SUB-EXPONENTIAL DECAY OF OPERATOR KERNELS FOR FUNCTIONS OF GENERALIZED SCHRÖDINGER OPERATORS

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Abstract. We study the decay at large distances of operator kernels of functions of generalized Schrödinger operators. We prove sub-exponential decay for functions in Gevrey classes and exponential decay for real analytic functions.

1. Introduction and result

In a previous paper [GK2], it was proven that operator kernels of smooth functions of generalized Schrödinger operators decay faster than any polynomial. Recall that the operator kernel of a bounded operator A acting on $\mathrm{L}^2(\mathbb{R}^d, \mathrm{d}q; \mathbb{C}^k)$ is defined as $\chi_x A \chi_y$, where

$$\chi_x(q) = \chi_0(q-x), \qquad q, x \in \mathbb{R}^d \,, \tag{1}$$

and χ_0 is the characteristic function of the cube centered at 0 with side 1. We use the standard notation $\langle x \rangle = (1+|x|^2)^{1/2}$, for $x \in \mathbb{R}^d$. Given a generalized Schrödinger operator H (see [GK2] and Remark 1.3 below) and a C^∞ function f with sufficiently fast decay at ∞, namely,

$$\int_{\mathbb{R}} |f^{(k)}(u)| \langle u \rangle^{k-1} \mathrm{d}u < \infty \ \text{ for } k = 0, 1, 2, \ldots \,, \tag{2}$$

it is shown in [GK2, Theorem 2] that for all $n = 0, 1, 2, \ldots$ one has

$$\|\chi_x f(H) \chi_y\| \le C_{n,f} \langle x - y \rangle^{-n}, \qquad x, y \in \mathbb{R}^d \tag{3}$$

(with a control on $C_{n,f}$ in terms of integrals as in (2)). Such kernel decay estimates are of interest from a purely mathematical point of view (see [Si, Da1, Da2]). Moreover, controls like (3) have already been quite useful in applications, e.g., [GK3, GK4, ElGr, Tc].

The purpose of this paper is to improve on (3) and show that if the C^∞ function f belongs to a Gevrey-like class (see Definition 1.1), then

$$\|\chi_x f(H) \chi_y\| \le C \mathrm{e}^{-c|x-y|^\alpha}, \qquad x, y \in \mathbb{R}^d \tag{4}$$

for some $\alpha \in (0, 1]$ depending on f. This extension was motivated by some applications which are discussed at the end of this section.

We recall that a function f belongs to the Gevrey class of order $a \ge 1$, denoted by $G^a(\mathbb{R})$ (e.g., [Ho]), if $f \in C^\infty(\mathbb{R})$ and for each compact subset $K \subset \mathbb{R}$ there exists a constant C_K such that

$$|f^{(k)}(u)| \le C_K (C_K (k+1)^a)^k, \qquad k = 0, 1, 2, \ldots, \ \ u \in K. \tag{5}$$

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Note that being Gevrey is thus a *local* property. The class $G^1(\mathbb{R})$ is the set of real analytic functions, and authors sometimes refer to Gevrey classes only for $a > 1$.

We emphasize that for any $a > 1$ the Gevrey class $G^a(\mathbb{R})$ contains compactly supported functions. (E.g., the function $f_a(x) = \exp(-(1-x^2)^{-\frac{1}{a-1}})$ for $|x| < 1$ and $f_a(x) = 0$ for $|x| \geq 1$ belongs to $G^a(\mathbb{R})$). In fact, most of the smooth cut-off functions which are commonly used can be chosen in $G^a(\mathbb{R})$ with a arbitrarily close to 1. We will denote the class of compactly supported functions in $G^a(\mathbb{R})$ by $G^a_{comp}(\mathbb{R})$. Of course many interesting functions are non-compactly supported, especially analytic ones, and we need to consider a larger set of functions.

To allow functions supported either on $\mathbb{R}$ or on a half-line, we combine the local Gevrey estimate (5) with the condition of decay at ∞ given by (2), and make the following definition. (By $C_{a,b,\ldots}$ we always denote a constant depending on $a, b, \ldots$.)

Definition 1.1. *Let I be an open interval and $a \geq 1$. A function $f \in C^\infty(\mathbb{R})$ is* L^1*-Gevrey of class a on I if for all $k = 0, 1, 2, \ldots$ we have*

$$\left\| f^{(k)} \langle u \rangle^{k-1} \right\|_{\mathrm{L}^1(I)} = \int_I |f^{(k)}(u)| \langle u \rangle^{k-1} \mathrm{d}u \leq C_{f,I} (C_{f,I}(k+1)^a)^k . \tag{6}$$

As far as the self-adjoint operator H is concerned, the sole property we shall require is that a suitable Combes-Thomas estimate [CoTh, GK2] holds. We state it as a definition.

Definition 1.2. *The self-adjoint operator H on $L^2(\mathbb{R}^d, \mathrm{d}q; \mathbb{C}^k)$ has property (CT) if there exists a constants C (depending on H and d) such that for all $z \notin \sigma(H)$ and all $x, y \in \mathbb{R}^d$ we have*

$$\|\chi_x (H - z)^{-1} \chi_y\| \leq \left(\tfrac{C}{\eta_z}\right) \exp\left(- \frac{C \eta_z}{1 + \eta_z + |z|} |x - y| \right) , \tag{7}$$

where $\eta_z = \mathrm{d}(z, \sigma(H))$.

Remark 1.3. *The inequality (7) is proven in* [GK2, Corollary 1] *for generalized Schrödinger operators, a class of semibounded second order partial differential operators of Mathematical Physics which includes the Schrödinger operator, magnetic Schrödinger operator, and the classical wave operators (i.e., acoustic operator, Maxwell operator, and other second order partial differential operators associated with classical wave equations). Specific examples of operators with property (CT) are given by the* Schrödinger operator

$$H = -\Delta + V^{(1)} + V^{(2)} \quad on \quad L^2(\mathbb{R}^d, \mathrm{d}x) , \tag{8}$$

with $0 \leq V^{(1)} \in \mathrm{L}^1_{\mathrm{loc}}$ and $V^{(2)}$ relatively form bounded with respect to $-\Delta$ with relative bound < 1, the acoustic operator

$$A = -\frac{1}{\sqrt{\kappa}} \nabla \cdot \frac{1}{\rho} \nabla \frac{1}{\sqrt{\kappa}} \quad on \quad \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x), \tag{9}$$

where $\kappa(x)$ is the compressibility and $\varrho(x)$ is the mass density, and the Maxwell operator

$$M = \frac{1}{\sqrt{\mu}} \nabla \times \frac{1}{\varepsilon} \nabla \times \frac{1}{\sqrt{\mu}} \quad on \quad \mathrm{L}^2(\mathbb{R}^3, \mathrm{d}x; \mathbb{C}^3), \tag{10}$$

where $\mu(x)$ is the magnetic permeability and $\varepsilon(x)$ is the dielectric constant, with $\kappa(x)$, $\varrho(x)$, $\mu(x)$, $\varepsilon(x)$ bounded, uniformly stricly positive, measurable functions.

Our results are stated in the following theorem.

Theorem 1.4. *Let H be a self-adjoint operator on $L^2(\mathbb{R}^d, \mathrm{d}q; \mathbb{C}^k)$ with property (CT), I an open interval containing the spectrum of H. Then given a real number $a \geq 1$ we have:*

(i): *if $a > 1$, then for each $a' > a$ there exists $C_{I,a,a'} > 0$, such that for all L^1-Gevrey function f of class a on I*

$$\|\chi_x f(H) \chi_y\| \leq C_{I,a,a'} (\mathrm{e} C_{f,I})^{3+|x-y|^{\frac{1}{a'}}} \exp\left(-\tfrac{a'-a}{4a'}|x-y|^{\frac{1}{a'}} \ln|x-y|\right) \tag{11}$$

for all $x, y \in \mathbb{R}^d$. As a consequence, for each $a' > a$, there exist constants $C_{f,I,a,a'} > 0$, $c_{a,a'} > 0$, such that

$$\|\chi_x f(H) \chi_y\| \leq C_{f,I,a,a'} \exp\left(-c_{a,a'}|x-y|^{\frac{1}{a'}}\right) \quad \text{for all } x, y \in \mathbb{R}^d . \tag{12}$$

(ii): *If $a = 1$, then there exist constants $C_I, c_I > 0$ such that for all L^1-Gevrey function f of class a on I*

$$\|\chi_x f(H) \chi_y\| \leq C_I \frac{C_{f,I}\langle x-y\rangle}{1+\ln C_{f,I}} \exp\left(-\frac{c_I}{C_{f,I}(1+\ln C_{f,I})}|x-y|\right) , \tag{13}$$

for all $x, y \in \mathbb{R}^d$.

Remark 1.5. *The mass $\frac{a'-a}{4a'}$ in (11) can be chosen arbitrarily close to $\frac{a'-a}{a'}$. This is done by modifying the proof by choosing $1 - s = \frac{1+\gamma(a'-a)}{a'}$ in (42) with γ close to zero.*

Remark 1.6. *If f has an analytic continuation to a region where one can use a suitable contour,* (13) *is a rather simple consequence of the Combes-Thomas estimate and the the Cauchy integral formula. In our theorem, no condition on the form of the domain of analyticity of f in $\mathbb{C}$ is assumed. We nevertheless obtain an exponential decay in $|x-y|$.*

To discuss the range of application of Theorem 1.4, and more precisely of Condition (6), we introduce the weighted Gevrey class $G^a_\gamma(I)$ in the following way: given $\gamma \in \mathbb{R}$, $a \geq 1$, and an open interval I, we say that $f \in G^a_\gamma(I)$ if $f \in C^\infty$ and there exists $C \geq 0$ such that

$$\sup_{u \in I} \left| f^{(k)}(u) \langle u \rangle^{k+\gamma} \right| \leq C(C(k+1)^a)^k, \qquad k = 0, 1, 2, \ldots . \tag{14}$$

The class $G^a_\gamma(I)$ is never trivial (*i.e.* reduced to 0): if $a > 1$ it contains all functions of $G^a_{comp}(\mathbb{R})$ which are supported in I, and if $a = 1$ it can be easily seen that for $t > 0$ and $z \notin I$, the functions

$$\exp(-tu^2) , \qquad \exp(-tu) , \qquad \exp(-iut) \exp(-u^2) , \qquad \frac{1}{u-z} ,$$

belong to $G^1_\gamma((\lambda, +\infty))$ for all λ and all γ for the first three, and to $G^1_1(\mathbb{R})$ for the last one.

Functions in $G^a_\gamma(I)$, $\gamma > 0$, clearly satisfy (6). Moreover, if $f_1 \in G^{a_1}_{\gamma_1}(I)$ and $f_2 \in G^{a_2}_{\gamma_2}(I)$, $\gamma_1, \gamma_2 \in \mathbb{R}$, $a_1, a_2 \geq 1$, then one can verify that

$$f_1 f_2 \in G^a_\gamma(I), \qquad a = \max(a_1, a_2), \quad \gamma = \gamma_1 + \gamma_2. \tag{15}$$

This is of interest since in applications we often have to consider cutoffs of the resolvent (see for instance [GK3]), that is, $f(H)$ with

$$f(u) = (1 - f_1(u))\frac{1}{u - \lambda_0}, \qquad f_1 \in C^\infty_{comp}(\mathbb{R}), \quad f_1 \equiv 1 \text{ near } \lambda_0 \in \mathbb{R}.$$

By choosing $f_1 \in G^a_{comp}(\mathbb{R})$ with a arbitrarily close to 1, we can deduce from our remarks that $f \in G^a_1(\mathbb{R})$, and hence get the sub-exponential decay of $\chi_x f(H)\chi_y$ by (12).

We end this section by discussing some applications of Theorem 1.4 to the theory of Schrödinger operators:

1. In the context of random Schrödinger operators, Theorem 1.4 can be used to extend the results of [GK3] to models where only a Wegner estimate with sub-exponential controls is available, as in the case of Bernoulli-Anderson models and polymers [CKM, KLS, DBG, DSS, JSBS]. In particular, it allows the definition of the strong insulator region in these cases, similar to what is done in [GK3] for models with a Wegner estimate with polynomial controls.
2. Still in the context of random Schrödinger operators, Theorem 1.4, combined with [GK1, Theorem 3.8], provides a simple and quick proof of sub-exponential decay of any order of the operator kernel of the Fermi projector $P_{E_F} = \chi_{(-\infty, E_F]}(H)$, provided the Fermi energy E_F lies inside the strong insulator region (equivalently, provided E_F lies in the region of applicability of the multiscale analysis).
3. In [GKT], an approximation lemma shows that the weight given to balls of radius ε by spectral measures of Hamiltonians which coincide on a region of size $\mathcal{O}(\varepsilon^{-1})$ are close up to a polynomial term in ε. Combining the methods of the present paper with the ones of [GKT] improves the latter by showing that the error term is actually sub-exponentially small. It moreover allows one to treat more general potentials, since they could be assumed to be sub-exponentially bounded rather than polynomially.
4. We note that if f is a real analytic function, then it follows from (13) that the moments $\|\langle X\rangle^{p/2}\mathrm{e}^{-iHt}f(H)\chi_0\|^2$ cannot grow faster than $(t(\ln t)^3)^p$. (This can be seen by cutting $\mathbb{R}^d$ in two regions: $F(|x|) \le t\ln t$ and $F(|x|) \ge t\ln t$, where $F(u) = u(\ln u)^{-2}$.)

2. Proof

In order to use the Combes-Thomas estimate (7), we need to express $f(H)$ in term of the resolvent $R(z) = (H-z)^{-1}$. This is done by the Helffer-Sjöstrand formula [HeSj, DiSj]

$$f(H) = \frac{1}{\pi}\iint \bar{\partial}\tilde{f}(u+iv)R(u+iv)\mathrm{d}u\mathrm{d}v, \qquad \bar{\partial} = \frac{1}{2}(\partial_u + i\partial_v) \tag{16}$$

where $\tilde{f}$ is a quasi-analytic extension of f, i.e., $\bar{\partial}\tilde{f} = \mathcal{O}(v)$ near $v = 0$, with conditions ensuring the convergence of the integral. Here we use a similar extension of f as the one in [GK2]. The only difference is that we add a smooth cutoff $\varphi(u)$ around the spectrum of H (we can thus allow for a larger class of functions f, like

e^{-x} if the operator H is bounded from below):

$$\tilde{f}_n(u+iv) = \varphi(u)\tau(v/\langle u\rangle)S_n\tilde{f}(u+iv), \tag{17}$$

$$\text{with} \quad S_n\tilde{f}(u+iv) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(u)(iv)^k, \tag{18}$$

where the function τ is smooth such that $\tau(t) = 1$ for $|t| \leq 1/2$ and $\tau(t) = 0$ for $|t| \geq 1$; the function φ is smooth as well and satisfies $\varphi(u) = 1$ near $\sigma(H)$, and $\operatorname{supp} \varphi \subset I$.

In all what follows, we set $\rho(u,v) = \varphi(u)\tau(v/\langle u\rangle)$. Then we have

$$\bar{\partial}\tilde{f}_n(u+iv) = \bar{\partial}\rho(u,v)\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(u)(iv)^k + \frac{\rho(u,v)}{2}\frac{1}{n!} f^{(n+1)}(u)(iv)^n. \tag{19}$$

Note that we have

$$\bar{\partial}\ \tau\left(v/\langle u\rangle\right) = \frac{1}{2\langle u\rangle}\left(-\frac{u}{\langle u\rangle}\frac{v}{\langle u\rangle}\tau'(v/\langle u\rangle) + i\tau'(v/\langle u\rangle)\right). \tag{20}$$

The main idea of the proof is to write

$$\begin{aligned} f(H) &= \frac{1}{\pi}\iint_{|v|<\epsilon\langle u\rangle} \bar{\partial}\tilde{f}_n(u+iv)R(u+iv)\mathrm{d}u\mathrm{d}v \qquad (21)\\ &\quad + \frac{1}{\pi}\iint_{|v|\geq\epsilon\langle u\rangle} \bar{\partial}\tilde{f}_n(u+iv)R(u+iv)\mathrm{d}u\mathrm{d}v \\ &= \mathrm{I} + \mathrm{II}, \qquad (22) \end{aligned}$$

with appropriate choices of ϵ and n.

In all what follows, the constants C, c may depend on I but never on ϵ, n, f nor x, y.

Lemma 2.1. *There exists $C, c > 0$ such that*

$$\begin{aligned} &\|\chi_x \ \mathrm{I} \ \chi_y\| \leq \qquad (23)\\ &\qquad Ce^{-c|x-y|}\sum_{k=0}^{n}\frac{1}{k!}\left\|f^{(k)}\langle u\rangle^{k-1}\right\|_{\mathrm{L}^1(I)} + \frac{C\epsilon^n}{n!}\left\|f^{(n+1)}\langle u\rangle^n\right\|_{\mathrm{L}^1(I)}, \end{aligned}$$

for all $n \geq 1$, $0 < \epsilon \leq 1/2$ and $x, y \in \mathbb{R}^d$. The constant c in the exponent depends only on the choice of φ (through its support and the distance of its support to the open interval I).

Proof. Assuming that $\epsilon \leq 1/2$ implies that the term involving τ' in (19) doesn't contribute. Actually we only have to consider $\varphi'(u)\tau(v/\langle u\rangle)S_n\tilde{f}$ and the last term of (19). We first estimate the contribution of the last term using the basic estimate

$\|R(u+iv)\| \le |v|^{-1}$ and we get

$$\left\| \iint_{|v|<\epsilon\langle u\rangle} \frac{\rho(u,v)}{n!} f^{(n+1)}(u)(iv)^n R(u+iv)\mathrm{d}u\mathrm{d}v \right\| \tag{24}$$
$$\le \frac{C}{n!}\int_I \int_{-\epsilon\langle u\rangle}^{\epsilon\langle u\rangle} \left| f^{(n+1)}(u)\right| |v|^{n-1}\mathrm{d}v\mathrm{d}u$$
$$\le \frac{2C}{n!}\int_I \left| f^{(n+1)}(u)\right| (\epsilon\langle u\rangle)^n \mathrm{d}u$$
$$\le \frac{2C}{n!}\epsilon^n \left\| f^{(n+1)}\langle u\rangle^n \right\|_{\mathrm{L}^1(I)} . \tag{25}$$

Note that we have not used the Combes-Thomas estimate for this part. Next we consider the term involving $\varphi'(u)$. This term is compactly supported outside $\sigma(H)$ with respect to $z=u+iv$, so that $\eta_z = \mathrm{dist}(z,\sigma(H))$ is uniformly bounded away from 0 and $\langle u\rangle$ is bounded. The Combes-Thomas estimate (7) implies that

$$\left\| \iint_{\frac{|v|}{\langle u\rangle}<\epsilon} \varphi'(u)\tau(v/\langle u\rangle) S_n\tilde{f}(u+iv)\chi_x R(u+iv)\chi_y \mathrm{d}u\mathrm{d}v \right\| \tag{26}$$
$$\le Ce^{-c|x-y|}\int_{\mathrm{supp}\ \varphi'}\int_{-\epsilon\langle u\rangle}^{\epsilon\langle u\rangle} \sum_{k=0}^{n} \frac{1}{k!}\left| f^{(k)}(u)\right| |v|^k \mathrm{d}v\mathrm{d}u$$
$$\le Ce^{-c|x-y|}\sum_{k=0}^{n}\frac{1}{k!}\left\| f^{(k)}\langle u\rangle^{k-1}\right\|_{\mathrm{L}^1(I)} , \tag{27}$$

where we used the fact that φ' is compactly supported and $\epsilon \le 1/2$. □

We consider now the estimates for II.

Lemma 2.2. *There exists $C, c>0$ such that*

$$\|\chi_x \text{ II } \chi_y\| \le \tag{28}$$
$$Ce^{-c|x-y|}\sum_{k=0}^{n}\frac{1}{k!}\left\| f^{(k)}\langle u\rangle^{k-1}\right\|_{\mathrm{L}^1(I)} + \frac{C}{n!}e^{-c\epsilon|x-y|}\left\| f^{(n+1)}\langle u\rangle^{n}\right\|_{\mathrm{L}^1(I)},$$

for all $\epsilon>0,\ n\ge 1,\ x,y\in\mathbb{R}^d$.

Proof. On the domain of integration that we are considering, one has $|v| \le \eta = \mathrm{dist}(u+iv,\sigma(H))$ and $\epsilon\langle u\rangle \le |v| \le \langle u\rangle$. Thus the Combes-Thomas estimate (7) reduces to

$$\|\chi_x R(u+iv)\chi_y\| \le \frac{C}{|v|}e^{-c\frac{|v|}{\langle u\rangle}|x-y|} \le \frac{C}{|v|}e^{-c\epsilon|x-y|}, \tag{29}$$

for all $x,y\in\mathbb{R}^d$, with $C,c>0$ constants independent of x,y,u,v. To evaluate $\|\chi_x \text{ II } \chi_y\|$ we use (19) and treat separately the sum and the last term. We use (29) only for the latter getting

$$\|\chi_x \text{ II } \chi_y\| \le \frac{1}{\pi}\iint |\bar\partial\rho(u,v)| \left\{\sum_{k=0}^{n}\frac{1}{k!}\left| f^{(k)}(u)\right| |v|^k\right\} \|\chi_x R(u+iv)\chi_y\|\mathrm{d}v\mathrm{d}u$$
$$+Ce^{-c\epsilon|x-y|}\int_I\int_{-\langle u\rangle}^{\langle u\rangle}\left\{\frac{1}{n!}\left| f^{(n+1)}(u)\right| |v|^{n-1}\right\}\mathrm{d}v\mathrm{d}u\,. \tag{30}$$

The last term clearly yields the second term in (28).

Now the term $|\bar\partial\rho| = |\varphi'\tau + \varphi\bar\partial\tau|$ will be yielding a factor $1/\langle u\rangle$, coming either from (20) or from the fact that φ' is compactly supported. In other terms, we will use the bound

$$|\bar\partial\rho(u,v)| \leq \frac{C}{\langle u\rangle}\,. \tag{31}$$

Moreover, on the support of $\bar\partial\rho$ the distance to the spectrum of H is uniformly bounded away from zero. Indeed, on the support of φ', a compact set disjoint from the spectrum of H, we have $\eta = \operatorname{dist}(u+iv, \sigma(H)) \geq C\langle u\rangle$ for some constant $C>0$. On the support of $\bar\partial\tau$ we have $\eta \geq |v| \geq \langle u\rangle/2$. As a consequence, it follows from by (7) that on the support of $\bar\partial\rho$ we have

$$\|\chi_x R(u+iv)\chi_y\| \leq \frac{C}{\langle u\rangle} e^{-c|x-y|}\,. \tag{32}$$

Plugging (31) and (32) into the first integral on the right hand side of (30) leads to

$$\begin{aligned}
&\int\!\!\int |\bar\partial\rho(u,v)| \left\{\sum_{k=0}^{n} \frac{1}{k!}\left|f^{(k)}(u)\right| |v|^k\right\} \|\chi_x R(u+iv)\chi_y\| \mathrm{d}v\mathrm{d}u \\
&\leq \quad Ce^{-c|x-y|}\int_I\int_{-\langle u\rangle}^{\langle u\rangle} \frac{1}{\langle u\rangle^2}\left\{\sum_{k=0}^{n}\frac{1}{k!}\left|f^{(k)}(u)\right||v|^k\right\}\mathrm{d}v\mathrm{d}u \qquad (33)\\
&\leq \quad Ce^{-c|x-y|}\sum_{k=0}^{n}\frac{1}{k!}\left\|f^{(k)}\langle u\rangle^{k-1}\right\|_{\mathrm{L}^1(I)}\,, \qquad (34)
\end{aligned}$$

giving the first term in (28). □

The next lemma gives an estimate on the following quantity which appears in the last two lemmas, after controling the norms by (6),

$$A_{n,a} := \sum_{k=0}^{n}\frac{1}{k!}C_{f,I}(C_{f,I}(k+1)^a)^k + \frac{1}{n!}C_{f,I}(C_{f,I}(n+2)^a)^{(n+1)}. \tag{35}$$

The result will motivate the choice of ϵ and n in the end.

Lemma 2.3. *For all $a \geq 1$, one has*

$$A_{n,a} \leq 3\mathrm{e}^{4a} n^a(\mathrm{e}C_{f,I})^{n+2} n^{(a-1)n}, \qquad \forall\ n\geq 1.$$

Proof. First note that $k! \geq (k/\mathrm{e})^k$ for $k\geq 1$, and estimate $\mathrm{e}^k C_{f,I}^k$ by $(\mathrm{e}C_{f,I})^n$ for $k\leq n$ (recall $C_{f,I}\geq 1$). Moreover, we have $(k+1)^{ak}/k^k \leq \mathrm{e}^a k^{(a-1)k}$ and, recalling $a\geq 1$, we can therefore estimate each term of the sum by the one for $k=n$. In a similar way $(n+2)^{a(n+1)}/n^n \leq \mathrm{e}^{4a}n^{a(n+1)-n}$. The result now follows easily. □

Proof of Theorem 1.4. The last three lemmas show clearly that we have to consider, with $K_{f,I} = \mathrm{e}C_{f,I}$,

$$n^a(K_{f,I})^{n+2}\exp\left((a-1)n\ln n - c|x-y|\right), \tag{36}$$

$$n^a(K_{f,I})^{n+2}\exp\left((a-1)n\ln n + n\ln\epsilon\right), \tag{37}$$

$$n^a(K_{f,I})^{n+2}\exp\left((a-1)n\ln n - c\epsilon|x-y|\right). \tag{38}$$

To simplify, and with no loss of generality, we may assume that the constants c appearing in (36) and (38) are the same. In both cases $a=1$ and $a>1$, we will choose $n = \mathcal{O}(|x-y|^a)$, thus if we can show that the exponential terms listed above have the decay announced in the theorem, we shall get the result.

Proof in the case $a = 1$. We choose $n \sim \delta|x-y|$ with $\delta > 0$ small enough and a fixed $\epsilon > 0$ small enough as well. Then (37) gives the following first condition:

$$\epsilon K_{f,I} < 1. \tag{39}$$

Now, by choosing n such that $\delta|x-y| - 1 \leq n+2 < \delta|x-y|$, (38) yields the second condition:

$$\delta \ln K_{f,I} - c\epsilon < 0\,. \tag{40}$$

Similarly, we obtain the third condition from (36),

$$\delta \ln K_{f,I} - c < 0. \tag{41}$$

The existence of $\epsilon, \delta > 0$ such that these three conditions are satisfied is clear since we may first choose $\epsilon < 1$ satisfying (39) and then δ small enough satisfying (40) thus (41) as well. We may choose ϵ such that $\epsilon K_{f,I} = \mathrm{e}^{-1}$, and $\delta = (c\epsilon)/(2\ln K_{f,I})$. Since $K_{f,I} \geq \mathrm{e}$, each term (36), (37), (38) is bounded by the bound of (37), namely $nK_{f,I}^2 \exp(-n)$. The estimate (13) follows. ☐

Proof in the case $a > 1$. In this part n and ϵ will both depend on $|x-y|$. Pick $a' > a$ as in the theorem. Precisely, we write $\nu = a' - a > 0$, and set

$$\begin{aligned} &\epsilon = |x-y|^{-s}, \qquad |x-y|^{\delta} \leq n < |x-y|^{\delta} + 1 \\ &\text{with} \qquad \delta = \frac{1}{a'}\,, \qquad 1 - s = \frac{1+\frac{1}{2}\nu}{a+\nu}\,. \end{aligned} \tag{42}$$

Note that $|x-y| \sim n^{1/\delta}$ and $\epsilon \leq n^{-s/\delta}$, moreover since we consider large $|x-y|$ we may assume that $|x-y| \geq (n/2)^{1/\delta}$. We now have for n large enough (depending only on s, δ, a),

$$\begin{aligned}
&(43) & (\text{Eq.}36) &\leq n^a (K_{f,I})^{n+2} \exp\left(-\frac{c}{2} n^{\frac{1}{\delta}} + (a-1) n \ln n\right) \\
&(44) & &\leq (K_{f,I})^{n+2} \exp\left(-\frac{c}{4} n^{\frac{1}{\delta}}\right) \\
&(45) & (\text{Eq.}37) &\leq n^a (K_{f,I})^{n+2} \exp\left(-(\frac{s}{\delta} - a + 1) n \ln n\right) \\
&(46) & &\leq (K_{f,I})^{n+2} \exp\left(-\frac{1}{2}(\frac{s}{\delta} - a + 1) n \ln n\right) \\
&(47) & (\text{Eq.}38) &\leq n^a (K_{f,I})^{n+2} \exp\left(-\frac{c}{2} n^{\frac{1-s}{\delta}} + (a-1) n \ln n\right) \\
&(48) & &\leq (K_{f,I})^{n+2} \exp\left(-\frac{c}{4} n^{\frac{1-s}{\delta}}\right),
\end{aligned}$$

provided

$$\frac{1-s}{\delta} > 1 \qquad \text{and} \qquad \frac{s}{\delta} - a + 1 > 0.$$

But (42) ensures the latter, since:

$$\frac{1-s}{\delta} = 1 + \nu \qquad \frac{s}{\delta} - a + 1 = \frac{1}{2}\nu\,. \tag{49}$$

Thus the estimate of the theorem depends only on the slowest term (46), and we obtain

$$\|\chi_x f(H)\chi_y\| \leq C_{a,a'}(K_{f,I})^{n+2} \exp\left(-\frac{a'-a}{4} n \ln n\right), \tag{50}$$

and we get the result since $n \sim |x-y|^{\delta} = |x-y|^{\frac{1}{a'}}$ and $K_{f,I} = \mathrm{e} C_{f,I}$. □

This finishes the proof of Theorem 1.4. □

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References

[CKM] Carmona, R., Klein, A., Martinelli, F.: Anderson localization for Bernoulli and other singular potentials. Commun. Math. Phys. **108**, 41-66 (1987)

[CoTh] Combes, J.M. , Thomas, L.: Asymptotic behavior of eigenfunctions for multi-particle Schrödinger operators. Commun. Math. Phys. **34**, 251-270 (1973)

[DSS] Damanik, D., Sims, R., Stolz, G.: Localization for one dimensional, continuum, Bernoulli-Anderson models. Duke Math. J. **114**, 59-100 (2002)

[Da1] Davies, E.B.: Kernel estimates for functions of second order elliptic operators. Quart. J. Math. Oxford (2) **39**, 37-46 (1988).

[Da2] Davies, E.B.: *Heat kernels and spectral theory.* Cambridge University Press, 1989

[DBG] De Bièvre, S., Germinet, F.: Dynamical localization for random dimer Schrödinger operator. J. Stat. Phys. **98**, 1135-1147 (2000)

[DiSj] Dimassi, M. , Sjöstrand, J.: *Spectral asymptotics in the semi-classical limit*, London Math. Soc., Lect. Note Series 268, Cambridge Univ. Press 1999

[ElGr] Elbau, P., Graf,G.M.: Equality of bulk and Edge Hall Conductance Revisited. Commun. Math. Phys. **229**, 415-432 (2002)

[GKT] Germinet, F., Kiselev, A., Tcheremchantsev, S.: Transfer matrices and transport for 1D Schrödinger operators with singular spectrum. In preparation

[GK1] Germinet, F., Klein, A.: Bootstrap Multiscale Analysis and Localization in Random Media. Commun. Math. Phys. **222**, 415-448 (2001)

[GK2] Germinet, F., Klein, A.: Operator kernel estimates for functions of generalized Schrödinger operators, Proc. Amer. Math. Soc. 131, 911-920 (2003)

[GK3] Germinet, F., Klein, A.: A characterization of the metal-insulator transport transition. Submitted (mparc 02-19)

[GK4] Germinet, F., Klein, A.: The Anderson metal-insulator transport transition. To appear in Contemp. Math.

[HeSj] Helffer, B., Sjöstrand, S.: Équation de Schrödinger avec champ magnétique et équation de Harper, in *Schrödinger operators*, H. Holden & A. Jensen eds., LNP 345, 118-197, 1989

[Ho] Hörmander, L.: *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis.* Second edition. Springer Study Edition. Springer-Verlag, Berlin, 1990

[JSBS] Jitomirskaya, S., Schulz-Baldes, H., Stolz, G.: Delocalization in random polymer models. To appear in Commun. Math. Phys.

[KLS] Klein, A., Lacroix, J., Speis, A.: Localization for the Anderson model on a strip with singular potentials. J. Funct. Anal. **94**, 135-155 (1990)

[Si] Simon, B.: Schrödinger semi-groups. Bull. Amer. Math. Soc. **7**, 447-526 (1982)

[Tc] Tcheremchantsev S.. In preparation

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