THERMODYNAMIC LIMITS FOR A QUANTUM CRYSTAL BY HEAT KERNEL METHODS.

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Abstract

We consider a d-dimensional quantum anharmonic crystal, where the interaction between the ions satisfies hypotheses, based on the idea that the ions are not too far from the points $\mathbb{Z}^d$, and that the interaction between them decreases exponentially with their distance. Under these conditions, we study carefully the heat kernel of the Hamiltonian related to each finite set of $\mathbb{Z}^d$, with constants in the inqualities that are independent of this set, (thus, improving an earlier result of Sjöstrand). Then, taking the limit when the finit set 'tends to $\mathbb{Z}^d$', we define as usual a Gibbs state on the algebra of quasilocal observables, proving the convergence in norm, with an exponential rate, for the usual limit defining the state. Then, proving first an exponential decay of the correlations for finite sets and passing to the limit, we prove some properties, (mixing, triviality at infinity when restricted to a suitable subalgebra), of our Gibbs state. The decay of correlation relies on the study of the heat kernel, and is itself used for estimating the rate of convergence in the thermodynamic limit. Another consequence of this decay of correlations is the continuity of the mean energy per site with respect of the temperature, but all the results in this paper are valid under some inequalities between the temperature, the coupling constant, and the Planck's constant, making impossible to let the temperature tend to 0 when the other parameters are fixed.

1. Introduction.

Let us consider a quantum d-dimensional lattice of particles, each of them moving in $\mathbb{R}^p$. For each finite subset Λ of $\mathbb{Z}^d$, we denote by $H_\Lambda(\varepsilon)$ the following differential operator in $(\mathbb{R}^p)^\Lambda$, depending on the Planck's constant h and on another small parameter ε (measuring the decay of interactions between particles of the lattice):

$$H_\Lambda(\varepsilon) \;=\; -\frac{h^2}{2} \sum_{\lambda\in\Lambda} \Delta_{x_\lambda} \;+\; V_{\Lambda,\varepsilon}(x) \tag{1.1}$$

where $x = (x_\lambda)_{\lambda\in\Lambda}$ denotes the variable of $(\mathbb{R}^p)^\Lambda$, each variable x_λ being in $\mathbb{R}^p$. We suppose that the potential $V_{\Lambda,\varepsilon} \in C^\infty((\mathbb{R}^p)^\Lambda)$ is of the following form, with $A \in C^\infty(\mathbb{R}^p, \mathbb{R})$ and $B_\lambda \in C^\infty(\mathbb{R}^{2p}, \mathbb{R})$

$$V_{\Lambda,\varepsilon}(x) \;=\; \sum_{\lambda\in\Lambda} A(x_\lambda) \;+\; \sum_{\substack{\lambda,\mu\in\Lambda \\ \lambda\neq\mu}} \varepsilon^{|\lambda-\mu|} B_{\lambda-\mu}(x_\lambda, x_\mu) \qquad x = (x_\lambda)_{\lambda\in\Lambda}. \tag{1.2}$$

We assume that, for some $C > 0$, $C^{-1}|x| \leq A(x) \leq C|x|$ for $|x| \geq C$, that all the B_λ are uniformly bounded, and that, for each integer $q \geq 1$, the derivatives of order q of A and B_λ are uniformly bounded. In (1.2), $|\lambda|$ is the ℓ^∞ norm in $\mathbb{Z}^d$.

The first part of this paper (Sect. 2 to 4) will be devoted to the description of the integral kernel of $e^{-tH_\Lambda(\varepsilon)}$, with inequalities where the constants are independent of Λ and ε. Sjöstrand began this study in [22].

We denote by ∇_{x_λ} the partial differential with respect to the variable $x_\lambda \in \mathbb{R}^p$, and by ∇_λ with respect to $(x_\lambda, y_\lambda) \in \mathbb{R}^{2p}$ $(\lambda \in \Lambda)$ of a C^∞ function f on $(\mathbb{R}^{2p})^\Lambda$. The norm of $\nabla_\lambda f(x)$ is its norm in $(\mathbb{R}^{2p})^\star$. We denote by $\mathrm{diam}(A)$ the diameter, for the norm ℓ^∞, of a subset A of $\mathbb{Z}^d$.

Theorem 1.1. *Under the previous hypotheses, the integral kernel $U_\Lambda(x,y,t,h,\varepsilon)$ of $e^{-tH_\Lambda(\varepsilon)}$ can be written under the form*

$$U_\Lambda(x,y,t,h,\varepsilon) \;=\; (2\pi t h^2)^{-p|\Lambda|/2}\; e^{-\frac{|x-y|^2}{2th^2}}\; e^{-\psi_\Lambda(x,y,t,h,\varepsilon)}. \tag{1.3}$$

where ψ_Λ is a C^∞ function in $(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda \times [0,+\infty[$, depending on the parameters $h \in]0,1]$ and $\varepsilon \in [0,1]$. Moreover, there exists a constant $\sigma_0 > 0$ and, for each integer m, there exists a constant $C_m > 0$ and a function $\gamma \to \varepsilon_m(\gamma)$ from $]0,1[$ in itself, with the following properties. For each finite subset Λ of $\mathbb{Z}^d$, and for each points $\lambda^{(1)}$, ... $\lambda^{(m)}$ in Λ, we have

$$\|\nabla_{\lambda^{(1)}}...\nabla_{\lambda^{(m)}}\psi_\Lambda\|_{L^\infty((\mathbb{R}^{2p})^\Lambda)} \le tC_m\ \varepsilon^{\gamma\ \mathrm{diam}(\{\lambda^{(1)},...\lambda^{(m)}\})} \qquad \mathit{if} \quad ht \le \sigma_0 \quad \mathit{and} \quad \varepsilon \le \varepsilon_m(\gamma). \tag{1.4}$$

Let us recall that Sjöstrand proved in [22] that, near the diagonal, the integral kernel U_Λ can be written in the form (1.3), and he proved that an approximation modulo $\mathcal{O}(h^\infty)$ of the function ψ_Λ satisfies inequalities which are probably equivalent to (1.4), but are written in a different form, using the concept of $0-$standard function, introduced in the same article.

The first application of our results on the heat kernel is a construction of Gibbs states, as thermodynamic limits, on the $C^\star$-algebra $\mathcal{A}$ of quasi-local observables on the lattice, and an estimationof the rate of convergence. Let us recall the definition of $\mathcal{A}$ (see B. Simon [21], section II.1, for more details). To each finite set Λ of $\mathbb{Z}^d$, we associate the Hilbert space $\mathcal{H}_\Lambda = L^2((\mathbb{R}^p)^\Lambda)$. If $\Lambda_1 \subseteq \Lambda_2$, we have a natural identification of $\mathcal{L}(\mathcal{H}_{\Lambda_1})$ into $\mathcal{L}(\mathcal{H}_{\Lambda_2})$. Classically, $\mathcal{A}$ is the closure of the union of (the equivalence classes of) all these $\mathcal{L}(\mathcal{H}_\Lambda)$. On this algebra $\mathcal{A}$, we shall define Gibbs states by the usual limit (1.6) below, taking a sequence Λ_n of finite subsets 'tending to $\mathbb{Z}^d$'. For simplicity, we shall consider, for each $n \ge 1$,

$$\Lambda_n = \{-n, \ldots, n\}^d. \tag{1.5}$$

The existence of limits like (1.6) below has been proved, in some other different situations, by many authors. Sometimes, it is necessary to take a subsequence of Λ_n, sometimes the RHS of (1.6) converges for each fixed A, but not in norm, and often probabilistic methods are used. In Theorem 1.2, other new things are the rate of convergence in norm, and the method of the proof, which relies (via Theorem 1.5 below) on the estimations of the heat kernel given by Theorem 1.1.

Recently, Minlos, Pechersky, Verbeure and Zagrebnov ([16] and [17]) proved the existence of Gibbs states on $\mathcal{A}$ by similar limits, but for potentials of a very different type, and by techniques of Feynmann integrals. S. Albeverio, Kondratiev, Pasurek and Röckner ([1] and [2]), also constructed Gibbs space by probabilistic methods.

We tried to give results as global as possible, but we need, as many authors (cf Minlos [15]), some inequalities between our three parameters: t, (the inverse of the absolute temperature), h and ε. We set $M(t) = \sup(t, 1)$.

Theorem 1.2. *Let Q_0 be a finite subset of $\mathbb{Z}^d$, and let A be an element of $\mathcal{L}(\mathcal{H}_{Q_0})$ (a local observable). (Thus, for n large enough, Λ_n contains Q_0, and A can be considered as an element of $\mathcal{L}(\mathcal{H}_{\Lambda_n})$). Then, with* *our hypotheses on the potentials, for each integer N, there exists a function $\gamma \to \varepsilon_0(\gamma, N)$ from $]0,1[$ in itself such that the following limit exists:*

$$\omega(A) \;=\; \lim_{n\to+\infty} \frac{tr\ (e^{-tH_{\Lambda_n}(\varepsilon)}A)}{tr\ (e^{-tH_{\Lambda_n}(\varepsilon)})} \tag{1.6}$$

if the following conditions are satisfied

$$ht \le \sigma_0, \qquad 0 < \varepsilon \le \varepsilon_0(\gamma, \sharp(Q_0)) \quad 0 < \gamma < 1 \qquad M(t)\varepsilon^\gamma \le \frac{1}{2} \tag{1.7}$$

where σ_0 is the constant of Theorem 1.1. Moreover, there exists a function $(h,t) \to K(t,h,N)$, bounded on each compact set of $]0,+\infty[\times]0,+\infty[$ such that, under the conditions (1.7),

$$\left|\omega(A) - \frac{tr\ (e^{-tH_{\Lambda_n}(\varepsilon)}A)}{tr\ e^{-tH_{\Lambda_n}(\varepsilon)}}\right| \le K(t,h,\mathrm{diam}(Q_0))\ (M(t)\varepsilon^\gamma)^{\frac{n}{10}}\|A\| \tag{1.8}$$

Let us remark that $|\omega(A)| \le \|A\|$, and therefore that the linear form $A \to \omega(A)$, defined on the qunion of all the $\mathcal{L}(\mathcal{H}_Q)$, can be extended to a state on $\mathcal{A}$. (It satisfies $\omega(I) = 1$ and $\omega(A^\star A) \ge 0$).

Our technique can be extended to observables which are not necessarily bounded, and therefore do not belong to the $C^\star-$algebra. We shall restrict ourselves to the case where A is the multiplication by a polynomially bounded function.

Theorem 1.3. *With the notations of Theorem 1.2, let $f \in C^\infty((\mathbb{R}^p)^{Q_0})$ be a function satisfying, for some positive integer m, and for some constant $N_m(f)$,*

$$|f(x_{Q_0})| \;\le\; N_m(f)\ (1 + |x_{Q_0}|)^m \qquad \forall x_{Q_0} \in (\mathbb{R}^p)^{Q_0}, \tag{1.9}$$

and A be the operator (in $L^2((\mathbb{R}^p)^{Q_0})$) of multiplication by f. (Then, if Λ contains Q_0, the operator $e^{-tH_\Lambda(\varepsilon)}A$ is well defined in $\mathcal{L}(\mathcal{H}_\Lambda)$, and of trace class). Then there exists a constant $\sigma_1 > 0$ and a function $\gamma \to \varepsilon_0(\gamma, m, \sharp(Q_0))$ from $]0,1[$ in itself such that the limit (1.6) exists if

$$h^2(t+t^2) \le \sigma_1, \qquad 0 < \varepsilon \le \varepsilon_0(\gamma, m, \sharp(Q_0)), \qquad \gamma \in]0,1[, \qquad M(t)\varepsilon^\gamma \le \frac{1}{2}.$$

For example, if $d = 1$, if $Q_0 = \{1,2\}$, and if A is the multiplication by the function $f(x_1,x_2) = x_2 - x_1$, we can think than $\omega(A)$ is related to the dilation coefficient of the crystal.

With our hypotheses, J. Sjöstrand [22] proved that the following limit (free energy) exists

$$P(t,h) \;=\; \lim_{n\to+\infty} \frac{1}{\sharp(\Lambda_n)}\ ln\ \left(tr\ \left(e^{-tH_{\Lambda_n}(h)}\right)\right) \tag{1.10}$$

He proved also that $P(t,h)$ has an expansion in powers of h when $h \to 0$, while $\varepsilon > 0$ is fixed. In the literature on solid state physics (Kittel [13] or Ashcroft-Mermin [3]), it appears that the partial derivative of $P(t,h)$ with respect of t is supposed to exist and to represent the mean energy $U(1/t,h)$ of the crystal (per site) at the temperature $1/t$. Therefore it seemed interesting to prove mathematically that this derivative exists and that

$$\frac{\partial P(t,h)}{\partial t} \;=\; -\lim_{n\to+\infty} \frac{1}{\sharp(\Lambda_n)} \frac{\left(tr\ (H_{\Lambda_n}(h)e^{-tH_{\Lambda_n}(h)}\right)}{\left(tr\ e^{-tH_{\Lambda_n}(h)}\right)} \tag{1.11}$$

Another application of our techniques is the proof of the following

Theorem 1.4. *With the preceding hypotheses, there exist constant ε_0 and $\sigma_1 > 0$ such that, if the positive parameters t, h and ε satisfy*

$$h^2(t+t^2) \le \sigma_1, \qquad 0 < \varepsilon < \varepsilon_0, \qquad M(t)\varepsilon^{\frac{1}{5}} \le \frac{1}{2}, \tag{1.12}$$

then $P(t,h,\varepsilon)$ is derivable with respect to t, and the derivative is given by (1.11) and is continuous as a function of its first variable in a neighborhood of t. Moreover, there exists a function $F(t)$, bounded on each compact set of $]0,+\infty[$, such that, under the conditions (1.12)

$$\left| \frac{\partial P(t,h)}{\partial t} + \frac{1}{\sharp(\Lambda_n)} \frac{\left(tr\ (H_{\Lambda_n}(h)e^{-tH_{\Lambda_n}(h)}\right)}{\left(tr\ e^{-tH_{\Lambda_n}(h)}\right)} \right| \le \frac{F(t)}{n} \tag{1.13}$$

This estimation shows that, for each h and ε (the last being small enough), the mean energy is a continuous function of the temperature $\frac{1}{t}$ in the interval in which (1.12) is satisfied.

The main step of the proof of Theorems 1.2, 1.3 and 1.4 is a result on the decay of the quantum correlation of two local observables. Let us define this notion. For each finite set Λ of $\mathbb{Z}^d$, and for each (bounded, or satisfying suitable other hypotheses) operator A in $\mathcal{H}_\Lambda$, we can define the 'mean value' of A as

$$E_{\Lambda,\varepsilon}(A) = \frac{tr\ \left(e^{-tH_\Lambda(\varepsilon)}A\right)}{tr\ \left(e^{-tH_\Lambda(\varepsilon)}\right)}\ . \tag{1.14}$$

If E_1 and E_2 are disjoints subsets of Λ, and if A (resp. B) is an operator in $\mathcal{H}_{E_1}$ (resp. in $\mathcal{H}_{E_2}$), we can consider A and B as commuting operators in $\mathcal{H}_\Lambda$, and define their quantum correlation as

$$cov_{\Lambda,\varepsilon}\ (A,B) = E_{\Lambda,\varepsilon}(AB) - E_{\Lambda,\varepsilon}(A)E_{\Lambda,\varepsilon}(B) \tag{1.15}$$

We denote by $\delta(E,F)$ the distance, for the ℓ^∞ norm, of two subsets E and F of $\mathbb{Z}^d$.

Theorem 1.5. *With the previous notations, we can find, for each integers N_1 and N_2, a function $(t,h) \to K(t,h,N_1,N_2)$, bounded on each compact set of $]0,+\infty[\times]0,+\infty[$, and a function $\gamma \to \varepsilon_0(\gamma,N_1,N_2)$ from $]0,1[$ to itself such that, for each disjoint subsets E_1 and E_2 of any box Λ of $\mathbb{Z}^d$, for each $A \in \mathcal{L}(\mathcal{H}_{E_1})$ and $B \in \mathcal{L}(\mathcal{H}_{E_2})$, we have,*

$$cov_{\Lambda,\varepsilon}(A,B) \le K\big(t,h,\mathrm{diam}(E_1),\mathrm{diam}(E_2)\big)\ \ \big((M(t)\varepsilon^\gamma\big)^{\frac{1}{5}\delta(E_1,E_2)}\|A\|\ \|B\|, \tag{1.16}$$

(where $M(t) = \sup(t,1)$), if

$$ht \le \sigma_0, \qquad 0 < \varepsilon \le \varepsilon_0(\gamma,\sharp(E_1),\sharp(E_2)), \qquad M(t)\varepsilon^\gamma \le \frac{1}{2}. \tag{1.17}$$

If B is the multiplication by a bounded C^∞ function, K may be chosen independent of $\mathrm{diam}(E_2)$ and ε_0 may be chosen independent of $\sharp(E_2)$. If both A and B are the multiplications by bounded C^∞ functions, ε_0 depends only on γ and we can take $K = 4\inf(t,1)\ \inf(\sharp(E_1),\sharp(E_2))$.

If, in the definition of $H_{\Lambda\varepsilon}$, the potential $V_{\Lambda\varepsilon}$ is replaced by a family, depending on an auxiliary parameter, such that the hypotheses of the introduction are satisfied uniformly, then the constants in the theorems 1.5 can be chosen independent of this parameter.

As a consequence, we see that the state ω defined in (1.6) satisfies the following mixing property, in the sense of B. Simon [21], Chapter III, Definition (III.1.21), adapted to the quantum case. For each pair of local observables $A \in \mathcal{L}(\mathcal{H}_{E_1})$ and $B \in \mathcal{L}(\mathcal{H}_{E_2})$, where E_1 and E_2 are finite sets of $\mathbb{Z}^d$, for each t, h and ε satisfying (1.7) (with $Q_0 = E_1 \cup E_2$), and (1.17), if $K(t, h, N_1, N_2)$ is the function of Theorem 1.5, we have

$$|\omega(AB) - \omega(A)\omega(B)| \leq K(t, h, \mathrm{diam}(E_1), \mathrm{diam}(E_2)) \; (M(t)\varepsilon^{\gamma})^{\frac{1}{5}\delta(E_1,E_2)} \|A\| \; \|B\|. \tag{1.18}$$

If A and B are multiplication by bounded functions, the constant K depends only on $\inf(\sharp(E_1), \sharp(E_2))$. Therefore, for each A and B in the algebra $\mathcal{A}$ of quasilocal observables, we have, denoting by τ_h the translation with respect to $h \in \mathbb{Z}^d$,

$$\lim_{|h| \to +\infty} \Big[\omega(A \circ \tau_h(B)) - \omega(A)\omega(\tau_h(B)) \Big] = 0,$$

which seems to be the good analogous of the Definition (III.1.21) of the *mixing property* in [21].

Another property, defined also in [21], is the *triviality at infinity*, which is characterized in [21] in the Theorem IV.1.4 (of Lanford-Ruelle, see the reference in [21]). If we restrict the state ω to the subalgebra $\mathcal{A}^{(0)}$ of $\mathcal{A}$, contructed in the same way as $\mathcal{A}$, but taking only, for each finite set Λ, the algebra $\mathcal{A}_{\Lambda}^{(0)}$ of multiplications by bounded functions on $(\mathbb{R}^p)^{\Lambda}$, and then taking the union and completion as before, then ω satisfies the condition of the Lanford-Ruelle theorem for this subalgebra $\mathcal{A}^{(0)}$: for each $A \in \mathcal{A}^{(0)}$ and for each $\varepsilon > 0$, there exists a finite set Λ of $\mathbb{Z}^d$ such that, for each finite set E of $\mathbb{Z}^d \setminus \Lambda$ and for each B in $\mathcal{A}_E^{(0)}$, we have

$$|\omega(AB) - \omega(A)\omega(B)| \leq \varepsilon \|B\|.$$

Similar results are proved in Helffer-Sjöstrand [12] in the case of classical mechanics, and with very different hypotheses on the potential, and by a different method. (See also [8], [9], [10], and Bach-Möller [4]). The Theorems 1.2 to 1.4 are consequences of the two last statements of Theorem 1.5, and the proof of this theorem uses Theorem 1.1.

In section 2, we shall prove the global existence of ψ_Λ, such that (1.3) is satisfied, in other words, such that ψ_Λ is the solution of the Cauchy problem in $(\mathbb{R}^p)^{\Lambda}$

$$\frac{\partial \psi_\Lambda}{\partial t} \; + \; \frac{x-y}{t} \; . \; \nabla_x \psi_\Lambda - \frac{h^2}{2} \Delta \psi_\Lambda \; = \; V_{\Lambda,\varepsilon}(x) \; - \; \frac{h^2}{2} |\nabla \psi_\Lambda|^2 \tag{1.19}$$

$$\psi_\Lambda(x, y, 0, h, \varepsilon) = 0 \tag{1.20}$$

We shall also prove preliminary estimates, in which the constants still depend on $\sharp(\Lambda)$, and a variant of the maximum principle, which are used in further sections. Section 4 is devoted to the proof of Theorem 1.1 (estimations (1.4)). In sections 5 and 6, we shall decompose ψ_Λ as a sum of terms associated to clusters (cubes) of Λ, with an estimation of each term. In section 7, we study the case where Λ is the union of two disjoint subsets Λ_1 and Λ_2. In section 8, we study the behavior at infinity of ψ_Λ. In Sections 9 and 10, we prove Theorem 1.5, and, in section 9, a variant in which A and B are multiplications by polynomially bounded functions. In Section 11, we prove Theorems 1.2 to 1.4.

In the case of a quadratic potential, our result follow from the explicit computations of C. Royer [19].

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2. Some Lemmas on non linear parabolic equations.

Here, we shall study the Cauchy problems of the form (1.19), (1.20). For the results given here, the parameter ε and the fact that the variables are indexed by the points of a cube play no role, and therefore we may rewrite the Cauchy problem with simpler notations, in which the space dimension will be denoted by n.

$$\frac{\partial\psi}{\partial t} \;+\; \frac{x-y}{t}\;.\;\nabla_x\psi - \frac{h^2}{2}\Delta\psi \;=\; V(x) \;-\; \frac{h^2}{2}|\nabla\psi|^2 \tag{2.1}$$

$$\psi(x,y,0,h) = 0 \tag{2.2}$$

where y is a point of $\mathbf{R}^n$,which plays the role of a parameter, like $h \in]0,1]$. We assume that V is a C^∞ function on $\mathbf{R}^n$, such that all its derivatives of order ≥ 1 are bounded. In this section the constants in the inequalities may depend on the dimension n, excepted when they are explicit, but they are independent on y. More precisely, the aim of this section is the following result.

Theorem 2.1. *Assume that V is a C^∞ real-valued in $\mathbb{R}^n$, such that all its derivatives of order ≥ 1 are bounded. Then, there exist a unique global classical solution ψ to (2.1), (2.2). Moreover, for all $t, h > 0$,*

$$\left\| \frac{\partial\psi}{\partial x_j}(\cdot,\cdot,t,h)\right\|_\infty \;\leq\; \frac{t}{2} \sup_{j\in\{1,\dots,n\}} \left\|\frac{\partial V}{\partial x_j}\right\|_\infty ,$$

and, for all $T, h > 0$, for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq 2$, there exists $M > 0$ (which may depend on the dimension n and on T) such that,

$$\|\partial_x^\alpha\psi(\cdot,\cdot,t,h)\|_\infty \leq M \qquad \forall\, t \in (0,T], \;.$$

The two first subsections are devoted to preliminaries to the proof, and section 2.3 to the end of the proof.

2.1. Linearized equation.

As a first step, we shall solve explicitely the linearized equation, with a given right hand side f,

$$\frac{\partial\psi}{\partial t} \;+\; \frac{x-y}{t}\nabla_x\psi - \frac{h^2}{2}\Delta\psi \;=\; f(x,t) \quad t>0 \qquad \psi(x,y,0) = 0 \tag{2.3}$$

We shall have also the analogous problem,with an initial data g at a point $t_0 > 0$

$$\frac{\partial\psi}{\partial t} \;+\; \frac{x-y}{t}\nabla_x\psi - \frac{h^2}{2}\Delta\psi \;=\; f(x,t) \quad t>t_0 \qquad \psi(x,y,t_0) = g(x) \tag{2.4}$$

For each s and t such that $0 < s < t$, for each x and y in $\mathbf{R}^n$, let us set

$$a(s,t) \;=\; \frac{t}{s(t-s)} \qquad m(x,y,s,t) \;=\; \left(1-\frac{s}{t}\right)y \;+\; \frac{s}{t}x \tag{2.5}$$

and, for each $h > 0$

$$G_h(x,x',y,s,t) \;=\; (2\pi h^2)^{-n/2}\; a(s,t)^{n/2} e^{-\frac{a(s,t)}{2h^2}|x'-m(x,y,s,t)|^2} \tag{2.6}$$

Proposition 2.2. *The solution of the Cauchy Problem (2.3) is given by*

$$\psi_h(x,y,t) \;=\; \int_{\substack{x'\in\mathbf{R}^n\\ 0<s<t}} G_h(x,x',y,s,t)\; f(x',s)\; ds dx'$$

The solution of the Cauchy Problem (2.4) is given by

$$\psi_h(x,y,t) \;=\; \int_{\mathbf{R}^n} G_h(x,x',y,t_0,t) g(x') dx' \;+\; \int_{\substack{x'\in\mathbf{R}^n\\ t_0<s<t}} G_h(x,x',y,s,t)\; f(x',s)\; ds dx'$$

The proof is standard. For further use, we shall point out some properties of the kernel G_h

$$G_h > 0 \qquad \int_{\mathbf{R}^n} G_h(x,x',y,s,t)\; dx' \;=\; 1 \tag{2.7}$$

There exists $C>0$, independent of all the parameters, such that

$$\int_{\mathbf{R}^n} |\partial_{x_j} G_h(x,x',y,s,t)| dx' \;\leq\; \frac{C}{h} \sqrt{\frac{s}{t(t-s)}} \tag{2.8}$$

For each $0 < s < t$, let us denote by $G_h(s,t)$ the integral operator in $\mathbb{R}^n$ with integral kernel $G_h(x,x',y,s,t)$. If f is smooth enough we have :

$$\frac{\partial}{\partial_{x_j}} \left(G_h(s,t) f\right) \;=\; \frac{s}{t} G_h(s,t)\; \frac{\partial f}{\partial x_j} \tag{2.9}$$

2.2. The maximum principle.

All the proofs will rely on the following variant of the maximum principle.

Proposition 2.3. *Let $y\in\mathbf{R}^n$, $(n\geq 1)$, let $a_k(x,t)$ $(1\leq k\leq n)$ be continuous bounded functions in $\mathbf{R}^n\times[0,T]$ $(T>0)$, and u be a function in $C(\mathbf{R}^n\times[0,T])\bigcap C^2(\mathbf{R}^n\times]0,T])$ such that u and $\frac{\partial u}{\partial x_j}$ $(1\leq j\leq n)$ are bounded in $\mathbf{R}^n\times]0,T])$ and $u(x,0)=0$. Assume that the function f defined by the following equality (where $h>0$) is bounded*

$$\frac{\partial u}{\partial t} \;+\; \frac{x-y}{t}.\nabla u \;-\; \frac{h^2}{2}\Delta u \;+\; \sum_{k=1}^{n} a_k(x,t)\frac{\partial u}{\partial x_k} = f(x,t) \tag{2.10}$$

Then we have, for each $t\in[0,T]$

$$\|u(.\,,t)\|_\infty \;\leq\; \int_0^t \|f(.,\; s)\|_\infty\; ds \tag{2.11}$$

where $\|\;\;\|_\infty$ denotes the norm of $L^\infty(\mathbf{R}^n)$.

Proof. Here y is fixed and it is often omit from the notations. Let $\chi\in C^\infty([0,+\infty))$ satisfying $\chi(x)=0$ if $x\geq 2$, $\chi(x)=1$ if $x\leq 1$, and $\|\chi\|_\infty$, $\|\chi'\|_\infty$, $\|\chi''\|_\infty\leq 1$. For $R\geq 1$, define $\chi_R(x,t)=\chi\left(\left(\frac{\|x-y\|}{Rt}\right)^2\right)$, for all $x\in\mathbb{R}^n$ and $t>0$. Let $L=\frac{x-y}{t}\cdot\nabla_x-\frac{h^2}{2}\Delta_x+\sum_{k=1}^n a_k(x,t)\frac{\partial}{\partial x_k}$. We have,

$$\left(\frac{\partial}{\partial t}+L\right)\chi_R u = f_R, \quad \text{on } \mathbb{R}^n\times[0,T], \tag{2.12}$$

where

$$f_R=\chi_R f - h^2\left(\nabla u\cdot\nabla\chi_R+\frac{1}{2}u\,\Delta\chi_R\right)+u\,a\cdot\nabla\chi_R+u\frac{\partial\chi_R}{\partial t}+u\frac{x-y}{t}\cdot\nabla\chi_R.$$

Our choice of χ_R implies that the sum of last two terms vanishes.

We define $K(x,y,s,t)$, $0 \le s \le t \le T$, $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, by

$$\left(\frac{\partial}{\partial t} + L\right) K(s,t) = 0, \qquad K(s,s) = f_R(s). \tag{2.13}$$

Fix $t_0 \in (0,T]$. Following (2.12)(2.13), we get for $t \in [t_0, T]$,

$$(\chi_R u)(t) - (\chi_R u)(t_0) = \int_{t_0}^{t} K(s,t)\, ds.$$

Let $U = \{x \in \mathbb{R}^n,\ \|x-y\| < \sqrt{2}RT\}$. Thus, U is an open bounded set of $\mathbb{R}^n$, with a smooth boundary. One may look at (2.12) and (2.13) with $x \in U$ instead of $x \in \mathbb{R}^n$. Since the coefficients of L are bounded and continuous on $U \times [t_0, T]$, the standard maximum is applied to (2.13). One get,

$$\|K(\cdot, y, s, t)\|_{L^\infty(\mathbb{R}^n)} \le \|f_R(\cdot, y, s)\|_{L^\infty(\mathbb{R}^n)}.$$

Consequently,

$$\|\chi_R u(\cdot, y, t)\|_{L^\infty(\mathbb{R}^n)} - \|\chi_R u(\cdot, y, t_0)\|_{L^\infty(\mathbb{R}^n)} \le \int_{t_0}^{t} \|f_R(\cdot, y, s)\|_{L^\infty(\mathbb{R}^n)}\, ds. \tag{2.14}$$

Furthermore,

$$\|f_R(\cdot, y, s) - \chi_R f(\cdot, y, s)\|_{L^\infty(\mathbb{R}^n)} \le Cnh^2 \int_{t_0}^{t} \frac{\|\chi'\|}{Rs} \sup_k \left\| \frac{\partial u}{\partial x_k} \right\|_\infty + \frac{\|\chi''\|}{2R^2 s^2} \|u\|_\infty + \frac{\|\chi'\|}{Rs} \sup_k \|a_k\|_\infty\, ds.$$

Thus, one get the inequality stated in Proposition 2.3, by first taking the limit in (2.14), as $R \to \infty$, and then, by taking the limit as $t_0 \to 0$.

2.3. End of the proof of Theorem 2.1.

In order to prove Theorem 2.1, we shall be interested in the following Cauchy Problem.

$$\frac{\partial \psi}{\partial t} + \frac{x-y}{t} \cdot \nabla_x \psi - \frac{h^2}{2} \Delta \psi = V(x) - \frac{h^2}{2} |\nabla_x \psi|^2,\ s \le t \le s + \tau, \tag{2.15}$$

$$\psi(x, y, s) = g(s).$$

In the sequel, y is a parameter and all the estimates shall be uniform in this parameter. Therefore, y is now omitted in the notations.

Formally, (2.15) is equivalent to the following integral equation

$$\psi_h(\cdot, t+s) = G_h(s, s+t) g(s) + \int_0^t G_h(s+\sigma, s+t) \left[V - \frac{h^2}{2} |\nabla_x \psi(s+\sigma)|^2 \right] d\sigma,\ 0 \le t \le \tau.$$

Set $u_j = \frac{\partial \psi}{\partial x_j}$. Let $\alpha \in \mathbb{N}^n$ with $|\alpha| \ge 1$, then write $\alpha = (0, \ldots, 1, \ldots, 0) + \delta$ (the choice of δ does not matter, only $|\delta|$ is relevant in the estimates) and set $u^{(\alpha)} = \partial_x^\alpha \psi = \partial_x^\delta \partial_{x_j} \psi$. Formally, the $u^{(\alpha)}$'s, $|\alpha| \ge 1$, satisfy the following integral equations for $t \in [0, \tau]$,

$$u^{(\alpha)}(t+s) = \left(\frac{s}{s+t}\right)^{|\alpha|} G_h(s, s+t) \partial_x^\alpha g(s) + \int_0^t \left(\frac{\sigma+s}{t+s}\right)^{|\alpha|} G_h(s+\sigma, s+t) \partial_x^\alpha V d\sigma - \cdots$$

$$\cdots \frac{h^2}{2} c_\alpha \sum_{l=1}^{n} \int_0^t \left(\frac{\sigma+s}{t+s}\right)^{\delta} \frac{\partial G_h}{\partial x_j}(s+\sigma, s+t) \Big[u_l(\sigma+s) \overline{u}_l^{(\delta)}(\sigma+s) + \cdots \tag{2.16}$$

$$\cdots \sum_{\beta+\gamma=\delta, 1\le|\beta|,|\gamma|\le|\delta|-1} \binom{\gamma}{\beta} u_l^{(\beta)}(\sigma+s)\overline{u}_l^{(\gamma)}(\sigma+s)\Big] d\sigma.$$

where $c_\alpha = 1$ if $\alpha = 0$ and $C_\alpha = 2$ if $\alpha \neq 0$.

In this subsection, C always denotes the real positive constant given in subsection 2.1.

Lemma 2.4. *Fix $k \geq 2$ and $T > 0$. Set $s \in [0, T]$. Let $\tau > 0$ be such that, $s + \tau \in [0, T]$ and*

$$0 < \tau < \left(16CnhT \sup_{\alpha\in\mathbb{N}^n, 1\le|\alpha|\le k} \|\partial_x^\alpha V\|_{L^\infty(\mathbb{R}^n)}\right)^{-2}.$$

Define (M_l) the finite sequence of real numbers, $M_1 = 1$, $M_2 \geq 1$, and for $3 \leq l \leq k$,

$$M_l = \sup_{|\delta|\le l-1} \sum_{\beta+\gamma=\delta,\ |\beta|,|\gamma|\neq 0} \binom{\gamma}{\beta} M_{|\beta|+1} M_{|\gamma|+1}. \tag{2.17}$$

Suppose that

$$\forall\, \alpha \in \mathbb{N}^n \textit{ with } 1 \le |\alpha| \le k,\ \|\partial_x^\alpha g(s)\|_{L^\infty(\mathbb{R}^n)} \le s M_{|\alpha|} \sup_{\beta\in\mathbb{N}^n, 1\le|\beta|\le k} \left\|\partial_x^\beta V\right\|_{L^\infty(\mathbb{R}^n)},$$

then, there exists $(u^{(\alpha)})_{1\le|\alpha|\le k}$ solutions to (2.3) on $[s, s+\tau]$. Moreover, we have

$$\left\|u^{(\alpha)}(\cdot,t)\right\|_{L^\infty(\mathbb{R}^n)} \le 2tM_{|\alpha|} \sup_{\beta\in\mathbb{N}^n, 1\le|\beta|\le k} \left\|\partial_x^\beta V\right\|_{L^\infty(\mathbb{R}^n)},\ \forall t \in [s, s+\tau]. \tag{2.18}$$

Proof of Lemma 2.4. Here n, T, k, s, τ are fixed. We shall prove Lemma 2.4 by iterations on $|\alpha| \in \{1, \ldots, k\}$. First consider (2.16) for $|\alpha| = 1$. In particular, $\delta = 0$ and (2.16) is a nonlinear system where the unknown functions are the $u^{(\alpha)}$'s for $|\alpha| = 1$. Define

$$E_{s,\tau} = \left\{u = (u^{(\alpha)})_{|\alpha|=1} /\ u^{(\alpha)} \in C^0(\mathbb{R}^n \times [s, s+\tau]) \text{ and } \|u\|_{E_{s,\tau}} < \infty\right\},$$

where

$$\|u\|_{E_{s,\tau}} = \sup_{(x,t)\in\mathbb{R}^n\times[s,s+\tau], |\alpha|=1} t^{-1}|u^{(\alpha)}(x,t)|.$$

For $u \in E_{s,\tau}$, let $F^{(\alpha)}(u)$ be the r.h.s. of (2.16) and set $F = (F^{(\alpha)})_{|\alpha|=1}$.

Define $V_k = \sup_{1\le|\beta|\le k} \left\|\partial_x^\beta V\right\|_{L^\infty(\mathbb{R}^n)}$. Let B be the ball of $E_{s,\tau}$ with radius $2V_k$ and centered at the origin. We shall prove that F has a unique fixed point in B. We first verify that B is stable under F.

Suppose that, $\forall\, |\alpha| = 1$, $\left\|u^{(\alpha)}(\cdot,t)\right\|_{L^\infty(\mathbb{R}^n)} \le 2tV_k$. Following (2.16), we get, $\forall\, x \in \mathbb{R}^n$, $\forall\, |\alpha| = 1$,

$$\begin{aligned}
|F^{(\alpha)}(u)(x, s+t)| &\le (s+t)V_k + \frac{nh^2}{2}\int_0^t \sqrt{\frac{s+\sigma}{(s+t)(t-\sigma)}}\frac{C}{h}\left(2V_k(s+\sigma)\right)^2 d\sigma \\
&\le (s+t)V_k + 4CnhTV_k^2(t+s)t^{\frac{1}{2}} \\
&\le 2(t+s)V_k,
\end{aligned}$$

if $0 \le t \le \tau$. This implies that $\|F(u)\|_{E_{s,\tau}} \le 2V_k$. Consequently, F maps B into itself.

Next, we remark that F is a contraction in B. Take $u, v \in E_{s,\tau}$. We have

$$(F^{(\alpha)}(u) - F^{(\alpha)}(v))(t+s) = \frac{h^2}{2}\sum_{l=1}^n \int_0^t \frac{\partial G}{\partial x_j}(\sigma+s, t+s)((v_l - u_l)(u_l + v_l))(s+\sigma)d\sigma.$$

Assume that $u, v \in B$. Then,

$$\|F^{(\alpha)}(u) - F^{(\alpha)}(v)(\cdot, t+s)\|_{L^\infty(\mathbb{R}^n)} \leq CnhV_k \|u-v\|_{E_{s,\tau}} \int_0^t \frac{(s+\sigma)^2}{\sqrt{t-\sigma}} d\sigma.$$

$$\leq 2ChnTt^{1/2} V_k \|u-v\|_{E_{s,\tau}} (t+s)$$

$$\leq A\|u-v\|_{E_{s,\tau}} (t+s)$$

with $A < 1$, if $t \leq \tau$. Consequently, there is $A < 1$ such that,

$$\|F(u) - F(v)\|_{E_{s,\tau}} \leq A\|u-v\|_{E_{s,\tau}}, \ \forall\, u, v \in B.$$

This implies the existence of $u \in B$ satisfying $F(u) = u$, that is to say, there are $u^{(\alpha)}(t)$, $|\alpha| = 1$, solutions to (2.16) with $|\alpha| = 1$, $\forall\, t \in [s, s+\tau]$ and satisfying (2.18) when $|\alpha| = 1$.

Next, set $1 \leq N \leq k$ and suppose that Lemma 2.4 holds replacing k by $N-1$. We shall prove that it also true when replacing k by N. Thus, (2.16) with $|\alpha| = N$ is a linear system with unknown functions $u^{(\alpha)}$, $|\alpha| = N$.

Set

$$E'_{s,\tau} = \left\{ v = (v^{(\alpha)})_{|\alpha|=N} / \ v^{(\alpha)} \in C^0(\mathbb{R}^n \times [s, s+\tau]) \text{ and } \|v\|_{E'_{s,\tau}} < \infty \right\},$$

where

$$\|v\|_{E'_{s,\tau}} = \sup_{(x,t)\in\mathbb{R}^n\times[s,s+\tau], |\alpha|=N} t^{-1} |v^{(\alpha)}(x,t)|.$$

For $v \in E'_{s,\tau}$, define $G^{(\alpha)}(v)$ as the r.h.s. of (2.16) and set $G = (G^{(\alpha)})_{|\alpha|=N}$.

Let B' be the ball of $E'_{s,\tau}$ with radius $2M_N V_k$, centered at the origin, where M_N is arbitrary if $N = 2$ and M_N is given in the statement of Lemma 2.4 if $N \geq 3$.

Take $v \in B'$, i.e., $\|v^{(\alpha)}(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq 2M_N V_k t, \ \forall\, t \in [s, s+\tau]$. Knowing that, $\|u^{(\beta)}(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq 2M_{|\beta|} V_k t$, $|\beta| \in \{1, \dots, N-1\}$, $\forall\, t \in [s, s+\tau]$, we see that ($|\delta| = N-1$),

$$|G^{(\alpha)}(v)(x, s+t)| \leq M_N V_k (s+t) + 4ChnV_k^2 \int_0^t \frac{(s+\sigma)^2}{\sqrt{t-\sigma}} \times \cdots$$

$$\cdots \times \left(M_N + \sum_{\beta+\gamma=\delta, \ |\beta|, |\gamma| \neq 0} \binom{\delta}{\beta} M_{|\beta|+1} M_{|\gamma|+1} \right) d\sigma$$

$$\leq M_N V_k (s+t) + 16 M_N Cnht^{1/2} (s+t) T V_k^2$$

$$\leq 2M_N V_k (t+s),$$

if $0 \leq t \leq \tau$. Consequently, G maps B' into itself. Similarly (since G is affine), G is a contraction in B' (and also in $E'_{s,\tau}$). Therefore, $G(v) = v$ for some $v \in B'$. Equivalently, there are $u^{(\alpha)}$, $|\alpha| = N$, solutions to (2.16) with $|\alpha| = N$ and (2.18) with $k = N$, on $[s, s+\tau]$. The proof of Lemma 2.4 is complete.

Proof of Theorem 2.1. Fix $k \geq 2$ and $T > 0$. Apply Lemma 2.4 with $s = 0$, $g = 0$ and $M_2 = 1$. This provides $u^{(\alpha)}$, $1 \leq |\alpha| \leq k$, defined for $t \in [0, \tau]$. Then, define ψ by

$$\psi_h(t) = \int_0^t G_h(\sigma, t) \left[V - \frac{h^2}{2} \sum_{|\alpha|=1} (|u^{(\alpha)}|)^2(\sigma) \right] d\sigma, \ \forall\, t \in [0, \tau]. \tag{2.19}$$

Using the integrability properties of the kernels G and $\nabla_x G$, one may check that ψ is a classical solution to (2.15), and ψ satisfies

$$\forall\, \alpha \in \mathbb{N}^n, \ 1 \leq |\alpha| \leq k, \ \partial_x^\alpha \psi(\cdot, t) = u^{(\alpha)}(\cdot, t), \quad \forall\, t \in [0, \tau].$$

In particular, $\|\partial_x^\alpha \psi(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq 2M_{|\alpha|}V_k t$. By differentiating the equation (2.15) with respect to x_j, we see that the function $u_j = t\frac{\partial\psi}{\partial x_j}$ satisfies

$$\frac{\partial u_j}{\partial t} + \frac{x-y}{t}\cdot\nabla u_j - \frac{h^2}{2}\Delta u_j + \sum_{k=1}^n h^2 \frac{\partial\psi}{\partial x_k}\frac{\partial u_j}{\partial x_k} = t\frac{\partial V}{x_j}.$$

Since u_j and $\frac{\partial u_j}{\partial x_k}$ $(1 \leq k \leq n)$ are bounded in $\mathbb{R}^n \times [0,\tau]$, Proposition 2.3 shows that $\|u_j(\cdot,t)\|_\infty \leq \int_0^t s\|\frac{\partial V(\cdot)}{\partial x_j}\|_\infty d\,s$. It follows the estimation

$$\|\frac{\partial\psi}{\partial x_j}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{t}{2}\|\frac{\partial V}{\partial x_j}\|_\infty.$$

We emphasize the dependence of M_l on M_2 by writing $M_l = f_l(M_2)$.

Next, set $s=\tau$, $g(\tau)=\psi(\tau)$, $M_2=2$. Since $f_l(2) \geq 2f_l(1)$ then $g(\tau)$ satisfies the hypothesis in Lemma 2.4, and we obtain $(u^{(\alpha)})$ defined on $[\tau,2\tau]$. We define ψ on $]\tau,2\tau]$ by,

$$\psi_h(\tau+t) = G(\tau,\tau+t)\psi(\tau) + \int_0^t G_h(\tau+\sigma,\tau+t)\left[V - \frac{h^2}{2}\sum_{|\alpha|=N}(u^{(\alpha)})^2(\tau+\sigma)\right]d\sigma,\ \forall\, t\in[0,\tau]. \tag{2.20}$$

Moreover, $\|\partial_x^\alpha \psi(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq 2M_{|\alpha|}V_k t$, for all $t \in]\tau,2\tau]$, for all $|\alpha| \in \{1,\ldots,k\}$, where $M_1 = 1, M_2 = 2, M_l = f_l(2)$, $3 \leq l \leq k$. Observe that (2.20) is a smooth extension of (2.19). To see this, apply Lemma 2.4 on $[\tau/2, 3\tau/2]$ and use the uniqueness of the fixed point in the proof of Lemma 2.4.

Consequently, (2.19)(2.20) define a solution ψ on $[0,2\tau]$ to (2.15). This solution satisfies

$$\|\partial_x^\alpha\psi(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq 2M_{|\alpha|}V_k t,\ 1\leq|\alpha|\leq k,\ \forall t\in[0,2\tau].$$

Iterate this process to find a solution ψ to (2.1) on $[0,T]$ verifying, for some M depending only on T,k,n,h,

$$\left\|\frac{\partial\psi}{\partial x_j}(\cdot,t)\right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{t}{2}\|\frac{\partial V}{\partial x_j}\|_{L^\infty(\mathbb{R}^n)},$$

and

$$\|\partial_x^\alpha\psi(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq MV_k,\ \forall t\in[0,T],\ 2\leq|\alpha|\leq k.$$

Since T and k are arbitrary, the proof of Theorem 2.1 is finished.

3. A lemma about linear evolution equations.

For each finite subset Λ of $\mathbb{Z}^d$, we are studying a function $\varphi \in C^\infty((\mathbb{R}^p)^\Lambda\times(\mathbb{R}^p)^\Lambda\times[0,+\infty[)$, depending of course on Λ, and also on two parameters $h>0$ and $\varepsilon>0$. We assume that φ satisfies

$$\frac{\partial\varphi}{\partial t} + \frac{x-y}{t}\,.\,\nabla_x\varphi - \frac{h^2}{2}\Delta_x\varphi + h^2(\nabla_x a,\nabla_x\varphi) = F(x,y,t,h,\varepsilon) \tag{3.1}$$

$$\varphi(x,y,0,h,\varepsilon)=0 \tag{3.2}$$

$$\varphi(x,y,t,h,\varepsilon)=\varphi(y,x,t,h,\varepsilon), \tag{3.3}$$

where a and F are also functions in $C^\infty((\mathbb{R}^p)^\Lambda\times(\mathbb{R}^p)^\Lambda\times[0,+\infty[)$, which depend on Λ and may depend on h and ε.

We assume that we already know that, for each fixed Λ, t, h and ε, all the derivatives of order ≥ 1 in x and y of φ are bounded in $(\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda$. We want to obtain more precise bounds for the derivatives, assuming similar bounds for the derivatives of a and F.

We assume that there exists a constant $\sigma_0 > 0$, that, for each $m \geq 1$, there exist a constant $K_m > 0$ and a function $\gamma \to \varepsilon_m(\gamma)$ from $]0,1[$ in itself, such that, for each sequence $\lambda^{(1)}, ..., \lambda^{(m)}$ in Λ, we have, if $ht \leq \sigma_0$ and $0 < \varepsilon \leq \varepsilon_m(\gamma)$,

$$\|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} a\| \leq K_m t \varepsilon^{\gamma \mathrm{diam}(\{\lambda^{(1)},...,\lambda^{(m)}\})} \tag{3.4}$$

We assume that we can associate to each finite subset E of $\mathbb{Z}^d$ a function $\rho(E) \geq 0$ such that in the right hand side of (3.1) the function F satisfies the following estimates. For each $m \geq 1$, and for each $\gamma \in]0,1[$, there exist $C_m(F)(t,h) > 0$ and $\varepsilon_m(\gamma) \in]0,1[$ such that, for each sequence $\lambda^{(1)}, ..., \lambda^{(m)}$ in Λ, we have, if $ht \leq \sigma_0$ and $0 < \varepsilon \leq \varepsilon_m(\gamma)$,

$$\|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} F\| \leq C_m(F)(t,h) \varepsilon^{\gamma \rho(\{\lambda^{(1)},...,\lambda^{(m)}\})}. \tag{3.5}$$

where $C_m(F)(t,h)$ may depend on other parameters, according to the applications.

We assume also that the function $\rho(E)$ has the following property. For each γ_1 and γ_2 such that $0 < \gamma_1 < \gamma_2 < 1$, there exists $\varepsilon_0(\gamma_1,\gamma_2) \in]0,1[$ such that, for each finite subsets A and B of $\mathbb{Z}^d$ ($A \neq \emptyset$), we have

$$\sum_{\mu \in \mathbb{Z}^d} \varepsilon^{\gamma_2 \mathrm{diam}(A \cup \{\mu\}) + \gamma_1 \rho(B \cup \{\mu\})} \leq 4^d \varepsilon^{\gamma_1 \rho(A \cup B)} \qquad \text{if} \quad \varepsilon \leq \varepsilon_0(\gamma_1,\gamma_2). \tag{3.6}$$

Examples. The function $\rho(E) = \mathrm{diam}(E)$ satisfies (3.6) and, if S is a given subset of $\mathbb{Z}^d$, the function $\rho(E) = \sup_{\lambda \in E} \delta(\lambda, S)$ also.

Notations. If I is a subset of $\mathbb{N}$, $(X^{(i)})_{(i \in I)}$ a map from I to $\mathbb{R}^{2p}$, and $(\lambda^{(i)})_{(i \in I)}$ a map from I to $\mathbb{Z}^d$, we set

$$X^I . \nabla_{\lambda_I} = \prod_{i \in I} (X^{(i)} . \nabla_{\lambda_I}).$$

If $I = \emptyset$, $(X^I . \nabla_{x_{\lambda_I}})$ is the identity. We define similarly $X^I . \nabla_{x_{\lambda_I}}$ if $X^{(i)} \in \mathbb{R}^p$. We denote also by λ_I the set of the elements $\lambda^{(i)}$ $(i \in I)$.

Proposition 3.1. *With these notations, for each integer $m \geq 1$, there exists another $K_m > 0$ and another $\varepsilon_m(\gamma) \in]0,1[$ such that, for each finite set Λ of $\mathbb{Z}^d$ for each $t > 0$, $h > 0$, $\varepsilon > 0$ and $\gamma \in]0,1[$ satisfying*

$$ht \leq \sigma_0, \qquad \varepsilon \leq \varepsilon_m(\gamma), \tag{3.7}$$

for each sequence $(\lambda^{(1)}, ..., \lambda^{(m)})$ in Λ, a function φ satisfying (3.1), (3.2) and (3.3) corresponding to these parameters satisfies

$$\|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} \varphi(.,.,t)\| \leq K_m \ \varepsilon^{\gamma \rho(\{\lambda^{(1)},...,\lambda^{(m)}\})} \sum_{j=1}^m \int_0^t C_j(F)(s,h) ds \tag{3.8}$$

If the hypothesis (3.5) is also satisfied for $m = 0$, we have also, under similar conditions,

$$\|\varphi(.,.,t)\| \leq \varepsilon^{\gamma \rho(\emptyset)} \int_0^t C_0(s,h) ds \tag{3.8'}$$

Let us set, for each integers m and m'

$$N_{m,m'}(t,\varepsilon,\gamma) = \sup_{\substack{(\lambda^{(1)},\dots,\lambda^{(m)})\in\Lambda^m \\ (\mu^{(1)},\dots,\mu^{(m')})\in\Lambda^{m'}}} \frac{\|\nabla_{x_{\lambda^{(1)}}}\dots\nabla_{x_{\lambda^{(m)}}}\ \nabla_{y_{\mu^{(1)}}}\dots\nabla_{y_{\mu^{(m')}}}\varphi(.,.,t)\|}{\varepsilon^{\gamma\rho(\{\lambda^{(1)},\dots,\lambda^{(m)},\mu^{(1)},\dots,\mu^{(m')}\})}} \tag{3.9}$$

For the proof of Proposition 3.1, we have to prove that, if $m+m'\geq 1$, for some constant $\varepsilon_{m,m'}(\gamma)$,

$$(P_{m,m'}) \qquad N_{m,m'}(t,\varepsilon,\gamma) \leq K_m \sum_{j=1}^{m}\int_0^t C_j(F)(s,h)ds \qquad \text{if} \quad ht\leq\sigma_0 \quad \text{and} \quad 0<\varepsilon\leq\varepsilon_{m,m'}(\gamma)$$

First step: $m\geq 1$ and $m'=0$. We assume that $m'=0$ and either that $m=1$, or that $m\geq 2$ and that $(P_{j,0})$ is proved for $1\leq j\leq m-1$, and we shall prove $(P_{m,0})$. Let $(\lambda^{(1)},\dots\lambda^{(m)})$ a sequence of points in Λ and $(X^{(1)},\dots,X^{(m)})$ a sequence of vectors of $\mathbb{R}^p$. Let u the function defined by

$$u(x,y,t) = (X^{\{1,\dots,m\}}.\nabla_{x_{\lambda_{\{1,\dots,m\}}}})\varphi(x,y,t,h,\varepsilon). \tag{3.10}$$

This function satisfies

$$\frac{\partial(t^m u)}{\partial t} + \frac{x-y}{t}\ .\ \nabla_x(t^m u) - \frac{h^2}{2}\Delta_x(t^m u) + h^2(\nabla_x a(x),\nabla_x(t^m u)) = \Phi_m \tag{3.11}$$

where the right hand side Φ_m can be written $\Phi_m = t^m(F_m - h^2 G_m)$ with

$$F_m = (X^{\{1,\dots,m\}}.\nabla_{x_{\lambda_{\{1,\dots,m\}}}})F, \tag{3.12}$$

and, denoting by $\mathcal{P}_m$ the set of couples (I,J) such that (I,J) is a partition of $\{1,\dots,m\}$, and $I\neq\emptyset$,

$$G_m = \sum_{(I,J)\in\mathcal{P}_m}\ \sum_{\mu\in\Lambda}\Big(\nabla_{x_\mu}(X^I.\nabla_{x_{\lambda_I}})a\ ,\nabla_{x_\mu}(X^J.\nabla_{x_{\lambda_J}})\varphi\Big). \tag{3.13}$$

By our hypotheses (3.5) on F, we can write

$$\|F_m\| \leq \prod_{j=1}^{m}|X^{(j)}|\ C_m(F)(t,h)\varepsilon^{\gamma\rho(\{\lambda^{(1)},\dots,\lambda^{(m)}\})}$$

if $ht\leq\sigma_0$ and $0<\varepsilon\leq\varepsilon_m(\gamma)$. Under similar conditions, we have also, by the hypotheses (3.4) on a and by the definition of $N_{m,0}(t,\varepsilon,\gamma)$,

$$\|G_m\| \leq K_m\ t\ \prod_{j=1}^{m}|X^{(j)}| \sum_{(I,J)\in\mathcal{P}_m} N_{|J|+1}(t,\varepsilon,\gamma)\sum_{\mu\in\Lambda}\varepsilon^{\frac{\gamma+1}{2}\mathrm{diam}(\lambda_I\cup\{\mu\})+\gamma\rho(\lambda_J\cup\{\mu\})}$$

By our hypothesis (3.6) on the function ρ, since $I\neq\emptyset$ if $(I,J)\in\mathcal{P}_m$, it follows that, under similar conditions,

$$\|G_m\| \leq K_m\ t\ \prod_{j=1}^{m}|X^{(j)}| \sum_{(I,J)\in\mathcal{P}_m} N_{|J|+1}(t,\varepsilon,\gamma)\varepsilon^{\gamma\rho(\{\lambda^{(1)},\dots,\lambda^{(m)}\})}.$$

For all $(I,J)\in\mathcal{P}_m$, we have $|J|+1\leq m$. If $|J|+1\leq m-1$,we can apply our induction hypothesis $(P_{|J|+1,0})$. We obtain, by a combination of these inequalities, the following bound for the RHS of (3.11), under conditions of type (3.7)

$$\|\Phi_m\| \leq K_m\ t^m\ \prod_{j=1}^{m}|X^{(j)}|\ \varepsilon^{\gamma\rho(\{\lambda^{(1)},\dots,\lambda^{(m)}\})}\ \left[\sum_{j=1}^{m-1} th^2\int_0^t C_j(F)(s,h)ds + C_m(F) + th^2 N_m(t,\varepsilon,\gamma)\right].$$

(By convention, the sum in the RHS is replaced by 0 if $m = 1$). By Proposition 2.3, it follows that

$$\|t^m u(.,.,t)\| \leq K_m \prod_{j=1}^{m} |X^{(j)}| \, \varepsilon^{\gamma\rho(\{\lambda^{(1)},\dots,\lambda^{(m)}\})} \int_0^t s^m \Big[\sum_{j=1}^{m-1} sh^2 \int_0^s C_j(F)(s',h)ds' + \dots$$

$$\dots + C_m(F)(s,h) + sh^2 N_m(s,\varepsilon,\gamma)\Big] ds.$$

In other words

$$N_m(t,\varepsilon,\gamma) \leq K_m \int_0^t \left[\sum_{j=0}^{m-1} h^2 \frac{t^2 - s^2}{2} C_j(F)(s,h) + C_m(F)(s,h) + sh^2 N_m(s,\varepsilon,\gamma) \right] ds.$$

The property $(P_{m,0})$ follows by Gronwall's Lemma, using the condition $ht \leq \sigma_0$ (the constants K_m depends on σ_0, which is fixed).

Second step : $m = 0$ and $m' \geq 1$. In this case, $(P_{0,m'})$ follows from the symmetry (3.3).

Third step: $m \geq 1$ and $m' \geq 1$. Given $m \geq 1$ and $m' \geq 1$, we assume that $P_{j,k}$ has been proved for all j and k such that $1 \leq j + k \leq m + m'$ and $j \leq m - 1$ or $k \leq m' - 1$, and we shall prove $P_{m,m'}$. For that, we introduce, for each points $\lambda^{(1)}, \dots, \lambda^{(m)}, \mu^{(1)}, \dots, \mu^{(m')}$ in Λ, for each vectors $X^{(1)}, \dots, X^{(m)}, Y^{(1)}, \dots, Y^{(m')}$ of $\mathbb{R}^p$, the following function

$$u(x,y,t,\varepsilon) = (X^{\{1,\dots,m\}}.\nabla_{x_{\lambda_{\{1,\dots,m\}}}}) \, (Y^{\{1,\dots,m'\}}.\nabla_{y_{\mu_{\{1,\dots,m'\}}}}) \varphi(x,y,t,h,\varepsilon) \tag{3.14}$$

This function satisfies

$$\frac{\partial(t^m u)}{\partial t} + \frac{x-y}{t} \, . \, \nabla_x(t^m u) - \frac{h^2}{2}\Delta_x(t^m u) + h^2(\nabla_x a \, . \, \nabla_x(t^m \, u)) = \Phi_{m,m'} \tag{3.15}$$

where the right hand side $\Phi_{m,m'}$ can be written on the form $\Phi_{m,m'} = t^m F_{m,m'} + t^{m-1}\Psi_{m,m'} - \frac{h^2}{2} t^m G_{m,m'}$ with the following notations:

$$F_{m,m'} = (X^{\{1,\dots,m\}}.\nabla_{x_{\lambda_{\{1,\dots,m\}}}}) \, (Y^{\{1,\dots,m'\}}.\nabla_{y_{\mu_{\{1,\dots,m'\}}}}) F \tag{3.16}$$

$$\Psi_{m,m'} = \sum_{k=1}^{m'} (X^{\{1,\dots,m\}}.\nabla_{x_{\lambda_{\{1,\dots,m\}}}}) \, (Y^{(1)}.\nabla_{y_{\mu^{(1)}}}) \dots (Y^{(k)}.\nabla_{x_{\mu^{(k)}}}) \dots (Y^{(m')}.\nabla_{y_{\mu^{(m')}}}) \varphi(x,y,t) \tag{3.17}$$

and, denoting now by $\mathcal{P}_{m,m'}$ the set of (I, I', J, J') such that (I, J) is a partition of $\{1, ..., m\}$, (I', J') a partition of $\{1, ..., m'\}$ and $I \cup I' \neq \emptyset$:

$$G_{m,m'} = \sum_{(I,I',J,J') \in \mathcal{P}_{m,m'}} \sum_{\nu \in \Lambda} \Big(\nabla_{x_\nu}(X^I.\nabla_{x_{\lambda_I}})(Y^{I'}.\nabla_{y_{\mu_{I'}}})a, \nabla_{x_\nu}(X^J.\nabla_{x_{\lambda_J}})(Y^{J'}.\nabla_{y_{\mu_{J'}}})\varphi \Big). \tag{3.18}$$

By the hypothesis (3.5) on F, we can write

$$\|F_{m,m'}\| \leq C_{m+m'}(F) \prod_{i \leq m} |X^{(i)}| \prod_{j \leq m'} |Y^{(j)}| \varepsilon^{\gamma\rho(\{\lambda^{(1)},\dots,\lambda^{(m)},\mu^{(1)},\dots,\mu^{(m')}\})}$$

if $ht \leq \sigma_0$ and $\varepsilon \leq \varepsilon_{m,m'}(\gamma)$. By definition, we can write

$$\|\Psi_{m,m'}\| \leq m' \prod_{i \leq m} |X^{(i)}| \prod_{j \leq m'} |Y^{(j)}| \, N_{m+1,m'-1}(t,\varepsilon,\gamma) \varepsilon^{\gamma\rho(\{\lambda^{(1)},\dots,\lambda^{(m)},\mu^{(1)},\dots,\mu^{(m')}\})}$$

By the hypotheses (3.4) on a, we can write, if $ht \leq \sigma_0$ and $\varepsilon \leq \varepsilon_{m+m'}(\frac{1+\gamma}{2})$

$$\|\nabla_{x_\nu}(X^I.\nabla_{x_{\lambda_I}})(Y^{I'}.\nabla_{y_{\mu_{I'}}})a\| \leq K_{m+m'}t \prod_{i\in I}|X^{(i)}| \prod_{i\in I'}|Y^{(i)}|\varepsilon^{\frac{\gamma+1}{2}\mathrm{diam}(\lambda_I\cup\mu_{I'}\cup\{\nu\})}$$

By definition of $N_{m,m'}(t,\varepsilon,\gamma)$, we can write

$$\|\nabla_{x_\nu}(X^J.\nabla_{x_{\lambda_J}})(Y^{J'}.\nabla_{y_{\mu_{J'}}})\varphi\| \leq \prod_{i\in J}|X^{(i)}| \prod_{i\in J'}|Y^{(i)}| \; N_{|J|+1,|J'|}(t,\varepsilon,\gamma)\varepsilon^{\gamma\rho(\lambda_J\cup\mu_{J'}\cup\{\nu\})}$$

By the hypothesis (3.6) on the function ρ, it follows, if moreover $\varepsilon \leq \varepsilon_0(\gamma, \frac{\gamma+1}{2})$, that

$$\|G_{m,m'}\| \leq 4^d K_{m+m'}t \prod_{i=1}^{m}|X^{(i)}| \prod_{i=1}^{m'}|Y^{(i)}| \sum_{(I,I',J,J')\in\mathcal{P}_{m,m'}} N_{|J|+1,|J'|}(t,\varepsilon,\gamma)\varepsilon^{\gamma\rho(\{\lambda^{(1)},\dots,\lambda^{(m)},\mu^{(1)},\dots,\mu^{(m')}\})}$$

For each (I,I',J,J') in $\mathcal{P}_{m,m'}$, we have $|J|+1+|J'| \leq m+m'$ and either $|J| \leq m-1$ or $|J'| \leq m'-1$.Therefore, we can apply the induction hypothesis to all the terms $H_{I,I',J,J'}$, excepted when $|J|+1 = m$ and $|J'| = m'$. Thus we obtain, for the RHS of (3.15), with some constants $K_{m,m'}$ (depending only on m, m' and d) and $\varepsilon_{m,m'}(\gamma)$:

$$\|\Phi_{m,m'}\| \leq K_{m,m'}t^m \prod_{i\leq m}|X^{(i)}| \prod_{j\leq m'}|Y^{(j)}| \; \varepsilon^{\gamma\rho(\{\lambda^{(1)},\dots,\lambda^{(m)},\mu^{(1)},\dots,\mu^{(m')}\})} \dots$$

$$\dots \left[\sum_{j=1}^{m+m'-1} h^2 t\int_0^t C_j(F)(s,h)ds + C_{m+m'}(F)(t,h) + th^2 N_{m,m'}(t,\varepsilon,\gamma)\right]$$

if $ht \leq \sigma_0$ and $\varepsilon \leq \varepsilon_{m,m'}(\gamma)$. By the maximum principle (Proposition 2.3), the function u defined in (3.10) satisfies, under these conditions

$$\|t^m u(.,.,t)\| \leq \; K_{m,m'} \prod_{i\leq m}|X^{(i)}| \prod_{j\leq m'}|Y^{(j)}| \; \varepsilon^{\gamma\rho(\{\lambda^{(1)},\dots,\lambda^{(m)},\mu^{(1)},\dots,\mu^{(m')}\})}\dots$$

$$\dots \int_0^t s^m \left[\sum_{j=1}^{m+m'_1} th^2\int_0^s C_j(F)(s',h)ds' + C_{m+m'}(F)(t,h) + sh^2 N_{m,m'}(s,\varepsilon,\gamma)\right] ds$$

In other words, under the same conditions,

$$N_{m,m'}(t,\varepsilon,\gamma) \leq \; K_{m,m'}\int_0^t \left[\sum_{j=1}^{m+m'-1} h^2\frac{t^2-s^2}{2}C_j(F)(s,h) + C_{m+m'}(s,h) + sh^2 N_{m,m'}(s,\varepsilon,\gamma)\right] ds$$

The property $(P_{m,m'})$ follows by Gronwall's lemma. The Proposition is proved for $m \geq 1$.

Case $m = 0$. Now, we assume that the hypothesis of the RHS F of (3.1) is also satisfied form $m = 0$, i.e. $\|F\| \leq C_0(F)\varepsilon^{\gamma\rho(\emptyset)}$ under the previous conditions. Then it follows directly from Proposition 2.3 that, under the same conditions, we have (3.8'). Now the Proposition is proved.

4. Proof of Theorem 1.1.

With the results of Section 2, we know that, for each finite subset Λ of $\mathbb{Z}^d$, and for h and ε small enough, the solution ψ_Λ of the Cauchy problem (1.16), (1.17) exists globally. Then the function U_Λ defined in (1.3) is the integral kernel of the operator $e^{-tH_\Lambda(\varepsilon)}$. We have, for the first order derivatives of ψ_Λ, the estimation given by (). By the hypotheses on the potential $V_{\Lambda,\varepsilon}$, we have, for some constant $C > 0$ depending only on the constants in the hypotheses, $\|\nabla_{x_\lambda} V_{\Lambda,\varepsilon}\| \leq C$ if $0 < \varepsilon \leq 2^{-d}$. Since H_Λ is self-adjoint, it follows that, for each finite set Λ, for each $\lambda \in \Lambda$, we have

$$\|\nabla_\lambda \psi_\Lambda(.,.,t,h,\varepsilon)\| \leq Ct \qquad \text{if} \quad 0 < \varepsilon \leq 2^{-d}. \tag{4.1}$$

Therefore, the statement of Theorem 1.1 is already proved for $m = 1$. The next step is the proof for $m = 2$, before the induction. We shall use the notation ∇_{x_λ} of the introduction.

Lemma 4.1. *There exists $\sigma_0 > 0$ and $C > 0$ such that, for each $\gamma \in]0,1[$, for each λ and $\mu \in \Lambda$, we have*

$$\|\nabla_{x_\lambda}\nabla_{x_\mu}\psi_\Lambda(.,.,t,h,\varepsilon)\| \leq Ct\varepsilon^{\gamma|\lambda-\mu|} \qquad \text{if} \quad ht \leq \sigma_0 \quad \text{and} \quad \varepsilon^{1-\gamma} \leq \frac{1}{2^d} \tag{4.2}$$

Proof. For each t, h, ε and $\gamma \in]0,1[$, let us denote by $S(t,h,\varepsilon,\gamma)$ the best constant such that, for each vector $X \in \mathbb{R}^p$, for each $\lambda \in \Lambda$ and for each sequence $(Y_\mu)_{\mu\in\Lambda}$ of vectors in $\mathbb{R}^p$, we have

$$\sum_{\mu\in\Lambda} \frac{\|(X.\nabla_{x_\lambda})(Y_\mu.\nabla_{x_\mu})\psi_\Lambda(.,.,t,h,\varepsilon)\|}{\varepsilon^{\gamma|\lambda-\mu|}} \quad \leq \quad S(t,h,\varepsilon,\gamma)\ |X| \ \sup_{\mu\in\Lambda}|Y_\mu|$$

This function is well defined by the results of section 2. For each $\lambda \in \Lambda$, for each sequence $(Y_\mu)_{\mu\in\Lambda}$, for each ε and $\gamma \in]0,1[$, the function

$$\varphi = \sum_{\mu\in\Lambda} \frac{(X.\nabla_{x_\lambda})(Y_\mu.\nabla_{x_\mu})\psi_\Lambda(.,.,t,h,\varepsilon)}{\varepsilon^{\gamma|\lambda-\mu|}}$$

satisfies the equation

$$\frac{\partial(t^2\varphi)}{\partial t} \ + \ \frac{x-y}{t}\ .\ \nabla_x(t^2\varphi) \ - \ \frac{h^2}{2}\Delta_x(t^2\varphi) + h^2(\nabla_x\psi_\Lambda)\ .\ (\nabla_x t^2\varphi) \ = \ t^2F - h^2t^2G$$

where

$$F = \sum_{\mu\in\Lambda} \frac{(X.\nabla_{x_\lambda})(Y_\mu.\nabla_{x_\mu})V_\Lambda(.,\varepsilon)}{\varepsilon^{\gamma|\lambda-\mu|}}$$

$$G = \sum_{(\mu,\nu)\in\Lambda^2} \frac{(\nabla_{x_\nu}(X.\nabla_{x_\lambda})\psi_\Lambda(.,\varepsilon)\ ,\ \nabla_{x_\nu}(Y_\mu.\nabla_{x_\mu})\psi_\Lambda(.,\varepsilon))}{\varepsilon^{\gamma|\lambda-\mu|}}$$

We see, by our hypotheses on V, that

$$|F| \leq C\ |X| \ \sup_{\mu\in\Lambda}|Y_\mu|$$

if $\varepsilon^{1-\gamma} \leq \frac{1}{2^d}$. By the definition of S, we have

$$|G| \leq S(t,h,\varepsilon,\gamma)^2|X| \ \sup_{\mu\in\Lambda}|Y_\mu|$$

Therefore, by the maximum principle, (Proposition 2.3),

$$\|t^2\varphi(.,.,t)\| \leq \ |X| \ \sup_{\mu\in\Lambda}|Y_\mu| \ \int_0^t [Cs^2 + h^2s^2S(s,h,\varepsilon,\gamma)^2]\ ds$$

In other words

$$t^2S(t,h,\varepsilon,\gamma) \leq \int_0^t s^2[C + h^2S(s,h,\varepsilon,\gamma)^2]\ ds \tag{4.3}$$

We shall prove, by a kind of version of Gronwall lemma, that

$$S(t,h,\varepsilon,\gamma) \leq 2Ct \qquad \text{if} \quad \varepsilon^{1-\gamma} \leq \frac{1}{2^d} \quad \text{and} \quad 4Ch^2t^2 \leq 1 \tag{4.4}$$

If this result were false, let us denote by t_0 the lower bound of the $t > 0$ such that $S(t,h,\varepsilon,\gamma) \geq 2Ct$. The only point which needs details is that $t_0 > 0$. If t_0 were 0, there would be a contradiction with the results of section 2, proving that, for some $K_\Lambda(\varepsilon)$ perhaps depending on Λ and ε, we have $S(t,h,\varepsilon,\gamma) \leq K_\Lambda(\varepsilon)$ and therefore, by (4.3), that $S(t,h,\varepsilon,\gamma) \leq t(C + h^2 K_\Lambda(\varepsilon)^2)$ and, applying again (4.3), that $S(t,h,\varepsilon,\gamma) \leq Ct + h^2 \frac{t^3}{3}(C + h^2 K_\Lambda(\varepsilon)^2)^2$, which proves that the lower bound t_0 cannot be 0. Supposing that the inequality in the left of (4.4) is true for $0 < t < t_0$ and false for $t = t_0$, the contradiction follows easily under the conditions of (4.4). The Lemma is proved.

For higher order derivatives, we remark that the equation (1.16) satisfied by the function $\varphi(x,y,t) = \psi_{\Lambda,\varepsilon}(x,y,t)$ can be written on the form (3.1), with $a = \frac{1}{2}\varphi = \frac{1}{2}\psi_{\Lambda,\varepsilon}$ and $F = V_{\Lambda,\varepsilon}$. Therefore, we shall estimate the higher order derivatives of $\psi_{\Lambda,\varepsilon}$ with the ideas and notations of Proposition 3.1, where the function $\rho(E)$, defined for each subset E of $\mathbb{Z}^d$, is its diameter. We already saw that this function satisfies (3.6). Let us define, for each integers m and m', a function $N_{m,m'}(t,\varepsilon,\gamma)$ like in (3.9), with our choice of the function ρ (therefore $\rho(\{\lambda^{(1)},\dots,\lambda^{(m)},\mu^{(1)},\dots,\mu^{(m')}\}) = \mathrm{diam}(\{\lambda^{(1)},\dots,\lambda^{(m)},\mu^{(1)},\dots,\mu^{(m')}\})$. For the proof of Theorem 1.1, it remains to prove, for each integers $m \geq 2$ and $m' \geq 0$,

$(P_{m,m'})$ *There exists* $C_{m,m'} > 0$ *and, for each* $\gamma \in]0,1[$, *there exists* $\varepsilon_{m,m'}(\gamma) \in]0,1[$ *such that*

$$N_{m,m'}(t,\varepsilon,\gamma) \leq C_{m,m'} t \qquad \text{if} \quad ht \leq \sigma_0 \qquad \text{and} \quad \varepsilon \leq \varepsilon_{m,m'}(\gamma)$$

First step : $m \geq 3$ *and* $m' = 0$. Let $m \geq 3$ and $\gamma \in]0,1[$. We assume that $P_{k,0}$ has been proved for all $k \leq m-1$. We define, for each points $\lambda^{(1)}$, ...$\lambda^{(m)}$ in Λ, for each vectors $X^{(1)}$, ... $X^{(m)}$ in $\mathbb{R}^p$, with the notations introduced before Proposition 3.1, a function $u(x,t,\varepsilon)$ like in (3.10) (with $\varphi = \psi_{\Lambda,\varepsilon}$). We remark that this function satisfies (for a fixed y) the equation (3.11), with $a = \frac{1}{2}\varphi = \frac{1}{2}\psi_{\Lambda,\varepsilon}$, and $\Phi_m = t^m(F_m - h^2 G_m)$ where F_m is defined like in (3.12) with $F = V_{\Lambda,\varepsilon}$, and G_m like in (3.13), with the same value of a.

We remark that our hypotheses on the potential $V_{\Lambda,\varepsilon}$ imply that, for each integer $m \geq 1$, there exists $C_m > 0$, independent on all the parameters, such that, for each points $\lambda^{(1)}$, ... $\lambda^{(m)}$ in Λ

$$\|\nabla_{x_{\lambda^{(1)}}} \dots \nabla_{x_{\lambda^{(m)}}} V_{\Lambda,\varepsilon}\| \leq C_m \varepsilon^{\mathrm{diam}(\{\lambda^{(1)},\dots\lambda^{(m)}\})} \qquad \text{if} \quad 0 < \varepsilon \leq 2^{-d} \tag{4.5}$$

(The last condition is for the case where all the $\lambda^{(i)}$ are the same). Therefore, there exists $K_m > 0$ such that the function F_m defined like in (3.12) with $F = V_{\Lambda,\varepsilon}$ satisfies

$$|F_m(x,\varepsilon)| \leq K_m |X^{(1)}|\dots|X^{(m)}| \varepsilon^{\mathrm{diam}(\{\lambda^{(1)},\dots\lambda^{(m)}\})} \qquad \text{if} \quad 0 < \varepsilon \leq \frac{1}{2}.$$

Using the fact that $a = \frac{1}{2}\varphi$, we can write

$$\frac{1}{2}(\nabla_x \varphi, \nabla_x u) \ + \ G_m = (\nabla_x \varphi, \nabla_x u) \ + \ G'_m \tag{4.6}$$

where, denoting by $\mathcal{P}'_m$ the set of couples (I,J) of subsets of $\{1,...,m\}$ such that (I,J) is a partition of $\{1,...,m\}$ and $I \neq \emptyset$, $J \neq \emptyset$, we define G'_m like G_m in (3.13), but with $\mathcal{P}_m$ replaced by $\mathcal{P}'_m$

$$G'_m = \frac{1}{2} \sum_{(I,J)\in\mathcal{P}'_m} \sum_{\mu\in\Lambda} \Big(\nabla_{x_\mu}(X^I.\nabla_{x_{\lambda_I}})\psi_{\Lambda,\varepsilon} \ , \nabla_{x_\mu}(X^J.\nabla_{x_{\lambda_J}})\psi_{\Lambda,\varepsilon} \Big).$$

By (4.6), the equation, similar to (3.11), satisfied by u defined like (3.10) can be rewritten

$$\frac{\partial(t^m u)}{\partial t} \ + \ \frac{x-y}{t} \ . \ \nabla_x(t^m u) - \frac{h^2}{2}\Delta_x(t^m u) + 2h^2(\nabla_x a(x), \nabla_x(t^m u)) \ = \ \Phi'_m \tag{4.7}$$

where $\Phi'_m = t^m(F_m - h^2 G'_m)$. By the definition (3.9) of $N_{m,0}$, we have, using also the notations before Proposition 3.1,

$$\|G'_m\| \leq \prod_{j=1}^m |X^{(j)}| \left[\sum_{(I,J)\in\mathcal{P}'_m} N_{|I|+1,0}(t,\varepsilon,\gamma) N_{|J|+1,0}(t,\varepsilon,\frac{1+\gamma}{2}) \sum_{\mu\in\mathbb{Z}^d} \varepsilon^{\gamma \mathrm{diam}(\lambda_I\cup\{\mu\}) + \frac{1+\gamma}{2}\mathrm{diam}(\lambda_J\cup\{\mu\})} \right]$$

We remark that, for each $(I,J)\in\mathcal{P}'_m$, $(m\geq 3)$, we have $|I|+1\leq m$, $|J|+1\leq m$, and that we cannot have simultaneously $|I|+1=m$ and $|J|+1=m$. For each term such that $|I|+1\leq m-1$ or $|J|+1\leq m-1$, we can apply the induction hypothesis. We obtain, using also (3.6), satisfied by the function $\rho(E)=\mathrm{diam}(E)$, if $ht\leq\sigma_0$ and $\varepsilon\leq\varepsilon_{m-1,0}(\frac{1+\gamma}{2})\leq\varepsilon_{m-1,0}(\gamma)$, and if $\varepsilon\leq\varepsilon_0(\gamma,\frac{1+\gamma}{2})$,

$$\|G'_m\| \leq 4^d \ \prod_{j=1}^m |X^{(j)}| \ \varepsilon^{\gamma\mathrm{diam}(\{\lambda^{(1)},\ldots,\lambda^{(m)}\})} \left[\sum_{\substack{(I,J)\in\mathcal{P}'_m \\ |I|+1\leq m-1, |J|+1\leq m-1}} C^2_{m-1,0} t^2 + \ldots \right.$$

$$\left. \ldots + 2 \sum_{\substack{(I,J)\in\mathcal{P}'_m \\ |I|+1=m, |J|+1\leq m-1}} C_{m-1,0}\, t N_{m,0,}(t,\varepsilon,\gamma) \right]$$

Therefore, with some other constant K_0, we have, for the right hand side of (4.7), under conditions similar to those of (P_m),

$$\|\Phi'_m\| \leq K_0 t^m \ \prod_{j=1}^m |X^{(j)}| \ \varepsilon^{\gamma\mathrm{diam}(\{\lambda^{(1)},\ldots\lambda^{(m)}\})} \Big(1 + h^2t^2 + h^2 t N_{m,0}(t,\varepsilon,\gamma)\Big)$$

By the maximum principle (Proposition 2.3) (which can be applied since, by the results of section 2, its hypotheses are satisfied), it follows that, under these conditions,

$$t^m \|\varphi(.,.,t)\| \leq K_0 \ \prod_{j=1}^m |X^{(j)}| \ \varepsilon^{\gamma\mathrm{diam}(\{\lambda^{(1)},\ldots\lambda^{(m)}\})} \int_0^t s^m \ \Big(1 + h^2 s^2 + h^2 s N_{m,0}(s,\varepsilon,\gamma)\Big) ds$$

In other words

$$t^m N_{m,0}(t,\varepsilon,\gamma) \leq K_0 \int_0^t s^m \Big(1 + h^2 s^2 + h^2 s N_{m,0}(s,\varepsilon,\gamma)\Big) ds$$

The property $P_{m,0}$ follows by Gronwall's Lemma, and therefore is proved for all m.

Second step : $m'\geq 1$. If $m=0$ and $m'\geq 1$ the property $P_{0,m'}$ follows from $P_{m',0}$ since $\psi_{\Lambda,\varepsilon}$ is symmetric. We organize the double induction in a following way. Given $m\geq 1$ and $m'\geq 1$, we assume that $P_{j,k}$ has been proved for all j and k such that $j+k\leq m+m'$ and $k\leq m'-1$, and we shall prove $P_{m,m'}$. For that, we introduce, for each points $\lambda^{(1)}, \ldots, \lambda^{(m)}, \mu^{(1)}, \ldots, \mu^{(m')}$ in Λ, for each vectors $X^{(1)}, \ldots, X^{(m)}, Y^{(1)}, \ldots, Y^{(m')}$ of $\mathbb{R}^p$, the function u defined like in (3.14), with $\varphi=\psi_\Lambda$. We remark that this function satisfies (for a fixed y) the equation (3.15), with $a=\frac{1}{2}\psi_\Lambda$, and, since V is independent of y, $\Phi_{m,m'} = t^{m-1}\Psi_{m,m'} - h^2 t^m G_{m,m'}$, where $\Psi_{m,m'}$ is defined in (3.17) and $G_{m,m'}$ in (3.18). Using the fact that $a=\frac{1}{2}\varphi$, we can write

$$\frac{1}{2}(\nabla_x\varphi, \nabla_x u) \ + \ G_{m,m'} = (\nabla_x\varphi,\nabla_x u) \ + \ G'_{m,m'} \tag{4.8}$$

where, denoting by $\mathcal{P}'_{m,m'}$ the set of (I,I',J,J') such that (I,J) is a partition of $\{1,...,m\}$, (I',J') is a partition of $\{1,...,m'\}$, $I\cup I'\neq\emptyset$, $J\cup J'\neq\emptyset$, we define $G'_{m,m'}$ like $G_{m,m'}$ in (3.18), but with $\mathcal{P}_{m,m'}$ replaced by $\mathcal{P}'_{m,m'}$, i.e.

$$G'_{m,m'}=\frac{1}{2}\sum_{(I,I',J,J')\in\mathcal{P}'_{m,m'}}\sum_{\nu\in\Lambda}\Big(\nabla_{x_\nu}(X^I.\nabla_{x_{\lambda_I}})(Y^{I'}.\nabla_{y_{\mu_{I'}}})\psi_\Lambda,\nabla_{x_\nu}(X^J.\nabla_{x_{\lambda_J}})(Y^{J'}.\nabla_{y_{\mu_{J'}}})\psi_\Lambda\Big). \tag{4.9}$$

By (4.8), the equation, similar to (3.15) satisfied by u defined in (3.14) can be rewritten on the form (4.7), where the RHS is now $\Phi'_{m,m'}=t^{m-1}\Psi_{m,m'}-h^2t^mG'_{m,m'}$.

By definition, we have

$$\|\Psi_{m,m'}\|\leq m'\prod_{i\leq m}|X^{(i)}|\prod_{j\leq m'}|Y^{(j)}|\ N_{m+1,m'-1}(t,\varepsilon,\gamma)\varepsilon^{\gamma\mathrm{diam}(\{\lambda^{(1)},...,\lambda^{(m)},\mu^{(1)},...,\mu^{(m')}\})}$$

and, by our induction hypothesis, we have, for some $K_0>0$ and $\varepsilon_0(\gamma)>0$,

$$\|\Psi_{m,m'}\|\leq K_0t\prod_{i\leq m}|X^{(i)}|\prod_{j\leq m'}|Y^{(j)}|\varepsilon^{\gamma\mathrm{diam}(\{\lambda^{(1)},...,\lambda^{(m)},\mu^{(1)},...,\mu^{(m')}\})}$$

if $ht\leq\sigma_0$ and $\varepsilon\leq\varepsilon_0(\gamma)$.

By the definition of $N_{m,m'}$, using also the condition (3.6) satisfied by $\rho(E)=\mathrm{diam}(E)$, if $\varepsilon\leq\varepsilon_0(\frac{1+\gamma}{2})\leq\varepsilon_0(\gamma)$,

$$\|G'_{m,m'}\|\leq\prod_{i\leq m}|X^{(i)}|\prod_{j\leq m'}|Y^{(j)}|\left[\sum_{(I,I',J,J')\in\mathcal{P}'_{m,m'}}N_{|I|+1,|I'|}(t,\varepsilon,\gamma)\ N_{|J|+1,|J'|}(t,\varepsilon,\frac{1+\gamma}{2})\dots\right.$$

$$\left.\dots\varepsilon^{\gamma\mathrm{diam}(\{\lambda^{(1)},...,\lambda^{(m)},\mu^{(1)},...,\mu^{(m')}\})}\right]$$

For all (I,I',J,J') in $\mathcal{P}_{m,m'}$, we have $|I|+|I'|+1\leq m+m'$, $|J|+|J'|+1\leq m+m'$, $|I'|+|J'|=m'\geq 1$, hence $|I'|$ or $|J'|$ is $\leq m'-1$. For each term such that $|I'|\leq m-1$ or $|J'|\leq m-1$, we can apply the induction hypothesis, and the property $(P_{m,m'})$ follows like in the first step, and therefore is proved for all m and m'.

5. Cluster decomposition of a function on the lattice.

Let $\Lambda=\prod_{j=1}^d[a_j,b_j]$ be a box of $\mathbb{Z}^d$. Let E be a finite dimensional vector space. We shall associate to each box Q contained in Λ (which may be all Λ, and also may be reduced to a single point) an operator T_Q in $C^\infty(E^\Lambda\times E^\Lambda)$ in the following way. First, we define a linear map π_Q in $E^\Lambda\times E^\Lambda$ by

$$\Big(\pi_Q(x,y)\Big)_\lambda=\begin{cases}(x_\lambda,y_\lambda) & if\quad \lambda\in Q\\ (0,y_\lambda-x_\lambda) & if\quad \lambda\notin Q\end{cases}\qquad\forall(x,y)\in E^\Lambda\times E^\Lambda. \tag{5.1}$$

For each $j\in\{1,...d\}$, we define two linear operators $P_Q^{(j,+)}$ and $P_Q^{(j,-)}$in $E^\Lambda\times E^\Lambda$ in the following way. If the box $Q\subseteq\Lambda$ is defined by

$$Q=\prod_{j=1}^d[\alpha_j,\beta_j], \tag{5.2}$$

we set :

$$\Big(P_Q^{(j,+)}(x,y)\Big)_\lambda=\begin{cases}(x_\lambda,y_\lambda) & if\quad \lambda_j\neq\beta_j\\ (0,y_\lambda-x_\lambda) & if\quad \lambda_j=\beta_j\end{cases}\qquad\forall(x,y)\in E^\Lambda\times E^\Lambda.$$

$$\left(P_Q^{(j,-)}(x,y)\right)_\lambda = \begin{cases} (x_\lambda, y_\lambda) & if \quad \lambda_j \neq \alpha_j \\ (0, y_\lambda - x_\lambda) & if \quad \lambda_j = \alpha_j \end{cases} \qquad \forall (x,y) \in E^\Lambda \times E^\Lambda.$$

Then we define our operator T_Q, for each function f in $C^\infty(E^\Lambda \times E^\Lambda)$ by

$$(T_Q f)(x,y) = \sum_{\substack{\sigma \in \{0,1\}^d \\ \tau \in \{0,1\}^d}} (-1)^{|\sigma|_1 + |\tau|_1} f\Big(\left(P_Q^{(1,+)}\right)^{\sigma_1} ... \left(P_Q^{(d,+)}\right)^{\sigma_d} \left(P_Q^{(1,-)}\right)^{\tau_1} ... \left(P_Q^{(d,-)}\right)^{\tau_d} \pi_Q \ (x,y)\Big)$$

Here $|\sigma|_1$ denotes the ℓ^1 norm of the vector $\sigma \in \mathbb{Z}^d$.

Of course, we may have $P_Q^{(j,+)} = P_Q^{(j,-)}$. In particular, if Q is reduced to a single point, we have

$$(T_Q f)(x,y) = f(\pi_Q(x,y)) - f(0, y-x) \tag{5.3}$$

Proposition 5.1. *For each function f in $C^\infty(E^\Lambda \times E^\Lambda)$, we have*
i) $T_Q f$ depends only on $x-y$ and on the variables x_λ and y_λ such that $\lambda \in Q$.

ii)
$$f(x,y) - f(0,y-x) = \sum_{Q \subseteq \Lambda} (T_Q f)(x,y)$$

iii) If f depends in a smooth way on a parameter θ, we have $T_Q(\frac{\partial f}{\partial \theta}) = \frac{\partial (T_Q f)}{\partial \theta}$.

The next Proposition will be useful later to estimate the functions $T_Q f$ and their derivatives, when f will be the solution of one of the Cauchy problems studied in the previous sections.

For each finite subset Λ of $\mathbb{Z}^d$, for each function $f \in C^\infty((\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda)$, for each vector $u = (u_\lambda)_{\lambda \in \Lambda}$ in $(\mathbb{R}^p)^\Lambda$, we set

$$(S_u f)(x,y) = f(x+u, y+u) - f(x,y) \qquad \forall f \in C^\infty(E^\Lambda \times E^\Lambda) \quad \forall (x,y) \in E^\Lambda \times E^\Lambda \tag{5.4}$$

and we shall denote by $\sigma(u)$ the following set

$$\sigma(u) = \{\lambda \in \Lambda, \qquad u_\lambda \neq 0\}. \tag{5.5}$$

In the next section, we shall apply the operators T_Q to the function ψ_Λ of Theorem 1.1, and we shall see that the diameter of Q for the ℓ^∞ norm, in other words, the greatest of the $\beta_j - \alpha_j$, if Q is defined by (5.2), will play an important role. We shall apply the following proposition with the corresponding integer j . For a box Q defined as (5.2), and for each $j \in \{1, ...d\}$, we shall set

$$B_-^{(j)}(Q) = \{\lambda \in Q, \qquad \lambda_j = \alpha_j\}, \qquad B_+^{(j)}(Q) = \{\lambda \in Q, \qquad \lambda_j = \beta_j\} \tag{5.6}$$

Proposition 5.2. *Let $\lambda^{(1)}$, ... $\lambda^{(m)}$ be a finite sequence of points in Λ. Let j be an integer such that $1 \leq j \leq d$. Then :*
i) If, in this sequence, there is no point in $B_+^{(1)}(Q) \cup B_-^{(1)}(Q)$ (in particular, if this sequence is void), we have

$$\|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} T_Q f\|_\infty \leq 4^d \sup_{\substack{\sigma(u) \subset B_+^{(1)}(Q) \\ \sigma(v) \subset B_-^{(1)}(Q)}} \|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} S_u S_v f\|_\infty \tag{5.7}$$

ii) If one at least of the $\lambda^{(k)}$ is in $B_+^{(1)}(Q)$, but none of them is in $B_-^{(1)}(Q)$, we have

$$\|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} T_Q f\|_\infty \leq 4^d \sup_{\sigma(v) \subset B_-^{(1)}(Q)} \|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} S_v f\|_\infty \tag{5.8}$$

iii) If one at least of the $\lambda^{(k)}$ is in $B_+^{(1)}(Q)$ and another one (perhaps the same) is in $B_-^{(1)}(Q)$, then we have

$$\|\nabla_{\lambda^{(1)}}...\nabla_{\lambda^{(m)}}T_Q f\|_\infty \leq 4^d \|\nabla_{\lambda^{(1)}}...\nabla_{\lambda^{(m)}} f\|_\infty \tag{5.9}$$

Proposition 5.3. *Let $V_{\Lambda,\varepsilon}(x)$ be of the form (1.2), where A and B_λ satisfy the hypotheses of the introduction. We identify $V_{\Lambda,\varepsilon}(x)$ with a function of (x,y) independent of y, and then $T_Q V_{\Lambda,\varepsilon}(x)$ is also independent of y. Then, we can write, with $C_m > 0$ independent of all the parameters, for each sequence $(\lambda^{(1)},\dots,\lambda^{(m)})$ of points of Λ,*

$$|\nabla_{x_{\lambda^{(1)}}}\dots\nabla_{x_{\lambda^{(m)}}} T_Q V_{\Lambda,\varepsilon}(x,t)| \leq \begin{cases} C_m \varepsilon^{\gamma \mathrm{diamQ}} & \text{if } \ \mathrm{diam}(Q) > 0 \quad \textit{and} \quad \varepsilon^{1-\gamma} \leq 2^{-d} \\ C_m(1+|x_Q|) & \text{if } \ \mathrm{diam}(Q) = 0 \quad \textit{and} \quad \varepsilon \leq 2^{-d} \end{cases} \tag{5.10}$$

Proof. If $\mathrm{diam}(Q) > 0$, (5.10) follows, by Proposition 5.2, from (4.5) if $\varepsilon \leq 2^{-d}$ in the case iii). For the case ii), we can write, for some constant $K_m > 0$,

$$\|\nabla_{x_{\lambda^{(1)}}}...\nabla_{x_{\lambda^{(m)}}} S_u V_{\Lambda,\varepsilon}\| \leq K_m \varepsilon^{\gamma \sup_{j\leq m} \delta(\lambda^{(j)},\sigma(u))} \qquad \text{if} \quad \varepsilon^{1-\gamma} \leq 2^{-d} \tag{5.11}$$

and apply (5.8). For the case i), we shall prove that, for some constant $K_m > 0$, for each $\gamma \in]0,1[$, we have, for each sequence $\lambda^{(1)}$, ...$\lambda^{(m)}$ in Λ,

$$\|\nabla_{x_{\lambda^{(1)}}}...\nabla_{x_{\lambda^{(m)}}} S_u S_v V_{\Lambda,\varepsilon}\| \leq K_m \varepsilon^{\gamma\delta(\sigma(u),\sigma(v))} \mathrm{inf}(\sharp\sigma(u),\sharp\sigma(v)) \qquad \text{if} \quad \varepsilon^{1-\gamma} \leq 2^{-d} \tag{5.12}$$

and if $\sigma(u)\cap\sigma(v) = \emptyset$. Under this condition, we have $S_u S_v A(x_\lambda) = 0$ for all λ, and we see that $S_u S_v B_{\lambda-\mu}(x_\lambda, x_\mu) = 0$ unless λ or μ is in $\sigma(u)$ and λ or μ is in $\sigma(v)$, which implies, in our case, that $\lambda \in \sigma(u)$ and $\mu \in \sigma(v)$, or the opposite. Then we remark that

$$\sum_{\lambda\in\sigma(u),\mu\in\sigma(v)} \varepsilon^{|\lambda-\mu|} \leq 4^d \varepsilon^{\gamma\delta(\sigma(u),\sigma(v))} \mathrm{inf}(\sharp\sigma(u),\sharp\sigma(v)) \qquad \text{if} \quad \varepsilon^{1-\gamma} \leq 2^{-d}.$$

and (5.12) follows, and (5.10) also, if we apply (5.7). When Q is reduced to a single point, (5.10) follows from (5.3) (which means here $(T_{\{\lambda\}}V)(x) = V(x'^{(\lambda)}) - V(0)$, where $x_\mu'^{(\lambda)}$ is equal to x_λ if $\mu = \lambda$ and to 0 if not).

6. Cluster decomposition of the function ψ_Λ .

The aim of this section is the proof of the following Proposition.

Proposition 6.1. *For each integer $m \geq 0$, and for any $\gamma \in [0,1[$, there exists $C_m > 0$ and $\varepsilon_m(\gamma) \in]0,1[$ with the following properties.*
i) If $m \geq 1$, for each finite subset Λ of $\mathbb{Z}^d$, for each points $\lambda^{(1)}$, ... $\lambda^{(m)}$ in Λ, for each box $Q \subseteq \Lambda$, we have,

$$\|\nabla_{\lambda^{(1)}}...\nabla_{\lambda^{(m)}}\Big(T_Q\psi_\Lambda\Big)\| \leq tC_m\varepsilon^{\gamma \mathrm{diam}(Q)} \qquad \textit{if} \quad ht \leq \sigma_0 \quad \textit{and} \quad \varepsilon \leq \varepsilon_m(\gamma) \tag{6.1}$$

ii) If $m = 0$, this result is also valid if $\mathrm{diam}(Q) \neq 0$.
iii) If one at least of the points $\lambda^{(j)}$ is not in Q, we have also

$$\|\nabla_{\lambda^{(1)}}...\nabla_{\lambda^{(m)}}\Big(T_Q\psi_\Lambda\Big)\| \leq tC_m\varepsilon^{\gamma \sup_{j\leq m}\delta(\lambda^{(j)},Q)} \qquad \text{if} \quad ht \leq \sigma_0 \quad \text{and} \quad \varepsilon \leq \varepsilon_m(\gamma) \tag{6.2}$$

The proof relies on the two following Lemmas.

Lemma 6.2. *For each integer $m \geq 1$, and for each $\gamma \in]0,1[$, there exists $C_m > 0$ and $\varepsilon_m(\gamma) \in]0,1[$ such that, for each u in $(\mathbb{R}^p)^\Lambda$, for each points $\lambda^{(1)}, \dots \lambda^{(m)}$ in Λ, the solution ψ_Λ of (1.16), (1.17) satisfies, if S_u is the operator defined in (5.4)*

$$\|\nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(m)}} S_u \psi_\Lambda\| \leq C_m \ t \ \varepsilon^{\gamma \sup_{j \leq m} \delta(\lambda^{(j)}, \sigma(u))} \qquad if \quad ht \leq \sigma_0 \qquad \varepsilon \leq \varepsilon_m(\gamma) \tag{6.3}$$

Proof. The function $\varphi = S_u \psi_\Lambda$ satisfies the following equation

$$\frac{\partial \varphi}{\partial t} \ + \ \frac{x-y}{t} \ . \ \nabla_x \varphi - \frac{h^2}{2} \Delta_x \varphi + h^2 (\nabla_x a, \nabla_x \varphi) \ = \ S_u V_{\Lambda, \varepsilon}$$

where

$$a(x,y,t) = \frac{1}{2} \Big(\psi_\Lambda(x+u, y+u, t) + \psi_\Lambda(x,y,t) \Big). \tag{6.4}$$

The function φ satisfies also (3.2) and (3.3) and, by Theorem 1.1, the function a satisfies the hypothesis (3.4). We define, for each non void finite subset E of $\mathbb{Z}^d$, $\rho(E)$ by

$$\rho(E) = \sup_{\lambda \in E} \delta(\lambda, \sigma(u)).$$

We already know that this function satisfies (3.6), (it was given as an example just after (3.6)). Using also (5.11), we see that all the hypotheses of Proposition 3.1 are satisfied, and Lemma 6.2 follows from this Proposition.

Lemma 6.3. *For each integer $m \geq 0$, and for each $\gamma \in]0,1[$, there exists $C_m > 0$ and $\varepsilon_m(\gamma) \in]0,1[$, such that, for each u and v in $(\mathbb{R}^p)^\Lambda$ such that $\sigma(u) \cap \sigma(v) = \emptyset$, for each points $\lambda^{(1)}, \dots \lambda^{(m)}$ in Λ, the solution ψ_Λ of (1.16), (1.17) satisfies*

$$\|\nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(m)}} S_u S_v \psi_\Lambda\| \leq C_m \ t \ \varepsilon^{\gamma \delta(\sigma(u), \sigma(v))} \inf \ \ (\sharp\sigma(u), \sharp\sigma(v)) \qquad if \quad ht \leq \sigma_0 \qquad \varepsilon \leq \varepsilon_m(\gamma). \tag{6.5}$$

Proof. The function $\varphi = S_u S_v \psi_\Lambda$ satisfies

$$\frac{\partial \varphi}{\partial t} \ + \ \frac{x-y}{t} \ . \ \nabla_x \varphi - \frac{h^2}{2} \Delta_x \varphi + h^2 (\nabla_x a, \nabla_x \varphi) \ = \ S_u S_v V_{\Lambda, \varepsilon} - h^2 (\nabla_x S_u \psi_\Lambda, \nabla_x S_v \psi_\Lambda) \tag{6.6}$$

where

$$a(x,y,t) = \frac{1}{2} \Big(\psi_\Lambda(x+u+v, y+u+v, t) + \psi_\Lambda(x+u, y+u, t) + \psi_\Lambda(x+v, y+v, t) - \psi_\Lambda(x,y,t) \Big).$$

The function φ satisfies also (3.2) and (3.3). By Theorem 1.1, the function a satisfies the estimations (3.4) under conditions of type (3.7). Now, the function $\rho(E)$ of Section 3 is independent of the subset E of $\mathbb{Z}^d$: we set $\rho(E) = \delta(\sigma(u), \sigma(v))$ for all $E \subset \mathbb{Z}^d$, and the condition (3.6) is obviously satisfied. Now we estimate the derivatives of the RHS of (6.6). (These inequalities are valid also for $m = 0$ (not like in Lemma 6.2), and therefore, by Proposition 3.1, (6.5) will be valid also for $m = 0$). The first term of the RHS is bounded by (5.12). For the second term, we can write, for each sequence $\lambda^{(1)}, \dots \lambda^{(m)}$ in Λ,

$$\nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(m)}} (\nabla_x S_u \psi_\Lambda, \nabla_x S_v \psi_\Lambda) = \sum_{(I,J) \in \mathcal{P}''_m} \sum_{\nu \in \Lambda} (\nabla_{x_\nu} \nabla^I S_u \psi_\Lambda, \nabla_{x_\nu} \nabla^J S_v \psi_\Lambda)$$

where, here $\mathcal{P}''_m$ denotes the set of all (I,J) such (I,J) is a partition of $\{1,...,m\}$, (one of them, I or J, may be empty), and for each subset I of $\{1,...,m\}$, we set $\nabla^I = \prod_{i\in I}\nabla_{\lambda^{(i)}}$. By Lemma 6.2, there exists $K_m > 0$ and a function $\varepsilon_m(\gamma)$ such that

$$\|\nabla_{x_\mu}\nabla^I S_u\psi_\Lambda\| \ \|\nabla_{x_\mu}\nabla^J S_v\psi_\Lambda)\| \le K_m t^2 \ \varepsilon^{\frac{1+\gamma}{2}[\delta(\mu,\sigma(u))+\delta(\mu,\sigma(v))]}$$

if $\varepsilon \le \varepsilon_m(\frac{1+\gamma}{2})$ and if $ht \le \sigma_0$. We see that

$$(6.7) \qquad \sum_{\nu\in\mathbb{Z}^d} \varepsilon^{\frac{1+\gamma}{2}[\delta(\nu,\sigma(u))+\delta(\nu,\sigma(v))]} \le 4^d \varepsilon^{\gamma\delta(\sigma(u),\sigma(v))} \ \inf(\sharp\sigma(u),\sharp\sigma(v)) \qquad \text{if} \quad \varepsilon^{1-\gamma} \le 2^{-2d}.$$

It follows that, if all the preceding conditions are satisfied, with another K_m,

$$\|\nabla_{\lambda^{(1)}}...\nabla_{\lambda^{(m)}}(\nabla_x S_u\psi_\Lambda, \nabla_x S_v\psi_\Lambda)\| \le K_m t^2 \varepsilon^{\gamma\delta(\sigma(u),\sigma(v))}.$$

Therefore, the condition (3.5) on the RHS F of (6.6) is satisfied with our function $\rho(E)$, and $C_m(F) = K_m[\inf(\sharp\ \sigma(u), \sharp\ \sigma(v)) + h^2t^2]$, where K_m is independent of u and v. Since we may assume that $ht \le 1$ and that $\inf(\sharp\ \sigma(u), \sharp\ \sigma(v)) \ge 1$ (otherwise the Lemma is trivial), we can omit the second term in the expression of $C_m(F)$. All the hypotheses of Proposition 3.1 are satisfied, and Lemma 6.3 is a consequence of it.

End of the proof of Proposition 6.1. For the proof of i) and ii), we may assume that Q is defined by (5.2) and that $\mathrm{diam}(Q) = \beta_1 - \alpha_1$. Then, by Proposition 5.2, if the sequence $(\lambda^{(1)}, ...\lambda^{(m)})$ is in the case iii) of Proposition 5.2, (with $j = 1$), the estimation (6.1) is a direct consequence of Theorem 1.1 since $\delta(B_+^{(1)}(Q), B_-^{(1)}(Q) = \mathrm{diam}(Q)$. In the case ii) of Proposition 5.2, or in the obviously similar case where the signs $+$ and $-$ are permuted, (6.1) follows from Lemma 6.2. In the case i) of Proposition 5.2, (6.1) follows from Lemma 6.3 since $\sharp B_\pm^{(1)}(Q) \le \mathrm{diam}(Q))^{d-1}$. For the point ii), we apply the point i) of Proposition 5.2 and Lemma (6.3) with $m = 0$. (We may apply it since, if $\mathrm{diam}(Q) \ne 0$, we have $\beta_1 - \alpha_1 > 0$ and therefore $B_+^{(1)}(Q) \cap B_-^{(1)}(Q) = \emptyset$). For the proof of iii), we may assume that Q is defined by (5.2) and that $\sup_j \delta(\lambda^{(j)}, Q) = \lambda_1^{(1)} - \beta_1 > 0$. In the case i) and ii) of Proposition 5.2, (6.2) follows from Lemma 6.2, and in the cases iii), (6.2) follows from Theorem 1.1.

7. Study of the union of two disjoints subsets.

If $\Lambda \subset \mathbb{Z}^d$ is the union of two disjoints subsets Λ_1 and Λ_2, we may identify $(\mathbb{R}^p)^\Lambda$ with $(\mathbb{R}^p)^{\Lambda_1} \times (\mathbb{R}^p)^{\Lambda_2}$, denoting the variable of $(\mathbb{R}^p)^\Lambda$ by $(x^{(1)}, x^{(2)})$, where $x^{(j)} = (x_\lambda)_{\lambda\in\Lambda_j}$. Thus, a function f (resp. g) in $C^\infty(\mathbb{R}^p)^{\Lambda_1})$ (resp. in $C^\infty(\mathbb{R}^p)^{\Lambda_2})$) is identified with a function on $C^\infty((\mathbb{R}^p)^\Lambda)$, independent of the variable $x^{(2)}$ (resp. of $x^{(1)}$). As in [], we set $(f \oplus g)(x^{(1)}, x^{(2)}) = f(x^{(1)}) + g(x^{(2)})$. For each $\theta \in [0,1]$, we set

$$(7.1) \qquad V_{\Lambda,\varepsilon}(\theta) = \theta V_{\Lambda,\varepsilon} + (1-\theta)(V_{\Lambda_1,\varepsilon} \oplus V_{\Lambda_2,\varepsilon}).$$

We denote by $H_\Lambda(\varepsilon,\theta)$ the Hamiltonian defined as in (1.1), with $V_{\Lambda,\varepsilon}$ replaced by $V_{\Lambda,\varepsilon}(\theta)$. Since this new potential is of the same type that $V_{\Lambda,\varepsilon}$, and satisfies the same hypotheses, we can associate to the new Hamiltonian a function $\psi_\Lambda(x,y,t,h,\varepsilon,\theta)$, solution of the Cauchy problem like (1.16), (1.17) in Λ, but with $V_{\Lambda,\varepsilon}$ replaced by this new potential, i.e. satisfying:

$$(7.2) \qquad \frac{\partial\psi_\Lambda(\theta)}{\partial t} + \frac{x-y}{t} \ . \ \nabla_x\psi_\Lambda(\theta) - \frac{h^2}{2}\Delta_x\psi_\Lambda(\theta) \ = \ V_{\Lambda,\varepsilon,\theta}(x) \ - \ \frac{h^2}{2}|\nabla_x\psi_\Lambda(\theta)|^2$$

and $\psi_\Lambda(x,y,0,h,\varepsilon,\theta) = 0$. We are now interested to the derivative of this function with respect to θ.

Proposition 7.1. *For each integer $m \ge 1$, there exists a constant $C_m > 0$, independent of all the sets and parameters, such that, with the previous notations, we have, for each points $\lambda^{(1)}, \ . \ . \ . \ \lambda^{(m)}$ in $\Lambda^{(1)}$,*

$$(7.3) \qquad \|\nabla_{\lambda^{(1)}}\dots\nabla_{\lambda^{(m)}}\psi_\Lambda(\theta)\| \le tC_m \ \varepsilon^{\gamma\mathrm{diam}(\{\lambda_1,...\lambda_m\})} \qquad \text{if} \quad ht \le \sigma_0, \quad \varepsilon \le \varepsilon_m(\gamma) \quad \text{and} \quad \theta \in [0,1]$$

and, under similar conditions, for each u and v in $(\mathbb{R}^p)^\Lambda$,

$$\|\nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(m)}} S_u \psi_\Lambda(\theta)\| \leq tC_m \quad \varepsilon^{\gamma \sup_{j\leq m} \delta(\lambda^{(j)}, \sigma(u))} \tag{7.3'}$$

$$\|\nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(m)}} S_u S_v \psi_\Lambda(\theta)\| \leq tC_m \quad \varepsilon^{\gamma \delta(\sigma(u), \sigma(v))} \tag{7.3''}$$

$$\|\nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(m)}} \frac{\partial \psi_\Lambda(\theta)}{\partial \theta}\| \leq tC_m \ \varepsilon^{\gamma \mathrm{diam}(\{\lambda_1, \dots \lambda_m\})} \tag{7.4}$$

$$\|\nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(m)}} \frac{\partial \psi_\Lambda(\theta)}{\partial \theta}\| \leq tC_m \varepsilon^{\gamma \sup_{j\leq m} \delta(\lambda^{(j)}, \Lambda_2)} \tag{7.5}$$

Proof of (7.3), (7.3') and (7.3"). The potential $V_{\Lambda,\varepsilon}(\theta)$ is of the same type that $V_{\Lambda,\varepsilon}$, and satisfies the same hypotheses, with constants independent of $\theta \in [0,1]$. Therefore, all the proofs of Theorem 1.1 and Lemmas 6.2 and 6.3 can be applied to the Cauchy problem (7.2), (1.17), and leads to the same bounds for its solution, i.e. (7.3), (7.3') and (7.3").

Proof of (7.4). We see that $\varphi = \frac{\partial \psi_\Lambda(\theta)}{\partial \theta}$ satisfies

$$\frac{\partial \varphi}{\partial t} + \frac{x-y}{t} \, . \, \nabla_x \varphi - \frac{h^2}{2} \Delta_x \varphi + h^2 \left(\nabla_x \psi_\Lambda(\theta), \nabla_x \varphi \right) = \sum_{\lambda \in \Lambda_1, \mu \in \Lambda_2} \varepsilon^{|\lambda - \mu|} B_{\lambda - \mu}(x_\lambda, x_\mu) \tag{7.6}$$

This equation is of the form (3.1), and $\frac{\partial \psi_\Lambda(\theta)}{\partial \theta}$ satisfies also (3.2) and (3.3). By (7.3), the function $a = \psi_\Lambda(\theta)$ satisfies the condition (3.4). We put, for each finite non void subset E of $\mathbb{Z}^d$, $\rho(E) = \mathrm{diam}(E)$. We already know that this function satisfies the condition (3.6). The RHS $F(x)$ of (7.6) satisfies the condition (3.5) with this function ρ, just like $V_{\Lambda,\varepsilon}$. Therefore $\frac{\partial \psi_\Lambda(\theta)}{\partial \theta}$ satisfies all the hypotheses of Proposition 3.1, and therefore, under conditions like (3.7), satisfies (3.8), which is equivalent, with our function ρ, to (7.4).

Proof of (7.5). The proof is almost the same, but we put now, for each finite non void subset E of $\mathbb{Z}^d$,

$$\rho(E) = \sup_{\lambda \in E} \delta(\lambda, \Lambda_2).$$

This function satisfies the condition (3.6). This follows from the inequality (6.4), applied now to $S = \Lambda_2$. The RHS $F(x)$ of (7.6) satisfies also the condition (3.5) with this new function ρ and, by Proposition 3.1, $\frac{\partial \psi_\Lambda(\theta)}{\partial \theta}$, under conditions like (3.7), satisfies (3.8), which is equivalent, with our new function ρ, to (7.5).

Now, we shall be interested to the cluster decomposition of $\frac{\partial \psi_\Lambda(\theta)}{\partial \theta}$.

Proposition 7.2. *For each integer $m \geq 0$, and for each $\gamma \in]0,1[$, there exists $C_m > 0$ and $\varepsilon_m(\gamma) \in]0,1[$ with the following properties. For each finite subset Λ of $\mathbb{Z}^d$, for each points $\lambda^{(1)}$, ... $\lambda^{(m)}$ in Λ, for each box $Q \subseteq \Lambda$, we have,*

$$\|\nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(m)}} \Big(T_Q \frac{\partial \psi_\Lambda}{\partial \theta} \Big)\| \leq tC_m \varepsilon^{\gamma \mathrm{diam}(Q)} \qquad \text{if} \quad ht \leq \sigma_0, \quad \varepsilon \leq \varepsilon_m(\gamma) \quad \text{and} \quad \theta \in [0,1] \tag{7.7}$$

If one at least of the points λ_j is not in Q, (hence $m \geq 1$) we have also, under similar conditions,

$$\|\nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(m)}} \Big(T_Q \frac{\partial \psi_\Lambda}{\partial \theta} \Big)\| \leq tC_m \varepsilon^{\gamma \sup_{j\leq m} \delta(\lambda^{(j)}, Q)} \tag{7.8}$$

If moreover

$$\Lambda_1 = \prod_{j=1}^{d} [a_j, b_j] \qquad \Lambda_2 \subset \{\lambda \in \mathbb{Z}^d, \quad \lambda_1 > b_1\} \qquad Q = \prod_{j=1}^{d} [\alpha_j, \beta_j] \subseteq \Lambda_1 \tag{7.9}$$

then we have, under similar conditions, where m *can be* 0

$$\| \nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} \Big(T_Q \frac{\partial \psi_\Lambda}{\partial \theta} \Big) \| \leq t C_m \varepsilon^{\gamma (b_1 - \beta_1)} \tag{7.10}$$

The proof relies on the two following Lemmas.

Lemma 7.3. *For each integer* $m \geq 0$*, and for each* $\gamma \in]0, 1[$*, there exists* $C_m > 0$ *and* $\varepsilon_m(\gamma) \in]0, 1[$ *such that, for each* u *in* $(\mathbb{R}^p)^\Lambda$*, for each points* $\lambda^{(1)}, \dots \lambda^{(m)}$ *in* Λ*,we have, if* $m \geq 1$*,*

$$\| \nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(m)}} S_u \frac{\partial \psi_\Lambda}{\partial \theta} \| \leq C_m \ t \ \varepsilon^{\gamma \sup_{j \leq m} \delta(\lambda^{(j)}, \sigma(u))} \qquad if \quad ht \leq \sigma_0 \qquad \varepsilon \leq \varepsilon_m(\gamma). \tag{7.11}$$

If $\sigma(u)$ *is contained in* Λ_1*, we have, under the previous conditions, (but here* m *can be* 0*)*

$$\| \nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(m)}} S_u \frac{\partial \psi_\Lambda(\theta)}{\partial \theta} \| \leq C t \varepsilon^{\gamma \delta(\sigma(u), \Lambda_2)} \sharp \sigma(u) \tag{7.12}$$

Proof of (7.11). The function $\varphi = S_u \frac{\partial \psi_\Lambda}{\partial \theta}$ satisfies

$$\frac{\partial \varphi}{\partial t} \ + \ \frac{x - y}{t} \ . \ \nabla_x \varphi - \frac{h^2}{2} \Delta_x \varphi \ + \ h^2 \ (\nabla_x \psi_\Lambda(\theta), \nabla_x \ \varphi) = \Phi \tag{7.13}$$

where

$$\Phi = S_u V_{\Lambda_1, \Lambda_2} - h^2 \Big(\nabla_x S_u \psi_\Lambda(\theta), \nabla_x \frac{\partial \psi_\Lambda}{\partial \theta}(x + u, y + u, t) \Big) \qquad V_{\Lambda_1, \Lambda_2} = \sum_{\lambda \in \Lambda_1, \mu \in \Lambda_2} \varepsilon^{|\lambda - \mu|} B_{\lambda - \mu}(x_\lambda, x_\mu)$$

This equation is of the form (3.1), and φ satisfies also (3.2) and (3.3). By (7.3), the function $a = \psi_\Lambda(\theta)$ satisfies the condition (3.4). We put, for each finite non void subset E of $\mathbb{Z}^d$, $\rho(E) = \sup_{\lambda \in E} \delta(\lambda, \sigma(u))$. We already know that it satisfies (3.6). By the form of V_{Λ_1, Λ_2}, we can write, for each sequence $\lambda^{(1)}, \dots, \lambda^{(m)}$ in Λ, under conditions similar to those of (7.11),

$$\| \nabla_{x_{\lambda^{(1)}}} ... \nabla_{x_{\lambda^{(m)}}} S_u V_{\Lambda_1, \Lambda_2} \| \leq K_m \varepsilon^{\gamma \sup_{j \leq m} \delta(\lambda^{(j)}, \sigma(u))}. \tag{7.14}$$

By (7.3'), we can write, for each point ν in Λ and for each sequence $\lambda^{(1)}, \dots, \lambda^{(k)}$ in Λ, under the conditions of (7.11)

$$\| \nabla_{x_\nu} \nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(k)}} S_u \psi_\Lambda(\theta) \| \leq C_k t \varepsilon^{\gamma \rho(\{\lambda^{(1)}, \dots, \lambda^{(k)}, \nu\})} \tag{7.15}$$

and by (7.4), under the same type of condition, for each sequence $\mu^{(1)}, \dots \mu^{(\ell)}$ in Λ

$$\| \nabla_{x_\nu} \nabla_{\mu^{(1)}} ... \nabla_{\mu^{(\ell)}} \frac{\partial \psi_\Lambda(\theta)}{\partial \theta} \| \leq C_k t \varepsilon^{\frac{\gamma + 1}{2} \mathrm{diam}(\{\mu^{(1)}, \dots, \mu^{(\ell)}, \nu\})} \tag{7.16}$$

By taking the product, using (3.6), and the summation on μ, we obtain, for the RHS Φ of (7.13), for each sequence $\lambda^{(1)}, \ldots, \lambda^{(m)}$ in Λ,

$$\|\nabla_{\lambda^{(1)}} \ldots \nabla_{\lambda^{(m)}} \Phi\| \leq C_m (1 + h^2 t^2) \varepsilon^{\gamma \sup_{j \leq m} \delta(\lambda^{(j)}, \sigma(u))}.$$

Therefore, (7.11) follows from Proposition 3.1.

Proof of (7.12). We have only to change the bounds for the RHS Φ of (7.13). Our new function $\rho(E)$ is independent of E. We set $\rho(E) = \delta(\sigma(u), \Lambda_2)$, and (3.6) is obviously satisfied. We can write, instead of (7.14), if $\sigma(u) \subset \Lambda_1$, under conditions similar to those of (7.11),

$$\|\nabla_{x_{\lambda^{(1)}}} \ldots \nabla_{x_{\lambda^{(m)}}} S_u V_{\Lambda_1, \Lambda_2}\| \leq K_m \varepsilon^{\gamma \delta(\sigma(u), \Lambda_2)} \sharp \sigma(u).$$

The estimate (7.15), with γ replaced by $\frac{\gamma+1}{2}$, implies

$$\|\nabla_{x_\nu} \nabla_{\lambda^{(1)}} \ldots \nabla_{\lambda^{(k)}} S_u \psi_\Lambda(\theta)\| \leq C_k t \varepsilon^{\frac{\gamma+1}{2} \delta(\nu, \sigma(u))}.$$

By (7.5), we can write, instead of (7.16)

$$\|\nabla_\nu \nabla_{\mu^{(1)}} \ldots \nabla_{\mu^{(\ell)}} \frac{\partial \psi_\Lambda(\theta)}{\partial \theta}\| \leq C_k t \varepsilon^{\gamma \delta(\nu, \Lambda_2)}$$

We remark that

$$\sum_{\nu \in \mathbb{Z}^d} \varepsilon^{\frac{\gamma+1}{2} \delta(\nu, \sigma(u)) + \gamma \delta(\nu, \Lambda_2)} \leq 4^d \varepsilon^{\gamma \delta(\sigma(u), \Lambda_2)} \sharp \sigma(u) \qquad \text{if} \quad \varepsilon^{\frac{1-\gamma}{2}} \leq 2^{-d}$$

The end of the proof is the same as for (7.11).

Lemma 7.4. *With the notations of Lemma 7.3, (but here m can be 0), we have, for each u and v in $(\mathbb{R}^p)^\Lambda$, under conditions similar to those of (7.11),*

$$\|\nabla_{\lambda^{(1)}} \ldots \nabla_{\lambda^{(m)}} S_u S_v \frac{\partial \psi_\Lambda}{\partial \theta}\| \leq C_m \ t \ \varepsilon^{\gamma \delta(\sigma(u), \sigma(v))} \inf \ (\sharp \sigma(u), \sharp \sigma(v)) \tag{7.17}$$

Proof. The new function $\varphi = S_u S_v \frac{\partial \psi_\Lambda(\theta)}{\partial \theta}$ satisfies an equation very similar to (7.13)

$$\frac{\partial \varphi}{\partial t} \ + \ \frac{x-y}{t} \ . \ \nabla_x \varphi - \frac{h^2}{2} \Delta_x \varphi \ + \ h^2 \ (\nabla_x \psi_\Lambda(\theta), \nabla_x \ \varphi) = \Phi \tag{7.18}$$

but the new RHS Φ is now defined by $\Phi = \Phi_1 - \Phi_2 - \Phi_3 - \Phi_4$, where

$$\Phi_1 = S_u S_v V_{\Lambda_1, \Lambda_2} \qquad \Phi_2 = h^2 \Big(\nabla_x S_u S_v \psi_\Lambda(\theta), \nabla_x \frac{\partial \psi_\Lambda(\theta)}{\partial \theta}(x+u+v, y+u+v, t) \Big) \tag{7.19}$$

$$\Phi_3 = h^2 \Big(\nabla_x S_u \psi_\Lambda(\theta), \nabla_x S_v \frac{\partial \psi_\Lambda(\theta)}{\partial \theta}(x+u, y+u, t) \Big) \qquad \Phi_4 = h^2 \Big(\nabla_x S_v \psi_\Lambda(\theta), \nabla_x S_u \frac{\partial \psi_\Lambda(\theta)}{\partial \theta}(x+v, y+v, t) \Big)$$

We have only to estimate the new Φ, setting now $\rho(E) = \delta(\sigma(u), \sigma(v))$. For each sequence $\lambda^{(1)}, \ldots, \lambda^{(m)}$ in Λ, $(m \geq 0)$, we can write, under the conditions of (7.11),

$$\|\nabla_{x_{\lambda^{(1)}}} \ldots \nabla_{x_{\lambda^{(m)}}} S_u V_{\Lambda_1, \Lambda_2}\| \leq K_m \varepsilon^{\gamma \delta(\sigma(u), \sigma(v))} \ \inf(\sharp \sigma(u), \sharp \sigma(v)). \tag{7.20}$$

By (7.3') and (7.3"), we can write, for each point ν in Λ and for each sequence $\lambda^{(1)}, \dots , \lambda^{(k)}$ in Λ, under the conditions of (7.11),

$$\|\nabla_{x_\nu}\nabla_{\lambda^{(1)}}...\nabla_{\lambda^{(k)}}S_uS_v\psi_\Lambda(\theta)\| \leq C_k t \inf\left[\varepsilon^{\frac{1+\gamma}{2}\delta(\nu,\sigma(u))}\ ,\ \varepsilon^{\frac{1+\gamma}{2}\delta(\sigma(u),\sigma(v))}\right] \leq C_k t\varepsilon^{\gamma\delta(\sigma(u),\sigma(v))+\frac{1-\gamma}{2}\delta(\nu,\sigma(u))}.$$

By (7.4), under the previous conditions, for each $\nu \in \Lambda$ and for each sequence $\mu^{(1)}, \dots \mu^{(\ell)}$ in Λ

$$\|\nabla_{x_\nu}\nabla_{\mu^{(1)}}...\nabla_{\mu^{(\ell)}}\frac{\partial\psi_\Lambda(\theta)}{\partial\theta}\| \leq C_k t$$

By multiplication and summation on $\nu \in \Lambda$, remarking that

$$\sum_{\nu\in\mathbb{Z}^d}\varepsilon^{\frac{1-\gamma}{2}\delta(\nu,\sigma(u))} \leq 4^d\sharp(\sigma(u)) \qquad \text{if} \quad \varepsilon^{1-\gamma} \leq 2^{-2d},$$

and applying the same argument to the vector v, we obtain, under the usual conditions, for each sequence $\lambda^{(1)}, \dots , \lambda^{(m)}$ in Λ,

$$\|\nabla_{x_{\lambda^{(1)}}}...\nabla_{x_{\lambda^{(m)}}}\Phi_2\| \leq K_m t^2h^2\varepsilon^{\gamma\delta(\sigma(u),\sigma(v))}\ \inf(\sharp\sigma(u),\sharp\sigma(v)) \tag{7.21}$$

By (7.3'), we can write, for each point ν in Λ and for each sequence $\lambda^{(1)}, \dots , \lambda^{(k)}$ in Λ, under the usual conditions,

$$\|\nabla_{x_\nu}\nabla_{\lambda^{(1)}}...\nabla_{\lambda^{(k)}}S_u\psi_\Lambda(\theta)\| \leq C_k t\varepsilon^{\frac{\gamma+1}{2}\delta(\nu,\sigma(u))}$$

By (7.11), we can write, for each $\nu \in \Lambda$ and for each sequence $\mu^{(1)}, \dots \mu^{(\ell)}$ in Λ

$$\|\nabla_{x_\nu}\nabla_{\mu^{(1)}}...\nabla_{\mu^{(\ell)}}\frac{\partial\psi_\Lambda(\theta)}{\partial\theta}\| \leq C_k t\varepsilon^{\frac{\gamma+1}{2}\delta(\nu,\sigma(v))}$$

By multiplication of the last two inequalities , and summation on $\nu \in \Lambda$, using (6.7), we obtain for Φ_3, and, permuting u and v, for Φ_4, estimations similar to (7.21). Then Lemma 7.4 follows from Proposition 3.1.

End of the proof of Proposition 7.2. We may assume that Q is defined like in (5.2).

Proof of (7.7). We may assume that $\mathrm{diam}(Q) = \beta_1 - \alpha_1$. If the sequence $(\lambda^{(1)}, ..., \lambda^{(m)})$ is in the case iii) of Proposition 5.2, (7.7) follows from (5.9) and (7.4) since $\delta(B_+^{(1)}(Q), B_-^{(1)}(Q)) = \mathrm{diam}(Q)$. If we are in the case ii) of Proposition 5.2, (7.7) follows from (5.8) and (7.11). In the case i) of Proposition 5.2, (7.7) follows from (5.7) and Lemma 7.4.(We remark that the sets $B_\pm^{(1)}(Q)$ of (5.6) satisfy $\sharp(B_\pm^{(1)}(Q)) \leq (\mathrm{diam}(Q))^{d-1}$ and we have to change the coefficient γ).

Proof of (7.8). We may assume that $\sup_{j\leq m}\delta(\lambda^{(j)}, Q) = \delta(\lambda^{(1)}, B_+^{(1)}(Q))$. If we are in the case iii) of Proposition 5.2, (7.8) follows from (5.9) and (7.4). In the case ii), it follows from (5.8) and (7.4), and in the case i), from (5.7) and (7.11).

Proof of (7.10). In the case iii) of Proposition 5.2, (7.10) follows from (5.9) and (7.5), in the case ii) from (5.8) and (7.5), and in the case i) from (5.7) and (7.12) since, in the situation (7.9), we have $\delta(B_+^{(1)}(Q), \Lambda_2) \geq b_1 - \beta_1$.

8. Semi-classical approximation and global behavior of the heat kernel

Our first goal is to determine the first term of a possible semiclassical asymptotic expansion of the function $\psi_\Lambda(x, y, t, h, \varepsilon)$ of Theorem 1.1. For each finite set Λ of $\mathbb{Z}^d$, we consider the following function

$$\psi_\Lambda^{(0)}(x, y, t, \varepsilon) = t\int_0^1 V_{\Lambda,\varepsilon}(y + \theta(x - y))d\theta. \tag{8.1}$$

We shall prove that it is the first term of the semi-classical approximation of ψ_Λ, and give a bound of derivatives of the error term.

Proposition 8.1. *There exists $C > 0$ with the following properties. For each finite set Λ of $\mathbb{Z}^d$, if ψ_Λ is the solution of (1.16), (1.17), and $\psi_\Lambda^{(0)}$ the function defined above, we have:*

$$\sup_{\lambda\in\Lambda} \|\nabla_\lambda(\psi_\Lambda - \psi_\Lambda^{(0)})\| \leq Ch^2(t^2+t^3) \qquad \text{if} \quad ht \leq \sigma_0 \quad \text{and} \quad \varepsilon \leq 4^{-d} \tag{8.2}$$

where σ_0 is the constant of Theorem 1.1.

Proof. We see that

$$\frac{\partial \psi_\Lambda^{(0)}}{\partial t} + \frac{x-y}{t} \cdot \nabla_x \psi_\Lambda^{(0)} = V_{\Lambda,\varepsilon} \tag{8.3}$$

and therefore that the function $\varphi = \psi_\Lambda - \psi_\Lambda^{(0)}$ satisfies the equation

$$\frac{\partial \varphi}{\partial t} + \frac{x-y}{t} \cdot \nabla_x \varphi - \frac{h^2}{2}\Delta_x \varphi = F \qquad F = \frac{h^2}{2}\Delta_x \psi_\Lambda^{(0)} - \frac{h^2}{2}|\nabla_x \psi_\Lambda|^2$$

We see that, for each λ in Λ, we can write, by Theorem 1.1, applied with $m \leq 2$ and, for example, $\gamma = \frac{1}{2}$, $\|\nabla_\lambda F(.,t)\| \leq K_m h^2(t+t^2)$ if the conditions of (8.2) are satisfied. Therefore, applying Proposition 3.1 with $\rho(E) = 0$ and $m = 1$, (then we use the hypothesis on F only for $m = 1$), we obtain (8.2).

If E is a subset of Λ, we denote by x_E the family of variables $(x_\lambda)_{\lambda\in E}$, and we denote obviously the variable of $(\mathbb{R}^p)^\Lambda$ by $x = (x_E, x_{\Lambda\setminus E})$.

In the rest of this section, we shall be sure that, for $t > 0$, the integral kernel $U_\Lambda(x,y,t)$ is in $\mathcal{S}((\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda)$ (which is certainly already well known), and then we shall estimate expectations of the form (1.14), where A is the multiplication by a polynomially bounded function, depending only, with the previous notations, of x_E, where E is a subset of Λ with a fixed number of elements, (while $\sharp(\Lambda)$ may tend to $+\infty$).

In the following Proposition, the ℓ^1 norm of a vector x, denoted by $|x|_1$, will play an important role. The answer to our question will be given by the following proposition.

Proposition 8.2. *With these notations, if ε is small enough, for each $t > 0$ and $h > 0$, the function $U_\Lambda(.,.,t)$ is in $\mathcal{S}((\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda)$. Moreover, there exist $\sigma_1 > 0$, $\varepsilon_1 > 0$ and, for each integers m and N, a function $t \to C(t,m,N) > 0$, bounded on each compact of $]0,+\infty[$, such that, for each subset E of a box Λ, for each $t > 0$, $h > 0$ and $\varepsilon > 0$ such that $h^2(t+t^2) \leq \sigma_1$ and $\varepsilon \leq \varepsilon_1$, we have*

$$\int_{(\mathbb{R}^p)^\Lambda} (1+|x_E|_1)^m U_\Lambda(x,x,t,h,\varepsilon)dx \leq C(t,m,\sharp E)\mathrm{Tr}(e^{-tH_\Lambda(\varepsilon)}). \tag{8.4}$$

The proof of this Proposition relies on the following Lemma.

Lemma 8.3. *There exists $\kappa > 0$, $C > 0$, $\varepsilon_1 > 0$ and $\sigma_1 > 0$ such that, for each subset E of Λ, we have*

$$\psi_\Lambda(x,y,t) \geq \psi_\Lambda(0,x_{\Lambda\setminus E},0,y_{\Lambda\setminus E},t) + \kappa t(|x_E|_1 + |y_E|_1) - Ct(\sharp E) \tag{8.5}$$

if $\varepsilon \leq \varepsilon_0$, $h^2(t+t^2) \leq \sigma_1$.

Proof of Lemma 8.3. First step. We shall prove the analogous of (8.5) for $\psi_\Lambda^{(0)}$, i.e. that there exists $\kappa > 0$, $C > 0$ and $\varepsilon_0 > 0$, such that, for each subset E of a box Λ of $\mathbb{Z}^d$, we have

$$\psi_\Lambda^{[0]}(x, y, t) \geq \psi_\Lambda^{(0)}(0, x_{\Lambda\setminus E}, 0, y_{\Lambda\setminus E}, t) + 2\kappa t(|x_E|_1 + |y_E|_1) - Ct(\sharp E) \qquad \text{if } \ \varepsilon \leq \varepsilon_0. \tag{8.6}$$

If A and B_Λ are the functions appearing in (1.2), we set

$$\widetilde{A}(x, y) = \int_0^1 A(y + \theta(x-y))\ d\theta \qquad \widetilde{B_\lambda}(x, y) = \int_0^1 B_\lambda(y + \theta(x-y))\ d\theta \tag{8.7}$$

Therefore, we can write :

$$\psi_\Lambda^{(0)}(x, y, t) - \psi_\Lambda^{(0)}(0, x_{\Lambda\setminus E}, 0, y_{\Lambda\setminus E}, t) = t(F_1 + F_2) \tag{8.8}$$

where

$$F_1 = \sum_{\lambda\in E} [\widetilde{A}(x_\lambda, y_\lambda) - \widetilde{A}(0, 0)] \tag{8.9}$$

$$\begin{aligned} F_2 = & \sum_{\lambda\in E,\ \mu\in\Lambda\setminus E} \varepsilon^{|\lambda-\mu|} \Big(\widetilde{B}_{\lambda-\mu}(x_\lambda, x_\mu, y_\lambda, y_\mu) - \widetilde{B}_{\lambda-\mu}(0, x_\mu, 0, y_\mu)\Big) + ... \\ & + \sum_{\lambda\in\Lambda\setminus E,\ \mu\in E} \varepsilon^{|\lambda-\mu|} \Big(\widetilde{B}_{\lambda-\mu}(x_\lambda, x_\mu, y_\lambda, y_\mu) - \widetilde{B}_{\lambda-\mu}(x_\lambda, 0, y_\lambda, 0)\Big) + ... \\ & ... + \sum_{\substack{\lambda,\mu\in E \\ \lambda\neq\mu}} \varepsilon^{|\lambda-\mu|} \Big(\widetilde{B}_{\lambda-\mu}(x_\lambda, x_\mu, y_\lambda, y_\mu) - \widetilde{B}_{\lambda-\mu}(0)\Big) \end{aligned} \tag{8.10}$$

There exists some K, depending only on p, such that

$$|x| + |y| \leq K \int_0^1 |y + \theta(x-y)| d\theta \qquad \forall (x, y) \in \mathbb{R}^p \times \mathbb{R}^p. \tag{8.11}$$

By the hypotheses on A, we can write, for some constants $C_1 > 0$ and $C_2 > 0$, $|x| \leq C_1(A(x) - A(0)) + C_2$ for all $x \in \mathbb{R}^p$, and therefore, by (8.7), (8.9) and (8.11):

$$|x_E|_1 + |y_E|_1 \leq K\big[C_1 F_1(x, y) + C_2 \sharp(E)\big]. \tag{8.12}$$

By (8.10), there exists $C_3 > 0$ such that $|F_2(x, y)| \leq C_3\ \varepsilon(|x_E|_1 + |y_E|_1)$ if $\varepsilon \leq 1/2$. Therefore, by (8.8) and (8.12), under the same condition:

$$t(|x_E|_1 + |y_E|_1) \leq KC_1\Big[\psi_\Lambda^{(0)}(x, y, t) - \psi_\Lambda^{(0)}(0, x_{\Lambda\setminus E}, 0, y_{\Lambda\setminus E}, t)\Big] + KC_1C_3\varepsilon t(|x_E|_1 + |y_E|_1) + KC_2 t(\sharp E).$$

If $KC_1C_3\varepsilon \leq \frac{1}{2}$, the inequality (8.6) follows.

Second step. If $\psi_\Lambda^{(0)}$ is the classical approximation defined in (8.1), we have, by Proposition 8.1 (inequality (8.2)), if $ht \leq \sigma_0$ and $\varepsilon \leq 4^{-d}$,

$$\Big|(\psi_\Lambda - \psi_\Lambda^{(0)})(x_E, x_{\Lambda\setminus E}, y_E, x_{\Lambda\setminus E}, t) - (\psi_\Lambda - \psi_\Lambda^{(0)})(0, x_{\Lambda\setminus E}, 0, x_{\Lambda\setminus E}, t)\Big| \leq Ch^2(t^2 + t^3)(|x_E|_1 + |y_E|_1).$$

The lemma follows from (8.6) and from this inequality.

Proof of Proposition 8.2. First claim. For any $h > 0$ and $\varepsilon \leq \varepsilon_0$, we can find $t_1 > 0$ such that the conditions of Lemma 8.3 are satisfied if $0 < t \leq t_1$. Therefore, by (1.3), by Theorem 1.1 and Lemma 8.3, the function $U_\Lambda(.,.,t)$ is in $\mathcal{S}((\mathbb{R}^p)^\Lambda \times (\mathbb{R}^p)^\Lambda)$ if $0 < t < t_1$. The same result follows for each t by composition.

Second claim. Let h, t and ε satisfying the conditions of Lemma 8.3. Then, we have

$$I := \int_{(\mathbb{R}^p)^\Lambda} (1+|x_E|_1)^m U_\Lambda(x,x,t,h,\varepsilon)dx = (2\pi t h^2)^{-(p\sharp\Lambda)/2} \int_{(\mathbb{R}^p)^\Lambda} (1+|x_E|_1)^m \; e^{-\psi_\Lambda(x,x,t,h,\varepsilon)} dx \leq ...$$

$$\ldots \leq (2\pi t h^2)^{-(p\sharp\Lambda)/2} \int_{(\mathbb{R}^p)^\Lambda} (1+|x_E|_1)^m \; e^{-\psi_\Lambda(0,x_{\Lambda\setminus E},0,x_{\Lambda\setminus E},t,h,\varepsilon)-\kappa t|x_E|_1 + Ct\sharp(E)} dx.$$

By Theorem 1.1, there exists $\kappa_1 > 0$ such that

$$|\psi_\Lambda(x,x,t) - \psi_\Lambda(0,x_{\Lambda\setminus E},0,x_{\Lambda\setminus E},t)| \leq \kappa_1 t |x_E|_1. \tag{8.13}$$

If we set

$$\Psi_1(t,m,\sharp(E)) = \int_{(\mathbb{R}^p)^E} (1+|x_E|_1)^m e^{Ct\sharp(E)-\kappa t|x_E|_1} dx_E \qquad \Psi_2(t,\sharp(E)) = \int_{(\mathbb{R}^p)^E} e^{-\kappa_1 t|x_E|_1} dx_E, \tag{8.14}$$

we can write

$$I \leq (2\pi t h^2)^{-(p\sharp\Lambda)/2} \; \frac{\Psi_1(t,m,\sharp(E))}{\Psi_2(t,\sharp(E))} \; \int_{(\mathbb{R}^p)^\Lambda} e^{-\psi_\Lambda(0,x_{\Lambda\setminus E},0,x_{\Lambda\setminus E},t)-\kappa_1 t|x_E|} dx.$$

By (8.13), it follows that

$$I \leq (2\pi t h^2)^{-(p\sharp\Lambda)/2} \; \frac{\Psi_1(t,m,\sharp(E))}{\Psi_2(t,\sharp(E))} \; \int_{(\mathbb{R}^p)^\Lambda} e^{-\psi_\Lambda(x,x,t)} dx = \frac{\Psi_1(t,m,\sharp(E))}{\Psi_2(t,\sharp(E))} \; \mathrm{Tr}(e^{-tH_\Lambda(\varepsilon)}).$$

The proposition is proved.

9. Quantum correlation of local multiplicative observables.

The aim of this section is the proof of Theorem 1.5 in the case where A and B are multiplication by bounded or, more generally, polynomially bounded functions. More precisely, for each box Λ in $\mathbb{Z}^d$, for each disjoint finite subsets E_1 and E_2, if A is the multiplication by a function $f \in C^\infty((\mathbb{R}^p)^{E_1})$, and B is the multiplication by a function $g \in C^\infty((\mathbb{R}^p)^{E_2})$, satisfying, for some constant $N(f)$ and $N(g)$

$$|f(x)| \leq N(f)(1+|x_{E_1}|)^{m_1} \qquad |g(x)| \leq N(g)(1+|x_{E_2}|)^{m_2}, \tag{9.1}$$

we shall estimate the correlation $cov_{\Lambda,\varepsilon}(A,B)$ defined in (1.10), (1.11).

We shall distinguish the case $m_1 = m_2 = 0$ because the constants are then more explicit. We set $m(t) = \inf(t,1)$ and $M(t) = \sup(t,1)$.

Proposition 9.1. *With the previous notations, there exists a function $\gamma \to \varepsilon_0(\gamma)$ in $]0,1[$, such that,*
a) if the conditions (9.1) are satisfied with $m_1 = m_2 = 0$, and if $\gamma \in]0,1[$, we have

$$cov_{\Lambda,\varepsilon}(A,B) \leq 4 \; m(t) \; \inf(\sharp(E_1),\sharp(E_2)) \; (M(t)\varepsilon^\gamma)^{\delta(E_1,E_2)} \; N(f) \; N(g) \tag{9.2}$$

if the following conditions are satisfied:

$$ht \leq \sigma_0, \qquad \varepsilon \leq \varepsilon_0(\gamma) \qquad M(t)\varepsilon^\gamma \leq \frac{1}{2}, \tag{9.3}$$

where σ_0 is the constant of Theorem 1.1.

b) There exists $\sigma_1 > 0$, independent of all the parameters, and, for each integers m_1, m_2, N_1 and N_2, a function $t \to K(m_1, m_2, N_1, N_2, t)$, bounded on each compact set of $]0, +\infty[$, such that, if the conditions (9.1) are satisfied with $m_1 \geq 0$ and $m_2 \geq 0$, if $\gamma \in]0, 1[$, we have

$$cov_{\Lambda,\varepsilon}(A, B) \leq m(t)\ K(m_1, m_2, \sharp(E_1), \sharp(E_2), t)\ (M(t)\varepsilon^{\gamma})^{\delta(E_1,E_2)} N(f)N(g) \tag{9.4}$$

if the following conditions are satisfied:

$$h^2(t + t^2) \leq \sigma_1, \qquad \varepsilon \leq \varepsilon_0(\gamma), \qquad M(t)\varepsilon^{\gamma} \leq \frac{1}{2}. \tag{9.5}$$

If $m_1 = 0$ and $m_2 > 0$, we can take K independent of $\sharp E_1$.

If, in the definition of $H_{\Lambda\varepsilon}$, the potential $V_{\Lambda\varepsilon}$ is replaced by a family, depending on an auxiliary parameter, such that the hypotheses of the introduction are satisfied uniformly, then the constants in Proposition 9.1 can be chosen independent of this parameter.

The first step of the proof of Proposition 9.1, which will also used in Section 10 for the case where A and B may be arbitrary bounded operators, will be an expression of the quantum correlation defined in (1.11), using operators in $L^2((\mathbb{R}^p)^{\Lambda} \times (\mathbb{R}^p)^{\Lambda})$.

Denoting by $X = (x', x'')$ the variable of $(\mathbb{R}^p)^{\Lambda} \times (\mathbb{R}^p)^{\Lambda}$, for each operator A in $L^2((\mathbb{R}^p)^{\Lambda})$, we denote by A' (resp. A'') the operator A, seen as an operator in $L^2((\mathbb{R}^p)^{\Lambda} \times (\mathbb{R}^p)^{\Lambda})$, acting only on the variable x' (resp. on x''). We denote by $\widetilde{H}_{\Lambda,\varepsilon}$ the operator in $(\mathbb{R}^p)^{\Lambda} \times (\mathbb{R}^p)^{\Lambda}$ defined by $\widetilde{H_{\Lambda,\varepsilon}} = H'_{\Lambda,\varepsilon} + H''_{\Lambda,\varepsilon}$. Then, an easy computation shows that

$$cov_{\Lambda,\varepsilon}(A, B) = \frac{1}{2}\ \frac{Tr\ \left(e^{-t\widetilde{H}_{\Lambda,\varepsilon}}(A' - A'')(B' - B'')\right)}{Tr\ \left(e^{-t\widetilde{H}_{\Lambda,\varepsilon}}\right)} \tag{9.6}$$

In the second step, we shall replace, in the numerator, the operator $e^{-t\widetilde{H}_{\Lambda,\varepsilon}}$ by another one, denoted by $T(E_1, E_2)$, which will satisfy better estimates. In order to define this new operator, we need some notations.

Notations. For each disjoint subsets E_1 and E_2 of Λ, let $G_+(E_1, E_2)$ and $G_-(E_1, E_2)$ the sets of maps τ from Λ to $\{0, 1\}$ defined by

$$\begin{aligned} G_+(E_1, E_2) &= \{\tau : \Lambda \longrightarrow \{0, 1\} \quad |\tau(\lambda) = 0\ \ \forall \lambda \in E_1, \quad \tau(\lambda) = 0\ \ \forall \lambda \in E_2\} \\ G_-(E_1, E_2) &= \{\tau : \Lambda \longrightarrow \{0, 1\}, \quad |\tau(\lambda) = 0\ \ \forall \lambda \in E_1, \quad \tau(\lambda) = 1\ \ \forall \lambda \in E_2\}. \end{aligned} \tag{9.7}$$

For each $\tau \in G_{\pm}(E_1, E_2)$, we denote by ϕ_τ the map in $(\mathbb{R}^p)^{\Lambda} \times (\mathbb{R}^p)^{\Lambda}$ defined by

$$\Big(\phi_\tau(x', x'')\Big)_{\lambda} = \begin{cases} (x'_{\lambda}, x''_{\lambda}) & if \quad \tau(\lambda) = 0 \\ (x''_{\lambda}, x'_{\lambda}) & if \quad \tau(\lambda) = 1 \end{cases} \qquad \forall X = (x', x'') \in (\mathbb{R}^p)^{\Lambda} \times (\mathbb{R}^p)^{\Lambda} \tag{9.8}$$

Let Φ_τ the corresponding operator in $L^2((\mathbb{R}^p)^{\Lambda} \times (\mathbb{R}^p)^{\Lambda})$. For each operator S in $L^2((\mathbb{R}^p)^{\Lambda} \times (\mathbb{R}^p)^{\Lambda})$, for each function $F(X)$ in $\Big((\mathbb{R}^p)^{\Lambda}\Big)^2$ or $G(X, Y)$ in $\Big((\mathbb{R}^p)^{\Lambda}\Big)^4$, and for each $\tau \in G_{\pm}$, we set

$$S^{(\tau)} = (\Phi_\tau)^{-1} S \Phi_\tau, \qquad F^{(\tau)}(X) = F(\phi_\tau(X)), \qquad G^{(\tau)}(X, Y) = F(\phi_\tau(X), \phi_\tau(Y)). \tag{9.9}$$

If $U_\Lambda(x, y, t)$ is the heat kernel of Theorem 1.1, and $\psi_\Lambda(x, y, t)$ the function appearing in its expression (1.3), we set, if $X = (x', x'')$, $Y = (y', y'')$

$$\widetilde{U}_{\Lambda}(X, Y, t) = U_{\Lambda}(x', y', t)\ U_{\Lambda}(x'', y'', t) \qquad \widetilde{\psi}_{\Lambda}(X, Y, t) = \psi_{\Lambda}(x', y', t) + \psi_{\Lambda}(x'', y'', t).$$

Therefore we have

$$\widetilde{U}_\Lambda(X,Y,t) = (2\pi h t^2)^{-p\sharp(\Lambda)} e^{-\frac{|X-Y|^2}{2th^2}} e^{-\widetilde{\psi}_\Lambda(X,Y,t)} \tag{9.10}$$

Proposition 9.2. *We can write, if E_1 and E_2 are disjoint subsets of a box Λ of $\mathbb{Z}^d$, if $A$and B are either multiplications by functions depending only on x_{E_1} and x_{E_2} and satisfying (9.1), or bounded operators in $\mathcal{L}(\mathcal{H}_{E_1})$ and $\mathcal{L}(\mathcal{H}_{E_2})$:*

$$cov_{\Lambda,\varepsilon}(A,B) = \frac{1}{2}\ \frac{Tr\ \Big(T(E_1,E_2)(A'-A'')(B'-B'')\Big)}{Tr\ \Big(e^{-t\widetilde{H}_{\Lambda,\varepsilon}}\Big)} \tag{9.11}$$

where

$$T(E_1,E_2) = \frac{1}{2\ \sharp(G_+(E_1,E_2))}\left[\sum_{\tau\in G_+(E_1,E_2)} e^{-t\widetilde{H}^\tau} - \sum_{\tau\in G_-(E_1,E_2)} e^{-t\widetilde{H}^\tau}\right]. \tag{9.12}$$

Moreover, the integral kernel K_{E_1,E_2} of the operator $T(E_1,E_2)$ defined in (9.12) can be written

$$K_{E_1,E_2}(X,Y,t) = \frac{1}{2\ \sharp(G_+(E_1,E_2))}\sum_{\tau\in(G_+\cup G_-)(E_1,E_2)} \mathrm{sgn}(\tau)\widetilde{U}_\Lambda^{(\tau)}(X,Y,t) \tag{9.13}$$

where $\mathrm{sgn}(\tau)=\pm 1$ *if* $\tau\in G_\pm(E_1,E_2)$.

Proof. The numerator of the fraction in (9.6), i.e.

$$N(\varepsilon) := \mathrm{Tr}\ \Big(e^{-t\widetilde{H}_{\Lambda,\varepsilon}}(A'-A'')(B'-B'')\Big)$$

satisfies

$$N(\varepsilon) = \mathrm{sgn}(\tau)Tr\ \Big(e^{-t\widetilde{H}^\tau_{\Lambda,\varepsilon}}(A'-A'')(B'-B'')\Big) \qquad \forall\tau\in(G_+\cup G_-)(E_1,E_2)$$

and therefore, since $\sharp\big(G_+(E_1,E_2)\big)=\sharp\big(G_-(E_1,E_2)\big)$,

$$N(\varepsilon) = \mathrm{Tr}\ \Big(T(E_1,E_2)(A'-A'')(B'-B'')\Big), \tag{9.14}$$

the equalities (9.11) and (9.12) follow, and the expression of the integral kernel is direct since $\widetilde{U}^\tau_\Lambda(X,Y,t)$ is the heat kernel of $\exp(-t\widetilde{H}^\tau_{\Lambda,\varepsilon})$.

The next step will be another expression of $K_{E_1,E_2}(X,X)$ the restriction to the diagonal of the integral kernel (9.13) of the operator $T(E_1,E_2)$ defined in (9.12), using the cluster decomposition of Section 5. For that, we need again some notations.

Notations. For each box $Q\subset\Lambda$, we write obviously

$$(T_Q\widetilde{\psi}_\Lambda)(X,Y,t) = (T_Q\psi_\Lambda)(x',y',t)+(T_Q\psi_\Lambda)(x'',y'',t) \qquad \text{if} \quad X=(x',x''), \qquad Y=(y',y'').$$

For each box Q, let us set

$$M_Q(t) = \sup_{X\in(\mathbb{R}^p)^Q} T_Q\widetilde{\psi}_\Lambda(X,X,t) \qquad g_Q(X,t) = e^{M_Q(t)-(T_Q\widetilde{\psi}_\Lambda)(X,X,t)}-1 \tag{9.15}$$

We denote by $\mathcal{A}(\Lambda)$ the following set: the elements of $\mathcal{A}(\Lambda)$ are the (possibly empty) sets of boxes, contained in Λ, and of diameter ≥ 1. For each set $A \in \mathcal{A}(\Lambda)$, we set

$$G_A(X,t) = \prod_{Q\in A} g_Q(X,t). \tag{9.16}$$

If the set A is empty, we set $G_A(X,t) = 1$. We set

$$\Phi_0(X,t) = \widetilde{\psi}_\Lambda(0,0,t) + \sum_{\mathrm{diam}(Q)=0} (T_Q\widetilde{\psi}_\Lambda)(X,X,t) \ + \sum_{\mathrm{diam}(Q)\geq 1} M_Q(t), \tag{9.17}$$

where the sums are taken on all boxes contained in Λ.

Now, if E_1 and E_2 are subsets of a box Λ of $\mathbb{Z}^d$, we denote by $\mathcal{A}_0(E_1,E_2,\Lambda)$ the following set: the elements of $\mathcal{A}_0(E_1,E_2,\Lambda)$ are the (possibly empty) sets of boxes contained in Λ, with diameter ≥ 1, containing a sequence $(Q_1,...,Q_p)$ $(p\geq 1)$ of boxes connecting E_1 and E_2, i.e. such that $E_1\cap Q_1 \neq \emptyset$, $Q_1\cap Q_2\neq\emptyset$, . . . $Q_{p-1}\cap Q_p\neq\emptyset$, and $Q_p \cap E_2 \neq \emptyset$.

Proposition 9.3. *Under the hypotheses of Proposition 9.2, and with these notations, we have*

$$K_{E_1,E_2}(X,X,t) = \frac{(2\pi t h^2)^{-p|\Lambda|}e^{-\Phi_0(X,t)}}{2\ \sharp(G_+(E_1,E_2))} \sum_{\tau\in(G_+\cup G_-)(E_1,E_2)} \mathrm{sgn}(\tau) \sum_{A\in\mathcal{A}_0(E_1,E_2,\Lambda)} (G_A)^{(\tau)}(X,t). \tag{9.18}$$

Proof. Since each function $(T_Q\psi_\Lambda)(x,x,t)$ depends only on the variables x_λ such that $\lambda\in Q$, (by Proposition 5.1 i)), we have the following implication, if τ and τ' are maps from Λ to $\{0,1\}$, with the notation (9.9),

$$\left.\begin{array}{c} \tau'(\lambda)=\tau(\lambda)\ \ \forall\lambda\in Q \\ or \\ \tau'(\lambda) = 1-\tau(\lambda)\ \ \forall\lambda\in Q \end{array}\right\} \Longrightarrow (T_Q\widetilde{\psi}_\Lambda)^{(\tau)}(X,X,t) = (T_Q\widetilde{\psi}_\Lambda)^{(\tau')}(X,X,t) \tag{9.19}$$

By (9.10), (9.15) and Proposition 5.1 (point ii), we can write

$$\widetilde{U}_\Lambda(X,X,t) = (2\pi t h^2)^{-p\sharp(\Lambda)}e^{-\Phi_0(X,t)} \prod_{\mathrm{diamQ}\geq 1}(1+g_Q(X,t)) \tag{9.20}$$

It follows that

$$\widetilde{U}_\Lambda(X,X,t) = (2\pi t h^2)^{-p\sharp(\Lambda)}e^{-\Phi_0(X,t)} \sum_{A\in\mathcal{A}(\Lambda)} G_A(X,t) \tag{9.21}$$

where $\mathcal{A}(\Lambda)$ is the set defined before the statement of the proposition. By (9.19) and Proposition 5.1 i), Φ_0, defined in (9.17), is not modified by the action of ϕ_τ $(\tau\in G_\pm(E_1,E_2))$, and we can write, by (9.13) and (9.21)

$$K_{E_1,E_2}(X,X,t) = \frac{(2\pi t h^2)^{-p\ \sharp(\Lambda)}e^{-\Phi_0(X,t)}}{2\ \sharp(G_+(E_1,E_2))} \sum_{\tau\in(G_+\cup G_-)(E_1,E_2)} sgn(\tau) \sum_{A\in\mathcal{A}(\Lambda)} (G_A)^{(\tau)}(X,t) \tag{9.22}$$

Let us show that, if $A\in\mathcal{A}$, but if $A\notin\mathcal{A}_0(E_1,E_2,\Lambda)$, we have

$$\sum_{\tau\in(G_+\cup G_-)(E_1,E_2)} sgn(\tau)(G_A)^{(\tau)}(X,t) = 0. \tag{9.23}$$

Let us denote by $\widehat{E}_2(A)$ the set of all points $\lambda \in \Lambda$ which are either in E_2, or connected to E_2 by a sequence of boxes in A, in other words, such that there exists a sequence of boxes $Q_1 \in A, \ . \ . \ . \ Q_p \in A$ $(p \geq 1)$, such that $E_2 \cap Q_1 \neq \emptyset$, $Q_j \cap Q_{j+1} \neq \emptyset$ $(1 \leq j \leq p-1)$, and $\lambda \in Q_p$. For each $\tau \in G_+(E_1, E_2)$, we define a map $b_A(\tau)$ from Λ to $\{0,1\}$ by by

$$\Big(b_A(\tau)\Big)(\lambda) = \begin{cases} 1-\tau(\lambda) & if \quad \lambda \in \widehat{E}_2(A) \\ \tau(\lambda) & if \quad \lambda \notin \widehat{E}_2(A) \end{cases}$$

Let us prove that, with our hypotheses on A, for each $\tau \in G_+(E_1, E_2)$, we have

$$G_A^{(\tau)}(X,t) = G_A^{(b_A(\tau))}(X,t) \tag{9.24}$$

For each box $Q \in A$, either Q is disjoint from $\widehat{E}_2(A)$, and we have $(b_A(\tau))(\lambda) = \tau(\lambda)$ for all $\lambda \in Q$, or it is contained in $\widehat{E}_2(A)$, and we have $(b_A(\tau))(\lambda) = 1 - \tau(\lambda)$ for all $\lambda \in Q$. By (9.19), for each box $Q \in A$, we have $g_Q^{(\tau)}(X,t) = g_Q^{(b_A(\tau))}(X,t)$, and (9.24) follows. If $\tau \in G_+(E_1, E_2)$, $b_A(\tau)$ is in $G_-(E_1, E_2)$ since E_2 is contained in $\widehat{E}_2(A)$ and E_1 is disjoint from $\widehat{E}_2(A)$ (otherwise A would be in $\mathcal{A}_0(E_1, E_2, \Lambda)$). Since the map b_A is a bijection between $G_+(E_1, E_2)$ and $G_-(E_1, E_2)$, the equality (9.23) follows from (9.24). Therefore, the Proposition follows from (9.22) and (9.23), valid if $A \in \mathcal{A}(\Lambda) \setminus \mathcal{A}_0(E_1, E_2, \Lambda)$.

For the proof of Proposition 9.1, we need estimations for $K^{\Delta}(X,t) = K_{E_1,E_2}(X,X,t)$, but in the next section, we shall need also estimations of its derivatives, and we shall give in the next Proposition the needed estimations, with the following notations.

For each subset E of a box Λ of $\mathbb{Z}^d$, for each function F, C^∞ on $(\mathbb{R}^{2p})^\Lambda$, and for each integer $m \geq 0$, we set

$$|F|_E^{(m)}(X) = \sup_{0 \leq k \leq m} \ \sup_{(\lambda^{(1)},\dots,\lambda^{(k)}) \in E^k} |\nabla_{\lambda^{(1)}} \dots \nabla_{\lambda^{(k)}} F(X)| \tag{9.25}$$

and, if F is bounded such as all its derivatives,

$$\|F\|_E^{(m)} = \sup_{X \in (\mathbb{R}^{2p})^\Lambda} |F|_E^{(m)}(X) \tag{9.26}$$

Let us recall that $m(t) = \inf(t,1)$ and $M(t) = \sup(t,1)$.

Proposition 9.4. *For each integers $m \geq 0$ and $N \geq 0$, there exist a constant $C_{m,N}$, and a function $\gamma \to \varepsilon_0(\gamma, m, N)$ from $]0,1[$ in itself such that, for each subsets E, E_1 and E_2 of a finite box $\Lambda$of $\mathbb{Z}^d$ such that E_1 and E_2 are disjoint, we can write*

$$|K^{\Delta}_{E_1,E_2}|_E^{(m)}(X,t) \leq \dots \tag{9.27}$$

$$\dots \leq m(t)\ M(t)^m \ \ C_{m,\sharp(E)}\big(M(t)\varepsilon^\gamma\big)^{\delta(E_1,E_2)} \ \frac{\inf(\sharp(E_1), \sharp(E_2))}{2\ \sharp(G_+(E_1,E_2))} \sum_{\tau \in (G_+ \cup G_-)(E_1,E_2)} \widetilde{U}_\Lambda^\tau(X,X,t,h,\varepsilon)$$

if

$$ht \leq \sigma_0, \qquad M(t)\varepsilon^\gamma \leq 1/2, \qquad \varepsilon \leq \varepsilon_0(\gamma, m, \sharp(E)). \tag{9.28}$$

Notations. Let us denote by $\mathcal{S}(E_1, E_2, \Lambda)$ the set of $A \in \mathcal{A}(\Lambda)$ whose elements can be ordered in a finite sequence $Q_1, ..\ Q_p$ $(p \geq 1)$, satisfying $E_1 \cap Q_1 \neq \emptyset$, $Q_1 \cap Q_2 \neq \emptyset, \ . \ . \ . \ Q_{p-1} \cap Q_p \neq \emptyset$, and $Q_p \cap E_2 \neq \emptyset$. (Therefore, a set $A \in \mathcal{A}(\Lambda)$ is in $\mathcal{A}_0(E_1, E_2, \Lambda)$ if and only if it contains a set in $\mathcal{S}(E_1, E_2, \Lambda)$). We denote by $\mathcal{B}(E, \Lambda)$ the set of $A \in \mathcal{A}(\Lambda)$ whose all elements have a non void intersection with E, and by by $\mathcal{C}(E, \Lambda)$

the set of $A \in \mathcal{A}(\Lambda)$ whose all elements have a void intersection with E. We shall denote by K_m a constant such that:

$$|uv|_E^{(m)}(X) \leq K_m |u|_E^{(m)}(X) |v|_E^{(m)}(X).$$

Proof of Proposition 9.4. By (9.18), the proposition will follow if we prove that, for some constant $C_{m,\sharp(E)}$ and $\varepsilon_0(\gamma, m, \sharp(E))$, independent on τ, we have, for each $\tau \in (G_+ \cup G_-)(E_1, E_2)$, under the conditions (9.28),

$$(2\pi t h^2)^{-p\ \sharp(\Lambda)} \sum_{A \in \mathcal{A}_0(E_1,E_2)} |e^{-\Phi_0(.,t)} (G_A)^{(\tau)}(.,t)|_E^{(m)}(X) \leq \tag{9.29}$$

$$\leq C_{m,\sharp(E)} \varepsilon^{\gamma \delta(E_1,E_2)}\ \inf(\sharp(E_1), \sharp(E_2))\ \widetilde{U}_\Lambda^{(\tau)}(X, X, t).$$

For each sets $A \in \mathcal{S}(E_1, E_2)$, $B \in \mathcal{B}(E)$ and $C \in \mathcal{C}(E)$, the set $A \cup B \cup C$ is in $\mathcal{A}_0(E_1, E_2)$. All the elements of $\mathcal{A}_0(E_1, E_2)$ can be obtained in this form, perhaps in several ways, and we have $G_{A\cup B\cup C} = G_A G_B G_C$ if A, B and C are disjoint. Remarking that all functions G_A are ≥ 0, and that, if $C \in \mathcal{C}(E)$, $G_C^{(\tau)}(X,t)$ is independent of the variable X_E, it follows that

$$\sum_{A \in \mathcal{A}_0(E_1,E_2)} |e^{-\Phi_0(.,t)} (G_A)^{(\tau)}|_E^{(m)}(X,t) \leq \tag{9.30}$$

$$\leq K_m^2 |e^{-\Phi_0(.,t)}|_E^{(m)}(X) \sum_{A,B,C} |(G_A)^{(\tau)}\ |_E^{(m)}(X,t)\ |(G_B)^{(\tau)}|_E^{(m)}(X,t)\ (G_C)^{(\tau)}(X,t),$$

where the sum is over all the triples (A, B, C), where A, B and C are disjoint sets of boxes in Λ such that A is in $\mathcal{S}(E_1, E_2)$, B in $\mathcal{B}(E)$ and C in $\mathcal{C}(E)$.

By Proposition 6.1, we can write $|e^{-\Phi_0(.,t)}|_E^{(m)}(X) \leq C_m\ M(t)^m\ e^{-\Phi_0(X,t)}$. By Proposition 6.1, point i), and by the definition (9.16) of G_A, for any sequence of boxes $(Q_1, ..\ Q_N)$ contained in Λ, not reduced to single points, we can write, if $ht \leq \sigma_0$ and $\varepsilon \leq \varepsilon_m(\frac{3+\gamma}{4})$, for some other $C_m > 1$:

$$|\big(g_{Q_1} \dots g_{Q_N}\big)^{(\tau)}(.,t)|_E^{(m)}(X) \leq (C_m)^N t^N \varepsilon^{\frac{3+\gamma}{4} \sum_{j=1}^N \mathrm{diam}(Q_j))} \exp\left[\sum_{j=1}^N \big(M_{Q_j}(t) - (T_{Q_j} \widetilde{\psi}_\Lambda)^{(\tau)}(X, X, t)\big)\right].$$

In other words, denoting, for each set A of boxes contained in Λ, and of diameter ≥ 1, by $L(A)$ the sum of their diameters, for the ℓ^∞ norm, we have,

$$|G_A^{(\tau)}\ (.,t)|_E^{(m)}(X) \leq m(t)\ L(A)^m\ \Big(C_m M(t) \varepsilon^{\frac{3+\gamma}{4}}\Big)^{L(A)}\ \exp\left[\sum_{Q\in A} \big(M_Q(t) - (T_Q \widetilde{\psi}_\Lambda)^{(\tau)}\ (X, X, t)\big)\right].$$

It follows, if $C_m \varepsilon^{\frac{1-\gamma}{4}} \leq 1/2$, that, with another K_m,

$$|G_A^{(\tau)}(.,t)|_E^{(m)}(X) \leq\ m(t) K_m\ (M(t) \varepsilon^{\frac{1+\gamma}{2}})^{L(A)}\ \exp\left[\sum_{Q\in A} \big(M_Q(t) - (T_Q \widetilde{\psi}_\Lambda)^{(\tau)}(X, X, t)\big)\right].$$

Therefore

$$\sum_{A \in \mathcal{A}_0(E_1,E_2)} |e^{-\Phi_0(.,t)} (G_A)^{(\tau)}|_E^{(m)}(X,t) \leq K'_m e^{-\Phi_0(X,t)}\ M(t)^m \dots$$

$$\sum_{\substack{A\in\mathcal{S}(E_1,E_2), B\in\mathcal{B}(E) \\ A\cap B=\emptyset}} (M(t) \varepsilon^{\frac{1+\gamma}{2}})^{(L(A)+L(B))} \exp\left[\sum_{Q\in A\cup B} \big(M_Q(t) - (T_Q \widetilde{\psi}_\Lambda)^{(\tau)}(X, X, t)\big)\right]\ \sum_{C\cap(A\cup B)=\emptyset} G_C^{(\tau)}(X, t),$$

where the last sum is taken on all the (possibly empty) sets C of boxes in Λ, of diameter ≥ 1, but not belonging to $A \cup B$. We remark that

$$\sum_{C \cap (A \cup B) = \emptyset} (G_C)^{(\tau)}(X,t) \;=\; \exp\left[\sum_{Q \notin A \cup B} \left(M_Q(t) - (T_Q \widetilde{\psi}_\Lambda)^{(\tau)}\ (X,X,t)\right)\right].$$

Therefore

$$\sum_{A \in \mathcal{A}_0(E_1,E_2)} |e^{-\Phi_0} G_A^{(\tau)}|_E^{(m)}(X,t) \leq m(t) K''_m\ e^{-\widetilde{\psi}_\Lambda^{(\tau)}(X,X,t)} \sum_{A \in \mathcal{S}(E_1,E_2)} (M(t)\varepsilon^{\frac{1+\gamma}{2}})^{L(A)} \sum_{B \in \mathcal{B}(E)} (M(t)\varepsilon^{\frac{1+\gamma}{2}})^{L(B)}.$$

The estimation (9.29) and, therefore, the Proposition, will follow from the next Lemma, applied to $\rho = M(t)\varepsilon^{\frac{1+\gamma}{2}}$, (remarking that the condition $\rho \leq \frac{1}{2K}$ is satisfied if $M(t)\varepsilon^\gamma \leq \frac{1}{2}$ and $K\varepsilon^{\frac{1-\gamma}{2}} \leq 1$).

Lemma 9.5. *There exists $K > 1$, depending only on the dimension d, such that we have the following implication, for each finite sets E, E_1 and E_2 of $\mathbb{Z}^d$:*

$$0 < \rho \leq \frac{1}{2K} \implies \sum_{A \in \mathcal{S}(E_1,E_2,\mathbb{Z}^d)} \rho^{L(A)} \;\leq 2\ (K\rho)^{\delta(E_1,E_2)}\ \inf(\sharp(E_1),\sharp(E_2)) \tag{9.31}$$

$$0 < \rho \leq \frac{1}{2K} \implies \sum_{B \in \mathcal{B}(E,\mathbb{Z}^d)} \rho^{L(A)} \;\leq 2\sharp(E) \tag{9.32}$$

Proof of (9.32). Let $\mathcal{B}_L(E)$ be the set of $B \in \mathcal{B}(E,\mathbb{Z}^d)$ such that $L(A) = L$. We shall prove that there exists $K > 0$, depending only on the dimension d, such that, for each finite subset E of $\mathbb{Z}^d$, for each integer $L \geq 0$, we have

$$\sharp(\mathcal{B}_L(E)) \;\leq \sharp(E)\ \ K^L \tag{9.33}$$

Let us consider first the case where E is reduced to a single point λ. There are at most $\left(\frac{R(R+1)}{2}\right)^d \leq R^{2d}$ possible boxes, of greatest side R, which contain λ. If R_1, ..., R_p ($p \geq 1$) is a finite sequence of integers ≥ 1, the number of possible sequences $(Q_1,...Q_p)$ of boxes such that $\lambda \in Q_j$ and $\mathrm{diam}(Q_j) = R_j$ $(1 \leq j \leq p)$ is $\leq \prod_{j=1}^p R_j^{2d}$. If we choose $K > 1$ such that $R^{2d} \leq K^R$ for all $R \geq 1$, and if we remember that the number of sequences $(R_1\ ,...R_N)$ such that $R_j \geq 1$ and $R_1 + ... + R_N = L$ is 2^{L-1},we see that

$$\sharp(\mathcal{B}_L(\{\lambda\})) \;\leq (2K)^L$$

and (9.33), and therefore (9.32), follow easily.

Proof of (9.31). Let $\mathcal{S}_L(E_1,E_2)$ be the set of $A \in \mathcal{S}(E_1,E_2,\mathbb{Z}^d)$ such that $L(A) = L$, ($L(A)$ being the sum of the diameters, for the ℓ^∞ norm, of all the boxes in A). We remark that, if $A \in \mathcal{S}_L(E_1,E_2)$, we have $L(A) \geq \delta(E_1,E_2)$. We shall prove that there exists $K > 0$, depending only on the dimension d, such that, for each disjoint subsets E_1 and E_2 of $\mathbb{Z}^d$, and for any integer $L \geq \delta(E_1,E_2)$, we have

$$\sharp(\mathcal{S}_L(E_1,E_2)) \;\leq K^L \inf((\sharp E_1),(\sharp E_2)) \tag{9.34}$$

Suppose that $\sharp(E_1) \leq \sharp(E_2)$. For each point $a \in E_1$, for each sequence $(R_1,...,R_p)$ of integers ≥ 1, let $\mathcal{S}(a;R_1,...R_p)$ be the set of sequences of boxes $(Q_1,...,Q_p)$ contained in Λ such that $a \in Q_1$, $Q_1 \cap Q_2 \neq \emptyset$, $...Q_{p-1} \cap Q_p \neq \emptyset$ and $\mathrm{diam}(Q_j) = R_j$ $(1 \leq j \leq p)$. If we remark that the number of boxes with greatest side R intersecting a given box of greatest side R_0 is less than $R^d(R_0+R)^d \leq (R_0+R)^{2d}$, we see that

$$\sharp(\mathcal{S}(a;R_1,...R_p)) \leq R_1^{2d}\ (R_1+R_2)^{2d} ...(R_{p-1}+R_p)^{2d} \leq e^{4d(R_1+...R_p)}.$$

Given $L \geq 1$, the number of sequences $(R_1, ..., R_p)$ such that $p \geq 1$, $R_j \geq 1$ and $R_1 + ... + R_p = L$ is 2^{L-1}. Therefore, the number of sequences of boxes $(Q_1, ..., Q_p)$ such that $a \in Q_1$, $Q_1 \cap Q_2 \neq \emptyset$, $...Q_{p-1} \cap Q_p \neq \emptyset$ and $\sum \mathrm{diam}(Q_j) = L$ is $\leq (2e^{4d})^L$, and $\sharp(\mathcal{S}_L(E_1, E_2)) \leq \sharp(E_1) K^L$, with $K = 2e^{4d}$. The case where $\sharp(E_2) \leq \sharp(E_1)$ being similar, (9.34) is proved, and (9.31) follows easily.

End of the proof of Proposition 9.1.

If both A and B are multiplications by polynomially bounded functions f and g satisfying (9.1), the numerator $N(\varepsilon)$ of (9.6), calculated in (9.14) satisfies classically

$$N(\varepsilon) = \int_{(\mathbb{R}^{2p})^\Lambda} K_{E_1,E_2}(X,X) \Big(f(x'_{E_1}) - f(x''_{E_1}) \Big) \Big(g(x'_{E_2}) - g(x''_{E_2}) \Big) \, dX$$

Therefore,

$$|N(\varepsilon)| \leq 4N(f)N(g) \int_{(\mathbb{R}^p)^\Lambda} |K_{E_1,E_2}(X,X)| \, (1 + |X_{E_1}|)^{m_1} (1 + |X_{E_2}|)^{m_2} dX$$

By Proposition 9.4, applied to $m = 0$ and $E = \emptyset$, if the conditions (9.28) are satisfied, we have

$$|N(\varepsilon)| \leq \frac{4N(f)N(g) \ \inf(\sharp(E_1), \sharp(E_2))}{2|G_+(E_1,E_2)|} m(t) \ (M(t)\varepsilon^\gamma)^{\delta(E_1,E_2)} \sum_{\tau \in G_+(E_1,E_2) \cup G_+(E_1,E_2)} I_\tau(\varepsilon, t) \tag{9.35}$$

where

$$I_\tau(\varepsilon, t) = \int_{(\mathbb{R}^{2p})^\Lambda} \widetilde{U}_\Lambda^\tau(X, X, t, h, \varepsilon)(1 + |X_{E_1}|)^{m_1}(1 + |X_{E_2}|)^{m_2} |dX.$$

If $m_1 = m_2 = 0$, we have $I_\tau(\varepsilon, t) = \mathrm{Tr}\Big(e^{-t\widetilde{H}_\Lambda^{(\tau)}(\varepsilon)}\Big)$ and the point a) of Proposition 9.1 is proved. In the other cases, by Proposition 8.2, there exists $\sigma_1 > 0$, $\varepsilon_0 > 0$, and, for each integers m_1, m_2, N_1, N_2, a function $t \to C(t, m_1, m_2, N_1, N_2)$ such that, for each h, t and ε satisfying $h^2(t + t^2) \leq \sigma_1$ and $\varepsilon \leq \varepsilon_0$, we have

$$I_\tau(\varepsilon, t) \leq C(t, m_1, m_2, \sharp E_1, \sharp E_2) \mathrm{Tr}\Big(e^{-t\widetilde{H}_\Lambda^{(\tau)}(\varepsilon)}\Big). \tag{9.36}$$

If $m_j = 0$, C is independent on $\sharp(E_j)$ $(j = 1, 2)$. The Proposition follows from (9.35) and (9.36).

10. Quantum correlation : general case.

The aim of this section is the proof of Theorem 1.5 in the general case. For that, we need some notations. If T is an integral operator in $L^2((\mathbb{R}^p)^\Lambda)$ with an integral kernel $K_T(x, y)$ in $\mathcal{S}((\mathbb{R}^{2p})^\Lambda)$, and if E is a subset of Λ, let us denote, with the notation $x = (x_E, x_{\Lambda\setminus E})$ already introduced, for each $x_{\Lambda\setminus E}$ in $(\mathbb{R}^p)^{\Lambda\setminus E}$, by $T(x_{\Lambda\setminus E})$ the operator ("partial trace") in $L^2((\mathbb{R}^p)^E)$ with integral kernel $(x_E, y_E) \to K_T(x_E, x_{\Lambda\setminus E}, y_E, x_{\Lambda\setminus E})$, in other words the operator defined by :

$$\Big(T(x_{\Lambda\setminus E}) f \Big)(x_E) = \int_{(\mathbb{R}^p)^E} K_T(x_E, x_{\Lambda\setminus E}, y_E, x_{\Lambda\setminus E}) f(y_E) dy_E \qquad \forall f \in L^2((\mathbb{R}^p)^E), \qquad \forall x \in E. \tag{10.1}$$

If S is a trace class operator in $L^2((\mathbb{R}^p)^E)$, we denote by $\|S\|_E^{(tr)}$ its trace norm. Let us recall that, with the previous notations, if E is a subset of a finite set Λ of $\mathbb{Z}^d$, if L is a bounded operator in $L^2((\mathbb{R}^p)^E)$, (identified to a bounded operator in $L^2((\mathbb{R}^p)^\Lambda)$, we have:

$$|\mathrm{Tr}(T \circ L)| \leq \|L\| \int_{(\mathbb{R}^p)^{\Lambda\setminus E}} \|T(x_{\Lambda\setminus E})\|_E^{(tr)} \ dx_{\Lambda\setminus E} \tag{10.2}$$

Theorem 1.5 will be a consequence of the following proposition, in which we set $X_E = (x'_E, x''_E)$:

Proposition 10.1. *With the previous notations, for each integer $N \geq 1$, there exist a function $(t,h) \to C(t,h,N)$, bounded on each compact set of $]0,+\infty[\times]0,+\infty[$, and a function $\gamma \to \varepsilon_0(\gamma, N)$ on $]0,1[$, with the following properties. With the notations of Theorem 1.5, for each disjoint subsets E_1 and E_2 of any box Λ of $\mathbb{Z}^d$, for each set $E \subseteq E_1 \cup E_2$, one can find two disjoint subsets F_1 and F_2 of Λ, containing E_1 and E_2, such that if $T(F_1,F_2)$ is the operator defined like in (9.12), with E_j replaced by F_j, if $\gamma \in]0,1[$, we have*

$$\int_{(\mathbb{R}^{2p})^{\Lambda\setminus E}} \|T(F_1,F_2)(X_{\Lambda\setminus E})\|_E^{(tr)} dX_{\Lambda\setminus E} \leq \tag{10.3}$$

$$\leq C(t,h,\sharp(E))\ (1+\mathrm{diam}(E_1)+\mathrm{diam}(E_2))^d\ (M(t)\varepsilon^\gamma)^{\frac{1}{5}\delta(E_1,E_2)} Tr\ \left(e^{-t\widetilde{H}_{\Lambda,\varepsilon}}\right)$$

if

$$ht \leq \sigma_0, \qquad 0<\varepsilon\leq \varepsilon_0(\gamma,\sharp(E)) \qquad M(t)\varepsilon^\gamma \leq \frac{1}{2}. \tag{10.4}$$

If $E\cap E_j = \emptyset$, we can omit $\mathrm{diam}(E_j)$ *in the sum in the RHS* $(j=1,2)$.

Proof that Theorem 1.5 is a consequence of Proposition 10.1. If A and B are operators in $\mathcal{L}(\mathcal{H}_{E_1})$ and $\mathcal{L}(\mathcal{H}_{E_2})$ like in the Theorem 1.5, we can use the expression given by the Proposition 9.2 for their correlation, and we can replace, in (9.11), the operator $T(E_1,E_2)$ by $T(F_1,F_2)$ if F_1 and F_2 are any disjoint sets containing E_1 and E_2. We make the choice of Proposition 10.1. In the general case of Theorem 1.5, we shall apply Proposition 10.1 to the operator $T(F_1,F_2)$ and to the set $E=E_1\cup E_2$. By (10.2), applied with the operator $T=T(F_1,F_2)$, to the set $E=E_1\cup E_2$, and to the operator $L=(A'-A'')(B'-B'')$, we have

$$|\mathrm{Tr}\ \Big(T(F_1,F_2)(A'-A'')(B'-B'')\Big)| \leq 4\|A\|\ \|B\| \int_{(\mathbb{R}^{2p})^{\Lambda\setminus E}} \|T(F_1,F_2)(X_{\Lambda\setminus E})\|_E^{(tr)} dX_{\Lambda\setminus E}$$

By Proposition 10.1, it follows that, for some function K, the estimation (1.16) is satisfied under the conditions (1.17).

If B is the multiplication by a bounded C^∞ function g on $(\mathbb{R}^p)^{E_2}$, we apply Proposition 10.1, also with the operator $T(F_1,F_2)$, but with the set $E=E_1$. By (10.2) applied with the operator $\widetilde{T} = T\circ(B'-B'')$, the set $E=E_1$, and the operator $L=A'-A''$, we have

$$|\mathrm{Tr}\ \Big(T(F_1,F_2)(A'-A'')(B'-B'')\Big)| \leq 2\|A\| \int_{(\mathbb{R}^{2p})^{\Lambda\setminus E}} \|(T(F_1,F_2)\circ(B'-B'')(X_{\Lambda\setminus E})\|_E^{(tr)} dX_{\Lambda\setminus E}$$

Since g depends only on x_{E_2}, and therefore is independent on x_E, we have, for each $X_{\Lambda\setminus E}$, $(T(F_1,F_2)\circ(B'-B''))(X_{\Lambda\setminus E}) = (g(x'_{\Lambda\setminus E}) - g(x''_{\Lambda\setminus E})T(F_1,F_2)(X_{\Lambda\setminus E})$ and therefore

$$|\mathrm{Tr}\ \Big(T(F_1,F_2)(A'-A'')(B'-B'')\Big)| \leq 4\|A\|\ \|g\|_{L^\infty} \int_{(\mathbb{R}^{2p})^{\Lambda\setminus E}} \|T(F_1,F_2)(X_{\Lambda\setminus E})\|_E^{(tr)} dX_{\Lambda\setminus E}$$

and Proposition 10.1 gives again the estimation (1.13), where the function K is independent of the diameter of E_2.

For the proof of Proposition 10.1, we shall make the following choice

$$F_j = \{\lambda\in\Lambda, \qquad \delta(\lambda,E_j)\leq \frac{2}{5}\delta(E_1,E_2)\} \qquad j=1,2. \tag{10.5}$$

These sets are disjoint.

In order to estimate the trace-norms, we shall need the following elementary result (cf D. Robert [18] for a proof), in which we use the notations (9.25) and (9.26), with $X=(x',x'')$ replaced here by (x,y).

Lemma 10.2. *If S is an operator in $L^2((\mathbb{R}^p)^E)$, with an integral kernel K_S in $\mathcal{S}((\mathbb{R}^p)^E \times (\mathbb{R}^p)^E)$, the operator S is in trace-class, and there exists a constant $C(n)$, depending only on $n = p(\sharp E)$, such that*

$$\|S\|_E^{(tr)} \leq C(n) \int_{(\mathbb{R}^p)^E \times (\mathbb{R}^p)^E} (1 + |x_E| + |y_E|)^{2n+2} |K_S|_E^{(3n+3)}(x_E, y_E) dx_E dy_E \tag{10.6}$$

By this Lemma and Proposition 8.2, Proposition 10.1 will be a consequence of the following one, in which we shall use the notations (9.25) and (9.26).

Proposition 10.3. *With the notations of Proposition 10.1, we have, with the choice (10.5) of F_1 and F_2, if $K_{F_1,F_2}(X, Y, t)$ is the integral kernel of $T(F_1, F_2)$, if $X_{\Lambda\setminus E} = Y_{\Lambda\setminus E}$, under conditions of the form (10.4),*

$$|K_{F_1,F_2}(.,.,t)|_E^{(m)}(X, Y) \leq \tag{10.7}$$

$$\leq \frac{F(|X_E - Y_E|, m, \sharp(E), t, h)\varepsilon^{\frac{2}{5}\delta(E_1,E_2)}}{\sharp(G_+(F_1, F_2)} (1 + \mathrm{diam}(E_1) + \mathrm{diam}(E_2))^d \sum_{\tau \in (G_+\cup G_-)(F_1,F_2)} \widetilde{U}_\Lambda^{(\tau)}(X, X, t)$$

where

$$F(|X_E - Y_E|, m, \sharp(E), t, h) = C_m(t, h, \sharp(E))\ e^{-\frac{|X_E - Y_E|_2^2}{2th}} e^{Ct|X_E - Y_E|_1} (1 + |X_E - Y_E|)^M \tag{10.8}$$

where $C_m(t, h, \sharp(E))$ is a function of (h, t), bounded on each compact set of $]0, +\infty[\times]0, +\infty[$, and M is an integer, depending also on m. If $E \cap E_j = \emptyset$, we can omit the term $\mathrm{diam}(E_j)$ in the sum in the RHS of (10.7) ($j = 0, 1$).

Proof. We introduce the following functions, for each $\tau \in G_\pm(F_1, F_2)$:

$$g_\Lambda^\tau(X, Y, t) = \Big(\widetilde{\psi}_\Lambda^\tau(X, Y, t) - \widetilde{\psi}_\Lambda(X, Y, t)\Big)\ -\ \Big(\widetilde{\psi}_\Lambda^\tau(X, X, t) - \widetilde{\psi}_\Lambda(X, X, t)\Big), \tag{10.9}$$

$$G(X, Y, t) = \frac{\widetilde{U}_\Lambda(X, Y, t)}{\widetilde{U}_\Lambda(X, X, t)}. \tag{10.10}$$

We can write, by (9.13), $K_{F_1,F_2}(X, Y, t) = K^{(1)}(X, Y, t) + K^{(2)}(X, Y, t)$, where

$$K^{(1)}(X, Y, t) = \frac{G(X, Y, t)}{2\sharp(G_+(F_1, F_2))} \sum_{\tau \in (G_+\cup G_-)(F_1,F_2)} sgn(\tau)\big(e^{-g_\Lambda^\tau(X,Y,t)} - 1\big)\ \widetilde{U}_\Lambda^{(\tau)}(X, X, t) \tag{10.11}$$

$$K^{(2)}(X, Y, t) =\ G(X, Y, t)\ K_{F_1,F_2}(X, X, t). \tag{10.12}$$

We set also

$$K_{F_1,F_2}^{\Delta}(X, t) = K_{F_1,F_2}(X, X, t), \qquad \widetilde{U}_{\Lambda,\Delta}^{(\tau)}(X, t) = \widetilde{U}_\Lambda^{(\tau)}(X, X, t)$$

For the function K_{F_1,F_2}^{Δ}, we have the estimation given by Proposition 9.4 (in which we replace E_1 and E_2 by F_1 and F_2). We remark that $\sharp(F_j) \leq (\mathrm{diam}(E_j) + \frac{4}{5}\delta(E_1, E_2))^d$ and that $\delta(F_1, F_2) \geq \frac{1}{5}\delta(E_1, E_2)$. Therefore, Proposition 9.4 gives, after changing γ, for some function $C(t, m, N)$

$$|K_{F_1,F_2}^{\Delta}|_E^{(m)}(X, t) \leq \tag{10.13}$$

$$\leq C(t,m,\sharp(E))\ \big(M(t)\varepsilon^{\gamma}\big)^{\frac{1}{5}\delta(E_1,E_2)}\ \frac{\inf(\mathrm{diam}(E_1),\mathrm{diam}(E_2))^d}{\sharp(G_+(F_1,F_2))}\quad \sum_{\tau\in(G_+\cup G_-)(F_1,F_2)} \widetilde{U}^{\tau}_{\Lambda}(X,X,t)$$

under conditions similar to (9.28).

By Theorem 1.1, we can write, for some other function $C(t,m,N)$,

$$|\widetilde{U}^{(\tau)}_{\Lambda,\Delta}(.,t)|^{(m)}_{E}(X)\leq C(t,m,\sharp(E))\widetilde{U}^{(\tau)}_{\Lambda}(X,X,t) \tag{10.14}$$

For the function $G(X,Y,t)$ defined in (10.10), we can write, by Theorem 1.1, if $X_{\Lambda\setminus E}=Y_{\Lambda\setminus E}$

$$|G(.,.,t)|^{(m)}_{E}(X,Y)\leq F(|X_E-Y_E|,m,\sharp(E),t,h) \tag{10.15}$$

where F is a function of the same form as (10.8).

It remains to estimate the factor $e^{-g^{\tau}_{\Lambda}(X,Y,t)}-1$ appearing in (10.11). First, we can write, by (10.9) and Theorem 1.1, if $X_{\Lambda\setminus E}=Y_{\Lambda\setminus E}$

$$|g^{\tau}_{\Lambda}(X,Y,t)|\leq Ct|X_E-Y_E|. \tag{10.16}$$

For a more precise study of g^{τ}_{Λ}, we need the following Lemma.

Lemma 10.4. *For each integer m, there exists a function $C(t,m)>0$ with the following properties. With the notations and hypotheses of Proposition 10.1, and with the choice (10.5) of F_1 and F_2, for each map τ in $(G_+\cup G_-)(F_1,F_2)$, we have, if $X_{\Lambda\setminus E}=Y_{\Lambda\setminus E}$, under conditions of the form (10.4),*

$$|\nabla_E\Big(\widetilde{\psi}^{\tau}_{\Lambda}-\widetilde{\psi}_{\Lambda}\Big)(.,.,t)|^{(m)}_{E}(X,Y)\leq C(t,m)\ \varepsilon^{\frac{\gamma}{5}\delta(E_1,E_2)}\ (1+|X_E-Y_E|)\ (1+\mathrm{diam}(E_1)+\mathrm{diam}(E_2))^d$$

If $E\cap E_j=\emptyset$, we can omit the term $\mathrm{diam}(E_j)$ *in the sum in the RHS ($j=1,2$).*

Proof. Let $\lambda^{(1)},\dots,\lambda^{(k)}$ $(1\leq k\leq m+1)$ be a sequence of points of E. Using the operators T_Q of Section 5, associated to all boxes contained in Λ, we have:

$$\Big(\widetilde{\psi}^{\tau}_{\Lambda}-\widetilde{\psi}_{\Lambda}\Big)(X,Y,t)=\Big(\widetilde{\psi}^{\tau}_{\Lambda}-\widetilde{\psi}_{\Lambda}\Big)(0,Y-X,t)+\sum_{Q\subseteq\Lambda}T_Q(\widetilde{\psi}^{\tau}_{\Lambda}-\widetilde{\psi}_{\Lambda})(X,Y,t).$$

Suppose that at least one of the $\lambda^{(j)}$ is in $E\cap E_1$. There are some small changes to do in the other case. For this case, we can write $\Big(\widetilde{\psi}^{\tau}_{\Lambda}-\widetilde{\psi}_{\Lambda}\Big)(0,Y-X,t)=0$ if $X_{\Lambda\setminus E_1}=Y_{\Lambda\setminus E_1}$. Therefore, we can write, by Theorem 1.1, if we suppose only that $X_{\Lambda\setminus E}=Y_{\Lambda\setminus E}$, (one of the $\lambda^{(j)}$ being in $E\cap E_1$):

$$|\nabla_{\lambda^{(1)}}...\nabla_{\lambda^{(k)}}\Big(\widetilde{\psi}^{\tau}_{\Lambda}-\widetilde{\psi}_{\Lambda}\Big)(0,Y-X,t)|\leq C_{k+1}t\ \varepsilon^{\gamma\delta(E_1,E_2)}(1+|X_{E\cap E_2}-Y_{E\cap E_2}|).$$

We divide the boxes $Q\subseteq\Lambda$ into three categories (where F_1 is defined in (10.5)):

$$A=\{Q\subseteq F_1\}$$

$$B=\{Q\subseteq\Lambda,\quad Q\not\subseteq F_1,\quad\text{and}\quad \delta(E_1,Q)\leq\frac{1}{5}\delta(E_1,E_2)\}$$

$$C=\{Q\subseteq\Lambda,\qquad \delta(E_1,Q)>\frac{1}{5}\delta(E_1,E_2)\}$$

If $Q \in A$, and $X_{\Lambda \setminus E_1} = Y_{\Lambda \setminus E_1}$, then, for all τ in $(G_+ \cup G_-)(E_1, E_2)$, we have $\Big(T_Q(\widetilde{\psi}^\tau_\Lambda - \widetilde{\psi}_\Lambda)\Big)(X, Y, t) = 0$. We have only to remember that $T_Q \widetilde{\psi}_\Lambda$ depends only on X_Q, Y_Q and $X - Y$. If we assume only that $X_{\Lambda \setminus E} = Y_{\Lambda \setminus E}$, it follows from Proposition 6.1, point iii), if $ht \leq \sigma_0$ and $\varepsilon \leq \varepsilon_m(\frac{3+\gamma}{4})$, that

$$|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(k)}} T_Q \Big(\widetilde{\psi}^\tau_\Lambda - \widetilde{\psi}_\Lambda\Big)(X, Y, t)| \leq t C_{k+1} \; |X_{E \cap E_2} - Y_{E \cap E_2}| \varepsilon^{\frac{3+\gamma}{4} \delta(Q, E_2)}$$

We have also, by Proposition 6.1, point i), under similar conditions,

$$|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(k)}} T_Q \Big(\widetilde{\psi}^\tau_\Lambda - \widetilde{\psi}_\Lambda\Big)(X, Y, t)| \leq t C_k \; \varepsilon^{\frac{3+\gamma}{4} \mathrm{diam}(Q)} \tag{10.16}$$

Since $\delta(Q, E_2) \geq \frac{3}{5} \delta(E_1, E_2)$, it follows that, if $Q \in A$,

$$|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(k)}} T_Q \Big(\widetilde{\psi}^\tau_\Lambda - \widetilde{\psi}_\Lambda\Big)(X, Y, t)| \leq C'_k t(1 + |X_E - Y_E|) \varepsilon^{\frac{1+\gamma}{10} \delta(E_1, E_2) + \frac{1}{3}\gamma \mathrm{diam}(Q)}$$

Since

$$\sum_{Q \in A} \varepsilon^{\frac{1}{3}\gamma \mathrm{diam}(Q)} \leq 4^d \sharp(F_1) \leq 4^d \Big(\mathrm{diam}(E_1) + \frac{4}{5} \delta(E_1, E_2)\Big)^d \qquad \text{if} \;\; \varepsilon^{\frac{1}{3}\gamma} \leq 4^{-d},$$

it follows that

$$|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(k)}} \sum_{Q \in A} T_Q \Big(\widetilde{\psi}^\tau_\Lambda - \widetilde{\psi}_\Lambda\Big)(X, Y, t)| \leq C''_k t(1 + |X_E - Y_E|) \; \varepsilon^{\frac{\gamma}{5} \delta(E_1, E_2)} (\mathrm{diam}(E_1))^d$$

If $Q \in B$, by Proposition 6.1 (point i)), we can use again (10.16) under the same conditions. We remark that, if $Q \in B$, we have $\mathrm{diam}(Q) \geq \frac{1}{5} \delta(E_1, E_2)$ and therefore

$$\sum_{Q \in B} \varepsilon^{\frac{3+\gamma}{4} \mathrm{diam}(Q)} \leq \; \varepsilon^{\frac{1+\gamma}{10} \delta(E_1, E_2)} \; 4^d \Big(\mathrm{diam}(E_1) + \frac{4}{5} \delta(E_1, E_2)\Big)^d$$

if $\varepsilon^{\frac{1-\gamma}{4}} \leq 4^{-d}$. It follows that, with another C_m, under this additional condition,

$$|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(k)}} \; \sum_{Q \in B} T_Q \Big(\widetilde{\psi}^\tau_\Lambda - \widetilde{\psi}_\Lambda\Big)(X, Y, t)| \leq t C_m \varepsilon^{\frac{1+\gamma}{10} \delta(E_1, E_2)} \; \Big(\mathrm{diam}(E_1) + \frac{4}{5} \delta(E_1, E_2)\Big)^d \leq$$

$$\leq t C'_m \varepsilon^{\frac{\gamma}{5} \delta(E_1, E_2)} \; (1 + \mathrm{diam}(E_1))^d$$

If $Q \in C$, it follows, by a combination of the points i) and iii) of Proposition 6.1 that, with another C_m, we have, under the previous conditions,

$$|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(k)}} T_Q \Big(\widetilde{\psi}^\tau_\Lambda - \widetilde{\psi}_\Lambda\Big)(X, Y, t)| \leq t C_m \varepsilon^{\frac{1-\gamma}{4} \mathrm{diam}(Q) + \frac{1+\gamma}{2} \delta(Q, E_1)}.$$

We remark that, by the definition of C

$$\sum_{Q \in C} \varepsilon^{\frac{1-\gamma}{4} \mathrm{diam}(Q) + \frac{1+\gamma}{2} \delta(Q, E_1)} \leq \varepsilon^{\frac{\gamma}{5} \delta(E_1, E_2)} \sum_{Q \subset \mathbb{Z}^d} \varepsilon^{(\frac{1-\gamma}{4})(\mathrm{diam}(Q) + \delta(Q, E_1))} \leq C \varepsilon^{\frac{\gamma}{5} \delta(E_1, E_2)} (1 + \mathrm{diam}(E_1))^d$$

if $\varepsilon^{1-\gamma} \leq 16^{-d}$. By all the preceding inequalities, we can write, under conditions similar to (10.4)

$$|\nabla_{\lambda^{(1)}} ... \nabla_{\lambda^{(k)}} \Big(\widetilde{\psi}^\tau_\Lambda - \widetilde{\psi}_\Lambda\Big)(X, Y, t)| \leq \leq C(t, m) \; \varepsilon^{\frac{\gamma}{5} \delta(E_1, E_2)} \; (1 + |X_E - Y_E|) \; (1 + \mathrm{diam}(E_1))^d$$

under the hypotheses of the Lemma, if at least one of the $\lambda^{(j)}$ is in $E\cap E_1$. If one of them is in $E\cap E_2$, which cannot happen if $E\cap E_2$ is void, we obtain, with some changes, the similar inequality with $\mathrm{diam}(E_1)$ replaced by $\mathrm{diam}(E_2)$ in the RHS. The Lemma is proved.

By the previous Lemma, we can write, if $X_{\Lambda\setminus E} = Y_{\Lambda\setminus E}$,

$$(10.17)\quad |e^{-g^{\tau}_{\Lambda}(.,.,t)}-1|^{(m)}_{E}(X,Y) \leq C(m,t)\, e^{Ct|X_E-Y_E|}\varepsilon^{\frac{\gamma}{5}\delta(E_1,E_2)}\,(1+|X_E-Y_E|)(1+\mathrm{diam}(E_1)+\mathrm{diam}(E_2))^d.$$

The Propositions 10.2, and, therefore, 10.1, follow from (10.11)-(10.15) and (10.17).

11. Proof of Theorems 1.2, 1.3 and 1.4.

Let Λ be any box of $\mathbb{Z}^d$ and split it into two disjoint boxes Λ_1 and Λ_2. Suppose that the splitting happens orthogonally to the j^{th} axis, then define $\Lambda_\perp$ as the box in $\mathbb{Z}^{d-1}$, projection of Λ orthogonally to the j^{th} axis. In the following, we shall assume that $j=1$ and that Λ, Λ_1 and Λ_2 can be defined more explicitly by

$$(11.1)\qquad \Lambda=\prod_{j=1}^{d}[\alpha_j,\beta_j] \qquad \Lambda_1=[\alpha_1,m]\times\Lambda_\perp \qquad \Lambda_2=[m+1,\beta_1]\times\Lambda_\perp \qquad \Lambda_\perp=\prod_{j=2}^{d}[\alpha_j,\beta_j]$$

where $\alpha_j\leq\beta_j$ and $\alpha_1<m<\beta_1$. Let π_j denotes the $j-$th projection from $\mathbb{Z}^d$ into $\mathbb{Z}$, and let $P_j(m)=\{\lambda\in\mathbb{Z}^d,\quad \lambda_j=m\}$, $(1\leq j\leq d,\ m\in\mathbb{Z})$.

As in Section 7, let $H_\Lambda(\varepsilon,\theta)$ be the Hamiltonian defined as in (1.1) with V replaced by $V_{\Lambda,\varepsilon}(\theta)$ defined in (7.1), $U_{\Lambda,\varepsilon}(\theta,x,y,t)$ the corresponding heat kernel, and $\psi_\Lambda(x,y,t,h,\varepsilon,\theta)$ the function appearing in its expression (1.3). We set $\varphi_\theta(x,t)=\psi_\Lambda(x,x,t,h,\varepsilon,\theta)$.

For the proof of Theorem 1.2, we consider a finite subset Q_0 of $\mathbb{Z}^d$, and an element A of $\mathcal{L}(\mathcal{H}_{Q_0})$. If a box Λ contains Q_0, A can be considered as an element of $\mathcal{L}(\mathcal{H}_\Lambda))$ and we shall study the mean value $E_{\Lambda,\varepsilon}(A)$ defined in (1.14). Theorem 1.2 will be a consequence of the following Proposition.

Proposition 11.1. *With the previous notations, we can write, for each boxes Λ, Λ_1 and Λ_2 of $\mathbb{Z}^d$ defined by (11.1), with $\alpha_1<m<\beta_1$, if Q_0 is contained in Λ_1, if $A\in\mathcal{L}(\mathcal{H}_{Q_0})$, we have, under conditions similar to (1.7),*

$$(11.2)\qquad |E_{\Lambda,\varepsilon}(A)-E_{\Lambda_1,\varepsilon}(A)|\leq K(t,h,\mathrm{diam}(Q_0))\ (M(t)\varepsilon^{\gamma})^{\frac{1}{5}\delta(Q_0,P_1(m))}\|A\|$$

Proof. First step. We shall give an expression of the LHS of (11.2). We set, with the notations of the beginning of this section, for each $\theta\in[0,1]$,

$$(11.3)\qquad R(\theta,x,y,t)=\frac{\partial\psi}{\partial\theta}(x_{Q_0},x_{\Lambda\setminus Q_0},y_{Q_0},x_{\Lambda\setminus Q_0},t)\ -\ \frac{\partial\psi}{\partial\theta}(x,x,t).$$

Let $B_Q(\theta)$ denotes the multiplication operator by the function $\left(T_Q\frac{\partial\psi}{\partial\theta}\right)_{|D}$, D being the diagonal $\{(x,y)\in(\mathbb{R}^p)^\Lambda\times(\mathbb{R}^p)^\Lambda\mid x=y\}$. For a given $K(x,y)$, in $\mathcal{S}((\mathbb{R}^p)^\Lambda\times(\mathbb{R}^p)^\Lambda)$, let $Op(K)$ be the operator with integral kernel $K(x,y)$. Then we shall prove that

$$(11.4)\qquad E_{\Lambda,\varepsilon}(A)-E_{\Lambda_1,\varepsilon}(A)=\int_0^1\frac{\mathrm{Tr}(Op(U_{\Lambda,\varepsilon}(\theta)R(\theta))\circ A)}{\mathrm{Tr}\,(e^{-tH(\theta)})}\,d\theta+\frac{1}{2}\sum_{Q\subseteq\Lambda}\int_0^1 cov(B_Q(\theta),A)\,d\theta$$

Here we define $cov_{\Lambda,\varepsilon,\theta}(B_Q, A)$ by (1.15), with $H_{\Lambda,\varepsilon}(\theta)$ as Hamiltonian, even if A and $B_Q(\theta)$ do not act on different sets of variables, (in this case, it is no more commutative). In order to prove (11.4), we set

$$F(\theta) = \frac{\mathrm{Tr}\left(e^{-tH_{\Lambda,\varepsilon}(\theta)}A\right)}{\mathrm{Tr}\, e^{-tH_{\Lambda,\varepsilon}(\theta)}}.$$

Consequently, $E_{\Lambda,\varepsilon}(A) - E_{\Lambda_1,\varepsilon}(A) \ = \ F(1) - F(0)$, and we shall calculate F'_θ. We have:

$$(\mathrm{Tr}(e^{-tH_{\Lambda,\varepsilon}(\theta)}))'_\theta = \mathrm{Tr}(e^{-tH_{\Lambda,\varepsilon}(\theta)}B),$$

where B is the multiplication operator by the function $\frac{\partial\psi}{\partial\theta}(x,x,t)$. We see that

$$(\mathrm{Tr}(e^{-tH_{\Lambda,\varepsilon}(\theta)}A))'_\theta = \mathrm{Tr}(Op(U_{\Lambda,\varepsilon}(\theta)T)\circ A)$$

with

$$T = \frac{\partial\psi}{\partial\theta}|_{D_0}, \qquad D_0 = \{(x,y) \in (\mathrm{I\!R}^p)^\Lambda \times (\mathrm{I\!R}^p)^\Lambda \mid x_{\Lambda\setminus Q_0} = y_{\Lambda\setminus Q_0}\}.$$

Moreover, $T(x,y) = \frac{\partial\psi}{\partial\theta}(x,x) + R_\theta(x,y)$, where R_θ is defined above. Following Proposition 5.1,

$$\frac{\partial\psi}{\partial\theta}|_D = \frac{\partial\psi}{\partial\theta}(0,0) + \sum_{Q\subseteq\Lambda}\left(T_Q\frac{\partial\psi}{\partial\theta}\right)_{|D}.$$

In particular,

$$(\mathrm{Tr}(e^{-tH_{\Lambda,\varepsilon}(\theta)}))'_\theta = \frac{\partial\psi}{\partial\theta}(0,0)\mathrm{Tr}(e^{-tH_{\Lambda,\varepsilon}(\theta)}) + \sum_Q \mathrm{Tr}(e^{-tH_{\Lambda,\varepsilon}(\theta)}B_Q(\theta)), \tag{11.6}$$

and

$$(\mathrm{Tr}(e^{-tH_{\Lambda,\varepsilon}(\theta)}A))'_\theta = \frac{\partial\psi}{\partial\theta}(0,0)\mathrm{Tr}(e^{-tH_{\Lambda,\varepsilon}(\theta)}A) + \mathrm{Tr}(Op(U_{\Lambda,\varepsilon}(\theta)R(\theta))\circ A) + \sum_{Q\subseteq\Lambda}\mathrm{Tr}(e^{-tH_{\Lambda,\varepsilon}(\theta)}B_Q(\theta)A). \tag{11.7}$$

We write $X = (x', x'') \in (\mathrm{I\!R}^p)^\Lambda \times (\mathrm{I\!R}^p)^\Lambda$ and denote by $H'(\theta), A', B'_Q$ (resp. $H''(\theta), A'', B''_Q$) the operators H_θ, A, B_Q seen as operators on $L^2((\mathrm{I\!R}^p)^\Lambda \times (\mathrm{I\!R}^p)^\Lambda)$ as in section 9. From (11.6) and (11.7), we derive that

$$F'_\theta(\theta) = \frac{\mathrm{Tr}(Op(U_\Lambda(\theta)R(\theta))\circ A)}{\mathrm{Tr}\left(e^{-tH_{\Lambda,\varepsilon}(\theta)}\right)} + \sum_{Q\subseteq\Lambda}\frac{\mathrm{Tr}\left(e^{-t(H'(\theta)+H''(\theta))}B'_QA'\right) - \mathrm{Tr}\left(e^{-t(H'(\theta)+H''(\theta))}B''_QA'\right)}{\mathrm{Tr}\left(e^{-t(H'(\theta)+H''(\theta))}\right)}.$$

It follows, by (9.6), (exchanging the role of ' and "), that

$$F'_\theta(\theta) = \frac{\mathrm{Tr}(Op(U_{\Lambda,\varepsilon}(\theta)R(\theta))\circ A)}{\mathrm{Tr}\left(e^{-tH_{\Lambda,\varepsilon}(\theta)}\right)} + \frac{1}{2}\sum_{Q\subseteq\Lambda} cov_{\Lambda,\varepsilon,\theta}(B_Q(\theta), A).$$

Therefore (11.4) is proved.

Second step. For each $x_{\Lambda\setminus Q_0}$ in $(\mathrm{I\!R}^p)^{\Lambda\setminus Q_0}$ and $t > 0$, let $\Phi(x_{\Lambda\setminus Q_0}, t)$ be the operator in $\mathcal{H}_{Q_0}$ with integral kernel $(x_{Q_0}, y_{Q_0}) \to (U_{\Lambda,\varepsilon}R)(\theta, x_{Q_0}, x_{\Lambda\setminus Q_0}, y_{Q_0}, x_{\Lambda\setminus Q_0}, t)$. By (10.2), we can write

$$|\mathrm{Tr}\left(Op(U_{\Lambda,\varepsilon}(\theta)R_\theta)\circ A\right)| \leq \|A\| \int_{(\mathrm{I\!R}^p)^{\Lambda\setminus Q_0}} \|\Phi(x_{\Lambda\setminus Q_0}, t)\|^{(tr)}_{Q_0}\, dx_{\Lambda\setminus Q_0}$$

By Lemma 10.2, we have, if $N = 3p\sharp(Q_0) + 3$, with the notations (9.25), (9.26), with $X = (x', x'')$ replaced by (x, y),

$$\|\Phi(x_{\Lambda\setminus Q_0}, t)\|_{Q_0}^{(tr)} \leq$$

$$\leq C_N \int_{(\mathbb{R}^{2p})^{Q_0}} (1 + |x_{Q_0}| + |y_{Q_0}|)^N \ |U_{\Lambda,\varepsilon}(\theta, ., t)|_{Q_0}^{(N)} \ |R_\theta(., t)|_{Q_0}^{(N)}(x_{Q_0}, x_{\Lambda\setminus Q_0}, y_{Q_0}, x_{\Lambda\setminus Q_0}) dx_{Q_0} dy_{Q_O}.$$

By (11.3) and by Proposition 7.1 (inequality (7.5)), we can write

$$|R_\theta(., t)|_{Q_0}^{(N)}(x_{Q_0}, x_{\Lambda\setminus Q_0}, y_{Q_0}, x_{\Lambda\setminus Q_0}) \leq tC_1 |x_{Q_0} - y_{Q_0}| \ \varepsilon^{\gamma\delta(Q_0, P_1(m))}.$$

By Theorem 1.1, we can write, if $x_{\Lambda\setminus Q_0} = y_{\Lambda\setminus Q_0}$,

$$|U_{\Lambda,\varepsilon}(\theta, ., t)|_{Q_0}^{(N)}(x, y) \leq C(t, h, \sharp(Q_0)) \ (1 + |x_{Q_0} - y_{Q_0}|)^M \ e^{-\frac{|x_{Q_0} - y_{Q_0}|^2}{2th}} \ e^{Ct|x_{Q_0} - y_{Q_0}|} \ U_{\Lambda,\varepsilon}(\theta, x, x, t).$$

Therefore, since $\int U_{\Lambda,\varepsilon}(\theta, x, x, t) dx = \mathrm{Tr}\,(e^{-tH_\Lambda(\theta)})$,

$$\left|\frac{\mathrm{Tr}(Op(U_\Lambda(\theta) R_\theta) \circ A)}{\mathrm{Tr}\,(e^{-tH_\Lambda(\theta)})}\right| \leq C(t, h, \sharp(Q_0)) \ \|A\| \ \varepsilon^{\gamma\delta(Q_0, P_1(m))}.$$

Now, we shall estimate the second term in (11.4). If we apply Proposition 7.2, first with inequality (7.7)), and then with (7.9), (7.10), we obtain:

$$\|B_Q\| \leq tC_0 \varepsilon^{\frac{1+\gamma}{2}\mathrm{diam}(Q)}, \qquad \|B_Q\| \leq tC_0 \varepsilon^{\gamma\delta(Q, P_1(m))}.$$

By Theorem 1.5 about the correlation of two operators, one of them being the multiplication by a bounded function, (which is applicable even if Q and Q_0 are not disjoint), we have

$$|cov(B_Q, A)| \leq K(t, h, \mathrm{diam}(Q_0)) \ ((M(t)\varepsilon^{\frac{1+\gamma}{2}})^{\frac{1}{5}\delta(Q_0, Q)} \|A\| \ \|B_Q\|,$$

and therefore, with another K

$$|cov(B_Q, A)| \leq K(t, h, \mathrm{diam}(Q_0))(M(t)\varepsilon^\gamma)^{\frac{1}{5}(\mathrm{diam}(Q) + \delta(Q, Q_0) + \delta(Q, P_1(m))} \ \varepsilon^{\frac{1-\gamma}{10}(\mathrm{diam}(Q) + \delta(Q, Q_0)}$$

We have $\mathrm{diam}(Q) + \delta(Q, Q_0) + \delta(Q, P_1(m)) \geq \delta(Q_0, P_1(m))$. By Lemma 11.2 below (point (11.8), applied with $\rho = \varepsilon^{\frac{1-\gamma}{10}}$) we have, if $0 < \rho \leq 2^{-d}$,

$$\sum_{Q\subseteq\Lambda} \rho^{\mathrm{diam}(Q) + \delta(Q, Q_0)} \leq \sum_{\lambda\in Q_0} \sum_{Q\subset\mathbb{Z}^d} \rho^{\mathrm{diam}(Q) + \delta(Q, \lambda)} \leq \sharp(Q_0) \left[\sum_{I\subset\mathbb{Z}} \rho^{\frac{1}{d}(\mathrm{diam}(I) + \delta(I, 0))}\right]^d \leq \sharp(Q_0) \ C(\frac{1}{2})^d$$

Therefore, if $\varepsilon^{\frac{1-\gamma}{10}} \leq 2^{-d}$, $|F'_\theta(\theta)| \leq K(t, h, \mathrm{diam}(Q_0)) \ \|A\|(M(t)\varepsilon^\gamma)^{\frac{1}{5}\delta(Q_0, P_1(m))}$. Thus, Proposition 11.1 is proved.

Theorem 1.2 follows easily, because, if Λ_n is defined by (1.5) and if Q_0 is fixed, we have $\delta(Q_0, P_j(m)) \geq \frac{m}{2}$ $(1 \leq j \leq d)$, if m is large enough, and Proposition 11.1, applied d times, proves that, if $m < n$ and m is large enough

$$|E_{\Lambda_m,\varepsilon}(A) - E_{\Lambda_n,\varepsilon}(A)| \leq K(t, h, \mathrm{diam}(Q_0)) \ (M(t)\varepsilon^\gamma)^{\frac{m}{10}} \|A\|.$$

Therefore, under the conditions (1.7), the sequence $E_{\Lambda_n,\varepsilon}(A)$ has a limit $\omega_{h,t,\varepsilon}(A)$, and we have, for n large enough, the estimation (1.8).

Proof of Theorem 1.3. It is a straightforward modification of the proof of Theorem 1.2. Let A be the multiplication operator by a function f, depending only on the variables x_λ, $\lambda \in Q_0$, and satisfying (1.9). We prove equality (11.4)) exactly like for Theorem 1.2. Since A is a multiplication, we have:

$$\mathrm{Tr}(Op(U_{\Lambda,\varepsilon}(\theta)R_\theta) \circ A) = \int_{(\mathbb{R}^p)^\Lambda} U_\Lambda(x,x,t,\theta)R_\theta(x,x,t)f(x)dx$$

We estimate R_θ as before, and we apply Proposition 8.2. We obtain, if Λ and Λ_1 are defined by (11.1) and $Q_0 \subset \Lambda_1$,

$$\frac{|\mathrm{Tr}(Op(U_\Lambda(\theta)R_\theta) \circ A)|}{\mathrm{Tr}\,(e^{-tH(\theta)})} \leq C(t,m,h,\sharp(E))\ N_m(f)\ \varepsilon^{\gamma\delta(Q,P_1(m))}$$

Instead of Theorem 1.5, we use Proposition 9.1, point b), remarking that B_Q is the multiplication by a bounded function, and we obtain

$$|cov(B_Q,A)| \leq K(t,m,\mathrm{diam}(Q_0))\ ((M(t)\varepsilon^{\frac{1+\gamma}{2}})^{\delta(Q_0,Q)}N_m(a)\ \ \|B_Q\|,$$

The rest of the proof is unchanged.

We used in the proof of Proposition 11.1, and we shall use again, the following Lemma.

Lemma 11.2. *For each $\rho \in]0,1[$, there exists a constant $C(\rho) > 0$ such that, for each intervals I and I' of $\mathbb{Z}$, we have:*

$$\sum_{\lambda\in\mathbb{Z}} \rho^{\delta(\lambda,I)} \leq C(\rho) + \sharp(I) \tag{11.8}$$

$$\sum_{\lambda\in\mathbb{Z}} \rho^{\delta(\lambda,I)+\delta(\lambda,I')} \leq (C(\rho) + \inf(\sharp(I),\sharp(I')) + \delta(I,I'))\ \rho^{\delta(I,I')} \tag{11.9}$$

Moreover, for each finite interval $\{\alpha,\dots,\beta\}$ of $\mathbb{Z}$, for each point m of this interval, we have the following inequalities, where the summmations are on all the finite subintervals I, I' and I'' of $\{\alpha,\dots,\beta\}$ or $\mathbb{Z}$.

$$\sum_{I\subset\mathbb{Z}} \rho^{\mathrm{diam}(I\cup\{m\})} \leq C(\rho) \tag{11.10}$$

$$\sum_{I,I'\subset\mathbb{Z}} \rho^{\mathrm{diam}(I)+\mathrm{diam}(I'\cup\{m\})+\delta(I,I')} \leq C(\rho) \tag{11.11}$$

$$\sum_{I,I'\subseteq\{\alpha,\dots,\beta\}} \rho^{\mathrm{diam}(I)+\mathrm{diam}(I')+\delta(I,I')} \leq C(\rho)\sharp(\{\alpha,...,\beta\}) \tag{11.12}$$

$$\sum_{I,I',I''\subset\mathbb{Z}} \rho^{\mathrm{diam}(I)+\mathrm{diam}(I')+\delta(I,I')+\mathrm{diam}(I''\cup\{m\})+\delta(I\cup I',I'')} \leq C(\rho) \tag{11.13}$$

$$\sum_{I,I',I''\subseteq\{\alpha,\dots,\beta\}} \rho^{\mathrm{diam}(I)+\mathrm{diam}(I')+\delta(I,I')+\mathrm{diam}(I'')+\delta(I\cup I',I'')} \leq C(\rho)\sharp(\{\alpha,...,\beta\}). \tag{11.14}$$

Proof. The proof of (11.8) and (11.9) are direct. In (11.10), (11.11) and (11.13), we may assume that $m = 0$. For (11.10), we remark that, if $I = [a, b]$, then $\mathrm{diam}(I \cup \{0\}) \geq \frac{1}{2}(|a| + |b|)$. For (11.11) and (11.13), we remark that, if $\rho \in]0, 1[$, there is $C(\rho) > 0$ such that, for each interval J of $\mathbb{Z}$, we have

$$\sum_{I \subset \mathbb{Z}} \rho^{\mathrm{diam}(I) + \delta(I,J)} \leq C(\rho)(1 + \mathrm{diam}(J))$$

(the sum is taken on all intervals I of $\mathbb{Z}$). By this inequality, (11.11) follows from (11.10) and (11.13) from (11.11). The point (11.12) (resp.11.14) is an easy consequence of (11.11) (resp. (11.13)). For example, in (11.12), we can decompose the LHS as a sum of terms S_m ($\alpha \leq m \leq \beta$), S_m being the sum of all terms corresponding to triples (I, I', I'') such that the lower bound of I' is m.

Proof of Theorem 1.4. For each finite subset Λ of $\mathbb{Z}^d$, let $\psi_\Lambda(x, y, t, h)$ be the function defined by Theorem 1.1, and set

$$X_\Lambda(t, h, \varepsilon) = \frac{\partial}{\partial t} \ln \left[\mathrm{Tr}(e^{-tH_{\Lambda,\varepsilon}}) \right] = \frac{\int_{\mathbb{R}^{p|\Lambda|}} e^{-\psi_\Lambda(x,x,t,h)} \frac{\partial \psi_\Lambda}{\partial t}(x, x, t, h)\, dx}{\int_{\mathbb{R}^{p|\Lambda_n|}} e^{-\psi_\Lambda(x,x,t,h)}\, dx}.$$

Theorem 1.4 will be a consequence of the following Proposition.

Proposition 11.3. *With these notations, there exists a constant $\sigma_1 > 0$ and a function $t \to F(t)$, bounded on each compact of $]0, +\infty[$, with the following properties. We can write, for each box Λ of $\mathbb{Z}^d$, split into two boxes Λ_1 and Λ_2 as in (11.1), for each $t > 0$, h and ε satisfying conditions of type (1.12),*

$$|X_\Lambda(t, h, \varepsilon) - X_{\Lambda_1}(t, h, \varepsilon) - X_{\Lambda_2}(t, h, \varepsilon)| \leq F(t)\ \sharp(\Lambda_\perp). \tag{11.15}$$

Proof. With the notations recalled in the beginning of this section, we set $\varphi_\theta(x, t) = \psi_\Lambda(x, x, t, h, \varepsilon, \theta)$, and

$$F(\theta) = \frac{\int_{\mathbb{R}^{p|\Lambda|}} e^{-\phi_\theta(x)} \frac{\partial \phi_\theta}{\partial t}(x)\, dx}{\int_{\mathbb{R}^{p|\Lambda|}} e^{-\phi_\theta(x)}\, dx}.$$

We can write

$$X_\Lambda - X_{\Lambda_1} - X_{\Lambda_2} = F(0) - F(1)$$

By computations, similar to those of Theorem 1.2, we find that F'_θ, the derivative of F w.r.t. θ, satisfies $F'_\theta(\theta) = F_1(\theta) + F_2(\theta)$, where

$$F_1(\theta) = \frac{\int_{\mathbb{R}^{p|\Lambda|}} e^{-\phi_\theta(x)} \frac{\partial^2 \phi_\theta(x)}{\partial t \partial \theta}\, dx}{\int_{\mathbb{R}^{p|\Lambda|}} e^{-\phi_\theta(x)}\, dx}$$

and, (defining $cov(A, B)$ for two operators A and B by (9.6) and identifying a function f and the operator of multiplication by f, which allows us to use the notation $cov(f, g)$ for two functions f and g)

$$F_2(\theta) = \frac{1}{2} cov\left(\frac{\partial \phi_\theta}{\partial t}, \frac{\partial \phi_\theta}{\partial \theta}\right). \tag{11.16}$$

Estimation of $F_1(\theta)$. Following (1.19), with V replaced by V_θ, and differentiating with respect to θ, we obtain

$$|\partial_t \partial_\theta \phi_\theta(x, t)| \leq \frac{h^2}{2} \sum_{\lambda \in \Lambda} |\Big(\Delta_{x_\lambda} \partial_\theta \psi_\theta\Big)(x, x, t)| + h^2 \sum_{\lambda \in \Lambda} |\nabla_{x_\lambda} \partial_\theta \psi_\theta(x, x, t)|\ |\nabla_{x_\lambda} \psi_\theta(x, x, t)| + |\partial_\theta V_\theta|.$$

We see (using Proposition 7.1, inequalities (7.3) and (7.5)) that the first two terms, in the RHS of the inequality above, are bounded, if $ht \leq \sigma_0$ and ε is small enough, by

$$Ch^2(t + t^2) \sum_{\lambda \in \Lambda} \varepsilon^{\frac{1}{2}|\lambda_1 - m|} \leq 4Ch^2(t + t^2)\ \sharp(\Lambda_\perp),$$

where C is independent of all the parameters. We remark that

$$\partial_\theta V_\theta = (V_{\Lambda_1} \oplus V_{\Lambda_2}) - V_\Lambda = \sum_{\lambda\in\Lambda_1,\mu\in\Lambda_2} \varepsilon^{|\lambda-\mu|} B_{\lambda-\mu}(x_\lambda, x_\mu)$$

and , if Λ_1 and Λ_2 are defined in (11.1), that we have $\lambda_1 \leq m \leq \mu_1$ and that $(\lambda_2, ..., \lambda_d)$ and $(\mu_2, ..., \mu_d)$ are in $\Lambda_\perp$ if λ is in Λ_1 and μ in Λ_2. Therefore

$$|\partial_\theta V_\theta(x)| \;\leq\; C \sum_{\lambda\in\Lambda_1,\mu\in\Lambda_2} \varepsilon^{|\lambda-\mu|} \;\leq\; 4^d \; C\sharp(\Lambda_\perp)$$

if $\varepsilon \leq 2^{-d}$. Hence $|\frac{\partial^2\phi_\theta(x)}{\partial t\partial\theta}| \leq C(1+th^2)\sharp(\Lambda_\perp)$ and therefore $|F_1(\theta)| \leq C(1+th^2)\ \sharp(\Lambda_\perp)$, provided that $ht \leq \sigma_0$ and ε are small enough.

Estimation of $F_2(\theta)$. By (1.19), setting $\varphi(x, t\theta) = \psi_\Lambda(\theta, x, x, t, h, \varepsilon)$, we have, omitting h and ε,

$$\frac{\partial\phi(x,t,\theta)}{\partial t} = \frac{h^2}{2}(\Delta_x\psi_\Lambda)(\theta,x,x,t) - \frac{h^2}{2}|(\nabla_x\psi_\Lambda(\theta,x,x,t)|^2 + V_\theta(x). \tag{11.17}$$

Using the cluster decomposition for ψ_θ, we get, by Proposition 5.1 ii),

$$\psi_\Lambda(\theta,x,y,t) = \psi_\Lambda(\theta,0,y-x,t) + \sum_{Q\subseteq\Lambda}(T_Q\psi_\Lambda)(\theta,x,y,t).$$

Now, we take the derivatives of this equality with respect to the variables x_λ ($\lambda \in \Lambda$), we restrict then to the diagonal, and we report into (11.17), we take also the derivative with respect to θ, and we report both in the expression (11.16) of $F_2(\theta)$. We remark that, if a function does not depend on x, its correlation with any other one vanishes. In order to write more shortly what we obtain, we introduce the following notations:

$$v_\lambda(t) = -(\nabla_{x_\lambda}\psi_\theta)(0,0,t) \qquad f_{\lambda,Q}(x,t) = (\Delta_{x_\lambda}T_Q\psi_\theta)(x,x,t) \qquad g_{\lambda,Q}(x,t) = (\nabla_{x_\lambda}T_Q\psi_\theta)(x,x,t)$$

$$h_Q(x,t) = (T_Q\frac{\partial}{\partial\theta}\psi_\theta)(x,x,t).$$

Thus, we obtain:

$$\begin{aligned} F_2(\theta) = \frac{h^2}{4}\sum_{\substack{\lambda\in\Lambda\\ Q,Q'\subseteq\Lambda}} cov(f_{\lambda,Q}, h_{Q'}) \; - h^2 \sum_{\substack{\lambda\in\Lambda\\ Q,Q'\subseteq\Lambda}} v_\lambda(t)\ .\ cov(g_{\lambda,Q}\ ,\ h_{Q'}) - \\ -\frac{h^2}{2}\sum_{\substack{\lambda\in\Lambda\\ Q,Q',Q''\subseteq\Lambda}} cov\Big(g_{\lambda,Q}.g_{\lambda,Q'}\ ,\ h_{Q''}\Big) + \sum_{Q,Q'\subseteq\Lambda} cov(T_QV_\theta, h_{Q'}). \end{aligned} \tag{11.18}$$

Now, we shall estimate all the terms. By Proposition 5.1 i), $f_{\lambda,Q}$, $g_{\lambda,Q}$ and h_Q depend only on x_Q. By a combination of the two points of Proposition 6.1, we have, if $ht \leq \sigma_0$ and $\varepsilon \leq \varepsilon_0(\frac{1}{2})$,

$$|f_{\lambda,Q}(x,t)| + |g_{\lambda,Q}(x,t)| \leq Ct\varepsilon^{\frac{1}{4}(\mathrm{diam}(Q)+\delta(\lambda,Q))} \tag{11.19}$$

By Theorem 1.1, we can write $|v_\lambda(t)| \leq Ct$. By Proposition 5.3, we can write, when $\varepsilon^{3/4} \leq 2^{-d}$,

$$|T_QV_\theta| \leq \begin{cases} C\varepsilon^{\frac{1}{4}\mathrm{diam}(Q)} & \text{if} \quad \sharp(Q) \geq 2 \\ C(1+|x_Q|) & \text{if} \quad \sharp(Q) = 1 \end{cases} \tag{11.20}$$

By a combination of the points (7.7) and (7.10) of Proposition 7.2, we have, if $ht \leq \sigma_0$ and $\varepsilon \leq \varepsilon_0(\frac{1}{2})$, for each box Q of Λ,

$$|h_Q(x,t)| \leq Ct\varepsilon^{\frac{1}{4}\big(\delta(Q,P_1(m))+\mathrm{diam}(Q)\big)}. \tag{11.21}$$

We apply Proposition 9.1 to estimate the four correlations in the expression of $F_2(\theta)$. The hypothesis (9.1) is satisfied with $m_1 = m_2 = 0$, excepted for the term $cov(T_Q V_\theta, h_{Q'})$ when Q is reduced to a single point: then (9.1) is satisfied with $m_1 = 1$ and $m_2 = 0$. Thus Proposition 9.1 is applicable, with $\gamma = \frac{1}{2}$, to all our correlations under the conditions (1.12), if we choose ε_0 and σ_1 small enough, (and we can choose them such that (11.19), (11.20)and (11.21) are valid under the same conditions). Under these conditions, we have, by Proposition 9.1a), (11.19) and (11.21),

$$\frac{h^2}{4}|cov(f_{\lambda,Q}, h_{Q'})| \leq th^2 \ (M(t)\varepsilon^{\frac{1}{2}} \ \sharp(Q) \ \|f_{\lambda,Q}\| \ \|h_{Q'}\| \ \leq \ Ch^2 t(M(t)\varepsilon^{\frac{1}{5}H(\lambda,Q,Q')}$$

where

$$H(\lambda, Q, Q') = \mathrm{diam}(Q) + \mathrm{diam}(Q') + \delta(Q, Q') + \delta(\lambda, Q) + \delta(Q', P_1(m))$$

By Lemma 11.2, for each ρ in $]0, 1[$, there exists $C(\rho) > 0$, (which we can consider as an increasing function of ρ), such that, for each box Λ with the notation (11.1), for each integer m, we have

$$\sum_{\substack{Q,Q'\subseteq\Lambda \\ \lambda\in\Lambda}} \rho^{H(\lambda,Q,Q')} \ \leq \ C(\rho) \ \sharp(\Lambda_\perp).$$

Applying this inequality to $\rho = M(t)\varepsilon^{\frac{1}{5}}$, we obtain, under the condition (1.12),

$$\frac{h^2}{4} \sum_{\substack{\lambda\in\Lambda \\ Q,Q'\subseteq\Lambda}} |cov(f_{\lambda,Q}, h_{Q'})| \leq Cth^2 \sharp(\Lambda_\perp).$$

The other terms in the expression (11.18) of $F_2(\theta)$ are bounded similarly, excepted the terms $cov(T_Q V_\theta, h_{Q'})$ such that $\sharp(Q) = 1$, for which we need Proposition 9.1 with $m_1 = 1$, $m_2 = 0$, and $\gamma = \frac{1}{2}$. The function $K(t, 1, 0, \sharp(Q))$ appearing in (9.4) is independent of $\sharp(Q')$ since $m_2 = 0$. If $t > 0$, and if h and ε satisfy (1.12), we obtain,

$$|cov(T_{\{\lambda\}} V_\theta, h_{Q'})| \leq G(t) \ \ (M(t)\varepsilon^{\frac{1}{4}})^{H(\lambda,Q')},$$

where $G(t) = K(t, 1, 0, 1)C^2$, C being the constant of (11.20) and (11.21), and $H(\lambda, Q') = \delta(\lambda, Q') + \mathrm{diam}(Q') + \delta(Q', P_1(m))$. By Lemma 11.2, there is an increasing function $\rho \to C(\rho)$ on $]0, 1[$ such that

$$\sum_{\lambda\in\Lambda, Q'\subseteq\Lambda} \rho^{H(\lambda,Q')} \ \leq \ C(\rho) \ \sharp(\Lambda_\perp).$$

If we apply that to $\rho = M(t)\varepsilon^{\frac{1}{4}}$, we obtain, under the condition (1.12),

$$\sum_{\lambda\in\Lambda, Q\subseteq\Lambda} |cov(T_{\{\lambda\}} V_\theta, h_{Q'})| \leq G(t)C(\frac{1}{2}) \ \sharp(\Lambda_\perp).$$

The Proposition is proved.

End of the proof of Theorem 1.4. Using Proposition 11.1 and the same arguments as Sjöstrand [17, Section 8, p.45-46], we conclude that, if $\Lambda_n = \{-n, \ldots, n\}^d$, and if the hypotheses (1.12) are satisfied, $\frac{X_{\Lambda_n}}{\sharp\Lambda_n}$ has a limit $U(t, h, \varepsilon)$ and that we can write, with another $F(t)$,

$$|U(t, h, \varepsilon) - \frac{X_{\Lambda_n}}{\sharp\Lambda_n}| \leq \frac{F(t)}{n}.$$

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