$

Indeed, the canonical anti-commutation relations (CAR) (5)-(8) yield

$$\begin{aligned} &\int \mathrm{d}\xi_2 \overline{G_+(\xi_1, \xi_2)} b_+(\xi_2) \Phi^{(\ell,m,n)} \\ &= \ (-1)^\ell \sum_{\alpha=1}^{m} (-1)^{\alpha+1} \langle G_+(\xi_1, .), g_\alpha \rangle \prod_{i=1}^{\ell} b_d^*(h_i) \otimes \prod_{\substack{j \in \{1, \ldots, m\} \\ j \neq \alpha}} b_+^*(g_j) \otimes \prod_{k=1}^{n} b_-^*(h_k)\Omega \end{aligned} \tag{11}$$

We have the identity $\langle G_+(\xi_1, .), g_\alpha \rangle = \sum_{\beta=1}^{\infty} \langle G_+, g_\beta \otimes g_\alpha \rangle \overline{g_\beta(\xi_1)}$ where $\langle G_+, g_\beta \otimes g_\alpha \rangle = \int_{\Sigma_+ \times \Sigma_+} \overline{G_+(\xi_1, \xi_2)} g_\beta(\xi_1) g_\alpha(\xi_2) \mathrm{d}\xi_1 \, \mathrm{d}\xi_2$.

Applying $b_+(\xi_1)$ to the right hand side of (11), using the CAR as above and integrating with respect to ξ_1 we get

$$
\begin{aligned}
A_+\Phi^{(\ell,m,n)} &= \sum_{\alpha=1}^{m-1}(-1)^{\alpha+1}\sum_{1\le\alpha<\beta\le m}\langle G_+, g_\alpha\otimes g_\beta\rangle \\
&(-1)^{\beta+1}\prod_{i=1}^{\ell} b_d^*(f_i)\otimes \prod_{\substack{j\in\{1,\dots,m\}\\ j\neq\alpha, j\neq\beta}} b_+^*(g_j)\otimes\prod_{k=1}^{n} b_+^*(h_k)\Omega \\
&+\sum_{\alpha=1}^{m-1}(-1)^{\alpha+1}\sum_{1\le\alpha<\beta\le m}\langle G_+, g_\beta\otimes g_\alpha\rangle \\
&(-1)^{\beta}\prod_{i=1}^{\ell} b_d^*(f_i)\otimes \prod_{\substack{j\in\{1,\dots,m\}\\ j\neq\alpha, j\neq\beta}} b_+^*(g_j)\otimes\prod_{k=1}^{n} b_-^*(h_k)\Omega\ ,
\end{aligned}
\tag{12}
$$

from which we deduce (10). Similarly, we have

$$
\begin{aligned}
A_-\Phi^{(\ell,m,n)} &= \sum_{1\le\gamma<\delta\le n}(-1)^{\gamma+\delta}\langle G_-^a, h_\gamma\otimes h_\delta\rangle \\
&\prod_{i=1}^{\ell} b_d^*(f_i)\otimes\prod_{j=1}^{m} b_+^*(g_j)\otimes \prod_{\substack{k\in\{1,\dots,n\}\\ k\neq\gamma, k\neq\delta}} b_+^*(h_k)\Omega\ ,
\end{aligned}
\tag{13}
$$

where

$$G_-^a(\xi_1,\xi_2) = G_-(\xi_1,\xi_2) - G_-(\xi_2,\xi_1)$$

and $\langle G_-^a, h_\gamma\otimes h_\delta\rangle = \int_{\Sigma_-\times\Sigma_-}\overline{G_-^a(\xi_1,\xi_2)}h_\gamma(\xi_1)h_\delta(\xi_2)\mathrm{d}\xi_1\,\mathrm{d}\xi_2$.

In a complete similar way, we also get

$$
\begin{aligned}
A_{+-}\Phi^{(\ell,m,n)} =&(-1)^m\sum_{\alpha=1}^{m}\sum_{\beta=1}^{n}(-1)^{\alpha+\beta}\langle G_{+-}, g_\alpha\otimes h_\beta\rangle \\
&\prod_{i=1}^{\ell} b_d^*(f_i)\otimes \prod_{\substack{j\in\{1,\dots,m\}\\ j\neq\alpha}} b_+^*(g_j)\otimes \prod_{\substack{k\in\{1,\dots,n\}\\ k\neq\beta}} b_-^*(h_k)\Omega
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
A_{-+}\Phi^{(\ell,m,n)} =&(-1)^m\sum_{\alpha=1}^{m}\sum_{\beta=1}^{n}(-1)^{\alpha+\beta}\langle G_{-+}, h_\beta\otimes g_\alpha\rangle \\
&\prod_{i=1}^{\ell} b_d^*(f_i)\otimes \prod_{\substack{j\in\{1,\dots,m\}\\ j\neq\alpha}} b_+^*(g_j)\otimes \prod_{\substack{k\in\{1,\dots,n\}\\ k\neq\beta}} b_-^*(h_k)\Omega
\end{aligned}
$$

where

$$\langle G_{+-}, g_\alpha \otimes h_\beta \rangle = \int_{\Sigma_+ \times \Sigma_-} \overline{G_{+-}(\xi_1,\xi_2)} g_\alpha(\xi_1) h_\beta(\xi_2) \mathrm{d}\xi_1 \, \mathrm{d}\xi_2$$

and

$$\langle G_{-+}, h_\beta \otimes g_\alpha \rangle = \int_{\Sigma_- \times \Sigma_+} \overline{G_{-+}(\xi_1,\xi_2)} h_\beta(\xi_1) g_\alpha(\xi_2) \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \ .$$

Using (14) we get, as for (12)

$$\begin{aligned} A_{-+-}\Phi^{(\ell,m,n)} = & \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} (-1)^{\alpha+\beta} \sum_{1 \le \gamma < \beta \le n} (-1)^{\gamma+1} \langle G^a_{-+-}, h_\gamma \otimes g_\alpha \otimes h_\beta \rangle \\ & \prod_{i=1}^{\ell} b^*_d(f_i) \otimes \prod_{\substack{j \in \{1,\dots,m\} \\ j \neq \alpha}} b^*_+(g_j) \otimes \prod_{\substack{k \in \{1,\dots,n\} \\ k \neq \beta, k \neq \gamma}} b^*_-(h_k)\Omega \end{aligned} \tag{15}$$

and

$$\begin{aligned} A_{+-+}\Phi^{(\ell,m,n)} = & \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} (-1)^{\alpha+\beta} \sum_{1 \le \gamma < \beta \le n} (-1)^{\gamma+1} \langle G^a_{+-+}, g_\gamma \otimes h_\alpha \otimes g_\beta \rangle \\ & \prod_{i=1}^{\ell} b^*_d(f_i) \otimes \prod_{\substack{j \in \{1,\dots,m\} \\ j \neq \gamma, j \neq \beta}} b^*_+(g_j) \otimes \prod_{\substack{k \in \{1,\dots,n\} \\ k \neq \alpha}} b^*_-(h_k)\Omega \end{aligned}$$

where

$$G^a_{\epsilon\epsilon'\epsilon}(\xi_1,\xi_2,\xi_3) = G_{\epsilon\epsilon'\epsilon}(\xi_1,\xi_2,\xi_3) - G_{\epsilon\epsilon'\epsilon}(\xi_3,\xi_2,\xi_1)$$

and

$$\langle G^a_{-+-}, h_\gamma \otimes g_\alpha \otimes h_\beta \rangle = \int_{\Sigma_- \times \Sigma_+ \times \Sigma_-} \overline{G^a_{-+-}(\xi_1,\xi_2,\xi_3)} h_\gamma(\xi_1) g_\alpha(\xi_2) h_\beta(\xi_3) \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \, \mathrm{d}\xi_3$$

$$\langle G^a_{+-+}, g_\gamma \otimes h_\alpha \otimes g_\beta \rangle = \int_{\Sigma_+ \times \Sigma_- \times \Sigma_+} \overline{G^a_{+-+}(\xi_1,\xi_2,\xi_3)} g_\gamma(\xi_1) h_\alpha(\xi_2) g_\beta(\xi_3) \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \, \mathrm{d}\xi_3$$

By (10) and (13), we finally obtain

$$\begin{aligned} A_{++--}\Phi^{(\ell,m,n)} = & \sum_{1 \le \alpha < \beta \le m} (-1)^{\alpha+\beta} \sum_{1 \le \gamma < \delta \le n} (-1)^{\gamma+\delta} \langle G^{aa}_{++--}, g_\alpha \otimes g_\beta \otimes h_\gamma \otimes h_\delta \rangle \\ & \prod_{i=1}^{\ell} b^*_d(f_i) \otimes \prod_{\substack{j \in \{1,\dots,m\} \\ j \neq \alpha, j \neq \beta}} b^*_+(g_j) \otimes \prod_{\substack{k \in \{1,\dots,n\} \\ k \neq \gamma, k \neq \delta}} b^*_-(h_k)\Omega \end{aligned} \tag{16}$$

and

$$A_{--++}\Phi^{(\ell,m,n)} = \sum_{1\le\gamma<\delta\le n}(-1)^{\gamma+\delta}\sum_{1\le\alpha<\beta\le m}(-1)^{\alpha+\beta}\langle G^{aa}_{--++}, h_\gamma\otimes h_\delta\otimes g_\alpha\otimes g_\beta\rangle$$
$$\prod_{i=1}^{\ell} b_d^*(f_i)\otimes\prod_{\substack{j\in\{1,\dots,m\}\\ j\neq\alpha, j\neq\beta}} b_+^*(g_j)\otimes\prod_{\substack{k\in\{1,\dots,n\}\\ k\neq\gamma,k\neq\delta}} b_-^*(h_k)\Omega$$

where

$$\begin{aligned} G^{aa}_{\epsilon\epsilon\epsilon'\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4) &= G_{\epsilon\epsilon\epsilon'\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4) - G_{\epsilon\epsilon\epsilon'\epsilon'}(\xi_1,\xi_2,\xi_4,\xi_3)\\ &\quad -G_{\epsilon\epsilon\epsilon'\epsilon'}(\xi_2,\xi_1,\xi_3,\xi_4) + G_{\epsilon\epsilon\epsilon'\epsilon'}(\xi_2,\xi_1,\xi_4,\xi_3)\ . \end{aligned}$$

We have the following proposition

Proposition 3.4. *We have, for $\epsilon\neq\epsilon'$*

(i) $\|A_\epsilon\| = \|A^*_\epsilon\| \le \|G^a_\epsilon\|_{L^2(\Sigma_\epsilon\times\Sigma_\epsilon)}$

(ii) $\|A_{\epsilon\epsilon'}\| = \|A^*_{\epsilon\epsilon'}\| \le \|G_{\epsilon\epsilon'}\|_{L^2(\Sigma_\epsilon\times\Sigma_{\epsilon'})}$

(iii) $\|A_{\epsilon\epsilon'\epsilon}\| = \|A^*_{\epsilon\epsilon'\epsilon}\| \le \|G^a_{\epsilon\epsilon'\epsilon}\|_{L^2(\Sigma_\epsilon\times\Sigma_{\epsilon'}\times\Sigma_\epsilon)}$

(iv) $\|A_{\epsilon\epsilon\epsilon'\epsilon'}\| = \|A^*_{\epsilon\epsilon\epsilon'\epsilon'}\| \le \|G^{aa}_{\epsilon\epsilon\epsilon'\epsilon'}\|_{L^2(\Sigma_\epsilon\times\Sigma_\epsilon\times\Sigma_{\epsilon'}\times\Sigma_{\epsilon'})}$

Proof. For the proof of (i) we restrict ourselves to A_+. The case of A_- is similar. Since $\left(\prod_{i=1}^{\ell} b_d^*(f_i)\otimes\prod_{\substack{j\in\{1,\dots,m\}\\ j\neq\alpha,j\neq\beta}} b_+^*(g_j)\otimes\prod_{k=1}^{n} b_+^*(h_k)\Omega\right)_{1\le\alpha<\beta\le m}$ is an orthonormal family, it follows from (10) that

$$\|A_+\Phi^{(\ell,m,n)}\|^2 = \sum_{1\le\alpha<\beta\le m}|\langle G_+, g_\alpha\otimes g_\beta\rangle|^2_{L^2(\Sigma_+\times\Sigma_+)} \le \|G^a_+\|^2_{L^2(\Sigma_+\times\Sigma_+)}\|\Phi^{(\ell,m,n)}\|^2\ .$$

In order to prove (i), it is enough to show that

$$\|A_+\Psi\|^2 \le \|G^a_+\|^2\|\Psi\|^2 \tag{17}$$

for every $\Psi\in\mathfrak{F}_{0,D}$.

The most significant finite linear combination of basis vectors is the following one

$$\Psi = \sum_{\mu=1}^{N}\sum_{\nu=1}^{P}\lambda_{\mu\nu}\prod_{i=1}^{\ell} b_d^*(f_i)\otimes\prod_{j=1}^{m-1} b_+^*(g_j)b_+^*(g_{m_\mu})\otimes\prod_{k=1}^{n-1} b_-^*(h_k)b_-^*(h_{n_\nu})\Omega \tag{18}$$

Here N and P are positive integers and we have $m-1<m_1<m_2<\ldots<m_N$ and $n-1<n_1<n_2<\ldots<n_P$. From now on we restrict ourselves to finite linear combination

of the form (18). The proof of inequality (17) is easier for any other finite linear combination of basis vectors. Note that $\|\Psi\|^2 = \sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\lambda_{\mu\nu}|^2$. By (10) we get

$$\begin{aligned}
A_+\Psi = &\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}\sum_{1\le\alpha<\beta\le m-1}(-1)^{\alpha+\beta}\lambda_{\mu\nu}\langle G_+^a, g_\alpha\otimes g_\beta\rangle \\
&\prod_{i=1}^{\ell} b_d^*(f_i)\otimes \prod_{\substack{j\in\{1,\dots,m-1\}\\ j\neq\alpha, j\neq\beta}} b_+^*(g_j)b_+^*(g_{m_\mu})\otimes\prod_{k=1}^{n-1} b_-^*(h_k)b_-^*(h_{n_\nu})\Omega \\
&+\sum_{\nu=1}^{P}\sum_{1\le\alpha\le m-1}(-1)^{\alpha+m}\left(\sum_{\mu=1}^{N}\lambda_{\mu\nu}\langle G_+^a, g_\alpha\otimes g_{m_\mu}\rangle\right) \\
&\prod_{i=1}^{\ell} b_d^*(f_i)\otimes \prod_{\substack{j\in\{1,\dots,m-1\}\\ j\neq\alpha}} b_+^*(g_j)\otimes\prod_{k=1}^{n-1} b_-^*(h_k)b_-^*(h_{n_\nu})\Omega\ .
\end{aligned} \tag{19}$$

The right hand side of (19) is a linear combination of vectors of an orthogonal family. Thus

$$\begin{aligned}
\|A_+\Psi\|^2 = &\sum_{1\le\alpha<\beta\le m-1}(\sum_{\nu=1}^{P}\sum_{\mu=1}^{N}|\lambda_{\mu\nu}|^2)|\langle G_+^a, g_\alpha\otimes g_\beta\rangle|^2 \\
&+\sum_{1\le\alpha\le m-1}\sum_{\nu=1}^{P}|\sum_{\mu=1}^{N}\lambda_{\mu\nu}\langle G_+^a, g_\alpha\otimes g_{m_\mu}\rangle|^2
\end{aligned}$$

By the Cauchy-Schwarz inequality we get

$$\begin{aligned}
\|A_+\Psi\|^2 \le &(\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\lambda_{\mu\nu}|^2)\Big[\sum_{1\le\alpha<\beta\le m-1}\left|\langle G_+^a, g_\alpha\otimes g_\beta\rangle\right|^2 \\
&+\sum_{1\le\alpha\le m-1}\sum_{\mu=1}^{N}|\langle G_+^a, g_\alpha\otimes g_{m_\mu}\rangle|^2\Big] \le \|G_+^a\|^2\|\Psi\|^2\ .
\end{aligned}$$

This concludes the proof of (i).

Since $\left(\prod_{i=1}^{\ell} b_d^*(f_i)\otimes\prod_{\substack{j\in\{1,\dots,m\}\\ j\neq\alpha}} b_+^*(g_j)\otimes\prod_{\substack{k\in\{1,\dots,n\}\\ k\neq\beta}} b_+^*(h_k)\Omega\right)_{\substack{1\le\alpha\le m\\ 1\le\beta\le n}}$ is an orthonormal family we have

$$\|A_{+-}\Phi^{(\ell,m,n)}\|^2 = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n}|\langle G_{+-}, g_\alpha\otimes h_\beta\rangle|^2 \le \|G_{+-}\|^2\|\Phi^{(\ell,m,n)}\|^2$$

Let Ψ be of the form (18). By (14) we get

$$\begin{aligned}
A_{+-}\Psi = (-1)^m \sum_{\mu=1}^{N}\sum_{\nu=1}^{P}\sum_{\alpha=1}^{m-1}\sum_{\beta=1}^{n-1}(-1)^{\alpha+\beta}\lambda_{\mu\nu}\langle G_{+-}, g_\alpha \otimes h_\beta\rangle \\
\prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j\in\{1,\dots,m-1\}\\ j\neq\alpha}} b_+^*(g_j) b_+^*(g_{m_\mu}) \otimes \prod_{\substack{k\in\{1,\dots,n-1\}\\ k\neq\beta}} b_-^*(h_k) b_-^*(h_{n_\nu})\Omega \\
+ (-1)^m \sum_{\mu=1}^{N}\sum_{\beta=1}^{n-1}(-1)^{\beta}\sum_{\nu=1}^{P}\lambda_{\mu\nu}\langle G_{+-}, g_{m_\mu}\otimes h_\beta\rangle \\
\prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{j=1}^{m-1} b_+^*(g_j) \otimes \prod_{\substack{k\in\{1,\dots,n-1\}\\ k\neq\beta}} b_-^*(h_k) b_-^*(h_{n_\nu})\Omega \\
+ (-1)^{m+n} \sum_{\nu=1}^{P}\sum_{\alpha=1}^{m-1}(-1)^{\alpha}\sum_{\mu=1}^{N}\lambda_{\mu\nu}\langle G_{+-}, g_\alpha\otimes h_{n_\nu}\rangle \\
\prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j\in\{1,\dots,m-1\}\\ j\neq\alpha}} b_+^*(g_j) b_+^*(g_{m_\mu}) \otimes \prod_{k=1}^{n-1} b_-^*(h_k)\Omega \\
+ (-1)^n \left(\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}\lambda_{\mu\nu}\langle G_{+-}, g_{m_\mu}\otimes h_{n_\nu}\rangle\right) \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{j=1}^{m-1} b_+^*(g_j) \otimes \prod_{k=1}^{n-1} b_-^*(h_k)\Omega
\end{aligned} \tag{20}$$

The right hand side of (20) is a linear combination of vectors of an orthogonal family. Thus we obtain

$$\begin{aligned}
\|A_{+-}\Psi\|^2 = \sum_{\mu=1}^{N}\sum_{\nu=1}^{P}\sum_{\alpha=1}^{m-1}\sum_{\beta=1}^{n-1}|\lambda_{\mu\nu}|^2|\langle G_{+-}, g_\alpha\otimes h_\beta\rangle|^2 + \sum_{\beta=1}^{n-1}\sum_{\nu=1}^{P}\Big|\sum_{\mu=1}^{N}\lambda_{\mu\nu}\langle G_{+-}, g_{m_\mu}\otimes h_\beta\rangle\Big|^2 \\
+ \sum_{\alpha=1}^{m-1}\sum_{\mu=1}^{N}\Big|\sum_{\mu=1}^{P}\lambda_{\mu\nu}\langle G_{+-}, g_\alpha\otimes h_{n_\nu}\rangle\Big|^2 + \Big|\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}\lambda_{\mu\nu}\langle G_{+-}, g_{m_\mu}\otimes h_{n_\nu}\rangle\Big|^2 .
\end{aligned}$$

By the Cauchy-Schwarz inequality we get

$$
\begin{aligned}
&\|A_{+-}\Psi\|^2 \\
&\le \sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\lambda_{\mu\nu}|^2\sum_{\alpha=1}^{m-1}\sum_{\beta=1}^{n-1}|\langle G_{+-},g_\alpha\otimes h_\beta\rangle|^2+\sum_{\beta=1}^{n-1}\sum_{\nu=1}^{P}\sum_{\mu=1}^{N}|\lambda_{\mu\nu}|^2\sum_{\mu=1}^{N}|\langle G_{+-},g_{m_\mu}\otimes h_\beta\rangle|^2 \\
&+\sum_{\alpha=1}^{m-1}\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\lambda_{\mu\nu}|^2\sum_{\nu=1}^{P}|\langle G_{+-},g_\alpha\otimes h_{n_\nu}\rangle|^2+\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\lambda_{\mu\nu}|^2\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\langle G_{+-},g_{m_\mu}\otimes h_{n_\nu}\rangle|^2 \\
&\le \sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\lambda_{\mu\nu}|^2\Bigg[\sum_{\alpha=1}^{m-1}\sum_{\beta=1}^{n-1}|\langle G_{+-},g_\alpha\otimes h_\beta\rangle|^2+\sum_{\beta=1}^{n-1}\sum_{\mu=1}^{N}|\langle G_{+-},g_{m_\mu}\otimes h_\beta\rangle|^2 \\
&+\sum_{\alpha=1}^{m-1}\sum_{\nu=1}^{P}|\langle G_{+-},g_\alpha\otimes h_{n_\nu}|^2+\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\langle G_{+-},g_{m_\mu}\otimes h_{n_\nu}\rangle|^2\Bigg]\le\|G_{+-}\|^2\|\Psi\|^2\ .
\end{aligned}
$$

The proof for A_{-+} is the same. This concludes the proof of (ii).

Since

$$
\left(\prod_{i=1}^{\ell}b_d^*(f_i)\otimes\prod_{\substack{j\in\{1,\ldots,m\}\\ j\neq\alpha}}b_+^*(g_j)\otimes\prod_{\substack{k\in\{1,\ldots,n\}\\ k\neq\gamma,k\neq\beta}}b_-^*(h_k)\Omega\right)_{\substack{1\le\alpha\le m\\ 1\le\gamma<\beta\le n}}
$$

is an orthonormal family, by (15) we get

$$
\|A_{-+-}\Phi^{(\ell,m,n)}\|^2=\sum_{\alpha=1}^{m}\sum_{\beta=1}^{n}\sum_{1\le\gamma<\beta\le n}|\langle G^a_{-+-},h_\gamma\otimes g_\alpha\otimes h_\beta\rangle|^2\le\|G^a_{-+-}\|^2\|\Phi^{(\ell,m,n)}\|^2
$$

Let now Ψ be of the form (18). By (15) we obtain

$$
\begin{aligned}
A_{-+-}\Psi = &\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}\lambda_{\mu\nu}\sum_{\alpha=1}^{m-1}\sum_{\beta=1}^{n-1}(-1)^{\alpha+\beta}\sum_{1\le\gamma<\beta}(-1)^{\gamma+1}\langle G^a_{-+-}, h_\gamma\otimes g_\alpha\otimes h_\beta\rangle \\
&\prod_{i=1}^{\ell} b_d^*(f_i)\otimes\prod_{\substack{j\in\{1,\dots,m-1\}\\ j\neq\alpha}} b_+^*(g_j)b_+^*(g_{m_\mu})\otimes\prod_{\substack{k\in\{1,\dots,n-1\}\\ k\neq\beta,k\neq\gamma}} b_-^*(h_k)b_-^*(h_{n_\nu})\Omega \\
&+\sum_{\nu=1}^{P}\sum_{\beta=1}^{n-1}(-1)^{m+\beta}\sum_{1\le\gamma<\beta}(-1)^{\gamma+1}\sum_{\mu=1}^{N}\lambda_{\mu\nu}\langle G^a_{-+-}, h_\gamma\otimes g_{m_\mu}\otimes\otimes h_\beta\rangle \\
&\prod_{i=1}^{\ell} b_d^*(f_i)\otimes\prod_{j=1}^{m-1} b_+^*(g_j)\otimes\prod_{\substack{k\in\{1,\dots,n-1\}\\ k\neq\beta,k\neq\gamma}} b_-^*(h_k)b_-^*(h_{n_\nu})\Omega \\
&+\sum_{\mu=1}^{N}\sum_{\alpha=1}^{m-1}(-1)^{\alpha+n}\sum_{1\le\gamma<n}(-1)^{\gamma+1}\sum_{\nu=1}^{P}\lambda_{\mu\nu}\langle G^a_{-+-}, h_\gamma\otimes g_\alpha\otimes h_{n_\nu}\rangle \\
&\prod_{i=1}^{\ell} b_d^*(f_i)\otimes\prod_{\substack{j\in\{1,\dots,m-1\}\\ j\neq\alpha}} b_+^*(g_j)b_+^*(g_{m_\mu})\otimes\prod_{\substack{k\in\{1,\dots,n-1\}\\ k\neq\gamma}} b_-^*(h_k)\Omega \\
&+(-1)^{m+n}\sum_{1\le\gamma\le n-1}(-1)^{\gamma+1}\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}\lambda_{\mu\nu}\langle G^a_{-+-}, h_\gamma\otimes g_{m_\mu}\otimes h_{n_\nu}\rangle \\
&\prod_{i=1}^{\ell} b_d^*(f_i)\otimes\prod_{j=1}^{m-1} b_+^*(g_j)\otimes\prod_{\substack{k\in\{1,\dots,n-1\}\\ k\neq\gamma}} b_-^*(h_k)\Omega
\end{aligned}
\tag{21}
$$

The right hand side of (21) is a linear combination of vectors of an orthogonal family. Thus

$$
\begin{aligned}
\|A_{-+-}\Psi\|^2 = &\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\lambda_{\mu\nu}|^2\sum_{\alpha=1}^{m-1}\sum_{\beta=1}^{n-1}\sum_{1\le\gamma<\beta}|\langle G^a_{-+-}, h_\gamma\otimes g_\alpha\otimes h_\beta\rangle|^2 \\
&+\sum_{\alpha=1}^{m-1}\sum_{1\le\gamma\le n-1}\sum_{\mu=1}^{N}\left|\sum_{\nu=1}^{P}\lambda_{\mu\nu}\langle G^a_{-+-}, h_\gamma\otimes g_\alpha\otimes h_{n_\nu}\rangle\right|^2 \\
&+\sum_{\beta=1}^{n-1}\sum_{1\le\gamma<\beta\le n-1}\sum_{\nu=1}^{P}\left|\sum_{\mu=1}^{N}\lambda_{\mu\nu}\langle G^a_{-+-}, h_\gamma\otimes g_{m_\mu}\otimes h_\beta\rangle\right|^2 \\
&+\sum_{1\le\gamma\le n-1}\left|\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}\lambda_{\mu\nu}\langle G^a_{-+-}, h_\gamma\otimes g_{m_\mu}\otimes h_{n_\nu}\rangle\right|^2
\end{aligned}
$$

By the Cauchy-Schwarz inequality we get

$$\|A_{-+-}\Psi\|^2 \leq \sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\lambda_{\mu\nu}|^2 \Bigg[\sum_{\alpha=1}^{m-1}\sum_{\beta=1}^{n-1}\sum_{1\leq\gamma<\beta}|\langle G^a_{-+-}, h_\gamma\otimes g_\alpha\otimes h_\beta\rangle|^2$$
$$+\sum_{\alpha=1}^{m-1}\sum_{1\leq\gamma\leq n-1}\sum_{\nu=1}^{P}|\langle G^a_{-+-}, h_\gamma\otimes g_\alpha\otimes h_{n_\nu}\rangle|^2$$
$$+\sum_{\beta=1}^{n-1}\sum_{1\leq\gamma<\beta}\sum_{\mu=1}^{N}|\langle G^a_{-+-}, h_\gamma\otimes g_{m_\mu}\otimes h_\beta\rangle|^2$$
$$+\sum_{\gamma=1}^{n-1}\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\langle G^a_{-+-}, h_\gamma\otimes g_{m_\mu}\otimes h_{n_\nu}\rangle|^2\Bigg] \leq \|G^a_{-+-}\|^2\|\Psi\|^2 \ .$$

The proof of the estimate for A_{-+-} is the same. This concludes the proof of (iii).

Since

$$\left(\prod_{i=1}^{\ell} b^*_d(f_i)\otimes \prod_{\substack{j\in\{1,\dots,m\}\\ j\neq\alpha,j\neq\beta}} b^*_+(g_j)\otimes \prod_{\substack{k\in\{1,\dots,n\}\\ k\neq\gamma,k\neq\delta}} b^*_-(h_k)\Omega\right)_{\substack{1\leq\alpha<\beta\leq m\\ 1\leq\gamma<\delta\leq n}}$$

is an orthonormal family, we get from (16)

$$\|A_{++--}\Phi^{(\ell,m,n)}\|^2 = \sum_{1\leq\alpha<\beta\leq m}\ \sum_{1\leq\gamma<\delta\leq n}|\langle G^{aa}_{++--}, g_\alpha\otimes g_\beta\otimes h_\gamma\otimes h_\delta\rangle|^2$$
$$\leq \|G^{aa}_{--++}\|^2\|\Phi^{(\ell,m,n)}\|^2$$

Let now Ψ be of the form (18). By(16) we then get

$$A_{++--}\Psi = \sum_{\mu=1}^{N}\sum_{\nu=1}^{P}\lambda_{\mu\nu}\sum_{1\leq\gamma<\delta\leq n-1}(-1)^{\gamma+\delta}\sum_{1\leq\alpha<\beta\leq m-1}(-1)^{\alpha+\beta}\langle G^{aa}_{++--}, g_\alpha\otimes g_\beta\otimes h_\gamma\otimes h_\delta\rangle$$
$$\prod_{i=1}^{\ell} b^*_d(f_i)\otimes \prod_{\substack{j\in\{1,\dots,m-1\}\\ j\neq\alpha,j\neq\beta}} b^*_+(g_j)b^*_+(g_{m_\mu})\otimes \prod_{\substack{k\in\{1,\dots,n-1\}\\ k\neq\gamma,k\neq\delta}} b^*_-(h_k)b^*_-(h_{n_\nu})\Omega \tag{22}$$

$$+\sum_{\mu=1}^{N}\sum_{1\le\gamma\le n-1}(-1)^{\gamma+n}\sum_{1\le\alpha<\beta\le m-1}(-1)^{\alpha+\beta}\sum_{\nu=1}^{P}\lambda_{\mu\nu}\langle G^{aa}_{++--}, g_\alpha\otimes g_\beta\otimes h_\gamma\otimes h_{n_\nu}\rangle$$
$$\prod_{i=1}^{\ell}b^*_d(f_i)\otimes\prod_{\substack{j\in\{1,\dots,m-1\}\\ j\neq\alpha, j\neq\beta}}b^*_+(g_j)b^*_+(g_{m_\mu})\otimes\prod_{\substack{j\in\{1,\dots,n-1\}\\ k\neq\gamma}}b^*_-(h_k)\Omega$$
$$+\sum_{\nu=1}^{P}\sum_{1\le\alpha\le m-1}(-1)^{\alpha+m}\sum_{1\le\gamma<\delta\le n-1}(-1)^{\gamma+\delta}\sum_{\mu=1}^{N}\lambda_{\mu\nu}\langle G^{aa}_{++--}, g_\alpha\otimes g_{m_\mu}\otimes h_\gamma\otimes h_\delta\rangle$$
$$\prod_{i=1}^{\ell}b^*_d(f_i)\otimes\prod_{\substack{j\in\{1,\dots,m-1\}\\ j\neq\alpha}}b^*_+(g_j)\otimes\prod_{\substack{k\in\{1,\dots,n-1\}\\ k\neq\gamma,k\neq\delta}}b^*_-(h_k)b^*_-(h_{n_\nu})\Omega$$
$$+\sum_{1\le\alpha\le m-1}(-1)^{\alpha+m}\sum_{1\le\gamma\le n-1}(-1)^{\gamma+n}\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}\lambda_{\mu\nu}\langle G^{aa}_{++--}, g_\alpha\otimes g_{m_\mu}\otimes h_\gamma\otimes h_{n_\nu}\rangle$$
$$\prod_{i=1}^{\ell}b^*_d(f_i)\otimes\prod_{\substack{j\in\{1,\dots,m-1\}\\ j\neq\alpha}}b^*_+(g_j)\otimes\prod_{\substack{k\in\{1,\dots,n-1\}\\ k\neq\gamma}}b^*_-(h_k)\Omega\ .$$

The right hand side of (22) is a linear combination of vectors of an orthogonal family. Thus

$$\begin{aligned}\|A_{++--}\Psi\|^2 &= \sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\lambda_{\mu\nu}|^2\sum_{1\le\alpha<\beta\le m-1}\sum_{1\le\gamma<\delta\le n-1}|\langle G^{aa}_{++--}, g_\alpha\otimes g_\beta\otimes h_\gamma\otimes h_\delta\rangle|^2\\ &+\sum_{1\le\gamma\le n-1}\sum_{1\le\alpha<\beta\le n-1}\sum_{\mu=1}^{N}\left|\sum_{\nu=1}^{P}\lambda_{\mu\nu}\langle G^{aa}_{++--}, g_\alpha\otimes g_\beta\otimes h_\gamma\otimes h_{n_\nu}\rangle\right|^2\\ &+\sum_{1\le\alpha\le m-1}\sum_{1\le\gamma<\delta\le n-1}\sum_{\nu=1}^{P}\left|\sum_{\mu=1}^{N}\lambda_{\mu\nu}\langle G^{aa}_{++--}, g_\alpha\otimes g_{m_\mu}\otimes h_\gamma\otimes h_\delta\rangle\right|^2\\ &+\sum_{1\le\alpha\le m-1}\sum_{1\le\gamma\le n-1}\left|\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}\lambda_{\mu\nu}\langle G^{aa}_{++--}, g_\alpha\otimes g_{m_\mu}\otimes h_\gamma\otimes h_{n_\nu}\rangle\right|^2\end{aligned}\tag{23}$$

By the Cauchy-Schwarz inequality and (23) we get

$$\begin{aligned}\|A_{++--}\Psi\|^2 \leq \sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\lambda_{\mu\nu}|^2 \sum_{1\leq\alpha<\beta\leq m-1}\ \sum_{1\leq\gamma<\delta\leq n-1} |\langle G^{aa}_{++--}, g_\alpha\otimes g_\beta\otimes h_\gamma\otimes h_\delta\rangle|^2 \\ + \sum_{1\leq\gamma\leq n-1}\ \sum_{1\leq\alpha<\beta\leq n-1}\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\lambda_{\mu\nu}|^2|\langle G^{aa}_{++--}, g_\alpha\otimes g_\beta\otimes h_\gamma\otimes h_{n_\nu}\rangle|^2 \\ + \sum_{1\leq\alpha\leq m-1}\ \sum_{1\leq\gamma<\delta\leq n-1}\sum_{\nu=1}^{P}\sum_{\mu=1}^{N}|\lambda_{\mu\nu}|^2\sum_{\mu=1}^{N}|\langle G^{aa}_{++--}, g_\alpha\otimes g_{m_\mu}\otimes h_\gamma\otimes h_\delta\rangle|^2 \\ + \sum_{1\leq\alpha\leq m-1}\ \sum_{1\leq\gamma\leq n-1}\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\lambda_{\mu\nu}|^2\sum_{\mu=1}^{N}\sum_{\nu=1}^{P}|\langle G^{aa}_{++--}, g_\alpha\otimes g_{m_\mu}\otimes h_\gamma\otimes h_{n_\nu}\rangle|^2\end{aligned}$$

which yields

$$\|A_{++--}\Psi\| \leq \|G^{aa}_{++--}\|\|\Psi\|$$

The proof for A_{--++} is the same as above. This concludes the proof of (iv).

Proposition 3.4 is thus proved.

□

Remark 3.5. *The following formal operators*

$$\int \mathrm{d}\xi_1\,\mathrm{d}\xi_2\,G_\epsilon(\xi_1,\xi_2)b^*_\epsilon(\xi_1)b^*_\epsilon(\xi_2)\ ,$$

$$\int \mathrm{d}\xi_1\,\mathrm{d}\xi_2\,G_{\epsilon\epsilon'}(\xi_1,\xi_2)b^*_\epsilon(\xi_1)b^*_{\epsilon'}(\xi_2)\ ,$$

$$\int \mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3\,G_{\epsilon\epsilon'\epsilon}(\xi_1,\xi_2,\xi_3)b^*_\epsilon(\xi_1)b^*_{\epsilon'}(\xi_2)b^*_\epsilon(\xi_3)\ ,$$

and

$$\int \mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3\,\mathrm{d}\xi_4\,G_{\epsilon\epsilon\epsilon'\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4)b^*_\epsilon(\xi_1)b^*_\epsilon(\xi_2)b^*_{\epsilon'}(\xi_3)b^*_{\epsilon'}(\xi_4)\ ,$$

are the ones associated respectively with A^*_ϵ, $A^*_{\epsilon,\epsilon'}$, $A^*_{\epsilon,\epsilon',\epsilon}$ and $A^*_{\epsilon\epsilon\epsilon'\epsilon'}$, as quadratic forms on $\mathfrak{F}_{0,D}\times\mathfrak{F}_{0,D}$.

We now investigate operators in $\mathfrak{F}_D$ built from a product of creation and annihilation operators. Let us introduce

$$\mathfrak{F}_{D,fin} = \Big\{\Psi = (\Psi^{(q,r,s)})_{q\geq 0, r\geq 0, s\geq 0};\ \Psi^{(q,r,s)} \text{ is in the Schwartz space and } \Psi^{(q,r,s)} = 0 \text{ for all but finitely many } (q,r,s)\Big\}\ .$$

$\mathfrak{F}_{D,fin}$ is a core for N_D. For $G_\epsilon \in L^2$, by mimicking the proof of [32, Theorem X.44], one can show that there exists two operators, denoted by B_ϵ ($\epsilon \in \{+,-\}$), such that B_ϵ are the unique operators in $\mathfrak{F}_D$ such that $\mathfrak{F}_{D,fin} \subset \mathfrak{D}(B_\epsilon)$ is a core for B_ϵ and

$$B_\epsilon = \int_{\Sigma_\epsilon \times \Sigma_\epsilon} \mathrm{d}\xi_1\, \mathrm{d}\xi_2 G_\epsilon(\xi_1,\xi_2) b_\epsilon^*(\xi_1) b_\epsilon(\xi_2)$$

and

$$B_\epsilon^* = -\int_{\Sigma_\epsilon \times \Sigma_\epsilon} \mathrm{d}\xi_1\, \mathrm{d}\xi_2 \overline{G_\epsilon(\xi_1,\xi_2)} b_\epsilon^*(\xi_1) b_\epsilon(\xi_2) \tag{24}$$

as quadratic forms on $\mathfrak{F}_{D,fin} \times \mathfrak{F}_{D,fin}$.

We then have the following proposition whose proof is borrowed from [18]

Proposition 3.6. *For $G_\epsilon \in L^2(\Sigma_\epsilon \times \Sigma_\epsilon)$ we have*

$$\|B_\epsilon\,(N_D+1)^{-1/2}\| \leq \|G\|_{L^2}\ , \tag{25}$$

and

$$\|(N_D+1)^{-1/2}\,B_\epsilon\| \leq \|G\|_{L^2}\ . \tag{26}$$

Proof. Since N_D is a self-adjoint operator, (26) follows from (24) and (25). We only investigate B_+ since the proof for B_- is the same.

Let $\Psi = (\Psi^{(q,r,s)})$ and $\Phi = (\Phi^{(q',r',s')}$ be two vectors in $\mathfrak{F}_{D,fin}$. We have

$$\begin{aligned}&\langle \Phi^{(q',r',s')}, B_+ \Psi^{(q,r,s)}\rangle_{\mathfrak{F}_D}\\ &\quad = \delta_{qq'}\delta_{rr'}\delta_{ss'} \int_{\Sigma_+ \times \Sigma_+} G_+(\xi_1,\xi_2) \langle b_+(\xi_1)\Phi^{(q,r,s)}, b_+(\xi_2)\Psi^{(q,r,s)}\rangle_{\mathfrak{F}_a^{(q,r-1,s)}} \mathrm{d}\xi_1\, \mathrm{d}\xi_2\end{aligned}$$

Thus $B_+\Psi^{(q,r,s)} \in \mathfrak{F}_a^{(q,r,s)}$ for every triple (q,r,s). Therefore, we only need to estimate $\langle \Phi^{(q,r,s)}, B_+\Psi^{(q,r,s)}\rangle$ for every triple (q,r,s). By the Fubini theorem we have

$$|\langle \Phi^{(q,r,s)}, B_+\Psi^{(q,r,s)}\rangle|^2 = \left| \int \left\langle b_+(\xi_1)\Phi^{(q,r,s)}, \int G_+(\xi_1,\xi_2) b_+(\xi_2)\Psi^{(q,r,s)} \mathrm{d}\xi_2 \right\rangle \mathrm{d}\xi_1 \right|^2$$

By the Cauchy-Schwarz inequality and the fact that $\|b_+(f)\| = \|f\|$, we get

$$|\langle \Phi^{(q,r,s)}, B_+\Psi^{(q,r,s)}\rangle|^2 \leq \left(\int_{\Sigma_+} \|b_+(\xi_1)\Phi^{(q,r,s)}\| \left(\int_{\Sigma_+} |G_+(\xi_1,\xi_2)|^2 \mathrm{d}\xi_2 \right)^{\frac{1}{2}} \mathrm{d}\xi_1 \right)^2 \|\Psi^{(q,r,s)}\|^2$$

Applying again Cauchy-Schwarz inequality and the definition of $b_+(\xi)$, we finally get

$$\begin{aligned}|\langle \Phi^{(q,r,s)}, B_+\Psi^{(q,r,s)}\rangle|^2 &\leq r\|G_+\|^2 \|\Phi^{(q,r,s)}\|^2 \|\Psi^{(q,r,s)}\|^2\\ &\leq \|G_+\|^2 \|\Phi^{(q,r,s)}\|^2 \|(N_D+1)^{\frac{1}{2}}\Psi^{(q,r,s)}\|^2\ .\end{aligned} \tag{27}$$

Since $B_+\Psi^{(q,r,s)} \in \mathfrak{F}_a^{(q,r,s)}$, we have

$$|\langle\Phi, B_+\Psi^{(q,r,s)}\rangle|^2 \leq \|G_+\|^2\|\Phi\|^2\|(N_D+1)^{\frac{1}{2}}\Psi^{(q,r,s)}\|^2 \ , \tag{28}$$

for every $\Phi \in \mathfrak{F}_{D,fin}$. Now, since $\mathfrak{F}_{D,fin}$ is dense in $\mathfrak{F}_D$, inequality (28) holds for all $\Phi \in \mathfrak{F}_D$ and all triples (q,r,s). Therefore, we have

$$\|B_+\Psi^{(q,r,s)}\|^2 \leq \|G_+\|^2\|(N_D+1)^{\frac{1}{2}}\Psi^{(q,r,s)}\|^2$$

which yields

$$\|B_+\Psi\| \leq \|G_+\| \ \|(N_D+1)^{\frac{1}{2}}\Psi\| \tag{29}$$

for every $\Psi \in \mathfrak{F}_{D,fin}$. Since $\mathfrak{F}_{D,fin}$ is a core for $(N_D+1)^{\frac{1}{2}}$, $\mathfrak{D}(B_+) \subset \mathfrak{D}((N_D+1)^{\frac{1}{2}})$ and (29) is still true for every $\Psi \in \mathfrak{D}((N_D+1)^{\frac{1}{2}})$. This concludes the proof of Proposition 3.6. $\square$

Remark 3.7. *Inequalities* (25) *and* (26) *are the best estimates that we can get. Indeed, set*

$$\Phi^{(l,m,n)} = \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{j=1}^{m} b_+^*(g_j) \otimes \prod_{k=1}^{n} b_-^*(h_k)\Omega$$

we have

$$\begin{aligned} B_+\Phi^{(l,m,n)} &= \sum_{\alpha=1}^{m} \langle G_+, g_\alpha \otimes g_\alpha\rangle \Phi^{(l,m,n)} \\ &+ \sum_{\alpha=1}^{m}(-1)^{\alpha+1} \sum_{\beta=m+1}^{\infty} \langle G_+, g_\beta \otimes g_\alpha\rangle \prod_{i=1}^{\ell} b_d^*(f_i) \otimes b_+^*(g_\beta) \prod_{\substack{j\in\{1,\dots,m\}\\ j\neq\alpha}} b_+^*(g_j) \otimes \prod_{k=1}^{n} b_-^*(h_k)\Omega \ . \end{aligned} \tag{30}$$

Two different vectors in the right hand side of (30) *are orthogonal. Therefore, we get*

$$\|B_+\Phi^{(\ell,m,n)}\|^2 = \left|\sum_{\alpha=1}^{m} \langle G_+, g_\alpha \otimes g_\alpha\rangle\right|^2 + \sum_{\alpha=1}^{m} \sum_{\beta=m+1}^{\infty} |\langle G_+, g_\beta \otimes g_\alpha\rangle|^2$$

from which we deduce

$$\|B_+\Phi^{(\ell,m,n)}\|^2 \leq m\|G_+\|^2\|\Phi^{(\ell,m,n)}\|^2 \leq \|G_+\|^2\|N_D^{\frac{1}{2}}\Phi^{(\ell,m,n)}\|^2 \ .$$

We now have the following lemmas

Lemma 3.8. *Let A (resp. B) be a positive self-adjoint operator in the Hilbert space $\mathfrak{H}_1$ (resp. $\mathfrak{H}_2$) with domain $\mathfrak{D}(A)$ (resp. $\mathfrak{D}(B)$).*

Then the operator $H_0 = A \otimes \mathbb{1} + \mathbb{1} \otimes B$ with domain $\mathfrak{D}(A \otimes \mathbb{1}) \cap \mathfrak{D}(\mathbb{1} \otimes B)$ is self-adjoint in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and we have

$$\begin{aligned}\|(A \otimes \mathbb{1})\,\psi\| &\leq \|H_0\,\psi\|\ ,\\ \|(\mathbb{1} \otimes B)\,\psi\| &\leq \|H_0\,\psi\|\ ,\end{aligned}$$

for every $\psi \in \mathfrak{D}(A \otimes \mathbb{1}) \cap \mathfrak{D}(\mathbb{1} \otimes B)$.

Proof. The operators $A \otimes \mathbb{1}$, $\mathbb{1} \otimes B$ and $A \otimes \mathbb{1} + \mathbb{1} \otimes B$ are essentially self-adjoint on $\mathfrak{D}(A) \otimes \mathfrak{D}(B)$, the algebraic tensor product of $\mathfrak{D}(A)$ and $\mathfrak{D}(B)$. (See [32, chap.VIII]). Let $\widetilde{H_0}$ be the self-adjoint extension of $A \otimes \mathbb{1} + \mathbb{1} \otimes B$ defined on $\mathfrak{D}(A) \otimes \mathfrak{D}(B)$. The lemma is proved if we show that $\mathfrak{D}(\widetilde{H_0})$ is contained in $\mathfrak{D}(A \otimes \mathbb{1}) \cap \mathfrak{D}(B \otimes \mathbb{1})$. Let $(u_n)_{n\geq 1}$ be a sequence such that $u_n \in \mathfrak{D}(A) \otimes \mathfrak{D}(B)$, $u_n \underset{n\to\infty}{\longrightarrow} u$ and $\widetilde{H_0}\,u_n \underset{n\to\infty}{\longrightarrow} w$ in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. We have

$$\begin{aligned}\|\widetilde{H_0}(u_n - u_m)\|^2 &= \|(A \otimes \mathbb{1} + \mathbb{1} \otimes B)\,(u_n - u_m)\|^2\\ &= \|(A \otimes \mathbb{1})\,(u_n - u_m)\|^2 + \|(\mathbb{1} \otimes B)\,(u_n - u_m)\|^2\\ &+ 2\|A^{1/2} \otimes B^{1/2}(u_n - u_m)\|^2\end{aligned}$$

Therefore

$$\begin{aligned}\|\widetilde{H_0}(u_n - u_m)\| &\geq \|(A \otimes \mathbb{1})\,(u_n - u_m)\|\ ,\\ \|\widetilde{H_0}(u_n - u_m)\| &\geq \|(\mathbb{1} \otimes B)\,(u_n - u_m)\|\ .\end{aligned}$$

Thus, the sequences $((A \otimes \mathbb{1})u_n)_{n\geq 1}$ and $((A \otimes B)u_n)_{n\geq 1}$ are convergent in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. Finally it follows from the self-adjointness of $A \otimes \mathbb{1}$ and $\mathbb{1} \otimes B$ that $u \in \mathfrak{D}(A \otimes \mathbb{1}) \cap \mathfrak{D}(\mathbb{1} \otimes B)$, $(A \otimes \mathbb{1})u_n \underset{n\to\infty}{\longrightarrow} (A \otimes \mathbb{1})u$, $(\mathbb{1} \otimes B)u_n \underset{n\to\infty}{\longrightarrow} (\mathbb{1} \otimes B)u$ and $\widetilde{H_0}\,u_n \underset{n\to\infty}{\longrightarrow} \widetilde{H_0}u = H_0 u = w$. Hence the lemma is proved. □

The proof of the following lemma is easy.

Lemma 3.9. *Let $E_0 = \inf_{(\gamma,n)\in\mathcal{D}_d} E_{\gamma,n} > 0$. We have*

$$E_0\|N_D\,\Psi\| \leq \|d\Gamma(H_D)\,\Psi\|$$

for any $\Psi \in \mathfrak{D}(d\Gamma(H_D))$.

Applying Lemma 3.8 to $A = d\Gamma(H_D)$ and $B = H_{ph}$ we get, from Lemma 3.9, the following result

Proposition 3.10. *For any $\Psi \in \mathfrak{D}(d\Gamma(H_D) \otimes \mathbb{1}) \cap \mathfrak{D}(\mathbb{1} \otimes H_{ph})$, we have*

$$\begin{aligned}E_0\|\,(N_D \otimes \mathbb{1})\,\Psi\| &\leq \|H_0\Psi\|\\ \|\,(\mathbb{1} \otimes H_{ph})\,\Psi\| &\leq \|H_0\Psi\|\end{aligned}$$

where $H_0 = d\Gamma(H_D) \otimes \mathbb{1} + \mathbb{1} \otimes H_{ph}$

3.1 Proof of Theorem 2.2

The proof of Theorem 2.2 is achieved by showing that $H_I^{(1)}$ and $H_I^{(2)} \otimes \mathbb{1}$ are relatively H_0-bounded.

We first treat $H_I^{(1)}$. The result follows from Lemma 3.11 and Corollary 3.12. Then we study all the quartic terms that appear in H_I^2, and the results are collected in Corollary 3.15. In both cases, details of proofs are given in the Appendix.

The interaction between electrons, positrons and transversal photons can be written in the following form (See Section 2.4)

$$H_I^{(1)} = \sum_{i=1}^{6} \sum_{\mu=1,2} \int \mathrm{d}^3k (v_i^{\mu}(k) \otimes a_{\mu}^*(k) + v_i^{\mu *}(k) \otimes a_{\mu}(k)) \tag{31}$$

where

$$\begin{aligned}
v_1^{\mu}(k) &= \sum_{\gamma,\gamma',n,\ell} G^{\mu}_{d,\gamma,\gamma',n,\ell}(k) b^*_{\gamma,n} b_{\gamma',l} \ , \\
v_2^{\mu}(k) &= \sum_{\gamma,\gamma',n} \int \mathrm{d}p \, G^{\mu}_{d,+,\gamma,\gamma',n}(p;k)(b^*_{\gamma,n} b_{\gamma',+}(p) + b^*_{\gamma,+}(p) b_{\gamma',n}) \ , \\
v_3^{\mu}(k) &= \sum_{\gamma,\gamma',n} \int \mathrm{d}p \, G^{\mu}_{d,-,\gamma,\gamma',n}(p;k)(b^*_{\gamma,n} b^*_{\gamma',-}(p) + b_{\gamma,-}(p) b_{\gamma',n}) \ , \\
v_4^{\mu}(k) &= \sum_{\gamma,\gamma'} \int \int \mathrm{d}p \, \mathrm{d}p' G^{\mu}_{+,-,\gamma,\gamma'}(p,p';k)(b^*_{\gamma,+}(p) b^*_{\gamma',-}(p') + b_{\gamma,-}(p) b_{\gamma',+}(p')) \ , \\
v_5^{\mu}(k) &= \sum_{\gamma,\gamma'} \int \int \mathrm{d}p \, \mathrm{d}p' G^{\mu}_{+,+,\gamma,\gamma'}(p,p';k) b^*_{\gamma,+}(p) b_{\gamma',+}(p') \ , \\
v_6^{\mu}(k) &= \sum_{\gamma,\gamma'} \int \int \mathrm{d}p \, \mathrm{d}p' G^{\mu}_{-,-,\gamma,\gamma'}(p,p';k) b^*_{\gamma,-}(p) b_{\gamma',-}(p') \ .
\end{aligned}$$

It follows from Proposition 3.4, Proposition 3.6 and Lemma 3.9 that $v_i^{\mu}(k)$, $i \neq 5,6$, $v_j^{\mu}(k)^*(N_D+1)^{-1/2}$, $(N_D+1)^{-1/2} v_j^{\mu}(k)$, $j = 5,6$ are bounded from $\mathfrak{F}_D$ into $\mathfrak{F}_D$.

For $\beta = 0,1$, $\mu = 1,2$ and $i \in \{1,2,\ldots,6\}$, we set

$$\begin{aligned}
a^{\mu}_{\beta,i} &= \left(\int_{\mathbb{R}^3} \omega(k)^{-\beta} \|v_i^{\mu}(k)\|^2 \mathrm{d}^3k \right)^{1/2} , \qquad i \neq 5,6 \\
a^{\mu}_{\beta,i} &= \left(\int_{\mathbb{R}^3} \omega(k)^{-\beta} \|v_i^{\mu}(k)(N_D+1)^{-1/2}\|^2 \mathrm{d}^3k \right)^{1/2} , \qquad i = 5,6.
\end{aligned} \tag{32}$$

$$\begin{aligned}
b_i^{\mu} &= \left(\int \omega(k)^{-1} \|v_i^{\mu}(k)\|^2 \mathrm{d}^3k \right)^{1/2} , \qquad i \neq 5,6 \\
b_i^{\mu} &= \left(\int \omega(k)^{-1} \|(N_D+1)^{-1/2} \, v_i^{\mu}(k)\|^2 \mathrm{d}^3k \right)^{1/2} , \qquad i = 5,6.
\end{aligned} \tag{33}$$

Lemma 3.11. *For every $\Psi \in \mathfrak{D}(H_0)$, we have*

$$\left\| \int \mathrm{d}^3k \, v_i^\mu(k)^* \otimes a_\mu(k) \, \Psi \right\| \le \frac{b_i^\mu}{\sqrt{E_0}} \|(H_0 + m_0c^2)\Psi\|, \tag{34}$$

$$\begin{aligned} \left\| \int \mathrm{d}^3k \, v_i^\mu(k) \otimes a_\mu^*(k) \, \Psi \right\|^2 \le & \left(\frac{(a_{1,i}^\mu)^2}{E_0} + \varepsilon \frac{(a_{0,i}^\mu)^2}{E_0^2} \right) \|(H_0 + m_0c^2)\Psi\|^2 \\ & + \frac{(a_{0,i}^\mu)^2}{4\varepsilon} \|\Psi\|^2 \end{aligned} \tag{35}$$

for every $\varepsilon > 0$.

Proof. In the appendix, we prove that, for every $\Psi \in \mathfrak{D}(H_0)$ and every $\varepsilon > 0$, we have

$$\left\| \int \mathrm{d}^3k \, v_i^\mu(k)^* \otimes a_\mu(k) \, \Psi \right\| \le b_i^\mu \, \|(N_D + 1)^{1/2} \otimes H_{ph}^{1/2} \Psi\|, \tag{36}$$

$$\begin{aligned} \left\| \int \mathrm{d}^3k \, v_i^\mu(k) \otimes a_\mu^*(k) \, \Psi \right\|^2 & \le (a_{1,i}^\mu)^2 \|(N_D + 1)^{1/2} \otimes H_{ph}^{1/2} \Psi\|^2 \\ & + (a_{0,i}^\mu)^2 \left[\varepsilon \|(N_D + 1) \otimes \mathbb{1} \, \Psi\|^2 + \frac{1}{4\varepsilon} \|\Psi\|^2 \right], \end{aligned} \tag{37}$$

which together with Lemma 3.8 and Lemma 3.9 give (34) and (35) by noting that $E_0 < m_0c^2$. ☐

Finally, from (31), (34) and (35), we get

Corollary 3.12. *The operator $H_I^{(1)}$ is $(H_0 + m_0c^2)$-bounded, with relative bound*

$$C_1' = \sum_{\mu=1,2} \sum_{i=1}^{6} \frac{a_{1,i}^\mu + b_{1,i}^\mu}{\sqrt{E_0}} \ .$$

We now treat the term $H_I^{(2)}$.

For $F^{(1)} \in L^2$, the following formal operator

$$\int \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \, \mathrm{d}\xi_3 \, \mathrm{d}\xi_4 F^{(1)}(\xi_1, \xi_2, \xi_3, \xi_4) b_+^*(\xi_1) b_-^*(\xi_2) b_+(\xi_3) b_-(\xi_4)$$

is defined as a quadratic form on $\mathfrak{F}_{D,fin} \times \mathfrak{F}_{D,fin}$ in the following way

$$\int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 \langle b_-(\xi_2) b_+(\xi_1) \Psi, b_+(\xi_3) b_-(\xi_4) \Phi \rangle$$

As for the quadratic terms, one can show that there exists an operator, denoted by $C^{(1)}$ such that $C^{(1)}$ is the unique operator in $\mathfrak{F}_D$ such that $\mathfrak{F}_{D,fin} \subset \mathfrak{D}(C^{(1)})$ is a core for $C^{(1)}$ and

$$C^{(1)} = \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 F^{(1)}(\xi_1, \xi_2, \xi_3, \xi_4) b_+^*(\xi_1) b_-^*(\xi_2) b_+(\xi_3) b_-(\xi_4)$$

and

$$(C^{(1)})^* = \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 \overline{F^{(1)}(\xi_3, \xi_4, \xi_1, \xi_2)} b_+^*(\xi_1) b_-^*(\xi_2) b_+(\xi_3) b_-(\xi_4)$$

as quadratic forms on $\mathfrak{F}_{D,fin} \times \mathfrak{F}_{D,fin}$. In the same way, for $F^{(2)}_\epsilon$, $F^{(3)}_{\epsilon,\epsilon'}$ and $F^{(4)}_{\epsilon,\epsilon'}$ in L^2, we define the following operators as quadratic forms on $\mathfrak{F}_{D,fin} \times \mathfrak{F}_{D,fin}$

$$C^{(2)}_\epsilon = \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 F^{(2)}_\epsilon(\xi_1, \xi_2, \xi_3, \xi_4) b_\epsilon^*(\xi_1) b_\epsilon^*(\xi_2) b_\epsilon(\xi_3) b_\epsilon(\xi_4) \ ,$$

$$(C^{(2)}_\epsilon)^* = \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 \overline{F^{(2)}_\epsilon(\xi_4, \xi_3, \xi_2, \xi_1)} b_\epsilon^*(\xi_1) b_\epsilon^*(\xi_2) b_\epsilon(\xi_3) b_\epsilon(\xi_4) \ ,$$

$$C^{(3)}_{\epsilon,\epsilon'} = \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 F^{(3)}_{\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3, \xi_4) b_\epsilon^*(\xi_1) b_\epsilon(\xi_2) b_{\epsilon'}(\xi_3) b_\epsilon(\xi_4) \ ,$$

$$(C^{(3)}_{\epsilon,\epsilon'})^* = -\int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 \overline{F^{(3)}_{\epsilon,\epsilon'}(\xi_4, \xi_2, \xi_3, \xi_1)} b_\epsilon^*(\xi_1) b_\epsilon^*(\xi_2) b_{\epsilon'}^*(\xi_3) b_\epsilon(\xi_4) \ ,$$

$$C^{(4)}_{\epsilon,\epsilon'} = \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 F^{(4)}_{\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3, \xi_4) b_\epsilon(\xi_1) b_\epsilon(\xi_2) b_{\epsilon'}(\xi_3) b_{\epsilon'}(\xi_4) \ ,$$

$$(C^{(4)}_{\epsilon,\epsilon'})^* = \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 \overline{F^{(4)}_{\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3, \xi_4)} b_\epsilon^*(\xi_1) b_\epsilon^*(\xi_2) b_{\epsilon'}^*(\xi_3) b_{\epsilon'}^*(\xi_4) \ ,$$

We now have the following proposition

Proposition 3.13.

(i) For $F^{(1)} \in L^2$ we have

$$\|C^{(1)}(N_D + 1)^{-1}\| \leq \frac{1}{\sqrt{2}} \|F^{(1)}\|$$

and

$$\|(C^{(1)})^*(N_D + 1)^{-1}\| \leq \frac{1}{\sqrt{2}} \|F^{(1)}\|$$

(ii) For $F^{(2)}_{\epsilon} \in L^2$ *we have*
$$\|C^{(2)}_{\epsilon}(N_D+1)^{-1}\| \leq \|F^{(2),a}_{\epsilon}\|$$
and
$$\|(C^{(2)}_{\epsilon})^*(N_D+1)^{-1}\| \leq \|F^{(2),a}_{\epsilon}\|$$
where $F^{(2),a}_{\epsilon}(\xi_1,\xi_2,\xi_3,\xi_4) = F^{(2)}_{\epsilon}(\xi_1,\xi_2,\xi_3,\xi_4) - F^{(2)}_{\epsilon}(\xi_1,\xi_2,\xi_4,\xi_3)$.

(iii) For $F^{(3)}_{\epsilon,\epsilon'} \in L^2$ *we have*
$$\|C^{(3)}_{\epsilon,\epsilon'}(N_D+1)^{-\frac{1}{2}}\| \leq \|F^{(3),a}_{\epsilon,\epsilon'}\|$$
and
$$\|(C^{(3)}_{\epsilon,\epsilon'})^*(N_D+1)^{-\frac{1}{2}}\| \leq \|F^{(3),a}_{\epsilon,\epsilon'}\|$$
where $F^{(3),a}_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4) = F^{(3)}_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4) - F^{(3)}_{\epsilon,\epsilon'}(\xi_1,\xi_4,\xi_3,\xi_2)$.

(iv) For $F^{(4)}_{\epsilon,\epsilon'} \in L^2$ *we have*
$$\|C^{(4)}_{\epsilon,\epsilon'}\| \leq \|F^{(4),aa}_{\epsilon,\epsilon'}\|$$
and
$$\|(C^{(4)}_{\epsilon,\epsilon'})^*\| \leq \|F^{(4),aa}_{\epsilon,\epsilon'}\|$$
where $F^{(4),aa}_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4) = F^{(4)}_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3,\xi_4) - F^{(4)}_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_4,\xi_3) - F^{(4)}_{\epsilon,\epsilon'}(\xi_2,\xi_1,\xi_3,\xi_4) + F^{(4)}_{\epsilon,\epsilon'}(\xi_2,\xi_1,\xi_4,\xi_3)$.

Proof. We mimick the proof of Proposition 3.6 by using Proposition 3.4. We first prove (i). Let $\Psi = (\Psi^{(q,r,s)})$ and $\Phi = (\Phi^{(q,r,s)})$ be two vectors of $\mathfrak{F}_{D,fin}$. We have

$$\begin{aligned}&\langle \Phi^{(q',r',s')}, C^{(1)}\Psi^{(q,r,s)}\rangle = \delta_{q,q'}\delta_{r,r'}\delta_{s,s'}\\ &\times \int \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\xi_4 F^{(1)}(\xi_1,\xi_2,\xi_3,\xi_4)\langle b_-(\xi_2)b_+(\xi_1)\Phi^{(q,r,s)}, b_+(\xi_3)b_-(\xi_4)\Psi^{(q,r,s)}\rangle\end{aligned}$$

Thus $C^{(1)}\Psi^{(q,r,s)} \in \mathfrak{F}^{(q,r,s)}_a$ for every triple (q,r,s). By the Fubini Theorem we get

$$\begin{aligned}&|\langle \Phi^{(q',r',s')}, C^{(1)}\Psi^{(q,r,s)}\rangle|\\ &= \left|\int\left\langle b_-(\xi_2)b_+(\xi_1)\Psi^{(q,r,s)}, \int F^{(1)}(\xi_1,\xi_2,\xi_3,\xi_4)b_+(\xi_3)b_-(\xi_4)\Psi^{(q,r,s)}\mathrm{d}\xi_3\mathrm{d}\xi_4\right\rangle \mathrm{d}\xi_1\mathrm{d}\xi_2\right|\end{aligned}$$

By the Cauchy-Schwarz inequality and Proposition 3.4(ii) we get

$$\begin{aligned}&|\langle \Phi^{(q,r,s)}, C^{(1)}\Psi^{(q,r,s)}\rangle|^2 \leq\\ &\left(\int \|b_-(\xi_2)b_+(\xi_1)\Psi^{(q,r,s)}\| \left(\int |F^{(1)}(\xi_1,\xi_2,\xi_3,\xi_4)|^2\mathrm{d}\xi_3\mathrm{d}\xi_4\right)^{\frac{1}{2}}\mathrm{d}\xi_1\mathrm{d}\xi_2\right)^2 \|\Psi^{(q,r,s)}\|^2\end{aligned}$$

Again by the Cauchy-Schwarz inequality and the definitions of $b_+(\xi)$ and $b_-(\xi)$ we finally get

$$\begin{aligned} |\langle \Phi^{(q,r,s)}, C^{(1)}\Psi^{(q,r,s)}\rangle|^2 &\leq rs\|F^{(1)}\|^2\|\Phi^{(q,r,s)}\|^2\|\Psi^{(q,r,s)}\|^2 \\ &\leq \frac{1}{2}\|F^{(1)}\|^2\|\Phi^{(q,r,s)}\|^2\|(N_D+1)^2\Psi^{(q,r,s)}\|^2 \ , \end{aligned}$$

which yields

$$\|C^{(1)}(N_D+1)^{-1}\| \leq \frac{1}{\sqrt{2}}\|F^{(1)}\|_2 \ .$$

The estimate for $(C^{(1)})^*$ follows directly from above and from

$$\langle (C^{(1)})^*\Phi^{(q,r,s)}, \Psi^{(q,r,s)}\rangle = \langle (\Phi^{(q,r,s)}, C^{(1)}\Psi^{(q,r,s)}\rangle$$

Let us now prove (ii). We consider only the case of $C_+^{(2)}$, since the proof for $C_-^{(2)}$ is the same. We have

$$\begin{aligned} &\langle \phi^{(q',r',s')}, C_+^{(2)}\Psi^{(q,r,s)}\rangle = \delta_{qq'}\delta_{rr'}\delta_{ss'} \\ &\times \int \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\xi_4 F_+^{(2)}(\xi_1,\xi_2,\xi_3,\xi_4)\langle b_+(\xi_2)b_+(\xi_1)\Phi^{(q,r,s)}, b_+(\xi_3)b_+(\xi_4)\Psi^{(q,r,s)}\rangle \end{aligned}$$

Thus $C_+^{(2)}\Psi^{(q,r,s)} \in \mathfrak{F}_a^{(q,r,s)}$ for every triple (q,r,s) and

$$\begin{aligned} &|\langle \Phi^{(q,r,s)}, C_+^{(2)}\Psi^{(q,r,s)}\rangle|^2 \\ =&\left|\int \left\langle b_+(\xi_2)b_+(\xi_1)\Phi^{(q,r,s)}, \int F_+^{(2)}(\xi_1,\xi_2,\xi_3,\xi_4)b_+(\xi_3)b_+(\xi_4)\Psi^{(q,r,s)}\right\rangle \mathrm{d}\xi_1\mathrm{d}\xi_2\right|^2 \end{aligned}$$

By the Cauchy-Schwarz inequality and Proposition 3.4(i) we obtain

$$\begin{aligned} &|\langle \Phi^{(q,r,s)}, C_+^{(2)}\Psi^{(q,r,s)}\rangle|^2 \\ \leq&\left(\int \|b_+(\xi_2)b_+(\xi_1)\Phi^{(q,r,s)}\| \left(\int |F_+^{(2),a}(\xi_1,\xi_2,\xi_3,\xi_4)|^2\mathrm{d}\xi_3\mathrm{d}\xi_4\right)^{\frac{1}{2}} \mathrm{d}\xi_1\mathrm{d}\xi_2\right)^2 \|\Psi^{(q,r,s)}\|^2 \end{aligned}$$

Applying again Cauchy-Schwarz inequality and the definition of $b_+(\xi)$ we get

$$\begin{aligned} |\langle \Phi^{(q,r,s)}, C_+^{(2)}\Psi^{(q,r,s)}\rangle|^2 \leq& r(r-1)\|F_+^{(2),a}\|^2\|\Phi^{(q,r,s)}\|^2\|\Psi^{(q,r,s)}\|^2 \\ &\leq \|F_+^{(2),a}\|^2\|\Phi^{(q,r,s)}\|^2\|(N_D+1)\Psi^{(q,r,s)}\|^2 \ . \end{aligned}$$

We conclude this item as in the proof of Proposition 3.6. We thus obtain

$$\|C_+^{(2)}(N_D+1)^{-1}\| \leq \|F_\epsilon^{(2),a}\| \ .$$

The proof for $C_-^{(2)}$, $(C_+^{(2)})^*$ and $(C_-^{(2)})^*$ is similar.

We now prove (iii). We have

$$\langle \Phi^{(q',r',s')}, C^{(3)}_{+-}\Psi^{(q,r,s)}\rangle = \delta_{q',q}\delta_{r',r-1}\delta_{s',s-1}$$
$$\times \int \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\xi_4 F^{(3)}_{+-}(\xi_1,\xi_2,\xi_3,\xi_4)\langle b_+(\xi_1)\Phi^{(q',r',s')}, b_+(\xi_2)b_-(\xi_3)b_+(\xi_4)\Psi^{(q,r,s)}\rangle$$

Thus $C^{(3)}_{+-}\Psi^{(q,r,s)} \in \mathfrak{F}_a^{q,r-1,s-1)}$ for $r>1$ and $s>1$ and $C^{(3)}_{+-}\Psi^{(q,r,s)} = 0$ for $r \le 1$ or $s \le 1$. We then have

$$|\langle \Phi^{(q,r-1,s-1)}, C^{(3)}_{+-}\Psi^{(q,r,s)}\rangle|^2$$
$$= \left|\int \left\langle b_+(\xi_1)\Phi^{(q,r-1,s-1)}, \int F^{(3)}_{+-}(\xi_1,\xi_2,\xi_3,\xi_4)b_+(\xi_2)b_-(\xi_3)b_+(\xi_4)\Psi^{(q,r,s)}\right\rangle\right|^2$$

Using Cauchy-Schwarz inequality and Proposition 3.4(iii) we obtain

$$|\langle \Phi^{(q,r-1,s-1)}, C^{(3)}_{+-}\Psi^{(q,r,s)}\rangle|^2$$
$$\le \left(\int \|b_+(\xi_1)\Phi^{(q,r-1,s-1)}\| \left(\int |F^{(3),a}_{+-}(\xi_1,\xi_2,\xi_3,\xi_4)|^2 \mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\xi_4\right)^{\frac{1}{2}} \mathrm{d}\xi_1\right)^2 \|\Psi^{(q,r,s)}\|^2$$

Thus, using Cauchy-Schwarz inequality and the definition of $b_+(\xi)$ we get

$$\begin{aligned}|\langle \Phi^{(q,r-1,s-1)}, C^{(3)}_{+-}\Psi^{(q,r,s)}\rangle|^2 &\le (r-1)\|F^{(3),a}_{+-}\|^2\|\Phi^{(q,r-1,s-1)}\|^2\|\Psi^{(q,r,s)}\|^2\\ &\le \|F^{(3),a}_{+-}\|^2\|\Phi^{(q,r-1,s-1)}\|^2\|(N_D+1)^{\frac{1}{2}}\Psi^{(q,r,s)}\|^2\end{aligned}$$

We conclude as in the proof of Proposition 3.6 that

$$\|C^{(3)}_{+-}(N_D+1)^{-\frac{1}{2}}\| \le \|F^{(3),a}_{+-}\| \ .$$

The proofs for $C^{(3)}_{\epsilon\epsilon'}$ and $(C^{(3)}_{\epsilon\epsilon'})^*$ are very similar.

Let us finally prove (iv). We have

$$\langle \Phi^{(q',r',s')}, C^{(4)}_{+-}\Psi^{(q,r,s)}\rangle = \delta_{q,q'}\delta_{r',r-2}\delta_{s',s-2}$$
$$\times \int \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\xi_4 F^{(4)}_{+-}(\xi_1,\xi_2,\xi_3,\xi_4)\langle \Phi^{(q',r',s')}, b_+(\xi_1)b_+(\xi_2)b_-(\xi_3)b_-(\xi_4)\Psi^{(q,r,s)}\rangle \ .$$

Thus $C^{(4)}_{+-}\Psi^{(q,r,s)} \in \mathfrak{F}_a^{(q,r-2,s-2)}$ for $r>2$ and $s>2$ and $C^{(4)}_{+-}\Psi^{(q,r,s)} = 0$ for $r \le 2$ or $s \le 2$. By Proposition 3.4(iv) we get for $r>2$ and $s>2$

$$|\langle \Phi^{(q,r-2,s-2)}, C^{(4)}_{+-}\Psi^{(q,r,s)}\rangle|^2 \le \|F^{(4),aa}_{+-}\|^2\|\Phi^{(q,r-2,s-2)}\|^2\|\Psi^{(q,r,s)}\|^2$$

Thus

$$\|C^{(4)}_{+-}\| \le \|F^{(4),aa}_{+-}\| \ .$$

The proof for $C^{(4)}_{\epsilon,\epsilon'}$ and $(C^{(4)}_{\epsilon\epsilon'})^*$ is similar.

This concludes the proof of Proposition 3.13. ☐

Remark 3.14. *One can show, as in Remark 3.7 that the estimates (i), (ii), (iii) and (iv) are the best one can get.*

As a consequence of Lemmas 3.8, 3.9 and Proposition 3.13 we obtain

Corollary 3.15. *For any $\Psi \in \mathfrak{D}(N_D)$ we have*

$$\|H_I^{(2)}\Psi\| \le \left(\frac{1}{\sqrt{2}}\|F^{(1)}\| + \|F_+^{(2),a}\| + \|F_-^{(2),a}\|\right)\|(N_D+1)\Psi\|$$
$$+\left(\|F_{+-}^{(3),a}\| + \|F_{-+}^{(3),a}\|\right)\|(N_D+1)^{\frac{1}{2}}\Psi\| + \left(\|F_{+-}^{(4),aa}\| + \|F_{-+}^{(4),aa}\|\right)\|\Psi\| \ .$$

Proof of Theorem 2.2. Supposing that $C_0 < \infty$, $C_1 < \infty$ and choosing g_1 and g_2 such that $\frac{|g_1|}{\sqrt{E_0}}C_1 + \frac{|g_2|}{E_0}(\frac{1}{\sqrt{2}}\|F^{(1)}\| + \|F_+^{(2),a}\| + \|F_-^{(2),a}\|) < 1$, Theorem 2.2 follows from Proposition 3.6, Corollary 3.12, Corollary 3.15 and Theorem V.4.3 of [27].

4 Proof of Theorem 2.3

Throughout this section we assume that the assumptions of Theorem 2.3 are satisfied. Let us introduce an infrared cutoff in the first part of the interaction

Let $H_{I,m}^{(1)}$ be the operator obtained from (31) by substituting

$$v_{i,m}^{\mu}(k) := 1_{\{k;\omega(k)\ge m\}}v_i^{\mu}(k) \ , m > 0 \ ,$$

for $v_i^{\mu}(k)$. We then define

$$H_m := H_0 + g_1 H_{I,m}^{(1)} + g_2 H_I^{(2)} \otimes \mathbb{1}.$$

Theorem 2.3 will be a simple consequence of the following result.

Theorem 4.1. *There exists $g_0 > 0$ such that for every (g_1, g_2) satisfying $|g_1| + |g_2| \le g_0$, the following properties hold.*

(i) For every $\Psi \in \mathfrak{D}(H_0)$, $H_m\Psi \underset{m\mapsto 0}{\longrightarrow} H\Psi$.

(ii) For every $m \in (0,1)$, H_m has a normalized ground state Φ_m.

(iii) Fix λ in (E_0, m_0c^2). For every $m > 0$, we have

$$\langle \Phi_m, P_{(-\infty,\lambda]}(d\Gamma(H_D)) \otimes P_{\Omega_{ph}}\Phi_m\rangle \ge 1 - \delta_{g_1,g_2} \ , \tag{38}$$

where δ_{g_1,g_2} tends to zero when $|g_1|+|g_2|$ tends to zero and $\delta_{g_1,g_2} < 1$ for $|g_1|+|g_2| \le g_0$.

Theorem 2.3 is easily deduced from Theorem 4.1 as follows: Let H_m and Φ_m be the same as in Theorem 4.1. Since $\|\Phi_m\| = 1$, there exists a subsequence (Φ_{m_k}) of (Φ_m) such that $w - \lim \Phi_{m_k} = \Phi$.

On the other hand, since $P_{(-\infty,\lambda]}(d\Gamma(H_D)) \otimes P_{\Omega_{ph}}$ is of finite rank for $\lambda \in (E_0, m_0c^2)$, it follows from (38) that

$$\langle \Phi, P_{(-\infty,\lambda]}(d\Gamma(H_D)) \otimes P_{\Omega_{ph}} \Phi\rangle \geq 1 - \delta_{g_1, g_2},$$

which implies that $\Phi \neq 0$.

Now, in order to conclude the proof of Theorem 2.3, it suffices to apply the following well known result [4]

Lemma 4.2. *Let T_n, $n \geq 1$, and T be self-adjoint operators on a Hilbert space $\mathfrak{K}$ having a common core $\mathfrak{D}$ such that, for all $\Phi \in \mathfrak{D}$, $T_n\Phi \to T\Phi$ as $n \to \infty$. We assume that, for every $n \geq 1$, T_n has a normalized ground state Φ_n with ground state energy E_n such that $\lim_{n\to\infty} E_n = E$ and $\omega - \lim_{n\mapsto+\infty} \Phi_n = \Phi \neq 0$. Then Φ is a ground state of T with ground state energy E.*

Theorem 4.1 will be proved in several steps. We first prove (i). We then prove (iii). Finally, by the periodic approximation method, we will prove the existence of a ground state for H_m, $m > 0$.

4.1 Proof of (i) of Theorem 4.1

Proof. For $m > 0$, set

$$\tilde{v}_i^{\mu,m}(k) = 1_{\{k;\omega(k)\leq m\}}(k)\, v_i^{\mu}(k).$$

We denote by $\tilde{a}_{\beta,j}^{\mu,m}$ (resp. $\tilde{b}_j^{\mu,m}$) the expression (32) and (33) that we obtain by substituting $\tilde{v}_j^{\mu,m}(k)$ for $v_j^{\mu}(k)$.

Since $H - H_m = g_1(H_I^{(1)} - H_{I,m}^{(1)})$, it follows from Lemma 3.11 that

$$\|(H - H_m)\Phi\| \leq g_1 \sum_{\mu=1,2} \sum_{j=0}^{6} \left(\left(\frac{\tilde{a}_{1,j}^{\mu,m} + \tilde{b}_j^{\mu,m}}{\sqrt{E_0}} + \frac{\tilde{a}_{0,j}^{\mu,m}}{E_0} \right) \|(H_0 + m_0c^2)\Phi\| + \frac{\tilde{a}_{0,j}^{\mu,m}}{4} \|\Phi\| \right)$$

for every $\Phi \in \mathfrak{D}(H_0)$.

By Lebesgue's Theorem, $(\tilde{a}_{1,j}^{\mu,m} + \tilde{a}_{0,j}^{\mu,m} + \tilde{b}_j^{\mu,m})$ tends to zero when m tends to zero. We then get, for every $\Phi \in \mathfrak{D}(H_0)$,

$$\lim_{m\mapsto 0} \|(H - H_m)\Phi\| = 0\ . \tag{39}$$

Thus the first statement of Theorem 4.1 is proved.

$\square$

4.2 Proof of (iii) of Theorem 4.1

The proof of (38) will be a consequence of the following lemmas.

Lemma 4.3. *For every $m \in (0,1)$, we have*

$$E_m := \inf_{\substack{\Phi \in \mathfrak{D}(H_0) \\ \|\Phi\|=1}} (H_m \Phi, \Phi) \leq 0$$

and

$$\sup_{m \in (0,\, 1)} |E_m| < \infty \ . \tag{40}$$

Proof. Since $\langle a_\mu(k)\Omega_{ph}, \Omega_{ph}\rangle = \langle \Omega_{ph}, a^*_\mu(k)\Omega_{ph}\rangle = 0$, we have

$$\langle H^{(1)}_{I,m} \Omega_D \otimes \Omega_{ph}, \Omega_D \otimes \Omega_{ph}\rangle = 0 \ ,$$

and

$$\langle (\mathbb{1} \otimes H_{ph})\, \Omega_D \otimes \Omega_{ph}, \Omega_D \otimes \Omega_{ph}\rangle = 0 \ .$$

Furthermore we also obviously have

$$\langle H^{(2)}_I \Omega_D,\, \Omega_D\rangle = 0 \ .$$

Hence we get

$$\langle H_m \Omega_D \otimes \Omega_{ph}, \Omega_D \otimes \Omega_{ph}\rangle \quad = \quad \langle (d\Gamma(H_D) \otimes \mathbb{1})\, \Omega_D \otimes \Omega_{ph}, \Omega_D \otimes \Omega_{ph}\rangle = 0$$

and

$$E_m = \inf_{\substack{\Phi \in \mathfrak{D}(H_0) \\ \|\Phi\|=1}} \langle H_m \Phi, \Phi\rangle \leq \langle H_m \Omega_D \otimes \Omega_{ph}, \Omega_D \otimes \Omega_{ph}\rangle = 0$$

By Lemma 3.11, which also holds when $v^\mu_i(k)$ is replaced by $v^\mu_{i,m}(k)$, we get for some finite constant C which does not depend on $m \in (0,1)$:

$$\|H^{(1)}_{I,m}\Phi\| \leq C\left(\|H_0 + m_0 c^2)\Phi\| + \|\Phi\|\right) \tag{41}$$

for every $\Phi \in \mathfrak{D}(H_0)$. From Corollary 3.15 we have, for some finite constant $\tilde{C}$:

$$\|(H^{(2)}_I \otimes \mathbb{1})\Phi\| \leq \tilde{C}\left(\|(H_0 + m_0 c^2)\Phi\| + \|\Phi\|\right) \tag{42}$$

for every $\Phi \in \mathfrak{D}(H_0)$. Thus (41), (42) and the Kato-Rellich Theorem yield (40). $\square$

In Section 4.3 below, we will show that, for $|g_1| + |g_2|$ sufficiently small, H_m has a ground state $\Phi_m(g_1, g_2)$, i.e. there exists, for every $m \in (0,1)$, a normalized solution to $H_m \Phi_m(g_1, g_2) = E_m \Phi_m(g_1, g_2)$.

We have

Lemma 4.4. *There exist $\widetilde{g_0} > 0$ and $\nu(\widetilde{g_0}) > 0$, independent of $m \in (0,1)$, such that*

$$\|(H_0 + m_0c^2)\Phi_m(g_1, g_2)\| \leq \nu(\widetilde{g_0}) \ ,$$

for every $m \in (0,1)$ and for every $(g_1,\, g_2)$ such that $|g_1| + |g_2| \leq \widetilde{g_0}$.

Proof. For simplicity, we will drop the dependence on (g_1, g_2) in $\Phi_m(g_1, g_2)$ by denoting $\Phi_m := \Phi_m(g_1, g_2)$. We have

$$\begin{aligned} H_0\Phi_m &= H_m\Phi_m - g_1\, H^{(1)}_{I,m}\Phi_m - g_2(H^{(2)}_I \otimes \mathbb{1})\Phi_m \\ &= E_m\Phi_m - g_1\, H^{(1)}_{I,m}\Phi_m - g_2(H^{(2)}_I \otimes \mathbb{1})\Phi_m \ . \end{aligned} \tag{43}$$

From (41), (42) and (43) we obtain

$$\|H_0\Phi_m\| \leq |E_m| + \left(|g_1|C + |g_2|\tilde{C}\right)\left[\|(H_0 + m_0c^2)\Phi_m\| + 1\right] \ ,$$

which implies

$$\|(H_0 + m_0c^2)\Phi_m\| \leq \left(m_0c^2 + |E_m| + |g_1|C + |g_2|\tilde{C}\right)\left(1 - |g_1|C - |g_2|\tilde{C}\right)^{-1}$$

To conclude, we apply Lemma 4.3, and we choose $\widetilde{g_0} > 0$ such that $|g_1|\, C + |g_2|\tilde{C} \leq 1/2$ for every (g_1, g_2) such that $|g_1| + |g_2| \leq \widetilde{g_0}$. □

Lemma 4.5. *We have*

$$\sum_{\mu=1,2} \|[1 \otimes a_\mu(k), H^{(1)}_{I,m}]\, \Phi\|^2 \leq \frac{G(k)^2}{E_0^2}[\|H_0\Phi\|^2 + \|\Phi\|^2]$$

for every $\Phi \in \mathfrak{D}(H_0)$ and for a.e. $k \in \mathbb{R}^3$. Here

$$\begin{aligned} G(k)^2 \ &= \sum_{\mu=1,2}\sum_{\gamma,\gamma',n,\ell} |G^\mu_{d,\gamma,\gamma',n,l}(k)|^2 + 4\sum_{\mu=1,2}\sum_{r=+,-}\sum_{\gamma,\gamma',n}\int |G^\mu_{d,r,\gamma,\gamma',n}(p;k)|^2 \mathrm{d}p \\ &+ \ 4\sum_{\mu=1,2}\sum_{\gamma,\gamma'}\int\!\!\int |G_{+,-,\gamma,\gamma'}(p,p';k)|^2 \mathrm{d}p\mathrm{d}p' \\ &+ \ \sum_{\mu=1,2}\sum_{r=+,-}\sum_{\gamma,\gamma'}\int\!\!\int |G_{r,r,\gamma,\gamma'}(p,p';k)|^2 \mathrm{d}p\mathrm{d}p' \end{aligned}$$

Proof. By using the commutation relations, a simple computation shows that

$$[\mathbb{1} \otimes a_\mu(k), H^{(1)}_{I,m}] = \sum_{i=1}^{6} v^\mu_{i,m}(k)$$

The lemma now follows from Proposition 3.4 and Proposition 3.6. □

Lemma 4.6. *There exists $C > 0$, independent of g_1 and g_2 for $|g_1| + |g_2| \leq \widetilde{g_0}$, such that*

$$(\mathbb{1} \otimes N_{ph}\Phi_m, \Phi_m) \leq C\ g_1^2 \ \left(\int_{\mathbb{R}^3} \frac{G(k)^2}{\omega(k)^2}\, \mathrm{d}^3 k \right),$$

for every $m \in (0,1)$. Here

$$N_{ph} := \sum_{\mu=1,2} \int_{\mathbb{R}^3} \mathrm{d}^3 k\, a_\mu^*(k)\, a_\mu(k)$$

is the operator number of photons.

Proof. One easily checks that we have

$$[\mathbb{1} \otimes a_\mu(k), d\Gamma(H_D) \otimes \mathbb{1}]\, \Psi = 0\ ,$$

$$[\mathbb{1} \otimes a_\mu(k), H_I^{(2)} \otimes \mathbb{1}]\Psi = 0\ ,$$

and

$$[\mathbb{1} \otimes a_\mu(k), \mathbb{1} \otimes H_{ph}]\, \Psi = \omega(k)\, (\mathbb{1} \otimes a_\mu(k))\, \Psi\ ,$$

for every $\Psi \in \mathfrak{D}(H_0)$ and for a.e. $k \in \mathbb{R}^3$.

We then obtain :

$$(H_m + \omega(k))\, \mathbb{1} \otimes a_\mu(k) - \mathbb{1} \otimes a_\mu(k)\, H_m = g_1 [H_{I,m}^{(1)}, \mathbb{1} \otimes a_\mu(k)]\ ,$$

for a.e. $k \in \mathbb{R}^3$.

Applying this equality to Φ_m and taking the scalar product with $\mathbb{1} \otimes a_\mu(k)\, \Phi_m$, we get

$$\|\mathbb{1} \otimes a_\mu(k)\Phi_m\| \leq \frac{|g_1|}{\omega(k)}\ \|[\mathbb{1} \otimes a_\mu(k), H_{I,m}]\, \Phi_m\| \tag{44}$$

for a.e. $k \in \mathbb{R}^3$.

Since

$$\langle \mathbb{1} \otimes N_{ph}\, \Phi_m, \Phi_m \rangle = \sum_{\mu=1,2} \int \|\mathbb{1} \otimes a_\mu(k)\, \Phi_m\|^2 \mathrm{d}^3 k$$

Lemma 4.6 follows from (44), Lemma 4.4 and Lemma 4.5. □

Lemma 4.7. *Fix λ in $(0, m_0 c_0^2)$. There exists $\delta_{g_1,g_2}(\lambda) > 0$ such that $\delta_{g_1,g_2}(\lambda)$ tends to zero when $|g_1| + |g_2|$ tends to zero and*

$$\langle P_D^\perp(\lambda) \otimes P_{\Omega_{ph}} \Phi_m, \Phi_m \rangle \leq \delta_{g_1,g_2}(\lambda)$$

for every $m \in (0,1)$. Here $P_D^\perp(\lambda) = \mathbb{1} - P_D(\lambda) = \mathbb{1}_{[\lambda,\ \infty)}(d\Gamma(H_D))$.

Proof. Using $P_{\Omega_{ph}} H_{ph} = 0$, we get

$$\begin{aligned} & (P_D(\lambda)^\perp \otimes P_{\Omega_{ph}})\,(H_m - E_m) \\ = \;& P_D(\lambda)^\perp (d\Gamma(H_D) - E_m) \otimes P_{\Omega_{ph}} + g_1 (P_D(\lambda)^\perp \otimes P_{\Omega_{ph}})\, H_{I,m}^{(1)} \\ & + g_2 \left(P_D(\lambda)^\perp \otimes P_{\Omega_{ph}}\right) H_I^{(2)} \otimes \mathbb{1}\,. \end{aligned} \tag{45}$$

From (45) we obtain

$$\begin{aligned} 0 \;=\;& (P_D^\perp(\lambda) \otimes P_{\Omega_{ph}})\,(H_m - E_m)\,\Phi_m \qquad (46) \\ =\;& P_D^\perp(\lambda)\,(d\Gamma(H_D) - E_m) \otimes P_{\Omega_{ph}} \Phi_m + g_1\; P_D^\perp(\lambda) \otimes P_{\Omega_{ph}} H_{I,m}^{(1)} \Phi_m \\ & + g_2 \left(P_D(\lambda)^\perp \otimes P_{\Omega_{ph}}\right) H_I^{(2)} \otimes \mathbb{1}\,. \end{aligned}$$

Since

$$d\Gamma(H_D) P_D(\lambda)^\perp \geq \lambda\; P_D^\perp(\lambda)\;,$$

by (46) and Lemma 4.3, we obtain

$$\begin{aligned} \langle P_D^\perp(\lambda) \otimes P_{\Omega_{ph}} \Phi_m, \Phi_m\rangle \leq -\,\lambda^{-1} \Big[& g_1 \Big\langle (P_D^\perp(\lambda) \otimes P_{\Omega_{ph}}) H_{I,m}^{(1)} \Phi_m, \Phi_m \Big\rangle \\ & + g_2 \Big\langle (P_D^\perp(\lambda) \otimes P_{\Omega_{ph}}) (H_I^{(2)} \otimes \mathbb{1}) \Phi_m, \Phi_m \Big\rangle \Big]\;, \end{aligned}$$

for every $m \in (0,1)$.

Lemma 4.7 then follows by remarking that Corollary 3.12 also holds for $H_{I,m}^{(1)}$ and by using Lemma 3.15 and Lemma 4.4. ☐

We are now able to prove the inequality (38), i.e., point (iii) of Theorem 4.1.

Proof. The equation (38) is equivalent to

$$\langle (P_D^\perp(\lambda) \otimes P_{\Omega_{ph}} + \mathbb{1} \otimes P_{\Omega_{ph}}^\perp)\, \Phi_m, \Phi_m\rangle \leq \delta_{g_1,g_2},$$

Since $P_{\Omega_{ph}}^\perp \leq N_{ph}$, it suffices to show that

$$\langle (P_D^\perp(\lambda) \otimes P_{\Omega_{ph}} + \mathbb{1} \otimes N_{ph}) \Phi_m, \Phi_m\rangle \leq \delta_{g_1,g_2},$$

which follows from Lemma 4.6 and Lemma 4.7.

From the fact that δ_{g_1,g_2} tends to zero when $|g_1| + |g_2|$ tends to zero, we deduce the existence of $\widetilde{g_{00}}$ such that $\delta_{g_1,g_2} < 1$ for $|g_1| + |g_2| \leq \widetilde{g_{00}}$. ☐

4.3 Proof of (ii) of Theorem 4.1: Existence of a ground state for H_m

From now on we mimic the proof in [2] and [8] (See also [22], [23] and [17]) and we use the same notations as in [8].

For $m > 0$, we set

$$\mathcal{A}_\ell = \{k \in \mathbb{R}^3 \ ; \ \omega(k) \geq m\}, \quad \mathcal{A}_s = \mathbb{R}^3 \setminus \mathcal{A}_\ell \ ,$$

and

$$h_{s/\ell} = L^2(\mathcal{A}_{s/\ell}) \ .$$

Let $\mathfrak{F}(h_{\ell/s})$ be the bosonic Fock spaces associated with $h_{\ell/s}$, with associated vacua $\Omega_{\ell/s}$.

Using the following lemma (See [8] and [14]),

Lemma 4.8. *Let h_i, $i = 1, 2$ be two Hilbert spaces. There exists a unitary operator from the Fermi-Dirac Fock space (resp. the photonic Fock space) over $h_1 \oplus h_2$ to*

$$\mathfrak{F}_D(h_1) \otimes \mathfrak{F}_D(h_2) \quad (\textit{resp. } \mathfrak{F}_{ph}(h_1) \otimes \mathfrak{F}_{ph}(h_2)).$$

We may identify H_{ph} (resp. $g_1 H_I^{(1)}$) with

$$H_{ph,\ell} \otimes \mathbb{1}_s + \mathbb{1}_\ell \otimes H_{ph,s}, \quad (\text{resp. } g_1 H_{I,m}^{(1)} \otimes \mathbb{1}_s).$$

Here

$$H_{ph,\ell/s} = \sum_{\mu=1,2} \int_{\mathcal{A}_{\ell/s}} \mathrm{d}^3 k \, \omega(k) \, a_\mu^*(k) \, a_\mu(k).$$

We set

$$H_{m,\ell} = d\Gamma(H_D) \otimes \mathbb{1}_\ell + \mathbb{1}_D \otimes H_{ph,\ell} + g_1 \, H_{I,m}^{(1)} + g_2 H_I^{(2)} \otimes \mathbb{1}_\ell \ .$$

In this representation, H_m appears as

$$H_m = H_{m,\ell} \otimes \mathbb{1}_s + \mathbb{1}_D \otimes \mathbb{1}_\ell \otimes H_{ph,s} \ . \tag{47}$$

We then have

Lemma 4.9. *$H_{m,\ell}$ has a ground state $\Phi_{m,\ell}$ if and only if H_m has a ground state $\Phi_m = \Phi_{m,\ell} \otimes \Omega_s$.*

Proof. In view of (47), one has

$$H_m(\Phi_{m,\ell} \otimes \Omega_s) = H_{m,\ell} \, \Phi_{m,\ell} \otimes \Omega_s \tag{48}$$

Furthermore, it results from (47) that

$$\sigma(H_m) \quad = \quad \overline{\sigma(H_{m,\ell}) + \sigma(H_{ph,s})} \quad = \quad [\inf \sigma(H_{m,\ell}), +\infty)$$

which, together with (48), gives the lemma. ∎

By Lemma 4.9, Theorem 4.1 then follows from

Theorem 4.10. *There exists $\widetilde{g_{000}}$ such that, for every (g_1, g_2) such that $|g_1| + |g_2| \leq \widetilde{g_{000}}$ and $m \in (0,1)$, $H_{m,\ell}$ has a ground state $\Phi_{m,\ell}$ at $E_m = \inf\sigma(H_m) = \inf\sigma(H_{m,\ell})$.*

Proof. Let $\varepsilon > 0$ be a parameter and $\gamma \in \mathbb{Z}^3$. We use the same decomposition of $\mathbb{R}^3$ into a disjoint union of "cubes" $(C_{\gamma,\varepsilon})_{\gamma\in\mathbb{Z}^3}$ as in [8].

For $F \in L^1_{loc}(\mathbb{R}^3)$, we define its ε-average by

$$\langle F\rangle_\varepsilon(k) = \varepsilon^{-3}\int_{C_{\gamma,\varepsilon}} F(k')\,\mathrm{d}^3k' \quad \text{for} \quad k \in C_{\gamma,\varepsilon}\ .$$

More generally, when $G \in L^2(\Sigma\times\mathbb{R}^3_k)$, we define its ε-average with respect to k by

$$\langle G\rangle_\varepsilon(k) := G_\varepsilon(\cdot,k) = \varepsilon^{-3}\int_{C_{\gamma,\varepsilon}} G(\cdot,k')\,\mathrm{d}^3k' \quad \text{for} \quad k \in C_{\gamma,\varepsilon}.$$

Let $H^\varepsilon_{I,m}$ be the operator obtained from $H^{(1)}_{I,m}$ by substituting $\langle v^\mu_{i,m}\rangle_\varepsilon(k)$ for $v^\mu_{i,m}(k)$.

Similarly, we define $H^\varepsilon_{ph,\ell/s}$ by substituting $\langle\omega(k)\rangle_\epsilon$ for $\omega(k)$ in $H_{ph,\ell/s}$.

A simple calculation shows that

$$|\omega(k) - \langle\omega\rangle_\varepsilon(k)| \leq \tilde{C}\,\varepsilon\,\omega(k)\ ,$$

where $\tilde{C}$ does not depend on $\gamma\in\mathbb{Z}^3$. It implies

$$\pm(H_{ph,\ell} - H^\varepsilon_{ph,\ell}) \leq \tilde{C}\,\varepsilon\,H_{ph,\ell}\ . \tag{49}$$

Set $H^\varepsilon_{m,\ell} = d\Gamma(H_D)\otimes\mathbb{1}_s + \mathbb{1}_D\otimes H^\varepsilon_{ph,\ell} + g_1 H^\varepsilon_{I,m} + g_2 H^{(2)}_I \otimes \mathbb{1}_l$. Combining (49) with Corollary 3.12 and using the fact that

$$H^\varepsilon_{m,\ell} - H_{m,\ell} = H^\varepsilon_{ph,\ell} - H_{ph,\ell} + g_1(H^\varepsilon_{I,m} - H^{(1)}_{I,m})\ ,$$

we obtain

$$\|(H^\varepsilon_{m,\ell} - H_{m,\ell})\,(H_{m,\ell} - E_m + 1)^{-1}\| \leq \text{const.}\sum_{\beta=0,1}\sum_{j=1}^{6}(a^{\mu,m,\varepsilon}_{\beta,j} + b^{\mu,m,\varepsilon}_j)\ , \tag{50}$$

where $a^{\mu,m,\varepsilon}_{\beta,j}$ (resp. $b^{\mu,m,\varepsilon}_j$) denotes the expression (32) (resp. (33)) that we obtain when we substitute $v^\mu_{j,m}(k) - \langle v^\mu_{j,m}\rangle_\varepsilon(k)$ for $v^\mu_j(k)$.

By Lebesgue's dominated convergence Theorem, the right hand side of (50) converges to zero as $\varepsilon\mapsto 0$.

On the other hand, by mimicking [7] and [8] one shows that there exists $\widetilde{g_{000}}$ such that the finite volume approximation $H^\varepsilon_{m,\ell}$ has discrete spectrum in $(-\infty, E_m + m_2)$, for $m_2 < m$ and for (g_1,g_2) such that $|g_1| + |g_2| \leq \widetilde{g_{000}}$.

Theorem 4.10 then follows from (50) and from the following lemma (see [24]). $\square$

Lemma 4.11. *Let $(T_n)_{n\geq 1}$ and T be bounded below self-adjoint operators on a Hilbert space. Suppose that*

i) $T_n \to T$ as $n \mapsto +\infty$ in the norm resolvent sense.

ii) T_n has purely discrete spectrum in $[\inf\sigma(T_n), \inf\sigma(T_n)+C)$, where C is a constant independent of n.

Then T has purely discrete spectrum in $[\inf\sigma(T), \inf\sigma(T)+C)$.

Proofs of Theorem 2.3 and Theorem 4.1. Choosing $g_0 = \min\{\widetilde{g_0}, \widetilde{g_{00}}, \widetilde{g_{000}}\}$, Theorem 4.1 follows from (39), Lemma 4.7 and Theorem 4.10. The proof of Theorem 2.3 is easily deduced from Theorem 4.1 as it is explained at the beginning of the Section 4.

5 Appendix

5.1 Proof of (36) and (37)

We keep the same notations as in Section 3.

Lemma 5.1. *Under the assumptions of Theorem 2.2, we have*

$$\| \int \mathrm{d}^3 k\, v_i^{\mu}(k)^* \otimes a_{\mu}(k)\,\Psi\| \leq b_i^{\mu} \|(N_D+1)^{1/2} \otimes H_{ph}^{1/2}\Psi\| \ , \tag{51}$$

and

$$\begin{aligned}\| \int \mathrm{d}^3 k\, v_i^{\mu}(k) \otimes a_{\mu}^*(k)\,\Psi\|^2 \ \ &\leq \ \ (a_{1,i}^{\mu})^2 \|(N_D+1)^{1/2} \otimes H_{ph}^{1/2}\Psi\|^2 \\ &\quad + (a_{0,i}^{\mu})^2 \left[\varepsilon \|(N_D+1)\otimes \mathbb{1}\,\Psi\|^2 + \frac{1}{4\varepsilon}\|\Psi\|^2\right], \end{aligned} \tag{52}$$

for every $\Psi \in \mathfrak{D}(H_0)$ and every $\varepsilon > 0$.

Proof. We only give the proof of (52) for $i=5$. The case of $i=6$ is quite similar.

The other cases, i.e. $i \neq 5, 6$ are simpler since $v_i^{\mu}(k)$ is bounded on $\mathfrak{F}_D$ for $i \neq 5, 6$.

In the following we omit the indexes μ and 5.

Set

$$a(v) = \int \mathrm{d}^3 k\, v^*(k) \otimes a(k)$$

$$a(v)^* = \int \mathrm{d}^3 k\, v(k) \otimes a^*(k)$$

Let $u \in \mathfrak{D}(H_0)$. For $k \in \mathbb{R}^3$, set

$$\Phi(k) = \omega(k)^{1/2}((N_D+1)^{1/2} \otimes a(k))\, u.$$

One has

$$\begin{aligned}
\int \|\Phi(k)\|^2 \mathrm{d}^3k &= \int \omega(k) \, \left\langle (N_D+1)^{1/2} \otimes a(k)u, (N_D+1)^{1/2} \otimes a(k)u \right\rangle \mathrm{d}^3k \\
&= \left\langle (N_D+1)^{1/2} \otimes \int \omega(k) a^*(k) a(k) \mathrm{d}^3k u, (N_D+1)^{1/2} \otimes \mathbb{1}\, u \right\rangle \\
&= \left\langle ((N_D+1)^{1/2} \otimes H_{ph})\, u, ((N_D+1)^{1/2} \otimes \mathbb{1})\, u \right\rangle \\
&= \|((N_D+1)^{1/2} \otimes H_{ph}^{1/2}) u\|^2 \qquad (53)
\end{aligned}$$

Using (53), we get

$$\begin{aligned}
\|a(v)u\|^2 &= \int \langle v^*(k) \otimes a(k)u, v^*(k') \otimes a(k')u \rangle \, \mathrm{d}^3k \, \mathrm{d}^3k' \\
&= \int \omega(k)^{-\frac{1}{2}} \omega(k')^{-\frac{1}{2}} \\
&\quad \times \left\langle v^*(k)((N_D+1)^{-\frac{1}{2}} \otimes \mathbb{1})\Phi(k), v^*(k')((N_D+1)^{-\frac{1}{2}} \otimes \mathbb{1})\Phi(k') \right\rangle \mathrm{d}^3k \, \mathrm{d}^3k' \\
&\leq \left[\int \omega(k)^{-\frac{1}{2}} \|v^*(k)(N_D+1)^{-\frac{1}{2}}\|_{\mathfrak{F}_D} \|\Phi(k)\| \, \mathrm{d}^3k \right]^2 .
\end{aligned}$$

Cauchy-Schwarz inequality now implies that

$$\|a(v)u\|^2 \leq \left(\int \omega(k)^{-1} \|v^*(k)\, (N_D+1)^{-1/2}\|^2_{\mathfrak{F}_D} \mathrm{d}^3k \right) \left(\int \|\Phi(k)\|^2 \mathrm{d}^3k \right),$$

which, together with (53) gives (51).

Let us now prove (52). We have

$$\|a^*(v)u\|^2 = \int \langle v(k) \otimes a^*(k)u, v(k') \otimes a^*(k')u \rangle \, \mathrm{d}^3k \, \mathrm{d}^3k' \ .$$

Using the commutation relations for a and a^* we obtain

$$\begin{aligned}
\|a^*(v)u\|^2 &= \int \langle v(k) \otimes a^*(k) a(k')u, v(k') \otimes \mathbb{1}\, u \rangle \, \mathrm{d}^3k \, \mathrm{d}^3k' \\
&\quad + \int \|(v(k) \otimes \mathbb{1})\, u\|^2 \mathrm{d}^3k \qquad (54)
\end{aligned}$$

The first term in the right hand side of (54) can be written in the following way

$$
\begin{aligned}
&\int \big\langle v(k)\otimes a(k')u, v(k')\otimes a(k)u\big\rangle \,\mathrm{d}^3k\,\mathrm{d}^3k' \\
&\quad = \int \omega(k)^{-\frac{1}{2}}\omega(k')^{-\frac{1}{2}} \\
&\qquad \times\Big\langle v(k)\left((N_D+1)^{-\frac{1}{2}}\otimes \mathbb{1}\right)\Phi(k'), v(k')\left((N_D+1)^{-\frac{1}{2}}\otimes \mathbb{1}\right)\Phi(k)\Big\rangle \mathrm{d}^3k\,\mathrm{d}^3k'
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int \big\langle v(k)\otimes a^*(k)a(k')u, v(k')\otimes \mathbb{1}\, u\big\rangle \,\mathrm{d}^3k\,\mathrm{d}^3k' \\
&\quad \leq \left[\int \omega(k)^{-1/2}\|v(k)(N_D+1)^{-1/2}\|_{\mathfrak{F}_D}\|\Phi(k)\|\mathrm{d}^3k\right]^2 \\
&\quad \leq \left(\int \omega(k)^{-1}\|v(k)(N_D+1)^{-1/2}\|^2_{\mathfrak{F}_D}\mathrm{d}^3k\right)\left(\int \|\Phi(k)\|^2\mathrm{d}^3k\right)
\end{aligned}
$$

which, together with (53), yields

$$
\begin{aligned}
&\int \big\langle v(k)\otimes a^*(k)a(k')u, v(k')\otimes \mathbb{1}\, u\big\rangle \,\mathrm{d}^3k\,\mathrm{d}^3k' \\
&\quad \leq \left(\int \omega(k)^{-1/2}\|v(k)(N_D+1)^{-1/2}\|^2_{\mathfrak{F}_D}\mathrm{d}^3k\right]^2 \left\|\left((N_D+1)^{1/2}\otimes H_{ph}^{1/2}\right)u\right\|^2 \\
&\quad = \left(\int \omega(k)^{-1}\|(N_D+1)^{-1/2}v(k)^*\|^2_{\mathfrak{F}_D}\mathrm{d}^3k\right) \left\|\left((N_D+1)^{1/2}\otimes H_{ph}^{1/2}\right)u\right\|^2 \,. \qquad (55)
\end{aligned}
$$

For the second term in the right hand side of (54), we write

$$
\begin{aligned}
&\int \|(v(k)\otimes \mathbb{1})\,u\|^2\mathrm{d}^3k \\
&\quad \leq \left(\int \|v(k)(N_D+1)^{-\frac{1}{2}}\|^2_{\mathcal{F}_D}\mathrm{d}^3k\right)\|((N_D+1)^{\frac{1}{2}}\otimes \mathbb{1})u\|^2 \\
&\quad = \left(\int \|v(k)(N_D+1)^{-\frac{1}{2}}\|^2_{\mathfrak{F}_D}\mathrm{d}^3k\right)((N_D+1)u,\, u)\;.
\end{aligned}
$$

Using for all $\varepsilon>0$, $ab\leq \varepsilon\, a^2+\frac{1}{4\varepsilon}\,b^2$, we get

$$
\begin{aligned}
&\int \|(v(k)\otimes \mathbb{1})\,u\|^2\mathrm{d}^3k \\
&\quad \leq \left(\int \|v(k)(N_D+1)^{-1/2}\|^2\mathrm{d}^3k\right)\left[\varepsilon\|(N_D+1)u\|^2+\frac{1}{4\varepsilon}\,\|u\|^2\right] \qquad (56)
\end{aligned}
$$

Finally, (54), (55) and (56) give (52). $\square$

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